Research Article

Factorizable Ordered Hypergroupoids with Applications

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In this study, we propose the concept of factorizable ordered hypergroupoids (semihypergroups) and present several of its properties. Our goal is to construct ordered hypergroupoid from blood groups together with some other information. Finally, we discuss right magnifying elements for further research.

1. Introduction and Preliminaries

Marty [1] proposed the notion of a hypergroup based on the multivalued operations. Hyperstructures have many applications in both pure and applied sciences. Some researchers have tried to discuss important biological phenomena in the framework of fuzzy hyperstructures. Hyperstructures were used in many disciplines such as theoretical physics, coding theory, and biology. Basic concepts and relevant applications concerning hyperstructure theory can be found in [2, 3].

Ordered semihypergroups are suggested by Heidari and Davvaz [4] and then investigated by Davvaz et al. in [5] (also, see [6–8]). The research about generalization of hyperideals in ordered hyperstructures is growing rapidly [9, 10]. In recent years, pseudoorders have received extensive attention in ordered hyperstructures [5, 8]. Using the notion of (weak) pseudoorder [5, 8], several examples of ordered semi (hyper) groups have been constructed in connection with ordered semihypergroups. Weak pseudoorders can remarkably support the constructions of ordered semihypergroups [8].

Breakable semihypergroups were firstly presented by Heidari and Cristea [11] in 2019. In a breakable semihypergroup, each nonempty subset is a subsemihypergroup. The cyclicity of the EL-hyperstructures is expressed in [12]. Recently, much attention has been paid to investigating the factorizable hyperstructures. In [13], Heidari and Cristea

have suggested the concept of factorizable semihypergroups by using the concept of factorizable semigroups [14]. In this regard, Munir et al. [15] initiated the study of factorizable hypergroupoids and discussed their properties. For future work, one could extend the existing works [13, 15] to the framework of fuzzy sets.

In this note, we offer basic concepts on factorizable ordered hypergroupoids. We show that if a right hyperideal R and a left hyperideal L form the factors of an ordered hypergroupoid S , then $R \cap L = (R \diamond L)$. Connection between regular and factorizable ordered semihypergroups is presented. Moreover, we prove that if an ordered hypergroupoid contains either a right (left) magnifying element, then it is factorizable.

Now, we describe several information on an ordered hypergroupoid (semihypergroup) $(S, \diamond, \leq)$.

A hyperoperation is a mapping $\diamond : S \times S \longrightarrow P^*(S)$, where $P^*(S)$ denotes the family of all nonempty subsets of S . The couple $(S, \diamond)$ is called a hypergroupoid. If $\emptyset \neq U, V \subseteq S$ and $x \in S$, then

$$U \diamond V = \bigcup_{\substack{u \in U \\ v \in V}} u \diamond v, \quad (1)$$

$$x \diamond U = \{x\} \diamond U,$$

$$U \diamond x = U \diamond \{x\}.$$

A semihypergroup is a hypergroupoid $(S, \diamond)$, where

$$(u \diamond v) \diamond w = u \diamond (v \diamond w), \quad \text{for all } u, v, w \in S. \quad (2)$$

A semihypergroup S is a hypergroup if the reproduction axiom is verified as

$$x \diamond S = S = S \diamond x, \quad \text{for all } x \in S. \quad (3)$$

A hyperoperation $\diamond$ on $\emptyset \neq S$ is extensive [12] if $\{a, b\} \subseteq a \diamond b$ for all $a, b \in S$.

Definition 1 (see [4, 5]). A hypergroupoid $(S, \diamond)$ is called an ordered hypergroupoid (semihypergroup) if

- (1) $(S, \leq)$ is an ordered set
- (2) $u \leq v$ implies $u \diamond w \leq v \diamond w$ and $w \diamond u \leq w \diamond v$, for all $u, v, w \in S$

If $\emptyset \neq A, B \subseteq S$, then

$$A < B \Leftrightarrow \forall a \in A, \exists b \in B; a \leq b. \quad (4)$$

Here, $\emptyset \neq K$ of an ordered hypergroupoid (semihypergroup) $(S, \diamond, \leq)$ is called a subhypergroupoid (subsemihypergroup) of S if $a \diamond b \subseteq K$ for every $a, b \in K$ and $\leq_K$ is the relation $\leq$ restricted to K . Set

$$[I] := \{x \in S \mid x \leq i \text{ for some } i \in I\}. \quad (5)$$

If $\emptyset \neq I, J \subseteq S$, then

- (1) $I \subseteq [I]$
- (2) If $I \subseteq J$, then $[I] \subseteq [J]$
- (3) $([I] \diamond [J]) = [I \diamond J]$

An ordered hypergroupoid $(S, \diamond, \leq)$ is regular if $w \in (w \diamond S \diamond w)$ for every $w \in S$.

Definition 2 (see [4, 5]). A set $\emptyset \neq K$ of an ordered hypergroupoid $(S, \diamond, \leq)$ is left (resp. right) hyperideal if

- (1) $S \diamond K \subseteq K$ (resp. $K \diamond S \subseteq K$)
- (2) $[K] \subseteq K$

Here, a hyperideal is a left hyperideal of S being right hyperideal.

2. Results and Discussion

We extend the definition in [15] to the ordered case. In this section, we pay attention to the factorizable ordered hypergroupoids that is obligatory to prove our proposing results.

Definition 3. An ordered hypergroupoid (semihypergroup) $(S, \diamond, \leq)$ is said to be factorizable if there exist two different proper subhypergroupoids (subsemihypergroups) U and V of S such that $S = (U \diamond V)$. The pair (U, V) is called a factorization of S , with factors U and V .

Example 1. Consider an ordered hypergroup $S = \{R, W\}$ with the following hyperoperation $\diamond$ (as shown in Table 1) and (partial) order relation $\leq$.

$$\leq := \{(R, R), (W, W)\}. \quad (6)$$

The subhypergroups $\{R\}$ and $\{W\}$ then form the factorization of S as

$$(\{R\} \diamond \{W\}) = \{S\} = S. \quad (7)$$

There are four main categories within the ABO blood group system: A, B, O , and AB . The detail of blood groups is presented in [15, 16], and here we will not repeat them again. An application of ordered hypergroupoid to blood groups is given.

Example 2. The ABO blood groups are giving in a set

$$S = \{A, B, AB, O\}. \quad (8)$$

Define the hyperoperation $\diamond$ (as shown in Table 2) and (partial) order relation $\leq$ on S as follows.

$$\begin{aligned} \leq := & \{(O, O), (O, A), (O, B), (O, AB), (A, A), (A, AB), \\ & (B, B), (B, AB), (AB, AB)\}. \end{aligned} \quad (9)$$

Then, $(S, \diamond, \leq)$ is an ordered hypergroupoid. The covering relation of S is given by

$$< = \{(O, A), (O, B), (A, AB), (B, AB)\}. \quad (10)$$

The Hasse diagram of S is shown in Figure 1.

It is easily seen that

- (1) Clearly, $O \leq A \leq AB$ and $O \leq B \leq AB$. From the figure, it can be obviously seen that O is the donor blood group and AB is the recipient blood group. Also, $A \leq AB$ shows that people with blood group A can donate blood to people with blood groups A and AB . Similarly, $B \leq AB$ shows that people with blood group B can donate blood to people with blood groups B and AB .
- (2) The subhypergroupoids $U = \{A, O\}$ and $V = \{B, O\}$ form the factorization of S as $(U \diamond V) = \{S\} = S$.
- (3) S is a simple ordered hypergroupoid. It can be seen that S is simple if and only if $S = (S \diamond x \diamond S)$ for every $x \in S$.

Theorem 1. Let $(S, \diamond, \leq)$ be an ordered hypergroupoid. If the right hyperideal R and the left hyperideal L form the factors of S , then

$$R \cap L = (R \diamond L). \quad (11)$$

Proof. As $R \diamond L \subseteq S \diamond L \subseteq L$ and $R \diamond L \subseteq R \diamond S \subseteq R$, we have $(R \diamond L) \subseteq (R \cap L)$. Now, we obtain

TABLE 1: Codominant alleles of a gene.

$\diamond$	R	W
R	R	$\{R, W\}$
W	$\{R, W\}$	W

TABLE 2: ABO blood group inheritance.

$\diamond$	A	B	AB	O
A	$\{A, O\}$	S	$\{A, B, AB\}$	$\{A, O\}$
B	S	$\{B, O\}$	$\{A, B, AB\}$	$\{B, O\}$
AB	$\{A, B, AB\}$	$\{A, B, AB\}$	$\{A, B, AB\}$	$\{A, B\}$
O	$\{A, O\}$	$\{B, O\}$	$\{A, O\}$	O

FIGURE 1: Figure of S for Example 1.

$$\begin{aligned}
 (R \diamond L) &\subseteq (R \cap L) \\
 &\subseteq (R) \cap (L) \\
 &= R \cap L.
 \end{aligned} \tag{12}$$

We need to show that $R \cap L \subseteq (R \diamond L)$. Since (R, L) is a factorization of S , it follows that $(R \diamond L) = S$. It implies that $R \cap L \subseteq S = (R \diamond L)$. $\square$

Theorem 2. If $(S, \diamond, \leq)$ is an ordered hypergroup, then S is regular.

Proof. Take any $x \in S$. Then,

$$(x \diamond S) = S = (S \diamond x). \tag{13}$$

So, $x \leq x \diamond a$ for some $a \in S$. Again, $a \leq b \diamond x$ for some $b \in S$. Thus,

$$x \leq x \diamond a \leq x \diamond (b \diamond x). \tag{14}$$

It implies that $x \in (x \diamond S \diamond x)$. Hence, S is regular. $\square$

Theorem 3. Let $(S, \diamond, \leq)$ be an ordered semihypergroup, where $\leq := \{(x, y) | x = y\}$. Let S be factorizable as $S = (U \diamond V)$, where U is a group and $|u \diamond v| = 1$ for all $u \in U$ and $v \in V$. If V is a regular ordered semihypergroup, then S is also a regular ordered semihypergroup.

Proof. Take any $x \in S = (U \diamond V)$. Then, $x \leq u \diamond v$ for some $u \in U$ and $v \in V$. By assumption, V is regular. Consider the

element $v \in V$ and assume that $v \leq v \diamond v' \diamond v$ for some $v' \in V$. So,

$$\begin{aligned}
 x &\leq u \diamond v \\
 &\leq u \diamond (v \diamond v' \diamond v) \\
 &\leq u \diamond v \diamond v' \diamond (u^{-1} \diamond u) \diamond v \\
 &= (u \diamond v) \diamond v' \diamond u^{-1} \diamond (u \diamond v).
 \end{aligned} \tag{15}$$

Thus, $x \leq x \diamond w \diamond x$ for some $w \in v' \diamond u^{-1} \subseteq S$. Hence, S is regular. $\square$

Corollary 1. Let $(S, \diamond, \leq)$ be an ordered semihypergroup, where $\leq := \{(x, y) | x = y\}$. Let S be factorizable as $S = (U \diamond V)$, where U is a group and $|u \diamond v| = 1$ for all $u \in U$ and $v \in V$. If V is an ordered hypergroup, then S is regular.

Proof. By Theorem 2, V is a regular ordered semihypergroup. Now, by Theorem 3, S is also regular. $\square$

An ordered hypergroupoid $(S, \diamond, \leq)$ is said to be extensive if $\{a, b\} \subseteq a \diamond b$ for all $a, b \in S$.

Corollary 2. Let $(S, \diamond, \leq)$ be an ordered semihypergroup, where $\leq := \{(x, y) | x = y\}$. Let S be factorizable as $S = (U \diamond V)$, where U is a group and $|u \diamond v| = 1$ for all $u \in U$ and $v \in V$. If V is an extensive ordered semihypergroup, then S is regular.

Proof. Let V be an extensive ordered semihypergroup. Then, $\{a, b\} \subseteq a \diamond b$ for all $a, b \in S$. So,

$$t \diamond S = \bigcup_{s \in S} t \diamond s = S, \tag{16}$$

for all $t \in S$. Hence, S is an ordered hypergroup. Now, by Corollary 1, S is regular. $\square$

Definition 4. An element x of an ordered hypergroupoid $(S, \diamond, \leq)$ is said to be a right (resp. left) magnifying element if there exists a proper subhypergroupoid U of S such that $S = (U \diamond x)$ (resp. $S = (x \diamond U)$).

Example 3. In Example 2, $U = \{A, O\}$ is a subhypergroupoid of S . We have

$$\begin{aligned}
 (U \diamond \{AB\}) &= (\{A, O\} \diamond \{AB\}) = (\{A, B, AB\}) = S, \\
 (U \diamond \{B\}) &= (\{A, O\} \diamond \{B\}) = (S) = S.
 \end{aligned} \tag{17}$$

Thus, AB and B are right magnifying elements of S . We consider next the subhypergroupoid $V = \{B, O\}$. Observe that if $V = \{B, O\}$, then AB and A are right magnifying elements of S .

Example 4. Given a hypergroup $S = \{a, b, c, d\}$, let the hyperoperation $\diamond$ be defined in Table 3.

Consider an ordered hypergroup $(S, \diamond, \leq)$ with the following (partial) order relation $\leq$:

TABLE 3: Table of $\diamond$ for Example 4.

$\diamond$	a	b	c	d
a	a	a	$\{a, b, c\}$	$\{a, b, d\}$
b	a	a	$\{a, b, c\}$	$\{a, b, d\}$
c	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{c, d\}$
d	$\{a, b, d\}$	$\{a, b, d\}$	$\{c, d\}$	$\{a, b, d\}$

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}. \quad (18)$$

Note that based on the data from Table 3, $U = \{a, b, c\}$ is a subhypergroup of S . If $x = d$, then

$$(U \diamond x) = (\{a, b, c\} \diamond d) = \{S\} = S. \quad (19)$$

Hence, d is a right (left) magnifying element of S .

An ordered hypergroupoid $(S, \diamond, \leq)$ is called cyclic, if there exists a generator $x \in S$ such that for all $s \in S$, there exists $n \in \mathbb{N}$ such that $s \in x^n$.

Proposition 1. *Let $(S, \diamond, \leq)$ be an ordered hypergroupoid. If S contains either a right (left) magnifying element x , then S is factorizable.*

Proof. We shall prove this for right magnifying element x . We consider the following cases.

Case 1. Let S be noncyclic.

If x is a right magnifying element of S , then $S = (U \diamond x)$, where U is a proper subhypergroupoid of S . Since S is noncyclic, it follows that S is factorizable because $S = (U \diamond V)$, where $V = \langle x \rangle$.

Case 2. Let S be cyclic and $S = \langle a \rangle$.

As $x \in S$ is a right magnifying element, $S = (U \diamond x)$, where U is a proper subhypergroupoid of S . Since U is a proper subhypergroupoid of S , we get $U = \langle a^k \rangle$, where $k \in \mathbb{N}$. Moreover, for $l \in \mathbb{N}$, let $x = a^l$. Hence,

$$\begin{aligned} S &= (U \diamond x) \\ &= (\langle a^k \rangle \diamond a^l) \\ &= (\langle a^{k+l} \rangle) \\ &= (\langle a^m \rangle), \end{aligned} \quad (20)$$

where $m = k + l$, which is a contradiction because $(\langle a^m \rangle)$ is a proper subhypergroupoid of S . Now, the proof is completed.

Definition 5. An ordered hypergroupoid (semihypergroup) $(S, \diamond, \leq)$ is said to be generalized factorizable if there exist two different proper subsets U and V of S such that $S = (U \diamond V)$. The pair (U, V) is called a generalized factorization of S , with generalized factors U and V .

Example 5

- (1) Consider the ordered hypergroupoid $S = \{A, B, AB, O\}$ defined in Example 2. The subsets $\{A\}$ and $\{B\}$ then form the generalized factorization of S as

$$(\{A\} \diamond \{B\}) = \{S\} = S. \quad (21)$$

Thus, S can be composed by the parents possessing only A and B blood groups.

- (2) Consider the ordered hypergroup $S = \{a, b, c, d\}$ defined in Example 4. The subsets $\{a, c\}$ and $\{b, d\}$ then form the generalized factorization of S as

$$(\{a, c\} \diamond \{b, d\}) = \{S\} = S. \quad (22)$$

3. Conclusions

In this paper, we have described factorizable ordered hypergroupoids. We have also shown some results in this respect. An application into the fields of blood groups and factorizable ordered hypergroupoid theory was briefly introduced. We finished our study with generalized factorizable ordered hypergroupoids in hope that other factorizations such as (m, n) -factorizable ordered hypergroupoids can be discussed in the future. In the future, one can study applications of factorization in DNA coding theory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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