Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 9734342, 11 pages
https://doi.org/10.1155/2021/9734342

Research Article

Periodic Oscillating Dynamics for a Delayed Nicholson-Type Model with Harvesting Terms

Changjin Xu,[1] **Wei Zhang**,[2] **and Peiluan Li**[3]

[1]*Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, China*
[2]*College of Economics and Statistics, Guangzhou University, Guangzhou 510006, China*
[3]*School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China*

Correspondence should be addressed to Wei Zhang; wzha19680317@126.com

Received 6 July 2020; Revised 7 January 2021; Accepted 18 January 2021; Published 5 February 2021

Academic Editor: Chuangxia Huang

In this manuscript, a delayed Nicholson-type model with linear harvesting terms is investigated. Applying coincidence degree theory, we establish a sufficient condition which guarantees the existence of positive periodic solutions for the delayed Nicholson-type model. By constructing suitable Lyapunov functions, a new criterion for the uniqueness and global attractivity of the periodic solution of the Nicholson-type delay system is obtained. The derived results of this article are completely new and complement some previous investigations.

1. Introduction

In 1954, Nicholson [1] and later in 1980, Gurney et al. [2] established the following Nicholson's blowfly model,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)}, \delta, p, \tau, \quad a \in (0,\infty), \tag{1}$$

to describe the population of the Australian sheep-blowfly *Lucilia cuprina*. In model (1), $x(t)$ denotes the size of the population at time t, p denotes the maximum per capita daily egg production rate, δ stands for the per capita daily adult death rate, $(1/a)$ represents the size at which the blowfly population reproduces at its maximum rate, and τ is the generation rate. Since then, model (1) and its revised versions have been extensively investigated. For example, So and Yu [3] analyzed the stability and uniform persistence of the discrete version of model (1), Kulenovic et al. [4] investigated the global attractivity of system (1), and Ding and Li [5] focused on the stability and bifurcation of the numerical discretization model of (1). For more details, we refer the reader to [6–21].

It is well known that oscillatory behavior of population densities is one characteristic phenomenon of the population [22]. Thus, there have been extensive results on the existence of periodic solutions for Nicholson's blowfly models. We refer the reader to [7, 11, 21–29]. In recent years, Berezansky et al. [30] investigated the global dynamics of the following Nicholson-type delay model:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -a_1x_1(t) + b_1x_2(t) + c_1x_1(t-\tau)e^{-x_1(t-\tau)}, \\[2ex] \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -a_2x_2(t) + b_2x_1(t) + c_2x_2(t-\tau)e^{-x_2(t-\tau)}, \end{cases} \tag{2}$$

with the initial conditions

$$x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \ \varphi_i(0) > 0, \tag{3}$$

where $\varphi_i(s) \in C([-\tau,0],[0,+\infty))$ and a_i, b_i, c_i, and τ are nonnegative constants, $i = 1, 2$. Taking into account the effect of periodically varying environment, Wang et al. [31] proposed the following nonautonomous Nicholson-type delay model:

$$\begin{cases} \dfrac{dx_1}{dt} = -\alpha_1(t)x_1(t) + \beta_1(t)x_2(t) + \sum_{j=1}^{m} c_{1j}(t)x_1\left(t-\tau_{1j}(t)\right)e^{-\gamma_{1j}(t)x_1\left(t-\tau_{1j}(t)\right)}, \\ \dfrac{dx_2}{dt} = -\alpha_2(t)x_2(t) + \beta_2(t)x_1(t) + \sum_{j=1}^{m} c_{2j}(t)x_2\left(t-\tau_{2j}(t)\right)e^{-\gamma_{2j}(t)x_2\left(t-\tau_{2j}(t)\right)}, \end{cases} \tag{4}$$

and focused on the existence and exponential convergence of positive almost periodic solutions for (4). Here, $\alpha_i, \beta_i, c_{ij}, \tau_{ij}$: $R \longrightarrow (0, +\infty)$ are almost periodic functions, and $i = 1, 2$ and $j = 1, 2, \ldots, m$. In 2011, Liu [22] studied the existence and uniqueness of positive periodic solutions of (4). In 2010, assuming that a harvesting function is a function of the delayed estimate of the true population, Berezansky et al. [32] established the following Nicholson-type delay system with a linear harvesting term:

$$\frac{dx}{dt} = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)} - \mathscr{H}x(t-\sigma), \delta, p, \tau, \quad a \in (0, \infty), \tag{5}$$

where $\delta, p, \tau, a, \mathscr{H}, \sigma \in (0, +\infty)$, $\mathscr{H}x(t-\sigma)$ denotes a linear harvesting term, $x(t)$ represents the size of the population at time t, p represents the maximum per capita daily egg production rate, δ represents the per capita daily adult death rate, $(1/a)$ represents the size at which the blowfly population reproduces at its maximum rate, and τ is the generation rate. Berezansky et al. [32] proposed an open problem: how about the dynamical behaviors of Nicholson's blowfly model with a linear harvesting term?

Inspired by Berezansky et al. [30], Wang et al. [31], and Berezansky et al. [32], Liu and Meng [33] proposed the following Nicholson's blowfly model with linear harvesting terms:

$$\begin{cases} \dfrac{dx_1}{dt} = -\alpha_1(t)x_1(t) + \beta_1(t)x_2(t) + \sum_{j=1}^{m} c_{1j}(t)x_1\left(t-\tau_{1j}(t)\right)e^{-\gamma_{1j}(t)x_1\left(t-\tau_{1j}(t)\right)} \\ -\mathscr{H}_1(t)x_1(t-\sigma_1(t)), \\ \dfrac{dx_2}{dt} = -\alpha_2(t)x_2(t) + \beta_2(t)x_1(t) + \sum_{j=1}^{m} c_{2j}(t)x_2\left(t-\tau_{2j}(t)\right)e^{-\gamma_{2j}(t)x_2\left(t-\tau_{2j}(t)\right)} \\ -\mathscr{H}_2(t)x_2(t-\sigma_2(t)), \end{cases} \tag{6}$$

where $\alpha_i, \beta_i, c_{ij}, \tau_{ij}, \sigma_i, H_i$: $R \longrightarrow (0, +\infty)$ are almost periodic functions and $i = 1, 2$ and $j = 1, 2, \ldots, m$. Liu and Meng [33] established some sufficient conditions to check the existence, uniqueness, and local exponential convergence of the positive almost periodic solution of (6).

Here, we would like to point out that periodic phenomenon plays an important role in characterizing the dynamical behavior of Nicholson's blowfly models. Thus, it is worthwhile to investigate the periodic solution of Nicholson's blowfly models. Up to now, there is no manuscript which handles this aspect on the periodic solution of model (6).

The principle objective of this manuscript is to find a set of sufficient conditions that guarantee the existence of at least a positive periodic solution for model (6) and by constructing a suitable Lyapunov function to investigate the stability of periodic solutions of model (6).

Let

$$\begin{aligned} \bar{l} &= \frac{1}{\omega}\int_0^{\omega} l(t)dt, \\ l^L &= \min_{t\in[0,\omega]} l(t), \\ l^M &= \max_{t\in[0,\omega]} l(t), \end{aligned} \tag{7}$$

where $l(t)$ is an $\omega(\omega > 0)$-continuous periodic function. In addition, the following assumptions are given:

(i) $(\mathscr{A}1)$ For $i = 1, 2$ and $j = 1, 2, \ldots, m, \alpha_i, \beta_i, c_{ij}, \gamma_{ij}, \mathscr{H}_i$: $R \longrightarrow (0, +\infty)$ and σ_i, τ_{ij}: $R \longrightarrow [0, +\infty)$ are all ω-periodic functions

(ii) $(\mathscr{A}2)$ $\alpha_i^M + \mathscr{H}_i^M < \sum_{j=1}^{m} c_{ij}^L$ $(i = 1, 2)$

(iii) $(\mathscr{A}3)$ $\alpha_1^L\alpha_2^L - \beta_1^+\beta_2^M > 0$

(iv) $(\mathscr{A}4)$ $\alpha_1^L - \beta_2^M - \sum_{j=1}^{m}(c_{1j}^M/e^2) - \mathscr{H}_1^M > 0, \alpha_2^L - \beta_1^M - \sum_{j=1}^{m} c_{2j}^M/e^2 - \mathscr{H}_2^M > 0$

The manuscript is planned as follows. In Section 2, we state the necessary preliminary results. We then establish, in Section 3, some simple criteria for the existence of positive periodic solutions of model (6) by coincidence degree theory [34]. The uniqueness and global attractivity of the positive periodic solution are displayed in Section 4. An example is given to illustrate our key results in Section 5.

2. Preliminaries

In this section, some related basic knowledge is displayed [34, 35].

Assume that $\mathscr{X}, \mathscr{Y}$ are normed vector spaces, $\mathscr{L}\colon \mathrm{Dom}\mathscr{L} \subset X \longrightarrow \mathscr{Y}$ is a linear mapping, and $\mathscr{N}\colon \mathscr{X} \longrightarrow \mathscr{X}$ is a continuous mapping. We call the mapping $\mathscr{L}$ a Fredholm mapping of index zero if $\dim\mathrm{Ker}\mathscr{L} = \mathrm{codim}\mathrm{Im}\mathscr{L} < +\infty$ and $\mathrm{Im}\mathscr{L}$ is closed in $\mathscr{Y}$. If $\mathscr{L}$ is a Fredholm mapping of index zero and $\exists$ continuous projectors $\mathscr{P}\colon \mathscr{X} \longrightarrow \mathscr{X}$ and $\mathscr{Q}\colon \mathscr{Y} \longrightarrow \mathscr{Y}$ which satisfy $\mathrm{Im}\mathscr{P} = \mathrm{Ker}\mathscr{L}, \mathrm{Im}\mathscr{L} = \mathrm{Ker}\mathscr{Q} = \mathrm{Im}(\mathscr{I} - \mathscr{Q})$, then $\mathscr{L}|\mathrm{Dom}\mathscr{L} \cap \mathrm{Ker}\mathscr{P}\colon (\mathscr{I} - \mathscr{P})\mathscr{X} \longrightarrow \mathrm{Im}\mathscr{L}$ is invertible [34, 35]. Let $K_{\mathscr{P}}$ denote the inverse of $\mathscr{L}|\mathrm{Dom}\mathscr{L} \cap \mathrm{Ker}\mathscr{P}$. If Ω is an open bounded subset of $\mathscr{X}$, we call the mapping $\mathscr{N}$ $\mathscr{L}$–compact on $\overline{\Omega}$ if $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ is bounded and $K_{\mathscr{P}}(\mathscr{I} - \mathscr{Q})\mathscr{N}\colon \overline{\Omega} \longrightarrow \mathscr{X}$ is compact. Since $\mathrm{Im}\mathscr{Q}$ is isomorphic to $\mathrm{Ker}\mathscr{L}$, $\exists$ isomorphisms $\mathscr{J}\colon \mathrm{Im}\mathscr{Q} \longrightarrow \mathrm{Ker}\mathscr{L}$ [34, 35].

Lemma 1 (see [34]). *Suppose that $\mathscr{L}$ is a Fredholm mapping of index zero and $\mathscr{N}$ is $\mathscr{L}$–compact on $\overline{\Omega}$. If*

(i) $\forall\lambda \in (0, 1)$, all solutions x of $\mathscr{L}x = \lambda\mathscr{N}x$ satisfy the following condition: $x \notin \partial\Omega$

(ii) $\mathscr{Q}\mathscr{N}x \neq 0, \forall x \in \mathrm{Ker}\mathscr{L} \cap \partial\Omega$, and $\deg\{\mathscr{J}\mathscr{Q}\mathscr{N}, \Omega \cap \mathrm{Ker}\mathscr{L}, 0\} \neq 0$,

then $\mathscr{L}x = \mathscr{N}x$ possesses at least one solution, which stays in $\mathrm{Dom}\mathscr{L} \cap \overline{\Omega}$

3. Existence of Positive Periodic Solutions

Theorem 2. *If $(\mathscr{A}1)$–$(\mathscr{A}3)$ are satisfied, then system (6) has at least one positive ω-periodic solution.*

Proof. Based on the practical significance of model (6), here we only discuss the positive solutions of model (6) $\forall t \geq 0$. Set

$$u_1(t) = \ln[x_1(t)], \quad u_2(t) = \ln[x_2(t)]. \tag{8}$$

In view of (6) and (8), one has

$$\begin{cases} \dfrac{du_1(t)}{dt} = -\alpha_1(t) + \beta_1(t)e^{u_2(t)-u_1(t)} \\ \quad + \displaystyle\sum_{j=1}^{m} c_{1j}(t)e^{u_1(t-\tau_{1j}(t))-u_1(t)-\gamma_{1j}(t)e^{u_1(t-\tau_{1j}(t))}} \\ \quad -\mathscr{H}_1(t)e^{u_1(t-\sigma_1(t))-u_1(t)}, \\ \dfrac{du_2(t)}{dt} = -\alpha_2(t) + \beta_2(t)e^{u_1(t)-u_2(t)} \\ \quad + \displaystyle\sum_{j=1}^{m} c_{2j}(t)e^{u_2(t-\tau_{2j}(t))-u_2(t)-\gamma_{2j}(t)e^{u_2(t-\tau_{2j}(t))}} \\ \quad -\mathscr{H}_2(t)e^{u_2(t-\sigma_2(t))-u_2(t)}. \end{cases} \tag{9}$$

Set $\mathscr{X} = \mathscr{Z} = \{u(t)\} = \{(u_1(t), u_2(t))^T | u(t) \in C(R, R^2), u(t+\omega) = u(t)\}$ and $\|u\| = \|(u_1(t), u_2(t))^T\| = \sum_{i=1}^{2} \max_{t\in[0,\omega]} |u_i(t)|$. Then, $\mathscr{X}$ and $\mathscr{Z}$ are Banach spaces.

$\mathscr{L}\colon \mathrm{Dom}\mathscr{L} \subset \mathscr{X} \longrightarrow \mathscr{Z}$ and $\mathscr{N}\colon \mathscr{X} \longrightarrow \mathscr{Z}$:

$$\mathscr{L}u = \frac{du}{dt}, \quad \mathscr{N}u = \begin{pmatrix} f_1(u_1(t), u_2(t)) \\ f_2(u_1(t), u_2(t)) \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} f_1(u_1(t), u_2(t)) &= -\alpha_1(t) + \beta_1(t)e^{u_2(t)-u_1(t)} \\ &\quad + \sum_{j=1}^{m} c_{1j}(t)e^{u_1(t-\tau_{1j}(t))-u_1(t)-\gamma_{1j}(t)e^{u_1(t-\tau_{1j}(t))}} \\ &\quad - \mathscr{H}_1(t)e^{u_1(t-\sigma_1(t))-u_1(t)}, \\ f_2(u_1(t), u_2(t)) &= -\alpha_2(t) + \beta_2(t)e^{u_1(t)-u_2(t)} \\ &\quad + \sum_{j=1}^{m} c_{2j}(t)e^{u_2(t-\tau_{2j}(t))-u_2(t)-\gamma_{2j}(t)e^{u_2(t-\tau_{2j}(t))}} \\ &\quad - \mathscr{H}_2(t)e^{u_2(t-\sigma_2(t))-u_2(t)}. \end{aligned} \tag{11}$$

Define $\mathscr{P}$ and $\mathscr{Q}$ as follows:

$$\mathscr{P}u = \frac{1}{\omega}\int_0^{\omega} u(t)dt, \quad \mathscr{Q}u = \frac{1}{\omega}\int_0^{\omega} u(t)dt, \quad u \in X, u \in \mathscr{Z}. \tag{12}$$

Hence, $\mathrm{Ker}\mathscr{L} = \{u \in \mathscr{X} | u = h \in R^2\}$ and $\mathrm{Im}\mathscr{L} = \{u \in \mathscr{Z} | \int_0^{\omega} u(t)dt = 0\}$ are closed in $\mathscr{X}$, and $\dim(\mathrm{Ker}\mathscr{L}) = 2 = \mathrm{codim}(\mathrm{Im}\mathscr{L})$. Then,

$$\mathscr{Q}\mathscr{N}u = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega \mathscr{F}(s)\mathrm{d}s_1 \\ \frac{1}{\omega}\int_0^\omega \mathscr{F}_2(s)\mathrm{d}s \end{pmatrix},$$

$$K_{\mathscr{P}}(\mathscr{I}-\mathscr{Q})\mathscr{N}u = \begin{pmatrix} \int_0^t \mathscr{F}_1(s)\mathrm{d}s \\ \int_0^t \mathscr{F}_2(s)\mathrm{d}s \end{pmatrix} - \begin{pmatrix} \frac{1}{\omega}\int_0^\omega\int_0^t \mathscr{F}(s)\mathrm{d}s\mathrm{d}t_1 \\ \frac{1}{\omega}\int_0^\omega\int_0^t \mathscr{F}_2(s)\mathrm{d}s\mathrm{d}t \end{pmatrix} - \begin{pmatrix} \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^\omega F_1(s)\mathrm{d}s \\ \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^\omega \frac{1}{\omega}\int_0^\omega\int_0^t F_2(s)\mathrm{d}s \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} \mathscr{F}_1(s) &= -\alpha_1(s) + \beta_1(t)e^{u_2(s)-u_1(s)} \\ &\quad + \sum_{j=1}^m c_{1j}(s)e^{u_1\left(s-\tau_{1j}(s)\right)-u_1(s)-\gamma_{1j}(s)e^{u_1\left(s-\tau_{1j}(s)\right)}} \\ &\quad - \mathscr{H}_1(s)e^{u_1\left(s-\sigma_1(s)\right)-u_1(s)}, \\ \mathscr{F}_2(s) &= -\alpha_2(s) + \beta_2(s)e^{u_1(s)-u_2(s)} \\ &\quad + \sum_{j=1}^m c_{2j}(s)e^{u_2\left(s-\tau_{2j}(s)\right)-u_2(s)-\gamma_{2j}(s)e^{u_2\left(s-\tau_{2j}(s)\right)}} \\ &\quad - \mathscr{H}_2(s)e^{u_2\left(s-\sigma_2(s)\right)-u_2(s)}. \end{aligned} \tag{14}$$

Clearly, $\mathscr{Q}\mathscr{N}$ and $K_{\mathscr{P}}(\mathscr{I}-\mathscr{Q})\mathscr{N}$ are continuous. We can easily check that $\overline{K_{\mathscr{P}}(\mathscr{I}-\mathscr{Q})\mathscr{N}(\overline{\Omega})}$ is compact $\forall\,\Omega\subset X$. In addition, $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ is bounded. So, $\mathscr{N}$ is $\mathscr{L}$−compact on $\overline{\Omega}\,\forall\,\Omega\subset\mathscr{X}$.

In view of $\mathscr{L}u=\lambda\mathscr{N}u, \lambda\in(0,1)$, one has

$$\begin{cases} \frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = \lambda\Big[-\alpha_1(t) + \beta_1(t)e^{u_2(t)-u_1(t)} \\ \quad + \sum_{j=1}^m c_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)-u_1(t)-\gamma_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)}} \\ \quad -\mathscr{H}_1(t)e^{u_1\left(t-\sigma_1(t)\right)-u_1(t)}\Big], \\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = \lambda\Big[-\alpha_2(t) + \beta_2(t)e^{u_1(t)-u_2(t)} \\ \quad + \sum_{j=1}^m c_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)-u_2(t)-\gamma_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)}} \\ \quad -\mathscr{H}_2(t)e^{u_2\left(t-\sigma_2(t)\right)-u_2(t)}\Big]. \end{cases} \tag{15}$$

Suppose that $u(t)=(u_1(t),u_2(t))^T\in\mathscr{X}$ is an arbitrary solution of system (15) for a certain $\lambda\in(0,1)$; according to (15), one has

$$\begin{cases} \int_0^\omega\Big[-\alpha_1(t) + \beta_1(t)e^{u_2(t)-u_1(t)} \\ \quad + \sum_{j=1}^m c_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)-u_1(t)-\gamma_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)}} \\ \quad -\mathscr{H}_1(t)e^{u_1\left(t-\sigma_1(t)\right)-u_1(t)}\Big]\mathrm{d}t = 0, \\ \int_0^\omega\Big[-\alpha_2(t) + \beta_2(t)e^{u_1(t)-u_2(t)} \\ \quad + \sum_{j=1}^m c_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)-u_2(t)-\gamma_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)}} \\ \quad -\mathscr{H}_2(t)e^{u_2\left(t-\sigma_2(t)\right)-u_2(t)}\Big]\mathrm{d}t = 0. \end{cases} \tag{16}$$

Then,

$$\int_0^\omega\Big[\beta_1(t)e^{u_2(t)-u_1(t)} + \sum_{j=1}^m c_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)-u_1(t)-\gamma_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)}} -\mathscr{H}_1(t)e^{u_1\left(t-\sigma_1(t)\right)-u_1(t)}\Big]\mathrm{d}t = \int_0^\omega \alpha_1(t)\mathrm{d}t = \overline{\alpha}_1\omega, \tag{17}$$

$$\int_0^\omega\Big[\beta_2(t)e^{u_1(t)-u_2(t)} + \sum_{j=1}^m c_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)-u_2(t)-\gamma_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)}} -\mathscr{H}_2(t)e^{u_2\left(t-\sigma_2(t)\right)-u_2(t)}\Big]\mathrm{d}t = \int_0^\omega \alpha_2(t)\mathrm{d}t = \overline{\alpha}_2\omega. \tag{18}$$

From (15), (17), and (18), we get

$$\begin{aligned} \int_0^\omega\left|\dot{u}_1(t)\right|\mathrm{d}t &= \lambda\int_0^\omega\Big| -\alpha_1(t) + \beta_1(t)e^{u_2(t)-u_1(t)} \\ &\quad + \sum_{j=1}^m c_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)-u_1(t)-\gamma_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)}} \\ &\quad - \mathscr{H}_1(t)e^{u_1\left(t-\sigma_1(t)\right)-u_1(t)}\Big|\mathrm{d}t, \\ &\le \int_0^\omega \alpha_1(t)|\mathrm{d}t + \int_0^\omega \Big|\beta_1(t)e^{u_2(t)-u_1(t)} \\ &\quad + \sum_{j=1}^m c_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)-u_1(t)-\gamma_{1j}(t)e^{u_1\left(t-\tau_{1j}(t)\right)}} \\ &\quad - \mathscr{H}_1(t)e^{u_1\left(t-\sigma_1(t)\right)-u_1(t)}\Big|\mathrm{d}t = 2\overline{\alpha}_1\omega, \end{aligned} \tag{19}$$

$$\begin{aligned} \int_0^\omega\left|\dot{u}_2(t)\right|\mathrm{d}t &= \lambda\int_0^\omega\Big| -\alpha_2(t) + \beta_2(t)e^{u_1(t)-u_2(t)} \\ &\quad + \sum_{j=1}^m c_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)-u_2(t)-\gamma_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)}} \\ &\quad - \mathscr{H}_2(t)e^{u_2\left(t-\sigma_2(t)\right)-u_2(t)}\Big|\mathrm{d}t, \\ &\le \int_0^\omega \left|\alpha_2(t)\right|\mathrm{d}t + \int_0^\omega \Big|\beta_2(t)e^{u_1(t)-u_2(t)} \\ &\quad + \sum_{j=1}^m c_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)-u_2(t)-\gamma_{2j}(t)e^{u_2\left(t-\tau_{2j}(t)\right)}} \\ &\quad - \mathscr{H}_2(t)e^{u_2\left(t-\sigma_2(t)\right)-u_2(t)}\Big|\mathrm{d}t = 2\overline{\alpha}_2\omega. \end{aligned} \tag{20}$$

Because $(x_1(t), x_2(t))^T \in \mathscr{X}$, $\exists \varrho_i, \epsilon_i \in [0,\omega]$ which satisfies

$$\begin{aligned} u_i(\varrho_i) &= \max_{t\in[0,\omega]} u_i(t), \\ u_i(\epsilon_i) &= \min_{t\in[0,\omega]} u_i(t), \\ u_i'(\epsilon_i) &= u_i'(\varrho_i) = 0, \quad i = 1, 2. \end{aligned} \tag{21}$$

Let

$$u_1(\epsilon_1 - \tau_{1j^*}(\epsilon_1)) = \max_{j=1,2,\ldots,m} u_1(\epsilon_1 - \tau_{1j}(\epsilon_1)). \tag{22}$$

Because $u_1(\epsilon_1 - \tau_{1j}(\epsilon_1)) - u_1(\epsilon_1) \geq 0$, $j = 1, 2, \ldots, m$, by (9) and (22), one has

$$\begin{aligned} &\alpha_1(\epsilon_1) + \mathscr{H}_1(\epsilon_1) e^{u_1(\epsilon_1 - \sigma_1(\epsilon_1)) - u_1(\epsilon_1)} \\ &\quad = \beta_1(\epsilon_1) e^{u_2(\epsilon_1) - u_1(\epsilon_1)} \\ &\qquad + \sum_{j=1}^m c_{1j}(\epsilon_1) e^{u_1(\epsilon_1 - \tau_{1j}(\epsilon_1)) - u_1(\epsilon_1) - \gamma_{1j}(\epsilon_1) e^{u_1(\epsilon_1 - \tau_{1j}(\epsilon_1))}} \\ &\quad \geq e^{-\gamma_1^M e^{u_1(\epsilon_1 - \tau_{1j^*}(\epsilon_1))}} \sum_{j=1}^m c_{1j}(\epsilon_1), \end{aligned} \tag{23}$$

which leads to

$$\alpha_1^M + \mathscr{H}_1^M e^{u_1(\epsilon_1 - \sigma_1(\epsilon_1)) - u_1(\epsilon_1)} \geq e^{-\gamma_1^M e^{u_1(\epsilon_1 - \tau_{1j^*}(\epsilon_1))}} \sum_{j=1}^m c_{1j}(\epsilon_1). \tag{24}$$

By $(\mathscr{A}2)$, we have

$$u_1(\epsilon_1 - \tau_{1j^*}(\epsilon_1)) \geq \ln\left[\frac{1}{\gamma_{1j}^M} \ln \frac{\sum_{j=1}^m c_{1j}^L}{\alpha_1^M + \mathscr{H}_1^M}\right]. \tag{25}$$

Set $\epsilon_1 - \tau_{1j^*}(\epsilon_1) = n\omega + \sigma$, where $\sigma \in [0,\omega]$ is an integer. By (19) and (25), one has

$$u_1(t) \geq u_1(\sigma) - \int_0^\omega |\dot{u}_1(t)| \mathrm{d}t \geq \ln\left[\frac{1}{\gamma_{1j}^M} \ln \frac{\sum_{j=1}^m c_{1j}^L}{\alpha_1^M + \mathscr{H}_1^M}\right] - 2\overline{\alpha}_1 \omega := \theta_1, \tag{26}$$

$\forall t \in R$. In a similar way, it follows from (20) and (21) that

$$u_2(t) \geq \ln\left[\frac{1}{\gamma_{2j}^M} \ln \frac{\sum_{j=1}^m c_{2j}^L}{\alpha_2^M + \mathscr{H}_2^M}\right] - 2\overline{\alpha}_2 \omega := \theta_2. \tag{27}$$

By (9) and (21), one gets

$$\begin{aligned} &\alpha_1(\varrho_1) + \mathscr{H}_1(\varrho_1) e^{u_1(\varrho_1 - \tau_1(\varrho_1)) - u_1(\varrho_1)} = \beta_1(\varrho_1) e^{u_2(\varrho_1) - u_1(\varrho_1)} \\ &\qquad + \sum_{j=1}^m c_{1j}(\varrho_1) e^{u_1(\varrho_1 - \tau_{1j}(\varrho_1)) - u_1(\varrho_1) - \gamma_{1j}(\varrho_1) e^{u_1(\varrho_1 - \tau_{1j}(\varrho_1))}} \\ &\quad = \beta_1(\varrho_1) e^{u_2(\varrho_1) - u_1(\varrho_1)} \\ &\qquad + \sum_{j=1}^m c_{1j}(\varrho_1) \frac{\gamma_{1j}(\varrho_1) e^{u_1(\varrho_1 - \tau_{1j}(\varrho_1))} e^{-\gamma_{1j}(\varrho_1) e^{u_1(\varrho_1 - \tau_{1j}(\varrho_1))}}}{\gamma_{1j}(\varrho_1) e^{u_1(\varrho_1)}}. \end{aligned} \tag{28}$$

Since $\sup_{v\geq 0}(v/e^v) = (1/e)$,

$$\begin{aligned} \alpha_1(\varrho_1) &\leq \alpha_1(\varrho_1) + \mathscr{H}_1(\varrho_1) e^{u_1(\varrho_1 - \tau_1(\varrho_1)) - u_1(\varrho_1)} \\ &\leq \beta_1(\varrho_1) e^{u_2(\varrho_1) - u_1(\varrho_1)} + \sum_{j=1}^m \frac{c_{1j}(\varrho_1)}{\gamma_{1j}(\varrho_1) e} \frac{1}{e^{u_1(\varrho_1)}}, \end{aligned} \tag{29}$$

which leads to

$$\alpha_1(\varrho_1) e^{u_1(\varrho_1)} - \beta_1(\varrho_1) e^{u_2(\varrho_2)} \leq \sum_{j=1}^m \frac{c_{1j}(\varrho_1)}{\gamma_{1j}(\varrho_1) e}. \tag{30}$$

Hence,

$$\alpha_1^L e^{u_1(\varrho_1)} - \beta_1^M e^{u_2(\varrho_2)} \leq \sum_{j=1}^m \frac{c_{1j}(\varrho_1)}{\gamma_{1j}(\varrho_1) e}. \tag{31}$$

In a similar way, one also gets

$$\alpha_2^L e^{u_2(\varrho_2)} - \beta_2^M e^{u_1(\varrho_1)} \leq \sum_{j=1}^m \frac{c_{2j}(\varrho_2)}{\gamma_{2j}(\varrho_2) e}. \tag{32}$$

In view of (31) and (32), one has

$$\begin{aligned} \begin{bmatrix} e^{u_1(\varrho_1)} \\ e^{u_2(\varrho_2)} \end{bmatrix} &= \begin{bmatrix} \alpha_1^L & -\beta_1^M \\ -\beta_2^M & \alpha_2^L \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1^L & -\beta_1^M \\ -\beta_2^M & \alpha_2^L \end{bmatrix} \begin{bmatrix} e^{u_1(\varrho_1)} \\ e^{u_2(\varrho_2)} \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1^L & -\beta_1^M \\ -\beta_2^M & \alpha_2^L \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1^L e^{u_1(\varrho_1)} - \beta_1^M e^{u_2(\varrho_2)} \\ \alpha_2^L e^{u_2(\varrho_2)} - \beta_2^M e^{u_1(\varrho_1)} \end{bmatrix} \\ &\leq \begin{bmatrix} \alpha_1^L & -\beta_1^M \\ -\beta_2^M & \alpha_2^L \end{bmatrix}^{-1} \begin{bmatrix} \sum_{j=1}^m \frac{c_{1j}(\varrho_1)}{\gamma_{1j}(\varrho_1) e} \\ \sum_{j=1}^m \frac{c_{2j}(\varrho_2)}{\gamma_{2j}(\varrho_2) e} \end{bmatrix} \\ &\leq \begin{bmatrix} \frac{\alpha_2^L}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} & \frac{\beta_1^L}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} \\ \frac{\beta_2^M}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} & \frac{\alpha_2^L}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^m \frac{c_{1j}(\varrho_1)}{\gamma_{1j}(\varrho_1) e} \\ \sum_{j=1}^m \frac{c_{2j}(\varrho_2)}{\gamma_{2j}(\varrho_2) e} \end{bmatrix} \\ &\cdot \begin{bmatrix} \frac{\alpha_2^L}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} \sum_{j=1}^m \frac{c_{1j}^M}{\gamma_{1j}^L e} + \frac{\beta_1^L}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} \sum_{j=1}^m \frac{c_{2j}^M}{\gamma_{2j}^L e} \\ \frac{\beta_2^M}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} \sum_{j=1}^m \frac{c_{1j}^M}{\gamma_{1j}^L e} + \frac{\alpha_2^L}{\alpha_1^L \alpha_2^L - \beta_1^M \beta_2^M} \sum_{j=1}^m \frac{c_{2j}^M}{\gamma_{2j}^L e} \end{bmatrix}. \end{aligned} \tag{33}$$

Then, one has

$$\begin{aligned} u_1(t) &\leq u_1(\varrho_1) \leq \theta_1^*, \\ u_2(t) &\leq u_2(\varrho_1) \leq \theta_2^*, \end{aligned} \tag{34}$$

where

$$\theta_1^* = \ln\left[\frac{\alpha_2^L}{\alpha_1^L\alpha_2^L-\beta_1^M\beta_2^M}\sum_{j=1}^m\frac{c_{1j}^M}{\gamma_{1j}^L e}+\frac{\beta_1^L}{\alpha_1^L\alpha_2^L-\beta_1^M\beta_2^M}\sum_{j=1}^m\frac{c_{2j}^M}{\gamma_{2j}^L e}\right],$$
$$\theta_2^* = \ln\left[\frac{\beta_2^M}{\alpha_1^L\alpha_2^L-\beta_1^M\beta_2^M}\sum_{j=1}^m\frac{c_{1j}^M}{\gamma_{1j}^L e}+\frac{\alpha_2^L}{\alpha_1^L\alpha_2^L-\beta_1^M\beta_2^M}\sum_{j=1}^m\frac{c_{2j}^M}{\gamma_{2j}^L e}\right]. \quad (35)$$

Let $\mathcal{M}_1 = \max\{|\theta_1|, |\theta_1^*|\}$ and $\mathcal{M}_2 = \max\{|\theta_2|, |\theta_2^*|\}$. Then, $\mathcal{M}_i\,(i = 1, 2)$ are independent of λ. Consider the equation $\mathcal{Q}\mathcal{N}u = 0$. If $u \in R^2$, then $\exists v_{1j}\,(j = 1, 2, \ldots, m)$ and $v_{2j}\,(j = 1, 2, \ldots, m)$ which satisfy

$$\mathcal{Q}\mathcal{N}u = \begin{bmatrix}\frac{1}{\omega}\int_0^\omega f(t)\mathrm{d}t_1\\ \frac{1}{\omega}\int_0^\omega f_2(t)\mathrm{d}t\end{bmatrix} = \left[-\overline{\alpha}_1+\overline{\beta}_1 e^{u_2-u_1}+\sum_{j=1}^m \overline{c}_{1j}e^{-\gamma_{1j}(v_{1j})e^{u_1}}-\overline{\mathcal{H}}_1 \quad -\overline{\alpha}_2+\overline{\beta}_2 e^{u_1-u_2}+\sum_{j=1}^m \overline{c}_{2j}e^{-\gamma_{2j}(v_{2j})e^{u_2}}-\overline{\mathcal{H}}_2\right]=0, \quad (36)$$

where

$$f_1(t) = -\alpha_1(t)+\beta_1(t)e^{u_2(t)-u_1(t)}+\sum_{j=1}^m c_{1j}(t)e^{u_1(t-\tau_{1j}(t))-u_1(t)-\gamma_{1j}(t)e^{u_1(t-\tau_{1j}(t))}} - \mathcal{H}_1(t)e^{u_1(t-\sigma_1(t))-u_1(t)}\Big],$$
$$f_2(t) = -\alpha_2(t)+\beta_2(t)e^{u_1(t)-u_2(t)}+\sum_{j=1}^m c_{2j}(t)e^{u_2(t-\tau_{2j}(t))-u_2(t)-\gamma_{2j}(t)e^{u_2(t-\tau_{2j}(t))}} - \mathcal{H}_2(t)e^{u_2(t-\sigma_2(t))-u_2(t)}\Big]. \quad (37)$$

Let $\mathcal{M}^* > 0$ be large enough. If $u^* = (u_1^*, u_2^*)^T$ is a solution of equation (36), then $\sum_{i=1}^2 |u_i^*| < \mathcal{M}^*$. We define ϕ: $\mathrm{Dom}\mathcal{L} \times [0,1] \longrightarrow \mathcal{X}$ by

$$\phi(u_1,u_2,\lambda) = \begin{bmatrix}-\overline{\alpha}_1+\sum_{j=1}^m \overline{c}_{1j}e^{-\gamma_{1j}(v_{1j})e^{u_1}}\\ -\overline{\alpha}_2+\sum_{j=1}^m \overline{c}_{2j}e^{-\gamma_{2j}(v_{2j})e^{u_2}}\end{bmatrix} + \lambda\begin{bmatrix}\overline{\beta}_1 e^{u_2-u_1}-\overline{\mathcal{H}}_1\\ \overline{\beta}_2 e^{u_1-u_2}-\overline{\mathcal{H}}_2\end{bmatrix}. \quad (38)$$

Obviously, $\phi(u_1, u_2, 1) = \mathcal{Q}\mathcal{N}u$. Similar to the analysis of model (15), if $(u_1, u_2)^T$ is a solution of $\phi(u_1, u_2, \lambda) = 0$, then $\exists$ two constants $\mathcal{M}_1^*, \mathcal{M}_2^* > 0$ which satisfy

$$|u_1| < \mathcal{M}_1^*, |u_2| < \mathcal{M}_2^*, \quad (39)$$

and $\mathcal{M}_i^*\,(i = 1, 2)$ are independent of λ. Set $\mathcal{M}_0^* = \sum_{i=1}^2 \mathcal{M}_i^*$, $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}^* + \mathcal{M}_0^*$, where $\mathcal{M}_0 = \sum_{i=1}^2 \mathcal{M}_i$. Let

$$\Omega = \left\{(u_1,u_2)^T \in R^2: \sum_{i=1}^2 |u_i| < \mathcal{M}\right\}. \quad (40)$$

Next, we will check that all assumptions of Lemma 1 hold true.

By $\mathcal{M} > \mathcal{M}_0 = \sum_{i=1}^2 \mathcal{M}_i$, one knows that if $(u_1, u_2)^T \in \partial\Omega \cap \ker\mathcal{L}$, then $\mathcal{L}u \neq \lambda\mathcal{N}u, \lambda \in (0, 1)$. So, assumption (a) of Lemma 1 holds true. By $\mathcal{M}^* < \mathcal{M}$, one knows that if $(u_1, u_2)^T \in \partial\Omega \cap \ker\mathcal{L}$, then $\phi(u_1, u_2, \lambda) \neq 0, \lambda \in (0, 1)$. Then, $\phi(u_1, u_2, 1) \neq 0$, i.e., $\mathcal{Q}\mathcal{N}u \neq 0$. Obviously, $\mathcal{Q}\mathcal{N}u = \phi(u_1, u_2, \lambda) = 0$ has a unique solution $(u_1^*, u_2^*)^T$. Set $\mathcal{J} = \mathcal{I}$: $\mathrm{Im}\mathcal{L} \longrightarrow \ker\mathcal{L}$. One knows that $\mathcal{J}\mathcal{Q}\mathcal{N}u = \mathcal{Q}\mathcal{N}u = 0$ has a unique solution. Then,

$$\begin{aligned}&\deg\{\mathcal{J}\mathcal{Q}\mathcal{N}(u_1,u_2)^T;\Omega\cap\ker\mathcal{L};0\}\\&=\deg\{\phi(u_1,u_2,1);\Omega\cap\ker\mathcal{L};0\}\\&=\deg\{\phi(u_1,u_2,0);\Omega\cap\ker\mathcal{L};0\}=\mathrm{sign}\left\{\det\begin{bmatrix}\sigma_1^* & 0\\ 0 & \sigma_2^*\end{bmatrix}\right\},\end{aligned} \quad (41)$$

where

$$\sigma_1^* = -\sum_{j=1}^m \overline{c}_{1j}e^{-\gamma_{1j}(v_{1j})e^{u_1^*}}\gamma_{1j}(v_{1j})e^{u_1^*},$$
$$\sigma_2^* = -\sum_{j=1}^m \overline{c}_{2j}e^{-\gamma_{2j}(v_{2j})e^{u_2^*}}\gamma_{2j}(v_{2j})e^{u_2^*}, \quad (42)$$

and $(u_1^*, u_2^*)^T$ is a unique solution of $\mathcal{Q}\mathcal{N}u = \phi(u_1, u_2, 0) = 0$. Then,

$$\begin{aligned}&\deg\{\mathcal{J}\mathcal{Q}\mathcal{N}(u_1,u_2,)^T;\Omega\cap\ker\mathcal{L};0\}\\&=\mathrm{sign}\left\{\sum_{j=1}^m \overline{c}_{1j}e^{-\gamma_{1j}(v_{1j})e^{u_1^*}}\gamma_{1j}(v_{1j})e^{u_1^*}\cdot\sum_{j=1}^m \overline{c}_{2j}e^{-\gamma_{2j}(v_{2j})e^{u_2^*}}\gamma_{2j}(v_{2j})e^{u_2^*}\right\}\\&=1\neq 0.\end{aligned} \quad (43)$$

Then, assumption (b) of Lemma 1 is true. So, $\mathcal{L}u = \mathcal{N}u$ has at least one solution $(u_1(t), u_2(t))^T$ in $\mathrm{Dom}\mathcal{L} \cap \overline{\Omega}$. Thus, $(x_1(t), x_2(t))^T = (e^{u_1(t)}, e^{u_2(t)})^T$ is an ω-positive periodic solution of model (6). The proof ends. □

4. Stability Behavior of Periodic Solutions

Assume that the varying delays become constants, i.e., $\sigma_1(t) = \sigma_1, \sigma_{2t} = \sigma_2$, $\tau_{1j}(t) = \tau_{1j}$, and $\tau_{2j}(t) = \tau_{2j}\,(j = 1, 2, \ldots, m)$.

Definition 1 (see [35]). A bounded positive solution $(x_1^*(t), x_2^*(t))^T$ of model (6) is said to be globally asymptotically stable if $\forall$ positive bounded solutions $(x_1(t), x_2(t))^T$ of model (6), the following equality holds:

$$\lim_{t\longrightarrow+\infty}\left[\sum_{i=1}^{2}\left|x_i(t)-x_i^*(t)\right|\right]=0. \quad (44)$$

Definition 2 (see [35, 36]). Assume that $\tilde{g}\in R$ and f is a nonnegative function defined on $[\tilde{g},+\infty)$, integrable on $[\tilde{g},+\infty)$, and uniformly continuous on $[\tilde{g},+\infty)$; then, $\lim_{t\longrightarrow+\infty}f(t)=0$.

Theorem 1. *If $(\mathscr{A}1)$–$(\mathscr{A}4)$ hold, then model (6) has a unique positive ω-periodic solution $(x_1^*(t), x_2^*(t))^T$, and this solution is global attractivity.*

Proof. According to Section 3, we know that there exists $\mathscr{T}>0$ such that

$$\begin{aligned} &e^{\theta_1}<x_1^*(t)\le e^{\theta_1^*},\\ &e^{\theta_2}<x_2^*(t)\le e^{\theta_2^*}, t>\mathscr{T}. \end{aligned} \quad (45)$$

Define

$$\mathscr{V}_1(t)=\sum_{i=1}^{2}\left|x_i^*(t)-x_i(t)\right|. \quad (46)$$

Then, $\forall t\ge T$,

$$\begin{aligned}
\mathscr{D}^+\mathscr{V}_1(t) &= \sum_{i=1}^{2}\left(\dot{x}_i^*(t)-\dot{x}_i(t)\right)\operatorname{sgn}\left(x_i^*(t)-x_i(t)\right)\\
&= \operatorname{sgn}\left(x_1^*(t)-x_1(t)\right)\left[-\alpha_1(t)\left(x_1^*(t)-x_1(t)\right)+\beta_1(t)\left(x_2^*(t)-x_2(t)\right)\right.\\
&\quad+\sum_{j=1}^{m}c_{1j}(t)\left(x_1^*\left(t-\tau_{1j}\right)e^{-\gamma_{1j}(t)/x_1^*\left(t-\tau_{1j}\right)}-x_1\left(t-\tau_{1j}\right)e^{-\gamma_{1j}(t)x_1\left(t-\tau_{1j}\right)}\right)\\
&\quad\left.-\mathscr{H}_1(t)\left(x_1^*\left(t-\tau_1(t)\right)-x_1\left(t-\tau_1\right)\right)\right]\\
&\quad+\operatorname{sgn}\left(x_2^*(t)-x_2(t)\right)\left[-\alpha_2(t)\left(x_2^*(t)-x_2(t)\right)+\beta_2(t)\left(x_1^*(t)-x_1(t)\right)\right.\\
&\quad+\sum_{j=1}^{m}c_{2j}(t)\left(x_2^*\left(t-\tau_{2j}\right)e^{-\gamma_{2j}(t)x_2^*\left(t-\tau_{2j}\right)}-x_2\left(t-\tau_{2j}\right)e^{-\gamma_{2j}(t)x_2\left(t-\tau_{2j}\right)}\right)\\
&\quad\left.-\mathscr{H}_2(t)\left(x_2^*\left(t-\tau_2\right)-x_2\left(t-\tau_2\right)\right)\right]\\
&\le -\alpha_1^L\left|x_1^*(t)-x_1(t)\right|+\beta_1^M\left|x_2^*(t)-x_2(t)\right|\\
&\quad+\sum_{j=1}^{m}\frac{c_{1j}^M}{e^2}\left|x_1^*\left(t-\tau_{1j}-x_1\left(t-\tau_{1j}\right)\right)\right|\\
&\quad+\mathscr{H}_1^M\left|x_1^*\left(t-\tau_1\right)-x_1\left(t-\tau_1\right)\right)\right|\\
&\quad-\alpha_2^L\left|x_2^*(t)-x_2(t)\right|+\beta_2^M\left|x_1^*(t)-x_1(t)\right|\\
&\quad+\sum_{j=1}^{m}\frac{c_{2j}^M}{e^2}\left|x_2^*\left(t-\tau_{2j}\right)-x_2\left(t-\tau_{2j}\right)\right|\\
&\quad+\mathscr{H}_2^M\left|x_2^*\left(t-\tau_2\right)-x_2\left(t-\tau_2\right)\right|.
\end{aligned} \quad (47)$$

Define

$$\begin{aligned}\mathcal{V}_2(t) &= \sum_{j=1}^{m}\frac{c_{1j}^M}{e^2}\int_{t-\tau_{1j}}^{t}\left|x_1^*(s)-x_1(s)\right|\mathrm{d}s+\sum_{j=1}^{m}\frac{c_{2j}^M}{e^2}\int_{t-\tau_{2j}}^{t}\left|x_2^*(s)-x_2(s)\right|\mathrm{d}s\\ &\quad+\mathcal{H}_1^M\int_{t-\tau_1}^{t}\left|x_1^*(s)-x_1(s)\right|\mathrm{d}s+\mathcal{H}_2^M\int_{t-\tau_2}^{t}\left|x_2^*(s)-x_2(s)\right|\mathrm{d}s.\end{aligned}\tag{48}$$

It follows from (6) that, $\forall t\ge\mathcal{T}$,

$$\begin{aligned}\mathcal{D}^+\mathcal{V}_2(t) &= \sum_{j=1}^{m}\frac{c_{1j}^M}{e^2}\left|x_1^*(t)-x_1(t)\right|-\sum_{j=1}^{m}\frac{c_{1j}^M}{e^2}\left|x_1^*\left(t-\tau_{1j}\right)-x_1\left(t-\tau_{1j}\right)\right|\\ &\quad+\sum_{j=1}^{m}\frac{c_{2j}^M}{e^2}\left|x_2^*(t)-x_2(t)\right|-\sum_{j=1}^{m}\frac{c_{2j}^M}{e^2}\left|x_2^*\left(t-\tau_{2j}\right)-x_2\left(t-\tau_{2j}\right)\right|\\ &\quad+\mathcal{H}_1^M\left|x_1^*(t)-x_1(t)\right|-\mathcal{H}_1^M\left|x_1^*(t-\tau_1)-x_1(t-\tau_1)\right|\\ &\quad+\mathcal{H}_2^M\left|x_2^*(t)-x_2(t)\right|-\mathcal{H}_2^M\left|x_2^*(t-\tau_2)-x_1(t-\tau_2)\right|.\end{aligned}\tag{49}$$

Set the following Lyapunov function:

$$\mathcal{V}(t)=\mathcal{V}(t)+\mathcal{V}_2(t).\tag{50}$$

It follows from (47), (49), and (50) that

$$\mathcal{D}^+\mathcal{V}(t)\le-\sum_{i=1}^{2}\eta_i\left|x_i^*(t)-x_i(t)\right|,\tag{51}$$

where

$$\begin{aligned}\eta_1 &= \alpha_1^L-\beta_2^M-\sum_{j=1}^{m}\frac{c_{1j}^M}{e^2}-\mathcal{H}_1^M,\\ \eta_2 &= \alpha_2^L-\beta_1^M-\sum_{j=1}^{m}\frac{c_{2j}^M}{e^2}-\mathcal{H}_2^M.\end{aligned}\tag{52}$$

According to (H4), $\exists$ constants $\alpha_i^*\,(i=1,2,)$ and $\mathcal{T}^*>\mathcal{T}$ which satisfy

$$\eta_i(t)\ge\alpha_i^*>0,\quad (i=1,2),\ \text{for}\ t\ge\mathcal{T}^*.\tag{53}$$

By (53), one has

$$\mathcal{V}(t)+\sum_{i=1}^{2}\int_{\mathcal{T}^*}^{t}\eta_i(t)\left|x_i^*(t)-x_i(t)\right|\mathrm{d}s\le\mathcal{V}(\mathcal{T}^*).\tag{54}$$

By (53) and (54), one gets

$$\sum_{i=1}^{2}\int_{T^*}^{t}\eta_i(t)\left|x_i^*(t)-x_i(t)\right|\mathrm{d}s\le V(T^*)<\infty,\quad \text{for}\ t\ge\mathcal{T}^*.\tag{55}$$

Because $x_i^*(t)\,(i=1,2)$ are bounded $\forall t\ge\mathcal{T}^*$, $|x_i^*(t)-x_i(t)|\,(i=1,2)$ are uniformly continuous on $[\mathcal{T}^*,\infty)$. Applying Barbalat's lemma [36], one has

$$\lim_{t\longrightarrow\infty}\left|x_i^*(t)-x_i(t)\right|=0,\quad i=1,2.\tag{56}$$

According to Theorems 7.4 and 8.2 of [37], one knows that $(x_1^*(t),x_2^*(t))^T$ of system (6) is uniformly asymptotically stable. We end the proof. □

Remark 1. Zhou [17] considered the positive periodic solution of the Nicholson-type delay model which is a special form of (6), Liu [22] investigated the existence and uniqueness of positive periodic solutions of the Nicholson-type delay model without linear harvesting terms, and Liu and Meng [33] studied the positive almost periodic solution for model (6). To the best of our knowledge, no author considers the problems of the positive periodic solution of model (6). All the results in [17, 22, 33] and the references therein cannot be applicable to prove that system (6) has a unique positive ω-periodic solution which is global attractivity. This implies that the results of this article are new and that they complement earlier investigations.

Remark 2. In [38], the authors dealt with the periodic solution of the ratio-dependent food-chain system with delays by applying coincidence degree theory. In this paper, we investigate the periodic solution of the continuous delayed Nicholson-type model with harvesting terms, and some inequality techniques are different from those in [38]. In [33, 39], the authors dealt with almost periodic solutions for Nicholson's blowfly model. They did not involve the periodic

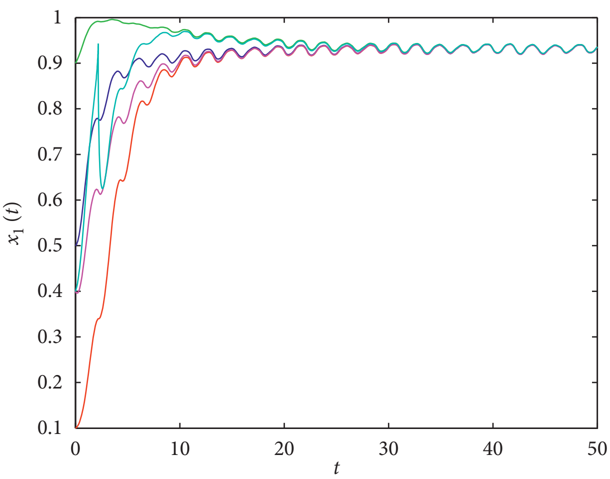

FIGURE 1: Computer simulation of model (57): t–x_1.

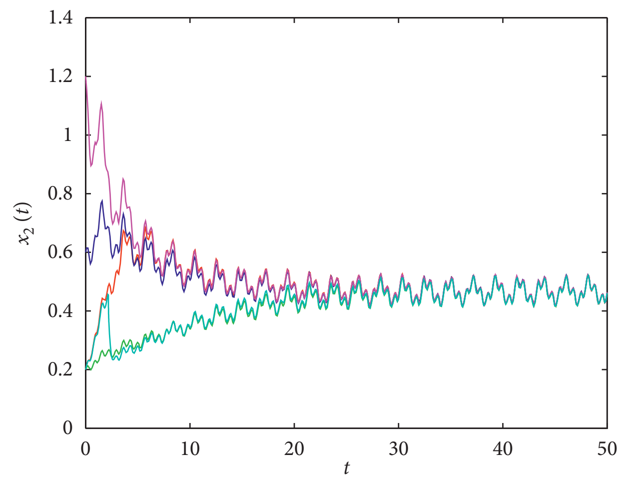

FIGURE 2: Computer simulation of model (57): t–x_2.

solution of the delayed Nicholson-type model. So, our works supplement the previous publications.

5. Software Simulations

Give the following model:

$$\begin{cases} \dfrac{dx_1}{dt} = -e^{-3}\left(0.1 + 0.3\sin^2 10t\right)x_1(t) + e^{-4}\left(0.2 + 0.1\cos^2 10t\right)x_2(t) \\ + e^{-2}\left(0.1 + 0.2\cos^2 10t\right)x_1(t-0.003)e^{-0.02x_1(t-0.003)} \\ + e^{-2}\left(0.2 + 0.1\sin^2 10t\right)x_1(t-0.002)e^{-0.01x_1(t-0.002)} \\ - e^{-3}\left(0.2 + 0.1\cos^2 10t\right)x_1(t-0.004), \\ \dfrac{dx_2}{dt} = -e^{-3}\left(0.1 + 0.2\cos^2 100t\right)x_2(t) + e^{-4}\left(0.2 + 0.2\sin^2 10t\right)x_1(t) \\ + e^{-2}\left(0.1 + 0.3\sin^2 10t\right)x_1(t-0.002)e^{-0.01x_1(t-0.002)} \\ + e^{-2}\left(0.1 + 0.2\sin^2 10t\right)x_1(t-0.001)e^{-0.02x_1(t-0.002)} \\ - e^{-3}\left(0.3 + 0.2\sin^2 10t\right)x_2(t-0.004). \end{cases} \tag{57}$$

Then,

$$\begin{aligned}
\alpha_1(t) &= e^{-3}\left(0.1 + 0.3\sin^2 10t\right),\\
\alpha_2(t) &= e^{-3}\left(0.1 + 0.2\cos^2 10t\right),\\
\beta_1(t) &= e^{-4}\left(0.2 + 0.1\cos^2 10t\right),\\
\beta_2(t) &= e^{-4}\left(0.2 + 0.2\sin^2 10t\right),\\
c_{11}(t) &= e^{-2}\left(0.1 + 0.2\cos^2 10t\right),\\
c_{12}(t) &= e^{-2}\left(0.2 + 0.1\sin^2 10t\right),\\
c_{21}(t) &= e^{-2}\left(0.1 + 0.3\sin^2 10t\right),\\
c_{22}(t) &= e^{-2}\left(0.1 + 0.2\sin^2 10t\right),\\
\mathscr{H}_1(t) &= e^{-3}\left(0.2 + 0.1\cos^2 10t\right),\\
\mathscr{H}_2(t) &= e^{-3}\left(0.3 + 0.2\sin^2 10t\right),\\
\gamma_{11}(t) &= 0.02,\\
\gamma_{12}(t) &= 0.01,\\
\gamma_{21}(t) &= 0.01,\\
\gamma_{22}(t) &= 0.02,\\
\sigma_1(t) &= 0.004,\\
\sigma_2(t) &= 0.004,\\
\tau_{11}(t) &= 0.003,\\
\tau_{12}(t) &= 0.002,\\
\tau_{21}(t) &= 0.002,\\
\tau_{22}(t) &= 0.001.
\end{aligned} \tag{58}$$

By direct computation, one has

$$\begin{aligned}
\alpha_1^M &= e^{-3}0.4,\\
\alpha_1^L &= e^{-3}0.1,\\
\alpha_2^M &= e^{-3}0.3,\\
\alpha_2^L &= e^{-3}0.1,\\
\beta_1^M &= e^{-4}0.3,\\
\beta_2^M &= e^{-4}0.4,\\
c_{11}^- &= e^{-2}0.1,\\
c_{12}^- &= e^{-2}0.2,\\
c_{21}^- &= e^{-2}0.1,\\
c_{22}^- &= e^{-2}0.1,\\
\mathscr{H}_1^M &= e^{-3}0.3,\\
\mathscr{H}_2^M &= e^{-3}0.5,\\
\gamma_{11}^M &= 0.02,\\
\gamma_{12}^M &= 0.01,\\
\gamma_{21}^M &= 0.01,\\
\gamma_{22}^M &= 0.02.
\end{aligned} \tag{59}$$

One can easily check that all the hypotheses $(\mathscr{H}1)$–$(\mathscr{H}4)$ are fulfilled. Thus, one can know that system (57) has a unique positive $(\pi/10)$-periodic solution which is uniformly asymptotically stable. These results are displayed in Figures 1 and 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors were supported by the National Natural Science Foundation of China (no. 61673008), Project of High-Level Innovative Talents of Guizhou Province ([2016]5651), Major Research Project of the Innovation Group of the Education Department of Guizhou Province ([2017]039), the Hunan Provincial Natural Science Foundation of China (no. 2020JJ4516), Key Project of Hunan Education Department (17A181), Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Changsha University of Science and Technology) (2018MMAEZD21), University Science and Technology Top Talents Project of Guizhou Province (KY[2018]047), and Guizhou University of Finance and Economics (2018XZD01).

References

[1] A. Nicholson, "An outline of the dynamics of animal populations," *Australian Journal of Zoology*, vol. 2, no. 1, pp. 9–65, 1954.

[2] W. S. C. Gurney, S. P. Blythe, and R. M. Nisbet, "Nicholson's blowflies revisited," *Nature*, vol. 287, no. 5777, pp. 17–21, 1980.

[3] J. W. H. So and J. S. Yu, "On the stability and uniform persistence of a discrete model of Nicholson's blowflies," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 1, pp. 233–244, 1995.

[4] M. R. S. Kulenovic, G. Ladas, and Y. G. Sficas, "Global attractivity in population dynamics," *Computer & Mathematics with Applications*, vol. 18, no. 10-11, pp. 925–928, 1989.

[5] X. H. Ding and W. X. Li, "Stability and bifurcation of numerical discretization Nicholson's blowflies equation with delay," *Discrete Dynamics in Nature and Society*, vol. 2006, p. 12, Article ID 01, 2006.

[6] S. H. Saker and B. G. Zhang, "Oscillation in a discrete partial delay Nicholson's blowflies model," *Mathematical and Computer Modelling*, vol. 36, no. 9-10, pp. 1021–1026, 2002.

[7] J. Li and C. Du, "Existence of positive periodic solutions for a generalized Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 221, no. 1, pp. 226–233, 2008.

[8] W.-T. Li and Y.-H. Fan, "Existence and global attractivity of positive periodic solutions for the impulsive delay Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 201, no. 1, pp. 55–68, 2007.

[9] B. G. Zhang and H. X. Xu, "A note on the global attractivity of a discrete model of nicholson's blowflies," *Discrete Dynamics in Nature and Society*, vol. 3, no. 1, pp. 51–55, 1999.
[10] J. Wei and M. Y. Li, "Hopf bifurcation analysis in a delayed Nicholson blowflies equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 60, no. 7, pp. 1351–1367, 2005.
[11] S. H. Saker and S. Agarwal, "Oscillation and global attractivity in a periodic Nicholson's blowflies model," *Mathematical and Computer Modelling*, vol. 35, no. 7-8, pp. 719–731, 2002.
[12] B. Liu, "Global dynamic behaviors for a delayed Nicholson's blowflies model with a linear harvesting term," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 45, no. 45, pp. 1–13, 2013.
[13] J. O. Alzabut, "Almost periodic solutions for an impulsive delay Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 234, no. 1, pp. 233–239, 2010.
[14] T. Faria, "Global asymptotic behaviour for a Nicholson model with patch structure and multiple delays," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 18, pp. 7033–7046, 2011.
[15] L. V. Hien, "Global asymptotic behaviour of positive solutions to a non-autonomous Nicholson's blowflies model with delays," *Journal of Biological Dynamics*, vol. 8, no. 1, pp. 135–144, 2014.
[16] P. Amster and A. Déboli, "Existence of positive -periodic solutions of a generalized Nicholson's blowflies model with a nonlinear harvesting term," *Applied Mathematics Letters*, vol. 25, no. 9, pp. 1203–1207, 2012.
[17] Q. Zhou, "The positive periodic solution for Nicholson-type delay system with linear harvesting terms," *Applied Mathematical Modelling*, vol. 37, no. 8, pp. 5581–5590, 2013.
[18] F. Long, "Positive almost periodic solution for a class of Nicholson's blowflies model with a linear harvesting term," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 2, pp. 686–693, 2012.
[19] H.-S. Ding and J. J. Nieto, "A new approach for positive almost periodic solutions to a class of Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 253, pp. 249–254, 2013.
[20] L. Duan and L. Huang, "Pseudo almost periodic dynamics of delay Nicholson's blowflies model with a linear harvesting term," *Mathematical Methods in the Applied Sciences*, vol. 38, no. 6, pp. 1178–1189, 2015.
[21] W. Chen and B. Liu, "Positive almost periodic solution for a class of Nicholson's blowflies model with multiple time-varying delays," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2090–2097, 2011.
[22] B. Liu, "The existence and uniqueness of positive periodic solutions of Nicholson-type delay systems," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3145–3151, 2011.
[23] Y. Chen, "Periodic solutions of delayed periodic Nicholson's blowflies model," *Canadian Applied Mathematics Quarterly*, vol. 11, pp. 1838–1844, 2003.
[24] H. Yang, "Weighted pseudo almost periodicity on neutral type CNNs involving multi-proportional delays and D operator," *AIMS Mathematics*, vol. 6, no. 2, pp. 1865–1879, 2021.
[25] C. Huang and Y. Tan, "Global behavior of a reaction-diffusion model with time delay and dirichlet condition," *Journal of Differential Equations*, vol. 271, pp. 186–215, 2021.
[26] X. Long, "Novel stability criteria on a patch structure Nicholson's blowflies model with multiple pairs of time-varying delays," *AIMS Mathematics*, vol. 5, no. 6, pp. 7387–7401, 2020.
[27] C. Huang, X. Zhao, J. Cao, and F. E. Alsaadi, "Global dynamics of neoclassical growth model with multiple pairs of variable delays," *Nonlinearity*, vol. 33, no. 12, pp. 6819–6834, 2020.
[28] C. Huang, H. Zhang, J. Cao, and H. Hu, "Stability and Hopf bifurcation of a delayed prey-predator model with disease in the predator," *International Journal of Bifurcation and Chaos*, vol. 29, no. 07, Article ID 1950091, 2019.
[29] C. X. Huang, H. D. Yang, and J. D. Cao, "Weighted pseudo almost periodicity of multi-proportional delayed shunting inhibitory cellular neural networks with D operator," *Discrete and Continuous Dynamical Systems Series S*, 2020.
[30] L. Berezansky, L. Idels, and L. Troib, "Global dynamics of Nicholson-type delay systems with applications," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 436–445, 2011.
[31] W. Wang, L. Wang, and W. Chen, "Existence and exponential stability of positive almost periodic solution for Nicholson-type delay systems," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 4, pp. 1938–1949, 2011.
[32] L. Berezansky, E. Braverman, and L. Idels, "Nicholson's blowflies differential equations revisited: main results and open problems," *Applied Mathematical Modelling*, vol. 34, no. 6, pp. 1405–1417, 2010.
[33] X. Liu and J. Meng, "The positive almost periodic solution for Nicholson-type delay systems with linear harvesting terms," *Applied Mathematical Modelling*, vol. 36, no. 7, pp. 3289–3298, 2012.
[34] R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer-Verlag, Berlin, China, 1997.
[35] C. Xu and M. Liao, "Periodicity in an stage-structured three-species predator-prey system with beddington-deangelis and hollinf iv functional response," *Asian-European Journal of Mathematics*, vol. 05, no. 02, p. 1250031, 2012.
[36] K. Goplsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht/Norwell, MA, USA, 1992.
[37] T. Yoshizawa, *Stability Theory by Lyapunov' Second Method*, Mathematics Society, Tokyo, Japan, 1966.
[38] C. J. Xu, M. X. Liao, and P. L. Li, "Periodic property and asymptotic behavior for a discrete ratio-dependent food-chain system with delays," *Discrete Dynamics in Nature and Society*, vol. 2020, p. 12, Article ID 9464532, 2020.
[39] C. Xu, M. X. Liao, M. Liao, P. Li, Q. Xiao, and S. Yuan, "A new method to investigate almost periodic solutions for an Nicholson's blowflies model with time-varying delays and a linear harvesting term," *Mathematical Biosciences and Engineering*, vol. 16, no. 5, pp. 3830–3840, 2019.