Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 9713521, 17 pages
https://doi.org/10.1155/2021/9713521

Research Article

European Option Pricing Formula in Risk-Aversive Markets

Shujin Wu and Shiyu Wang

Key Laboratory of Advanced Theory and Application in Statistics and Data Science, MOE School of Statistics, East China Normal University, Shanghai 200062, China

Correspondence should be addressed to Shujin Wu; sjwu@stat.ecnu.edu.cn

Received 18 May 2021; Accepted 20 July 2021; Published 31 July 2021

Academic Editor: Bosheng Song

In this study, using the method of discounting the terminal expectation value into its initial value, the pricing formulas for European options are obtained under the assumptions that the financial market is risk-aversive, the risk measure is standard deviation, and the price process of underlying asset follows a geometric Brownian motion. In particular, assuming the option writer does not need the risk compensation in a risk-neutral market, then the obtained results are degenerated into the famous Black–Scholes model (1973); furthermore, the obtained results need much weaker conditions than those of the Black–Scholes model. As a by-product, the obtained results show that the value of European option depends on the drift coefficient μ of its underlying asset, which does not display in the Black–Scholes model only because $\mu = r$ in a risk-neutral market according to the no-arbitrage opportunity principle. At last, empirical analyses on Shanghai 50 ETF options and S&P 500 options show that the fitting effect of obtained pricing formulas is superior to that of the Black–Scholes model.

1. Introduction

The option pricing theory began in 1900 when the French mathematician Louis Bachelier deduced an option pricing formula under the assumption that underlying asset prices follow a Brownian motion with zero drift. Since then, lots of researchers have contributed to the theory. Black and Scholes [1] present the very famous option pricing formula (i.e., Black–Scholes model) in a risk-neutral market and according to the no-arbitrage opportunity principle. Merton [2] shows the Black–Scholes-type model can be derived from weaker assumptions than in their original formulation and present some pricing methods for non-European options. Bakshi et al. [3] first derive an option pricing model that allows volatility, interest rates, and jumps to be stochastic. Gârleanu et al. [4] model demand-pressure effects on option prices. The model shows that demand pressure in one option contract increases its price by an amount proportional to the variance of the unhedgeable part of the option. Cai and Kou [5] propose a jump diffusion model for asset prices whose jump sizes have a mixed-exponential distribution, which is a weighted average of exponential distributions but with possibly negative weights, and then they extend the analytical tractability of the Black–Scholes model to alternative models. Bernarda and Czadob [6] investigate the pricing of basket options and more generally of complex exotic contracts depending on multiple indices. Their approach assumes that the underlying assets evolve as dependent $\mathrm{GARCH}(1, 1)$ processes. The dependence among the assets is modeled using a copula based on pair copula constructions. Bandi and Bertsimas [7] combine robust optimization and the idea of ε−arbitrage to propose a tractable approach to price a wide variety of options. Bao et al. [8] present a method that there is a possibility to get statistical arbitrage from Black–Scholes's option price.

In the last five years, there are still many researchers contributing to the theory of option pricing. Moretto et al. [9] study option pricing under deformed Gaussian distributions. Leippold and Scharer [10] develop a stochastic liquidity model, and they investigate discrete-time option pricing with stochastic liquidity. Hoka and Chanb [11] develop an option pricing method based on Legendre series expansion of the density function, and approximation formulas for pricing European type options are derived.

Davison and Mamba [12] obtain a solution of the Black–Scholes equation with a nonsmooth boundary condition using symmetry methods. Willems [13] derives a series expansion for the price of a continuously sampled arithmetic Asian option in the Black–Scholes setting. The expansion is based on polynomials that are orthogonal with respect to the log-normal distribution. More literature studies can refer to Liu et al. [14], Friz et al. [15], Dubinsky et al. [16], Huh [17], Liu et al. [18], Siddiqi [19] and their studies.

Although there are a huge number of literature studies on option pricing, they almost assume the financial markets are risk-neutral and complete, especially since Black and Scholes [1]. However, according to the theory and empirical analysis of risk, real financial markets are risk-aversive and incomplete. That is, investors need risk compensation for risky assets, and many risky assets cannot be duplicated by any portfolio constructed in real financial markets.

In this study, using the method of discounting the terminal expectation value into its initial value, we obtain European option pricing formula under the assumptions that the financial market is risk-aversive, the risk measure is standard deviation, and the price process of underlying asset follows a geometric Brownian motion. In particular, if the option writer does not need the risk compensation in a risk-neutral market, then our obtained results are degenerated into the Black–Scholes model [1]; furthermore, our obtained results need much weaker conditions than those in the Black–Scholes model. At last, we take the Shanghai 50 ETF options, the first floor option in the Chinese financial market, and S&P 500 options as samples to compare the fitting effect. The empirical analyses show that the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

2. The Black–Scholes Formula

In this study, we will investigate European option pricing and compare our results with those of Black and Scholes [1]. Thus, we first retell the main results of Black & Scholes [1].

Black and Scholes [1] present nine assumptions in the market for the security and for the option and then obtain their famous option pricing formula.

Assumption 1. Security price satisfies a geometric Brownian motion (GBM) model, where its drift coefficient and diffusion coefficient are constant through time. That is, the security price satisfies stochastic differential equation:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}B_t, \tag{1}$$

where μ and σ are constant and $\sigma > 0$.

Assumption 2. The short-term interest rate r is known and is constant through time. That is, the risk-free bond price satisfies ordinary differential equation:

$$\frac{\mathrm{d}P_t}{P_t} = r\mathrm{d}t, \tag{2}$$

where r is constant.

Assumption 3. The security pays no dividends or other distributions.

Assumption 4. There are no transaction costs in buying or selling the security or the option.

Assumption 5. The security can be continuously transacted.

Assumption 6. The amount of security can be arbitrarily divided.

Assumption 7. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

Assumption 8. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

Assumption 9. There is no-arbitrage opportunity.

When the above assumptions all hold, Black and Scholes [1] derived the pricing formula for European options, which is the Black–Scholes model.

Theorem 1 (see [1]). *If Assumptions 1 to 9 hold, then the values of European call option and European put option follow as*

$$C(S_0, K, r, \sigma, \tau) = S_0\Phi(d_2) - Ke^{-r\tau}\Phi(d_1), \tag{3}$$

$$P(S_0, K, r, \sigma, \tau) = Ke^{-r\tau}\Phi(-d_1) - S_0\Phi(-d_2), \tag{4}$$

where S_0 is the initial price of underlying asset, K is the strike price of option, r is the short-term interest rate, σ is the diffusion coefficient of underlying asset, τ is the left expiration time of option, $\Phi(\cdot)$ is the cumulative density function of standard normal distribution, and

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right), \\ d_2 &= d_1 + \sigma\sqrt{\tau}. \end{aligned} \tag{5}$$

3. European Option Pricing in Risk-Aversive Markets

Black–Scholes model and its modified versions have some defects. In fact, because real financial markets are incomplete, an option may not be duplicated constantly, so its value deduced by the asset duplication method and no-arbitrage principle may lose the deductive basis. On the other hand, real financial markets are risk-aversive. Option seller undertakes the total risk and option buyer has no any risk, so option seller needs a reasonable risk compensation according to the theory of risk. In this section, we will deduce the option pricing formula in risk-aversive markets only under three assumptions, i.e., Assumptions 1, 2, and 4 in

Section 2, which is reasonably far more than the Black–Scholes model and its modified versions.

In risk-aversive markets, assume the price process of some risky asset X by $\{X_t, t \geq 0\}$. Then the value of European call option at expiration time T with underlying asset X and strike price K follows as

$$(X_T - K)^+, \quad (6)$$

here and in the sequel the operator $(\cdot)^+ = \max\{0, \cdot\}$. The value of European put option at expiration time T with underlying asset X and strike price K follows as

$$(K - X_T)^+. \quad (7)$$

Note that no matter call option or put option, it is always its seller undertakes the total risk and its buyer has no any risk. According to risk theory, the seller reasonably requires some risk compensation $\lambda\rho(X)$, where $\lambda \geq 0$ is the risk-compensation coefficient, $\rho(\cdot)$ is the risk measure, and X is the risk size. After the seller has received the reasonable risk compensation, the seller takes risky asset as equivalent risk-free bond, so it follows from (6) that the value at time t of European call option with underlying asset X, strike price K, and expiration time T follows as

$$\begin{aligned} C_t(X, K, T, r, \lambda) = e^{-r(T-t)}\{E[(X_T - K)^+] \\ + \lambda\rho((X_T - K)^+)\}, \quad \forall t \leq T, \end{aligned} \quad (8)$$

where $r \geq 0$ is the risk-free rate during $[t, T]$. Analogically, it follows from (7) that the value of European put option with underlying asset X, strike price K, and expiration time T follows as

$$\begin{aligned} P_t(X, K, T, r, \lambda) = e^{-r(T-t)}\{E[(K - X_T)^+] \\ + \lambda\rho((K - X_T)^+)\}, \quad \forall t \leq T. \end{aligned} \quad (9)$$

In conclusion, we obtain the following proposition from (8) and (9).

Proposition 1. *In risk-aversive market, assume that the risk measure is $\rho(\cdot)$ and the risk-compensation coefficient is $\lambda \geq 0$, and assuming a European option with underlying asset X, strike price K, and expiration time T, then its call-option value at time t follows as*

$$C_t(X, K, T, r, \lambda) = e^{-r(T-t)}\{E[(X_T - K)^+] + \lambda\rho((X_T - K)^+)\}, \quad (10)$$

and its put-option value at time t follows as

$$P_t(X, K, T, r, \lambda) = e^{-r(T-t)}\{E[(K - X_T)^+] + \lambda\rho((K - X_T)^+)\}, \quad (11)$$

where $r \geq 0$ is the risk-free rate during $[t, T]$ and $t \leq T$.

In order to obtain a closed-form solution to Proposition 1, in the following, we always assume the price process of underlying asset follows some geometric Brownian motion model (1), and the risk measure is the standard deviation, i.e., $\rho(Z) = \mathrm{std}(Z)$ for any risk variable Z.

Using Proposition 1, we can deduce the value of European call option in risk-aversive markets and under Assumptions 1, 2, and 4.

Theorem 2. *In a risk-aversive market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0, risk-free interest rate is r through the time, the risk-compensation factor is $\lambda \geq 0$, and the risk measure is standard deviation, then the value of European call option with strike price K and left expiration time τ follows as*

$$\begin{aligned} C(S_0, K, r, \sigma, \tau, \lambda) = S_0 e^{(\mu-r)\tau}\Phi(d_2) - Ke^{-r\tau}\Phi(d_1) + \lambda e^{-r\tau} \cdot \mathrm{sqrt}\Big\{S_0^2 e^{2\mu\tau}\Big(e^{\sigma^2\tau}\Phi(d_3) - \Phi^2(d_2)\Big) - KS_0 e^{\mu\tau}\Phi(d_2)\Phi(-d_1) \\ + K^2\Phi(d_1)\Phi(-d_1)\Big\}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{S_0}{K} + \left(\mu - \frac{1}{2}\sigma^2\right)\tau\right), \\ d_m &= d_1 + (m-1)\sigma\sqrt{\tau}, \quad m = 2, 3. \end{aligned} \quad (13)$$

The Proof of Theorem 2 refers to Appendix A.

If a financial market is risk-neutral, then investors treat expected return and deterministic return equally, so expectation yield μ equals risk-free yield r, that is, $\mu = r$. Otherwise, if $\mu \neq r$, there will exist arbitrage opportunity. Furthermore, in a risk-neutral financial market, the risk-compensation factor equals zero, that is, $\lambda = 0$. Thus, it yields from Theorem 1 that we have the following corollary.

Corollary 1. *In a risk-neutral market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0, and risk-free yield is r through the time, then the value of European call option with strike price K and left expiration time τ follows as*

$$C(S_0, K, r, \sigma, \tau) = S_0\Phi(d_2) - Ke^{-r\tau}\Phi(d_1), \quad (14)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right),$$
$$d_2 = d_1 + \sigma\sqrt{\tau}. \quad (15)$$

In the following, we will deduce the pricing formula of European put option. Using Proposition 1, we will construct the pricing formula of European put option in risk-aversive markets and under Assumptions 1, 2, and 4.

Theorem 3. *In a risk-aversive market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0, risk-free yield is r through the time, the risk-compensation factor is $\lambda \geq 0$, and the risk measure is standard deviation, then the value of European put option with strike price K and left expiration time τ follows as*

$$P(S_0, K, r, \sigma, \tau, \lambda) = Ke^{-r\tau}\Phi(-d_1) - S_0e^{(\mu-r)\tau}\Phi(-d_2) + \lambda e^{-r\tau} \cdot \text{sqrt}\left\{S_0^2e^{2\mu\tau}\left(e^{\sigma^2\tau}\Phi(-d_3) - \Phi^2(-d_2)\right) - KS_0e^{\mu\tau}\Phi(-d_2)\Phi(d_1) + K^2\Phi(-d_1)\Phi(d_1)\right\}, \quad (16)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{S_0}{K} + \left(\mu - \frac{1}{2}\sigma^2\right)\tau\right),$$
$$d_m = d_1 + (m-1)\sigma\sqrt{\tau}, \quad m = 2, 3. \quad (17)$$

The Proof of Theorem 3 refers to Appendix B.

According to the analysis before Corollary 1, if a financial market is risk-neutral, then $\mu = r$ and $\lambda = 0$. Furthermore, it yields from Theorem 3 that we have the following corollary.

Corollary 2. *In a risk-neutral market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0, and risk-free yield is r through the time, then the value of European put option with strike price K and left expiration time τ follows as*

$$P(S_0, K, r, \sigma, \tau) = Ke^{-r\tau}\Phi(-d_1) - S_0\Phi(-d_2), \quad (18)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right),$$
$$d_2 = d_1 + \sigma\sqrt{\tau}. \quad (19)$$

Remark 1. Although Corollaries 1 and 2 obtain the same values as those in Theorem 1 for European call option and European put option, Corollaries 1 and 2 need much weaker conditions than those of Theorem 1. That is, Corollaries 1 and 2 improve Theorem 1.

In fact, if Assumptions 1, 2, and 4 in Section 2 hold, Corollaries 1 and 2 hold. However, the conditions that Theorem 1 (i.e., the Black–Scholes model) holds are Assumptions 1 to 9 in Section 2.

4. Empirical Analysis of Shanghai 50 ETF Options

In the section, we will present empirical analysis on Shanghai 50 ETF options, the first floor option in the Chinese financial market, and use the data of September 3 and 4, 2018, to compare the fitting effect of our pricing model and the Black–Scholes pricing model. All used data come from the CSMAR Database, which includes actual option price C or P, trading date t, exercise date T, strike price K, and current price of underlying asset S_0. In addition, the database also includes the historical volatility of Shanghai 50 ETF $\sigma = 0.1994$ on September 3, 2018, and $\sigma = 0.2005$ on September 4, 2018, and the 1-year deposit benchmark interest rate of Chinese Central Bank $r = 1.5\%$, which is chosen as the reference level of risk-free interest rate in the Chinese financial market. All data analyses in the following are worked out by the software MATLAB R2018b.

4.1. Parameter Estimation. There are 48 call options on September 3, 2018, and the current price of underlying asset $S_0 = 2.512$. We take the annual average return rate of the last month as the drift coefficient of the underlying asset μ. According to simple computation, we obtain $\mu = 2.557\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2018/09/03$, and then we work out d_1, d_2, and d_3 by Theorem 2; see Table 1 for detailed data, where C is the actual closing price of call option, T is the expiration time of call option, K is the strike price of call option, S_0 is the initial price of underlying asset, τ is the left expiration time of call option, and d_1, d_2, and d_3 are the parameters in Theorem 2. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = 0.0077963$ by the least square method with $R^2 = 0.7058$.

There are 60 put options on September 3, 2018, and the current price of underlying asset $S_0 = 2.512$. Similarly, we obtain $\mu = 2.93\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2018/09/03$, and then we work out d_1, d_2, and d_3 by Theorem 3; see Table 2 for detailed data, where P is

Table 1: Parameter estimation of d_1, d_2, and d_3 for call options.

C	T	K	S_0	τ	d_1	d_2	d_3
0.0030	2018/09/26	2.75	2.512	0.0630	−1.8013	−1.7512	−1.7012
0.0073	2018/09/26	2.70	2.512	0.0630	−1.4347	−1.3847	−1.3346
0.0146	2018/09/26	2.65	2.512	0.0630	−1.0613	−1.0112	−0.9612
0.0150	2018/10/24	2.75	2.512	0.1397	−1.2038	−1.1293	−1.0547
0.0159	2018/12/26	2.95	2.512	0.3123	−1.4263	−1.3149	−1.2035
0.0220	2018/12/26	2.90	2.512	0.3123	−1.2729	−1.1615	−1.0501
0.0228	2018/10/24	2.70	2.512	0.1397	−0.9576	−0.8831	−0.8086
0.0275	2018/09/26	2.60	2.512	0.0630	−0.6807	−0.6307	−0.5806
0.0287	2018/12/26	2.85	2.512	0.3123	−1.1169	−1.0054	−0.8940
0.0350	2018/10/24	2.65	2.512	0.1397	−0.7068	−0.6323	−0.5578
0.0372	2018/12/26	2.80	2.512	0.3123	−0.9580	−0.8466	−0.7352
0.0451	2018/09/26	2.55	2.512	0.0630	−0.2928	−0.2427	−0.1927
0.0484	2018/12/26	2.75	2.512	0.3123	−0.7964	−0.6849	−0.5735
0.0500	2018/10/24	2.60	2.512	0.1397	−0.4513	−0.3767	−0.3022
0.0601	2018/12/26	2.70	2.512	0.3123	−0.6317	−0.5203	−0.4088
0.0695	2018/09/26	2.50	2.512	0.0630	0.1028	0.1529	0.2029
0.0698	2018/10/24	2.55	2.512	0.1397	−0.1908	−0.1162	−0.0417
0.0742	2019/03/27	2.80	2.512	0.5616	−0.7049	−0.5555	−0.4061
0.0756	2018/12/26	2.65	2.512	0.3123	−0.4640	−0.3525	−0.2411
0.0878	2019/03/27	2.75	2.512	0.5616	−0.5844	−0.4349	−0.2855
0.0938	2018/12/26	2.60	2.512	0.3123	−0.2930	−0.1816	−0.0701
0.0944	2018/10/24	2.50	2.512	0.1397	0.0749	0.1495	0.2240
0.0980	2018/09/26	2.45	2.512	0.0630	0.5064	0.5565	0.6066
0.1026	2019/03/27	2.70	2.512	0.5616	−0.4616	−0.3121	−0.1627
0.1149	2018/12/26	2.55	2.512	0.3123	−0.1188	−0.0073	0.1041
0.1190	2019/03/27	2.65	2.512	0.5616	−0.3365	−0.1870	−0.0376
0.1247	2018/10/24	2.45	2.512	0.1397	0.3460	0.4205	0.4950
0.1361	2018/09/26	2.40	2.512	0.0630	0.9184	0.9684	1.0185
0.1380	2019/03/27	2.60	2.512	0.5616	−0.2090	−0.0596	0.0899
0.1390	2018/12/26	2.50	2.512	0.3123	0.0589	0.1704	0.2818
0.1592	2018/10/24	2.40	2.512	0.1397	0.6226	0.6971	0.7717
0.1623	2019/03/27	2.55	2.512	0.5616	−0.0791	0.0704	0.2198
0.1671	2018/12/26	2.45	2.512	0.3123	0.2402	0.3517	0.4631
0.1770	2018/09/26	2.35	2.512	0.0630	1.3390	1.3891	1.4391
0.1861	2019/03/27	2.50	2.512	0.5616	0.0534	0.2029	0.3523
0.1968	2018/10/24	2.35	2.512	0.1397	0.9051	0.9796	1.0541
0.1981	2018/12/26	2.40	2.512	0.3123	0.4253	0.5367	0.6481
0.2110	2019/03/27	2.45	2.512	0.5616	0.1886	0.3381	0.4875
0.2209	2018/09/26	2.30	2.512	0.0630	1.7687	1.8187	1.8688
0.2297	2018/12/26	2.35	2.512	0.3123	0.6142	0.7256	0.8371
0.2360	2018/10/24	2.30	2.512	0.1397	1.1936	1.2681	1.3427
0.2421	2019/03/27	2.40	2.512	0.5616	0.3266	0.4761	0.6255
0.2660	2018/12/26	2.30	2.512	0.3123	0.8072	0.9186	1.0300
0.2704	2018/09/26	2.25	2.512	0.0630	2.2078	2.2578	2.3079
0.2767	2019/03/27	2.35	2.512	0.5616	0.4675	0.6169	0.7664
0.3074	2018/12/26	2.25	2.512	0.3123	1.0044	1.1158	1.2273
0.3163	2018/09/26	2.20	2.512	0.0630	2.6567	2.7068	2.7568
0.3450	2019/03/27	2.25	2.512	0.5616	0.7585	0.9079	1.0574

the actual closing price of put option, T is the expiration time of put option, K is the strike price of put option, S_0 is the initial price of underlying asset, τ is the left expiration time of put option, and d_1, d_2, and d_3 are the parameters in Theorem 3. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = 0.013306$ by the least square method with $R^2 = 0.6218$.

In addition, we obtain the estimation of risk-free rate $r = 1.768\%$ by minimizing the mean square error of the Black–Scholes model on September 3, 2018. Note that $r =$ 1.768% is a little higher than the 1-year deposit benchmark interest rate of Chinese Central Bank 1.5%, so it is very reasonable that we take $r = 1.768\%$ as the risk-free rate. Furthermore, it is far advantageous for the Black–Scholes model to improve its fitting effect.

4.2. Comparison of Pricing Effect. In the section, we will compare the fitting effect of our pricing formulas with the Black–Scholes model.

Table 2: Parameter estimation of d_1, d_2, and d_3 for put options.

P	T	K	S_0	τ	d_1	d_2	d_3
0.0015	2018/09/26	2.20	2.512	0.0630	2.6567	2.7068	2.7568
0.0027	2018/09/26	2.25	2.512	0.0630	2.2078	2.2578	2.3079
0.0055	2018/09/26	2.30	2.512	0.0630	1.7687	1.8187	1.8688
0.0115	2018/09/26	2.35	2.512	0.0630	1.3390	1.3891	1.4391
0.0165	2018/10/24	2.30	2.512	0.1397	1.1936	1.2681	1.3427
0.0205	2018/09/26	2.40	2.512	0.0630	0.9184	0.9684	1.0185
0.0247	2018/12/26	2.20	2.512	0.3123	1.2061	1.3175	1.4289
0.0255	2018/10/24	2.35	2.512	0.1397	0.9051	0.9796	1.0541
0.0324	2018/12/26	2.25	2.512	0.3123	1.0044	1.1158	1.2273
0.0343	2018/09/26	2.45	2.512	0.0630	0.5064	0.5565	0.6066
0.0371	2018/10/24	2.40	2.512	0.1397	0.6226	0.6971	0.7717
0.0432	2018/12/26	2.30	2.512	0.3123	0.8072	0.9186	1.0300
0.0528	2018/10/24	2.45	2.512	0.1397	0.3460	0.4205	0.4950
0.0531	2018/09/26	2.50	2.512	0.0630	0.1028	0.1529	0.2029
0.0565	2018/12/26	2.35	2.512	0.3123	0.6142	0.7256	0.8371
0.0608	2019/03/27	2.25	2.512	0.5616	0.7585	0.9079	1.0574
0.0730	2019/03/27	2.30	2.512	0.5616	0.6114	0.7609	0.9103
0.0733	2018/10/24	2.50	2.512	0.1397	0.0749	0.1495	0.2240
0.0734	2018/12/26	2.40	2.512	0.3123	0.4253	0.5367	0.6481
0.0791	2018/09/26	2.55	2.512	0.0630	−0.2928	−0.2427	−0.1927
0.0902	2019/03/27	2.35	2.512	0.5616	0.4675	0.6169	0.7664
0.0920	2018/12/26	2.45	2.512	0.3123	0.2402	0.3517	0.4631
0.0989	2018/10/24	2.55	2.512	0.1397	−0.1908	−0.1162	−0.0417
0.1074	2019/03/27	2.40	2.512	0.5616	0.3266	0.4761	0.6255
0.1117	2018/09/26	2.60	2.512	0.0630	−0.6807	−0.6307	−0.5806
0.1140	2018/12/26	2.50	2.512	0.3123	0.0589	0.1704	0.2818
0.1244	2019/03/27	2.45	2.512	0.5616	0.1886	0.3381	0.4875
0.1284	2018/10/24	2.60	2.512	0.1397	−0.4513	−0.3767	−0.3022
0.1401	2018/12/26	2.55	2.512	0.3123	−0.1188	−0.0073	0.1041
0.1483	2019/03/27	2.50	2.512	0.5616	0.0534	0.2029	0.3523
0.1498	2018/09/26	2.65	2.512	0.0630	−1.0613	−1.0112	−0.9612
0.1686	2018/12/26	2.60	2.512	0.3123	−0.2930	−0.1816	−0.0701
0.1750	2019/03/27	2.55	2.512	0.5616	−0.0791	0.0704	0.2198
0.1908	2018/09/26	2.70	2.512	0.0630	−1.4347	−1.3847	−1.3346
0.2003	2018/12/26	2.65	2.512	0.3123	−0.4640	−0.3525	−0.2411
0.2007	2018/10/24	2.70	2.512	0.1397	−0.9576	−0.8831	−0.8086
0.2009	2019/03/27	2.60	2.512	0.5616	−0.2090	−0.0596	0.0899
0.2278	2018/12/26	2.70	2.512	0.3123	−0.6317	−0.5203	−0.4088
0.2312	2019/03/27	2.65	2.512	0.5616	−0.3365	−0.1870	−0.0376
0.2370	2018/09/26	2.75	2.512	0.0630	−1.8013	−1.7512	−1.7012
0.2440	2018/10/24	2.75	2.512	0.1397	−1.2038	−1.1293	−1.0547
0.2635	2019/03/27	2.70	2.512	0.5616	−0.4616	−0.3121	−0.1627
0.2657	2018/12/26	2.75	2.512	0.3123	−0.7964	−0.6849	−0.5735
0.2828	2018/09/26	2.80	2.512	0.0630	−2.1613	−2.1112	−2.0612
0.2976	2019/03/27	2.75	2.512	0.5616	−0.5844	−0.4349	−0.2855
0.3089	2018/12/26	2.80	2.512	0.3123	−0.9580	−0.8466	−0.7352
0.3323	2018/09/26	2.85	2.512	0.0630	−2.5149	−2.4648	−2.4148
0.3330	2019/03/27	2.80	2.512	0.5616	−0.7049	−0.5555	−0.4061
0.3477	2018/12/26	2.85	2.512	0.3123	−1.1169	−1.0054	−0.8940
0.3810	2018/09/26	2.90	2.512	0.0630	−2.8623	−2.8123	−2.7622
0.3939	2018/12/26	2.90	2.512	0.3123	−1.2729	−1.1615	−1.0501
0.4315	2018/09/26	2.95	2.512	0.0630	−3.2039	−3.1538	−3.1037
0.4405	2018/12/26	2.95	2.512	0.3123	−1.4263	−1.3149	−1.2035
0.4810	2018/09/26	3.00	2.512	0.0630	−3.5396	−3.4896	−3.4395
0.5813	2018/09/26	3.10	2.512	0.0630	−4.1947	−4.1447	−4.0946
0.6809	2018/09/26	3.20	2.512	0.0630	−4.8290	−4.7789	−4.7289
0.7798	2018/09/26	3.30	2.512	0.0630	−5.4438	−5.3937	−5.3437
0.8793	2018/09/26	3.40	2.512	0.0630	−6.0402	−5.9901	−5.9401
0.9790	2018/09/26	3.50	2.512	0.0630	−6.6193	−6.5692	−6.5192
1.0807	2018/09/26	3.60	2.512	0.0630	−7.1821	−7.1320	−7.0820

4.2.1. Comparison of Pricing Effect on Call Options. There are 62 call options on September 4, 2018, and the current price of underlying asset $S_0 = 2.55$. Based on the estimated parameters $\mu = 2.557\%$, $\sigma = 0.2005$, $r = 1.768\%$, $\lambda = 0.0077963$ for call options, the call options of Shanghai 50 ETF on September 4, 2018, are priced by our obtained pricing formula in Theorem 3 and the Black–Scholes model, respectively; see Table 3 for detailed data, where T is the expiration time of call option, K is the strike price of call option, S_0 is the initial price of underlying asset, τ is the left expiration time of call option, d_1, d_2, and d_3 are the parameters in Theorem 3, C_1 is the value of call option computed by the Black–Scholes model, C_2 is the value of call option computed by Theorem 3, and C is the actual closing price of call option.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E\left(\widehat{\left|C_1 - C\right|}\right) &= 0.0065574, \mathrm{Var}\left(\widehat{\left|C_1 - C\right|}\right) = 0.0045044, \\ E\left(\widehat{\left|C_2 - C\right|}\right) &= 0.0032696, \quad \mathrm{Var}\left(\widehat{\left|C_2 - C\right|}\right) = 0.001929, \end{aligned} \tag{20}$$

where C is the actual closing price of call option, C_1 is the value of call option computed by the Black–Scholes model, and C_2 is the value of call option computed by Theorem 3. It is obvious that $E(\widehat{|C_2 - C|}) < E(\widehat{|C_1 - C|})$. In the following, we will support the statement by the hypothesis test (i.e., t-test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} H_0&: E\left(\left|C_2 - C\right|\right) - E\left(\left|C_1 - C\right|\right) \geq 0, \\ H_1&: E\left(\left|C_2 - C\right|\right) - E\left(\left|C_1 - C\right|\right) < 0, \end{aligned} \tag{21}$$

The t−statistics equals −5.9311 with degree of freedom 61, and its p value is 7.6261×10^{-8}. Thus, we accept H_1, i.e., $E(|C_2 - C|) < E(|C_1 - C|)$. That is, the prices of call options computed by Theorem 3 are far nearer to their actual prices than those computed by the Black–Scholes model.

4.2.2. Comparison of Pricing Effect on Put Options. There are 62 put options on September 4, 2018, and the current price of underlying asset $S_0 = 2.55$. Based on the estimated parameters $\mu = 2.557\%$, $\sigma = 0.2005$, $r = 1.768\%$, and $\lambda = 0.013306$ for put options, the put options of Shanghai 50 ETF on September 4, 2018, are priced by our pricing formula in Theorem 2 and Black–Scholes model, respectively; see Table 4 for detailed data, where T is the expiration time of put option, K is the strike price of put option, S_0 is the initial price of underlying asset, τ is the left expiration time of put option, d_1, d_2, and d_3 are the parameters in Theorem 3, P_1 is the value of put option computed by the Black–Scholes model, P_2 is the value of put option computed by Theorem 3, and P is the actual closing price of put option.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E\left(\widehat{\left|P_1 - P\right|}\right) &= 0.00836, \mathrm{Var}\left(\widehat{\left|P_1 - P\right|}\right) = 0.0056946, \\ E\left(\widehat{\left|P_2 - P\right|}\right) &= 0.0044808, \mathrm{Var}\left(\widehat{\left|P_2 - P\right|}\right) = 0.0029812, \end{aligned} \tag{22}$$

where P is the actual closing price of put option, P_1 is the value of put option computed by the Black–Scholes model, and P_2 is the value of put option computed by Theorem 3. It is obvious that $E(\widehat{|P_2 - P|}) < E(\widehat{|P_1 - P|})$. In the following, we will support the statement by the hypothesis test (i.e., t-test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} H_0&: E\left(\left|P_2 - P\right|\right) - E\left(\left|P_1 - P\right|\right) \geq 0, \\ H_1&: E\left(\left|P_2 - P\right|\right) - E\left(\left|P_1 - P\right|\right) < 0, \end{aligned} \tag{23}$$

The t−statistics equals −4.7567 with degree of freedom 61, and its p value is 6.2304×10^{-6}. Thus, we accept H_1, i.e., $E(|P_2 - P|) < E(|P_1 - P|)$. That is, the prices of put options computed by Theorem 3 are far nearer to their actual prices than those computed by the Black–Scholes model.

Therefore, our pricing formulas in Theorem 2 and Theorem 3 have less absolute errors than those of the Black–Scholes model for both call options and put options. That is, the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

5. Empirical Analysis of S&P 500 Options

In the section, we will present empirical analysis on S&P 500 options and use the data of April 1 and 2, 2019 to compare the fitting effect of our pricing model and the Black–Scholes pricing model. All used data come from the Chicago Board Options Exchange, which includes actual option price C or P, trading date t, exercise date T, strike price K, and current price of underlying asset S_0. We consider the out-of-money put and call options, which are more liquid and actively traded than in-the-money options. And observations with trading volume below average, prices less than \$0.5 or left expiration time less than 10 days or longer than 360 days are discarded. The annualized historical volatility $\sigma = 0.1174$ based on closing prices of the underlying asset over the last month. According to the Board of Governors of the Federal Reserve System (https://www.federalreserve.gov/releases/h15/data.htm), the annualized risk-free interest rate is 2.43% on April 1 and 2.42% on April 2, 2019.

5.1. Parameter Estimation. We consider the daily logarithmic returns of S&P 500 index closing prices from January 2015 to December 2018. The augmented Dickey–Fuller test shows the Dickey–Fuller statistic equals −10.283 with lag order 10 and p value is 0.01, which indicates that the time series of returns is stationary. Furthermore, the autocorrelogram shows the coefficients of autocorrelation mostly fall within double standard deviations (see Figure 1). Thus, we accept that the time series of daily logarithmic returns of S&P 500 index closing prices are stationary and independent. We take the annual average return rate based on 252

Table 3: Pricing results by our pricing formula and the Black–Scholes model for call options.

T	K	S_0	τ	d_1	d_2	d_3	C_1	C_2	C
2018/09/26	3.00	2.55	0.0603	−3.2949	−3.2457	−3.1965	0.0005	0.0000	0.0005
2018/12/26	2.40	2.55	0.3096	0.5586	0.6702	0.7817	0.2239	0.2087	0.2200
2018/12/26	2.35	2.55	0.3096	0.7473	0.8589	0.9705	0.2601	0.2451	0.2555
2018/09/26	2.90	2.55	0.0603	−2.6062	−2.5570	−2.5077	0.0017	0.0002	0.0007
2019/03/27	2.50	2.55	0.5589	0.1525	0.3024	0.4523	0.2079	0.1883	0.2036
2019/03/27	2.55	2.55	0.5589	0.0204	0.1703	0.3202	0.1816	0.1626	0.1773
2019/03/27	2.65	2.55	0.5589	−0.2362	−0.0863	0.0636	0.1364	0.1188	0.1353
2018/12/26	2.60	2.55	0.3096	−0.1589	−0.0473	0.0643	0.1107	0.0963	0.1067
2018/09/26	2.20	2.55	0.0603	3.0060	3.0552	3.1044	0.3548	0.3520	0.3510
2018/09/26	2.85	2.55	0.0603	−2.2529	−2.2036	−2.1544	0.0030	0.0006	0.0012
2019/03/27	2.20	2.55	0.5589	1.0053	1.1552	1.3051	0.4160	0.3955	0.4070
2018/12/26	2.30	2.55	0.3096	0.9401	1.0517	1.1632	0.2991	0.2846	0.2925
2018/10/24	2.35	2.55	0.1370	1.1108	1.1850	1.2592	0.2264	0.2166	0.2255
2018/10/24	2.30	2.55	0.1370	1.4006	1.4748	1.5490	0.2698	0.2610	0.2653
2018/10/24	2.70	2.55	0.1370	−0.7601	−0.6859	−0.6117	0.0356	0.0258	0.0300
2018/12/26	2.20	2.55	0.3096	1.3386	1.4501	1.5617	0.3840	0.3706	0.3817
2018/09/26	2.50	2.55	0.0603	0.4090	0.4582	0.5074	0.0905	0.0800	0.0879
2018/09/26	2.95	2.55	0.0603	−2.9535	−2.9042	−2.8550	0.0010	0.0001	0.0005
2018/12/26	2.95	2.55	0.3096	−1.2909	−1.1794	−1.0678	0.0229	0.0150	0.0197
2018/09/26	2.80	2.55	0.0603	−1.8933	−1.8441	−1.7948	0.0052	0.0015	0.0019
2018/12/26	2.85	2.55	0.3096	−0.9818	−0.8703	−0.7587	0.0371	0.0272	0.0341
2018/10/24	2.55	2.55	0.1370	0.0101	0.0843	0.1585	0.0901	0.0780	0.0854
2018/10/24	2.40	2.55	0.1370	0.8271	0.9013	0.9755	0.1862	0.1753	0.1837
2018/09/26	3.10	2.55	0.0603	−3.9610	−3.9118	−3.8626	0.0001	0.0000	0.0004
2018/12/26	2.70	2.55	0.3096	−0.4972	−0.3856	−0.2740	0.0733	0.0604	0.0700
2019/03/27	2.35	2.55	0.5589	0.5653	0.7152	0.8651	0.3016	0.2811	0.2985
2019/03/27	2.40	2.55	0.5589	0.4249	0.5748	0.7246	0.2679	0.2476	0.2628
2018/12/26	2.75	2.55	0.3096	−0.6616	−0.5501	−0.4385	0.0588	0.0469	0.0550
2019/03/27	2.25	2.55	0.5589	0.8554	1.0053	1.1552	0.3758	0.3552	0.3679
2019/03/27	2.80	2.55	0.5589	−0.6035	−0.4536	−0.3038	0.0856	0.0707	0.0810
2018/12/26	2.65	2.55	0.3096	−0.3296	−0.2181	−0.1065	0.0905	0.0768	0.0871
2018/09/26	2.30	2.55	0.0603	2.1029	2.1521	2.2014	0.2571	0.2528	0.2538
2018/09/26	3.60	2.55	0.0603	−6.9988	−6.9496	−6.9003	0.0000	0.0000	0.0004
2019/03/27	2.75	2.55	0.5589	−0.4833	−0.3334	−0.1835	0.1004	0.0846	0.0988
2019/03/27	2.30	2.55	0.5589	0.7088	0.8587	1.0086	0.3376	0.3170	0.3316
2018/09/26	2.40	2.55	0.0603	1.2383	1.2875	1.3368	0.1658	0.1583	0.1641
2018/09/26	2.25	2.55	0.0603	2.5494	2.5986	2.6479	0.3055	0.3022	0.3040
2019/03/27	2.60	2.55	0.5589	−0.1091	0.0408	0.1907	0.1578	0.1394	0.1549
2018/10/24	2.45	2.55	0.1370	0.5492	0.6234	0.6976	0.1497	0.1380	0.1461
2018/12/26	2.90	2.55	0.3096	−1.1377	−1.0262	−0.9146	0.0292	0.0203	0.0266
2018/10/24	2.50	2.55	0.1370	0.2770	0.3512	0.4254	0.1175	0.1054	0.1135
2018/10/24	2.75	2.55	0.1370	−1.0074	−0.9332	−0.8590	0.0252	0.0167	0.0195
2018/09/26	3.30	2.55	0.0603	−5.2311	−5.1819	−5.1327	0.0000	0.0000	0.0003
2018/12/26	2.50	2.55	0.3096	0.1927	0.3043	0.4158	0.1607	0.1455	0.1572
2018/12/26	2.45	2.55	0.3096	0.3738	0.4854	0.5969	0.1907	0.1754	0.1870
2018/10/24	2.60	2.55	0.1370	−0.2516	−0.1774	−0.1031	0.0675	0.0559	0.0623
2018/09/26	2.70	2.55	0.0603	−1.1545	−1.1053	−1.0560	0.0153	0.0081	0.0103
2018/12/26	2.80	2.55	0.3096	−0.8232	−0.7116	−0.6000	0.0469	0.0359	0.0445
2018/09/26	2.55	2.55	0.0603	0.0067	0.0559	0.1052	0.0621	0.0512	0.0590
2018/09/26	3.50	2.55	0.0603	−6.4265	−6.3773	−6.3280	0.0000	0.0000	0.0003
2018/09/26	2.65	2.55	0.0603	−0.7747	−0.7255	−0.6763	0.0253	0.0164	0.0206
2018/09/26	2.35	2.55	0.0603	1.6660	1.7152	1.7645	0.2101	0.2045	0.2077
2018/09/26	2.45	2.55	0.0603	0.8194	0.8686	0.9179	0.1254	0.1162	0.1232
2018/12/26	2.25	2.55	0.3096	1.1371	1.2487	1.3602	0.3405	0.3265	0.3330
2018/10/24	2.65	2.55	0.1370	−0.5083	−0.4340	−0.3598	0.0495	0.0387	0.0442
2018/09/26	2.75	2.55	0.0603	−1.5272	−1.4780	−1.4288	0.0090	0.0037	0.0048
2018/09/26	3.40	2.55	0.0603	−5.8376	−5.7884	−5.7392	0.0000	0.0000	0.0003
2019/03/27	2.70	2.55	0.5589	−0.3609	−0.2110	−0.0611	0.1173	0.1006	0.1162
2018/09/26	3.20	2.55	0.0603	−4.6060	−4.5568	−4.5076	0.0000	0.0000	0.0003
2018/12/26	2.55	2.55	0.3096	0.0152	0.1268	0.2383	0.1341	0.1192	0.1311
2018/09/26	2.60	2.55	0.0603	−0.3878	−0.3386	−0.2893	0.0405	0.0303	0.0377
2019/03/27	2.45	2.55	0.5589	0.2873	0.4372	0.5871	0.2367	0.2166	0.2319

Table 4: Pricing results by our pricing formula and the Black–Scholes model for put options.

T	K	S_0	τ	d_1	d_2	d_3	P_1	P_2	P
2018/12/26	2.50	2.55	0.3096	0.1927	0.3043	0.4158	0.0963	0.0839	0.0983
2019/03/27	2.30	2.55	0.5589	0.7088	0.8587	1.0086	0.0583	0.0478	0.0635
2019/03/27	2.25	2.55	0.5589	0.8554	1.0053	1.1552	0.0463	0.0364	0.0520
2018/12/26	2.20	2.55	0.3096	1.3386	1.4501	1.5617	0.0186	0.0104	0.0205
2018/12/26	2.90	2.55	0.3096	−1.1377	−1.0262	−0.9146	0.3631	0.3568	0.3618
2018/09/26	2.30	2.55	0.0603	2.1029	2.1521	2.2014	0.0050	0.0007	0.0030
2018/09/26	3.20	2.55	0.0603	−4.6060	−4.5568	−4.5076	0.6473	0.6471	0.6437
2018/10/24	2.70	2.55	0.1370	v0.7601	−0.6859	−0.6117	0.1831	0.1703	0.1724
2018/09/26	2.50	2.55	0.0603	0.4090	0.4582	0.5074	0.0440	0.0278	0.0359
2018/09/26	2.65	2.55	0.0603	−0.7747	−0.7255	−0.6763	0.1282	0.1140	0.1191
2019/03/27	2.55	2.55	0.5589	0.0204	0.1703	0.3202	0.1521	0.1413	0.1569
2018/10/24	2.60	2.55	0.1370	−0.2516	−0.1774	−0.1031	0.1162	0.1005	0.1054
2018/09/26	2.20	2.55	0.0603	3.0060	3.0552	3.1044	0.0012	0.0000	0.0010
2019/03/27	2.40	2.55	0.5589	0.4249	0.5748	0.7246	0.0887	0.0775	0.0939
2019/03/27	2.60	2.55	0.5589	−0.1091	0.0408	0.1907	0.1780	0.1677	0.1829
2018/09/26	2.85	2.55	0.0603	−2.2529	−2.2036	−2.1544	0.3010	0.2980	0.2955
2018/12/26	2.40	2.55	0.3096	0.5586	0.6702	0.7817	0.0605	0.0475	0.0607
2019/03/27	2.45	2.55	0.5589	0.2873	0.4372	0.5871	0.1074	0.0962	0.1121
2019/03/27	2.65	2.55	0.5589	−0.2362	−0.0863	0.0636	0.2062	0.1967	0.2094
2019/03/27	2.80	2.55	0.5589	−0.6035	−0.4536	−0.3038	0.3040	0.2973	0.3078
2018/09/26	2.45	2.55	0.0603	0.8194	0.8686	0.9179	0.0277	0.0140	0.0219
2018/12/26	2.95	2.55	0.3096	−1.2909	−1.1794	−1.0678	0.4061	0.4013	0.4056
2018/12/26	2.25	2.55	0.3096	1.1371	1.2487	1.3602	0.0257	0.0161	0.0271
2018/09/26	2.80	2.55	0.0603	−1.8933	−1.8441	−1.7948	0.2541	0.2490	0.2486
2018/10/24	2.65	2.55	0.1370	−0.5083	−0.4340	−0.3598	0.1478	0.1332	0.1339
2018/12/26	2.75	2.55	0.3096	−0.6616	−0.5501	−0.4385	0.2446	0.2341	0.2387
2018/12/26	2.85	2.55	0.3096	−0.9818	−0.8703	−0.7587	0.3217	0.3140	0.3201
2018/09/26	2.70	2.55	0.0603	−1.1545	−1.1053	−1.0560	0.1668	0.1557	0.1585
2018/09/26	3.50	2.55	0.0603	−6.4265	−6.3773	−6.3280	0.9470	0.9468	0.9443
2019/03/27	2.35	2.55	0.5589	0.5653	0.7152	0.8651	0.0724	0.0614	0.0784
2018/10/24	2.55	2.55	0.1370	0.0101	0.0843	0.1585	0.0889	0.0728	0.0788
2018/12/26	2.70	2.55	0.3096	−0.4972	−0.3856	−0.2740	0.2096	0.1979	0.2079
2018/10/24	2.75	2.55	0.1370	−1.0074	−0.9332	−0.8590	0.2219	0.2110	0.2120
2018/12/26	2.45	2.55	0.3096	0.3738	0.4854	0.5969	0.0776	0.0640	0.0782
2018/10/24	2.40	2.55	0.1370	0.8271	0.9013	0.9755	0.0333	0.0204	0.0260
2018/09/26	3.10	2.55	0.0603	−3.9610	−3.9118	−3.8626	0.5474	0.5472	0.5451
2018/10/24	2.50	2.55	0.1370	0.2770	0.3512	0.4254	0.0661	0.0503	0.0569
2018/09/26	3.30	2.55	0.0603	−5.2311	−5.1819	−5.1327	0.7472	0.7470	0.7420
2018/10/24	2.30	2.55	0.1370	1.4006	1.4748	1.5490	0.0149	0.0063	0.0108
2018/12/26	2.35	2.55	0.3096	0.7473	0.8589	0.9705	0.0463	0.0343	0.0467
2018/09/26	2.40	2.55	0.0603	1.2383	1.2875	1.3368	0.0166	0.0062	0.0118
2018/09/26	3.40	2.55	0.0603	−5.8376	−5.7884	−5.7392	0.8471	0.8469	0.8453
2018/09/26	2.60	2.55	0.0603	−0.3878	−0.3386	−0.2893	0.0943	0.0779	0.0844
2018/09/26	2.25	2.55	0.0603	2.5494	2.5986	2.6479	0.0025	0.0002	0.0015
2018/10/24	2.35	2.55	0.1370	1.1108	1.1850	1.2592	0.0226	0.0118	0.0178
2019/03/27	2.70	2.55	0.5589	−0.3609	−0.2110	−0.0611	0.2368	0.2280	0.2402
2018/09/26	2.35	2.55	0.0603	1.6660	1.7152	1.7645	0.0094	0.0023	0.0065
2018/09/26	2.95	2.55	0.0603	−2.9535	−2.9042	−2.8550	0.3982	0.3974	0.3921
2018/10/24	2.45	2.55	0.1370	0.5492	0.6234	0.6976	0.0476	0.0330	0.0387
2018/12/26	2.65	2.55	0.3096	−0.3296	−0.2181	−0.1065	0.1772	0.1645	0.1753
2018/12/26	2.55	2.55	0.3096	0.0152	0.1268	0.2383	0.1212	0.1073	0.1210
2019/03/27	2.75	2.55	0.5589	−0.4833	−0.3334	−0.1835	0.2694	0.2616	0.2738
2018/09/26	2.75	2.55	0.0603	−1.5272	−1.4780	−1.4288	0.2091	0.2012	0.2011
2018/09/26	3.60	2.55	0.0603	−6.9988	−6.9496	−6.9003	1.0469	1.0467	1.0452
2019/03/27	2.20	2.55	0.5589	1.0053	1.1552	1.3051	0.0363	0.0272	0.0423
2018/12/26	2.80	2.55	0.3096	−0.8232	−0.7116	−0.6000	0.2821	0.2729	0.2800
2018/12/26	2.30	2.55	0.3096	0.9401	1.0517	1.1632	0.0348	0.0239	0.0353
2018/09/26	2.90	2.55	0.0603	−2.6062	−2.5570	−2.5077	0.3492	0.3476	0.3430
2018/09/26	2.55	2.55	0.0603	0.0067	0.0559	0.1052	0.0661	0.0489	0.0573
2019/03/27	2.50	2.55	0.5589	0.1525	0.3024	0.4523	0.1285	0.1174	0.1330
2018/12/26	2.60	2.55	0.3096	−0.1589	−0.0473	0.0643	0.1477	0.1342	0.1468
2018/09/26	3.00	2.55	0.0603	−3.2949	−3.2457	−3.1965	0.4477	0.4473	0.4459

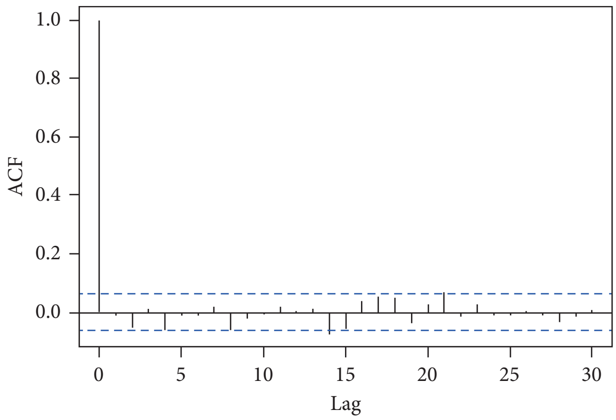

Figure 1: The autocorrelogram of the daily logarithmic returns of S&P 500 index closing prices.

effective trading days as the drift coefficient of the underlying asset μ.

There are 41 call options on April 1, 2019, and the current price of underlying asset $S_0 = 2867.19$. According to simple computation, we obtain $\mu = 4.932\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2019/04/01$, and then we work out d_1, d_2, and d_3 by Theorem 2; see Table 5 for detailed data, where C is the actual closing price of call option, τ is the left expiration time of call option, K is the strike price of call option, and d_1, d_2, and d_3 are the parameters in Theorem 2. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = -0.0246533$ by the least square method with $R^2 = 0.9532$.

There are 62 put options on April 1, 2019, and the current price of underlying asset $S_0 = 2867.19$. Similarly, we obtain $\mu = 4.932\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2019/04/01$, and then we work out d_1, d_2, and d_3 by Theorem 3; see Table 6 for detailed data, where P is the actual closing price of put option, τ is the left expiration time of put option, K is the strike price of put option, and d_1, d_2, and d_3 are the parameters in Theorem 3. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = 0.0269767$ by the least square method with $R^2 = 0.9561$.

5.2. Comparison of Pricing Effect. In the section, we will compare the fitting effect of our pricing formulas with the Black–Scholes model.

5.2.1. Comparison of Pricing Effect on Call Options. There are 25 call options on April 2, 2019, and the current price of underlying asset $S_0 = 2867.24$. Based on the estimated parameters $\mu = 4.932\%$, $\sigma = 0.1174$, $r = 2.42\%$, and $\lambda = -0.0246533$ for call options, the call options of S&P 500 on April 2, 2019, are priced by our obtained pricing formula in Theorem 1 and the Black–Scholes model, respectively; see Table 7 for detailed data, where K is the strike price of call option, τ is the left expiration time of call option, d_1, d_2, and d_3 are the parameters in Theorem 3, C is the actual closing price of call option, C_1 is the value of call option computed by the Black–Scholes model, and C_2 is the value of call option computed by Theorem 1.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E\left(\widehat{\left|C_1 - C\right|}\right) &= 6.902386, \operatorname{Var}\left(\widehat{\left|C_1 - C\right|}\right) = 6.810772, \\ E\left(\widehat{\left|C_2 - C\right|}\right) &= 5.156575, \operatorname{Var}\left(\widehat{\left|C_2 - C\right|}\right) = 9.726556, \end{aligned} \tag{24}$$

where C is the actual closing price of call option, C_1 is the value of call option computed by the Black–Scholes model, and C_2 is the value of call option computed by Theorem 2. It is obvious that $E(\widehat{|C_2 - C|}) < E(\widehat{|C_1 - C|})$. In the following, we will support the statement by the hypothesis test (i.e., test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} &H_0\colon E\left(\left|C_2 - C\right|\right) - E\left(\left|C_1 - C\right|\right) \geq 0, \\ &H_1\colon E\left(\left|C_2 - C\right|\right) - E\left(\left|C_1 - C\right|\right) < 0, \end{aligned} \tag{25}$$

the t−statistics equals −2.1465 with degree of freedom 24, and its p value is 0.01854. Thus, we accept H_1, i.e., $E(|C_2 - C|) < E(|C_1 - C|)$. That is, the prices of call options computed by Theorem 2 are far nearer to their actual prices than those computed by the Black–Scholes model.

5.2.2. Comparison of Pricing Effect on Put Options. There are 39 put options on April 2, 2019, and the current price of underlying asset $S_0 = 2867.24$. Based on the estimated parameters $\mu = 4.932\%$, $\sigma = 0.1174$, $r = 2.42\%$, and $\lambda = 0.0269767$ for put options, the put options of S&P 500 on April 2, 2019, are priced by our obtained pricing formula in Theorem 3 and the Black–Scholes model, respectively; see Table 8 for detailed data, where K is the strike price of put option, τ is the left expiration time of put option, d_1, d_2, and d_3 are the parameters in Theorem 3, P is the actual closing price of put option, P_1 is the value of put option computed by the Black–Scholes model, and P_2 is the value of put option computed by Theorem 3.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E\left(\widehat{\left|P_1 - P\right|}\right) &= 7.850124, \operatorname{Var}\left(\widehat{\left|P_1 - P\right|}\right) = 29.945065, \\ E\left(\widehat{\left|P_2 - P\right|}\right) &= 3.492115, \operatorname{Var}\left(\widehat{\left|P_2 - P\right|}\right) = 6.764690, \end{aligned} \tag{26}$$

where P is the actual closing price of put option, P_1 is the value of put option computed by the Black–Scholes model, and P_2 is the value of put option computed by Theorem 3. It is obvious that $E(\widehat{|P_2 - P|}) < E(\widehat{|P_1 - P|})$. In the following, we will support the statement by the hypothesis test (i.e., t-test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} &H_0\colon E\left(\left|P_2 - P\right|\right) - E\left(\left|P_1 - P\right|\right) \geq 0, \\ &H_1\colon E\left(\left|P_2 - P\right|\right) - E\left(\left|P_1 - P\right|\right) < 0, \end{aligned} \tag{27}$$

Table 5: Parameter estimation of d_1, d_2, and d_3 for call options.

C	τ	K	d_1	d_2	d_3
24.40	0.046575	2870	0.039320	0.064659	0.089997
21.50	0.046575	2875	−0.029375	−0.004037	0.021302
10.60	0.046575	2900	−0.371070	−0.345732	−0.320393
4.50	0.046575	2925	−0.709832	−0.684493	−0.659155
3.85	0.046575	2930	−0.777237	−0.751898	−0.726559
3.14	0.046575	2935	−0.844526	−0.819188	−0.793849
1.67	0.046575	2950	−1.045710	−1.020372	−0.995033
0.65	0.046575	2975	−1.378754	−1.353416	−1.328077
0.60	0.046575	2980	−1.445027	−1.419689	−1.394350
46.15	0.126027	2870	0.104771	0.146452	0.188133
44.00	0.126027	2875	0.063010	0.104691	0.146371
31.00	0.126027	2900	−0.144713	−0.103032	−0.061351
19.35	0.126027	2925	−0.350653	−0.308972	−0.267291
7.00	0.126027	2975	−0.757303	−0.715622	−0.673941
3.73	0.126027	3000	−0.958073	−0.916392	−0.874711
3.30	0.126027	3005	−0.998026	−0.956345	−0.914664
2.50	0.126027	3015	−1.077733	−1.036052	−0.994371
66.70	0.221918	2870	0.152504	0.207814	0.263123
63.80	0.221918	2875	0.121033	0.176343	0.231652
48.60	0.221918	2900	−0.035505	0.019805	0.075114
36.70	0.221918	2925	−0.190699	−0.135390	−0.080080
28.32	0.221918	2940	−0.283181	−0.227871	−0.172561
27.70	0.221918	2950	−0.344573	−0.289263	−0.233954
13.03	0.221918	3000	−0.648446	−0.593137	−0.537827
6.00	0.221918	3050	−0.947297	−0.891987	−0.836678
3.10	0.221918	3100	−1.241288	−1.185978	−1.130669
0.85	0.221918	3200	−1.815306	−1.759996	−1.704686
28.30	0.298630	2975	−0.377841	−0.313680	−0.249519
21.90	0.298630	3000	−0.508267	−0.444106	−0.379945
17.00	0.375342	3050	−0.637911	−0.565980	−0.494048
42.20	0.471233	3000	−0.313762	−0.233165	−0.152567
15.10	0.471233	3100	−0.720596	−0.639998	−0.559401
133.50	0.720548	2875	0.279420	0.379083	0.478746
69.70	0.720548	3000	−0.147614	−0.047950	0.051713
33.36	0.720548	3100	−0.476619	−0.376956	−0.277292
27.50	0.720548	3125	−0.557212	−0.457549	−0.357885
15.70	0.720548	3200	−0.795178	−0.695515	−0.595852
7.00	0.720548	3300	−1.103934	−1.004271	−0.904607
75.30	0.797260	3000	−0.109289	−0.004455	0.100380
18.70	0.797260	3200	−0.724912	−0.620077	−0.515243
52.44	0.969863	3100	−0.319343	−0.203716	−0.088089

Table 6: Parameter estimation of d_1, d_2, and d_3 for put options.

P	τ	K	d_1	d_2	d_3
0.35	0.046575	2400	7.097458	7.122797	7.148136
0.42	0.046575	2410	6.933361	6.958699	6.984038
0.40	0.046575	2420	6.769942	6.795281	6.820619
0.40	0.046575	2435	6.526076	6.551415	6.576753
0.47	0.046575	2440	6.445121	6.470460	6.495799
0.46	0.046575	2450	6.283708	6.309047	6.334385
0.55	0.046575	2455	6.203248	6.228587	6.253926
0.54	0.046575	2485	5.723905	5.749243	5.774582
0.62	0.046575	2490	5.644577	5.669915	5.695254
0.56	0.046575	2500	5.486398	5.511737	5.537076
0.72	0.046575	2515	5.250313	5.275652	5.300990
0.74	0.046575	2525	5.093704	5.119042	5.144381
0.75	0.046575	2550	4.704878	4.730217	4.755555
0.80	0.046575	2600	3.938534	3.963872	3.989211

Table 6: Continued.

P	τ	K	d_1	d_2	d_3
0.93	0.046575	2605	3.862711	3.888050	3.913389
1.38	0.046575	2640	3.335995	3.361334	3.386673
1.50	0.046575	2650	3.186787	3.212126	3.237464
1.80	0.046575	2675	2.816217	2.841555	2.866894
2.44	0.046575	2700	2.449093	2.474432	2.499770
2.55	0.046575	2705	2.376076	2.401415	2.426754
2.65	0.046575	2710	2.303195	2.328533	2.353872
3.03	0.046575	2720	2.157833	2.183172	2.208511
3.25	0.046575	2725	2.085353	2.110692	2.136030
4.50	0.046575	2750	1.724935	1.750274	1.775612
0.55	0.126027	2000	8.769770	8.811451	8.853132
1.15	0.126027	2275	5.678836	5.720517	5.762198
1.50	0.126027	2350	4.900655	4.942336	4.984017
1.95	0.126027	2400	4.395546	4.437227	4.478907
2.20	0.126027	2425	4.146924	4.188605	4.230285
3.20	0.126027	2500	3.416152	3.457833	3.499514
5.10	0.126027	2575	2.706983	2.748664	2.790345
6.05	0.126027	2600	2.475176	2.516857	2.558538
6.38	0.126027	2610	2.383077	2.424758	2.466439
8.55	0.126027	2650	2.018175	2.059856	2.101537
10.31	0.126027	2675	1.792898	1.834579	1.876260
12.35	0.126027	2700	1.569717	1.611398	1.653079
14.35	0.126027	2725	1.348593	1.390274	1.431955
18.07	0.126027	2750	1.129488	1.171169	1.212850
22.54	0.126027	2775	0.912366	0.954047	0.995728
27.70	0.126027	2800	0.697191	0.738872	0.780553
32.85	0.126027	2825	0.483929	0.525610	0.567291
41.25	0.126027	2850	0.272547	0.314227	0.355908
1.30	0.221918	2000	6.682380	6.737690	6.793000
5.39	0.221918	2375	3.575320	3.630630	3.685939
5.80	0.221918	2400	3.385999	3.441308	3.496618
9.26	0.221918	2500	2.647935	2.703245	2.758554
11.74	0.221918	2550	2.289903	2.345212	2.400522
12.90	0.221918	2565	2.183861	2.239171	2.294480
13.55	0.221918	2575	2.113511	2.168820	2.224130
15.32	0.221918	2600	1.938823	1.994133	2.049442
17.70	0.221918	2625	1.765807	1.821117	1.876426
19.60	0.221918	2650	1.594431	1.649740	1.705050
26.23	0.221918	2700	1.256476	1.311786	1.367095
28.70	0.221918	2715	1.156310	1.211619	1.266929
31.25	0.221918	2725	1.089839	1.145148	1.200458
35.95	0.221918	2750	0.924723	0.980033	1.035342
46.70	0.221918	2800	0.598948	0.654257	0.709567
59.32	0.221918	2850	0.278939	0.334248	0.389558
63.55	0.221918	2860	0.215611	0.270920	0.326230
64.70	0.221918	2865	0.184030	0.239340	0.294649
46.60	0.298630	2750	0.847874	0.912035	0.976196
57.50	0.298630	2800	0.567041	0.631203	0.695364

Table 7: Pricing results by our pricing formula and the Black–Scholes model for call option.

K	τ	d_1	d_2	d_3	C	C_1	C_2
2870	0.043836	0.036511	0.061093	0.085675	21.05	28.26	30.77
2875	0.043836	−0.034298	−0.009716	0.014866	18.15	25.85	26.25
2880	0.043836	−0.104985	−0.080403	−0.055821	18.20	23.58	21.99
2890	0.043836	−0.245990	−0.221408	−0.196826	13.70	19.45	14.27
2900	0.043836	−0.386509	−0.361927	−0.337345	8.50	15.87	7.65
2870	0.123288	0.103532	0.144758	0.185983	45.00	50.06	54.77
2875	0.123288	0.061310	0.102535	0.143760	40.85	47.55	50.91

Table 7: Continued.

K	τ	d_1	d_2	d_3	C	C_1	C_2
2895	0.123288	−0.106850	−0.065625	−0.024399	31.68	38.36	36.73
2900	0.123288	−0.148708	−0.107483	−0.066258	29.50	36.27	33.49
2925	0.123288	−0.356924	−0.315698	−0.274473	18.10	27.00	19.26
2940	0.123288	−0.481000	−0.439775	−0.398549	14.10	22.35	12.25
2950	0.123288	−0.563367	−0.522141	−0.480916	11.50	19.61	8.19
2955	0.123288	−0.604445	−0.563220	−0.521995	10.45	18.34	6.35
2870	0.219178	0.151657	0.206624	0.261591	64.00	69.14	75.39
2875	0.219178	0.119990	0.174957	0.229924	63.80	66.58	71.78
2900	0.219178	−0.037523	0.017444	0.072411	47.50	54.73	54.98
2950	0.219178	−0.348517	−0.293550	−0.238583	26.75	35.58	27.81
3000	0.219178	−0.654284	−0.599317	−0.544350	13.30	21.94	8.99
3025	0.219178	−0.805261	−0.750294	−0.695327	8.75	16.88	2.42
2900	0.295890	0.018662	0.082528	0.146394	62.77	67.32	69.15
2900	0.372603	0.062039	0.133708	0.205376	76.38	78.74	81.78
3100	0.372603	−0.868516	−0.796848	−0.725179	9.40	18.99	3.50
2900	0.468493	0.105947	0.186310	0.266673	91.40	91.87	96.10
3000	0.468493	−0.315908	−0.235545	−0.155182	41.26	52.22	44.38
3100	0.468493	−0.723929	−0.643566	−0.563203	15.00	27.13	12.45

Table 8: Pricing results by our pricing formula and the Black–Scholes model for put option.

K	τ	d_1	d_2	d_3	P	P_1	P_2
2550	0.043836	4.845658	4.870240	4.894822	0.59	0.00	0.05
2575	0.043836	4.448776	4.473358	4.497940	0.70	0.00	0.14
2600	0.043836	4.055729	4.080311	4.104893	0.81	0.00	0.35
2650	0.043836	3.280846	3.305428	3.330010	1.20	0.01	1.59
2660	0.043836	3.127625	3.152207	3.176790	1.40	0.02	2.08
2675	0.043836	2.898871	2.923453	2.948035	1.64	0.04	3.02
2700	0.043836	2.520449	2.545031	2.569613	1.95	0.14	5.27
2300	0.123288	5.474086	5.515311	5.556536	1.30	0.00	0.01
2400	0.123288	4.441720	4.482946	4.524171	1.95	0.00	0.14
2500	0.123288	3.451504	3.492730	3.533955	3.26	0.01	1.11
2590	0.123288	2.593606	2.634832	2.676057	5.40	0.20	4.65
2600	0.123288	2.500131	2.541356	2.582581	6.00	0.27	5.32
2615	0.123288	2.360589	2.401814	2.443040	6.20	0.41	6.45
2625	0.123288	2.268005	2.309230	2.350456	7.10	0.53	7.29
2670	0.123288	1.855696	1.896922	1.938147	9.20	1.62	11.98
2675	0.123288	1.810314	1.851539	1.892764	9.40	1.82	12.60
2700	0.123288	1.584666	1.625892	1.667117	11.50	3.13	15.93
2725	0.123288	1.361099	1.402324	1.443549	14.30	5.16	19.69
2775	0.123288	0.920051	0.961277	1.002502	20.50	12.41	28.36
2800	0.123288	0.702499	0.743724	0.784950	25.85	18.18	33.26
2825	0.123288	0.486881	0.528106	0.569331	31.50	25.71	38.61
2850	0.123288	0.273162	0.314387	0.355612	39.00	35.19	44.52
2200	0.219178	4.988269	5.043237	5.098204	2.60	0.00	0.03
2400	0.219178	3.405298	3.460265	3.515232	5.80	0.02	1.16
2500	0.219178	2.662636	2.717603	2.772570	9.00	0.22	4.11
2540	0.219178	2.373857	2.428824	2.483791	11.00	0.54	6.28
2550	0.219178	2.302373	2.357340	2.412307	11.95	0.67	6.94
2575	0.219178	2.124882	2.179849	2.234816	13.20	1.10	8.80
2600	0.219178	1.949106	2.004073	2.059040	14.80	1.76	10.99
2650	0.219178	1.602568	1.657535	1.712502	19.95	4.13	16.46
2725	0.219178	1.094832	1.149799	1.204766	29.92	12.23	27.74
2825	0.219178	0.439168	0.494135	0.549103	51.10	37.78	49.91
2850	0.219178	0.278879	0.333846	0.388814	58.00	47.68	57.10
2865	0.219178	0.183379	0.238346	0.293314	65.20	54.36	61.80
2425	0.295890	2.819513	2.883379	2.947245	10.76	0.16	3.16
2500	0.295890	2.342589	2.406455	2.470321	14.75	0.70	6.54
2675	0.295890	1.283205	1.347071	1.410937	32.50	9.91	24.37
2850	0.295890	0.290978	0.354844	0.418710	69.79	55.32	64.91
2860	0.295890	0.236134	0.300000	0.363866	76.22	59.70	68.22

The t−statistics equals −4.4919 with degree of freedom 38, and its p value is 0.00019. Thus, we accept H_1, i.e., $E(|P_2 - P|) < E(|P_1 - P|)$. That is, the prices of put options computed by Theorem 3 are far nearer to their actual prices than those computed by the Black–Scholes model.

Therefore, our pricing formulas in Theorems 2 and 3 have less absolute errors than those of the Black–Scholes model for both call options and put options. That is, the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

6. Conclusion

In this study, we obtain the pricing formula of European options, including European call option and European put option, in a risk-aversive market. Corollaries of our obtained results improve the Black–Scholes model owning to its much weaker conditions. It follows from our obtained results that European option value depends on the drift coefficient μ of its underlying security, which does not display in the Black–Scholes model only because $\mu = r$ in a risk-neutral financial market according to the no-arbitrage opportunity principle. Empirical analyses show that the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

Appendix

A. Proof of Theorem 2

In order to prove Theorem 2, we need the following lemma.

Lemma A.1. *Assuming $\ln X \sim N(\mu, \sigma^2)$, then for any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that*

$$E\left[X^m 1_{\{X \geq K\}}\right] = e^{m\mu + (1/2)m^2\sigma^2} \Phi\left(\frac{1}{\sigma}\left(\ln \frac{1}{K} + \mu + m\sigma^2\right)\right). \tag{A.1}$$

Proof. If $\ln X \sim N(\mu, \sigma^2)$, denote the density function of X by $f(x; \mu, \sigma)$, and then

$$f(x; \mu, \sigma) = \begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right\}, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{A.2}$$

For any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that

$$\begin{aligned} E\left[X^m 1_{\{X \geq K\}}\right] &= \int_{-\infty}^{+\infty} x^m 1_{\{x \geq K\}} f\left(x; \mu, \sigma^2\right) \mathrm{d}x, \\ &= \int_{K}^{+\infty} x^m \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\} \mathrm{d}x \\ &= \int_{\ln K}^{+\infty} e^{my} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y - \mu)^2}{2\sigma^2}\right\} \mathrm{d}y \\ &= e^{m\mu + (1/2)m^2\sigma^2} \int_{\ln K}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp \\ &\quad \left\{-\frac{\left[y - \left(\mu + m\sigma^2\right)\right]^2}{2\sigma^2}\right\} \mathrm{d}y \\ &= e^{m\mu + (1/2)m^2\sigma^2} \int_{1/\sigma\left[\ln K - \left(\mu + m\sigma^2\right)\right]}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \mathrm{d}z \\ &= e^{m\mu + (1/2)m^2\sigma^2} \left(1 - \Phi\left(\frac{1}{\sigma}\left[\ln K - \left(\mu + m\sigma^2\right)\right]\right)\right) \\ &= e^{m\mu + (1/2)m^2\sigma^2} \Phi\left(\frac{1}{\sigma}\left(\ln \frac{1}{K} + \mu + m\sigma^2\right)\right). \end{aligned} \tag{A.3}$$

The proof is complete. □

Noting that the underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$ and the current price of underlying asset is S_0, it follows from It $\hat{\text{o}}$ formula that

$$S_\tau = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau\right\}, \tag{A.4}$$

where $W = \{W_t, t \geq 0\}$ is the standard Wiener process, so

$$\ln S_\tau \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right). \tag{A.5}$$

It yields from Lemma A.1 and (A.5) that

$$E\left[1_{\{S_\tau \geq K\}}\right] = \Phi(d_1), \tag{A.6}$$

$$E\left[S_\tau 1_{\{S_\tau \geq K\}}\right] = S_0 e^{\mu\tau} \Phi(d_2), \tag{A.7}$$

$$E\left[S_\tau^2 1_{\{S_\tau \geq K\}}\right] = S_0^2 e^{\left(\mu + \sigma^2\right)\tau} \Phi(d_3). \tag{A.8}$$

It follows from (A.6) to (A.8) that

$$\begin{aligned} E[(S_\tau - K)^+] &= E\left[(S_\tau - K)1_{\{S_\tau \ge K\}}\right], \\ &= E\left[S_\tau 1_{\{S_\tau \ge K\}}\right] - KE\left[1_{\{S_\tau \ge K\}}\right] \\ &= S_0 e^{\mu\tau}\Phi(d_2) - K\Phi(d_1), \end{aligned} \tag{A.9}$$

$$\begin{aligned} E\left[((S_\tau - K)^+)^2\right] &= E\left[(S_\tau - K)^2 1_{\{S_\tau \ge K\}}\right], \\ &= E\left[S_\tau^2 1_{\{S_\tau \ge K\}}\right] - 2KE\left[S_\tau 1_{\{S_\tau \ge K\}}\right] \\ &\quad + K^2 E\left[1_{\{S_\tau \ge K\}}\right] \\ &= S_0^2 e^{(\mu+\sigma^2)\tau}\Phi(d_3) - 2KS_0 e^{\mu\tau}\Phi(d_2) \\ &\quad + K^2\Phi(d_1). \end{aligned} \tag{A.10}$$

Furthermore, it obtains from (A.9) and (A.10) that

$$\begin{aligned} \operatorname{std}((S_\tau - K)^+) &= \operatorname{sqrt}\left\{E\left[((S_\tau - K)^+)^2\right] - E^2[(S_\tau - K)^+]\right\}, \\ &= \operatorname{sqrt}\left\{S_0^2 e^{2\mu\tau}\left(e^{\sigma^2\tau}\Phi(d_3) - \Phi^2(d_2)\right) - KS_0 e^{\mu\tau}\Phi(d_2)\Phi(-d_1) + K^2\Phi(d_1)\Phi(-d_1)\right\}. \end{aligned} \tag{A.11}$$

Thus, it yields from Proposition 1, (A.9), and (A.11) that

$$\begin{aligned} C(S_0, K, r, \sigma, \tau, \lambda) &= e^{-r\tau}\{E[(S_\tau - K)^+] + \lambda \cdot \operatorname{std}((S_\tau - K)^+)\}, \\ &= S_0 e^{(\mu-r)\tau}\Phi(d_2) - Ke^{-r\tau}\Phi(d_1) + \lambda e^{-r\tau}\cdot \operatorname{sqrt}\left\{S_0^2 e^{2\mu\tau}\left(e^{\sigma^2\tau}\Phi(d_3) - \Phi^2(d_2)\right) - KS_0 e^{\mu\tau}\Phi(d_2)\Phi(-d_1) \right. \\ &\quad \left. + K^2\Phi(d_1)\Phi(-d_1)\right\}. \end{aligned} \tag{A.12}$$

The proof is complete. □

B. Proof of Theorem 3

In order to prove Theorem 3, we first present another lemma as follows.

Lemma B.1. *Assuming $\ln X \sim N(\mu, \sigma^2)$, then for any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that*

$$E\left[X^m 1_{\{X \le K\}}\right] = e^{m\mu + (1/2)m^2\sigma^2}\Phi\left(-\frac{1}{\sigma}\left(\ln\frac{1}{K} + \mu + m\sigma^2\right)\right). \tag{B.1}$$

Proof. If $\ln X \sim N(\mu, \sigma^2)$, denote the density function of X by $f(x; \mu, \sigma)$, and then

$$f(x; \mu, \sigma) = \begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}}\exp\left\{-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right\}, & x > 0, \\ 0, & x \le 0. \end{cases} \tag{B.2}$$

For any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that

$$\begin{aligned}
E\left[X^m 1_{\{X\le K\}}\right] &= \int_{-\infty}^{+\infty} x^m 1_{\{x\le K\}} f\left(x;\mu,\sigma^2\right)\mathrm{d}x,\\
&= \int_{-\infty}^{K} x^m \frac{1}{x\sigma\sqrt{2\pi}}\exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}\mathrm{d}x\\
&= \int_{-\infty}^{\ln K} e^{my}\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}\mathrm{d}y\\
&= e^{m\mu+(1/2)m^2\sigma^2}\int_{-\infty}^{\ln K} e^{my}\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{\left[y-\left(\mu+m\sigma^2\right)\right]^2}{2\sigma^2}\right\}\mathrm{d}y\\
&= e^{m\mu+(1/2)m^2\sigma^2}\int_{-\infty}^{1/\sigma\left[\ln K-\left(\mu+m\sigma^2\right)\right]}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}\mathrm{d}z\\
&= e^{m\mu+(1/2)m^2\sigma^2}\Phi\left(\frac{1}{\sigma}\left[\ln K-\left(\mu+m\sigma^2\right)\right]\right)\\
&= e^{m\mu+(1/2)m^2\sigma^2}\Phi\left(-\frac{1}{\sigma}\left(\ln\frac{1}{K}+\mu+m\sigma^2\right)\right).
\end{aligned} \tag{B.3}$$

The proof is complete. □

Noting that the underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$ and the current price of underlying asset is S_0, it follows from Itô formula that

$$S_\tau = S_0\exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\tau+\sigma W_\tau\right\}, \tag{B.4}$$

where $W=\{W_t, t\ge 0\}$ is a standard Wiener process, so

$$\ln S_\tau \sim N\left(\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)\tau,\sigma^2\tau\right). \tag{B.5}$$

It yields from Lemma B.1 and (B.5) that

$$E\left[1_{\{S_\tau\le K\}}\right] = \Phi(-d_1), \tag{B.6}$$

$$E\left[S_\tau 1_{\{S_\tau\le K\}}\right] = S_0 e^{\mu\tau}\Phi(-d_2), \tag{B.7}$$

$$E\left[S_\tau^2 1_{\{S_\tau\le K\}}\right] = S_0^2 e^{\left(\mu+\sigma^2\right)\tau}\Phi(-d_3). \tag{B.8}$$

It follows from (B.6) to (B.8) that

$$\begin{aligned}
E\left[(K-S_\tau)^+\right] &= E\left[(K-S_\tau)1_{\{S_\tau\le K\}}\right],\\
&= KE\left[1_{\{S_\tau\le K\}}\right]-E\left[S_\tau 1_{\{S_\tau\le K\}}\right]\\
&= K\Phi(-d_1)-S_0e^{\mu\tau}\Phi(-d_2),
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
E\left[\left((K-S_\tau)^+\right)^2\right] &= E\left[(K-S_\tau)^2 1_{\{S_\tau\le K\}}\right],\\
&= E\left[S_\tau^2 1_{\{S_\tau\le K\}}\right]-2KE\left[S_\tau 1_{\{S_\tau\le K\}}\right]\\
&\quad + K^2E\left[1_{\{S_\tau\le K\}}\right]\\
&= S_0^2e^{\left(\mu+\sigma^2\right)\tau}\Phi(-d_3)-2KS_0e^{\mu\tau}\Phi(-d_2)\\
&\quad + K^2\Phi(-d_1).
\end{aligned} \tag{B.10}$$

Furthermore, it obtains from (B.9) and (B.10) that

$$\begin{aligned}
\mathrm{std}\left((K-S_\tau)^+\right) &= \mathrm{sqrt}\left\{E\left[\left((K-S_\tau)^+\right)^2\right]-E^2\left[(K-S_\tau)^+\right]\right\},\\
&= \mathrm{sqrt}\left\{S_0^2e^{2\mu\tau}\left(e^{\sigma^2\tau}\Phi(-d_3)-\Phi^2(-d_2)\right)-KS_0e^{\mu\tau}\Phi(-d_2)\Phi(d_1)+K^2\Phi(-d_1)\Phi(d_1)\right\}.
\end{aligned} \tag{B.11}$$

Thus, it yields from (B.9) and (B.11) that

$$\begin{aligned}
P(S_0,K,r,\sigma,\tau,\lambda) &= e^{-r\tau}\left\{E\left[(K-S_\tau)^+\right]+\lambda\cdot\mathrm{std}\left((K-S_\tau)^+\right)\right\},\\
&= Ke^{-r\tau}\Phi(-d_1)-S_0e^{(\mu-r)\tau}\Phi(-d_2)+\lambda e^{-r\tau}\cdot\mathrm{sqrt}\left\{S_0^2e^{2\mu\tau}\left(e^{\sigma^2\tau}\Phi(-d_3)-\Phi^2(-d_2)\right)-KS_0e^{\mu\tau}\Phi(-d_2)\Phi(d_1)\right.\\
&\quad\left.+K^2\Phi(-d_1)\Phi(d_1)\right\}.
\end{aligned} \tag{B.12}$$

The proof is complete. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (no. 11471120).

References

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

[2] R. C. Merton, "Theory of rational option pricing," *The Bell Journal of Economics and Management Science*, vol. 4, no. 1, pp. 141–183, 1973.

[3] G. Bakshi, C. Cao, and Z. Chen, "Empirical performance of alternative option pricing models," *The Journal of Finance*, vol. 52, no. 5, pp. 2003–2049, 1997.

[4] N. Gârleanu, L. H. Pedersen, and A. M. Poteshman, "Demand-based option pricing," *The Review of Financial Studies*, vol. 22, no. 10, pp. 4259–4299, 2009.

[5] N. Cai and S. G. Kou, "Option pricing under a mixed-exponential jump diffusion model," *Management Science*, vol. 57, no. 11, pp. 2067–2081, 2011.

[6] C. Bernarda and C. Czadob, "Multivariate option pricing using copulae," *Applied Stochastic Models in Business and Industry*, vol. 29, pp. 509–526, 2013.

[7] C. Bandi and D. Bertsimas, "Robust option pricing," *European Journal of Operational Research*, vol. 239, no. 3, pp. 842–853, 2014.

[8] S. Bao, S. Chen, W. A. Zheng, and Y. Zhou, "Forecasting semi-stationary processes and statistical arbitrage," *Statistical Theory and Related Fields*, vol. 4, no. 2, pp. 179–189, 2020.

[9] E. Moretto, S. Pasquali, and B. Trivellato, "Option pricing under deformed Gaussian distributions," *Physica A: Statistical Mechanics and its Applications*, vol. 446, pp. 246–263, 2016.

[10] M. Leippold and S. Schärer, "Discrete-time option pricing with stochastic liquidity," *Journal of Banking & Finance*, vol. 75, pp. 1–16, 2017.

[11] J. Hok and T. L. Chan, "Option pricing with Legendre polynomials," *Journal of Computational and Applied Mathematics*, vol. 322, pp. 25–45, 2017.

[12] A. H. Davison and S. Mamba, "Symmetry methods for option pricing," *Communications in Nonlinear Science and Numerical Simulation*, vol. 47, pp. 421–425, 2017.

[13] S. Willems, "Asian option pricing with orthogonal polynomials," *Quantitative Finance*, vol. 19, no. 4, pp. 605–618, 2019.

[14] Y. Liu, X. Chen, and D. A. Ralescu, "Uncertain currency model and currency option pricing," *International Journal of Intelligent Systems*, vol. 30, no. 1, pp. 40–51, 2015.

[15] P. Friz, S. Gerhold, and A. Pinter, "Option pricing in the moderate deviations regime," *Mathematical Finance*, vol. 28, no. 3, pp. 962–988, 2018.

[16] A. Dubinsky, M. Johannes, A. Kaeck, and N. J. Seeger, "Option pricing of earnings announcement risks," *The Review of Financial Studies*, vol. 32, no. 2, pp. 646–687, 2019.

[17] J. Huh, "Pricing options with exponential Lévy neural network," *Expert Systems With Applications*, vol. 127, pp. 128–140, 2019.

[18] X. Liu, Y. Cao, C. Ma, and L. Shen, "Wavelet-based option pricing: an empirical study," *European Journal of Operational Research*, vol. 272, no. 3, pp. 1132–1142, 2019.

[19] H. Siddiqi, "Anchoring-adjusted option pricing models," *Journal of Behavioral Finance*, vol. 20, no. 2, pp. 139–153, 2019.