Research Article

New Existence of Solutions for Fractional Integro-Differential Equations with Nonseparated Boundary Conditions

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Received 2 April 2021; Revised 7 May 2021; Accepted 11 June 2021; Published 24 June 2021

Academic Editor: Xinan Hao

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Results reported in this article prove the existence and uniqueness of solutions for a class of nonlinear fractional integro-differential equations supplemented by nonseparated boundary value conditions. We consider a new norm to establish the existence of solution via Krasnoselskii fixed point theorem; however, the uniqueness results are obtained by applying the contraction mapping principle. Some examples are provided to illustrate the results.

1. Introduction

Fractional differential equations have been an important tool to describe many problems and processes in different fields of science. In fact, fractional models are more realistic than the classical models. Fractional differential equations appear in many fields such as physics, economics, image processing, blood flow phenomena, aerodynamics, and so on. For more details about fractional differential equations and their applications, we provide the following references [1–13].

Recently, fractional integro-differential equations were investigated by many researchers in different problems, and a lot of papers were published in this matter (see, for example, [14–16]).

Furthermore, many boundary conditions were considered for the fractional-order integro-differential equations; some of these conditions are the classical, periodic, anti-periodic, nonlocal, multipoint, and the integral boundary conditions (for details, see [17–22]).

On the other hand, many papers have considered the nonseparated boundary conditions as they are a very important class of boundary value conditions (we refer the readers to [23–28]).

Motivated by the above discussion, in this paper, we establish the existence and uniqueness of solutions for a class of fractional integro-differential equations with nonseparated boundary value conditions as follows:

$$\begin{cases} {}^c D^\alpha ({}^c D^\beta x)(t) = f(t, x(t), \phi x(t), \psi x(t)), & t \in [0, 1], \\ x(0) = \lambda_1 x(1), \\ x'(0) = \lambda_2 x'(1), \\ {}^c D^\beta x(0) = \lambda_3 {}^c D^\beta x(1), \\ {}^c D^{\beta+1} x(0) = \lambda_4 {}^c D^{\beta+1} x(1), \end{cases} \quad (1)$$

where $1 < \alpha \leq 2$, $1 < \beta \leq 2$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} - \{0, 1\}$, ${}^c D^\beta$, ${}^c D^\alpha$ are Caputo's fractional derivatives, $f: [0; 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a given continuous function, and

$$\begin{aligned}\phi x(t) &= \int_0^t \lambda(t, s)x(s)ds, \\ \psi x(t) &= \int_0^t \delta(t, s)x(s)ds,\end{aligned}\quad (2)$$

where $\lambda, \delta: [0, 1] \times [0, 1] \rightarrow [0, +\infty)$, with $\phi^* = \sup_{t \in [0, 1]} |\int_0^t \lambda(t, s)ds| < \infty$, $\psi^* = \sup_{t \in [0, 1]} |\int_0^t \delta(t, s)ds| < \infty$.

Our motivation comes from the fact that not many papers have considered the existence and uniqueness results of nonlinear integro-differential equations with non-separated boundary conditions. On top of that, we show the existence results under some weak conditions. The main results in this paper can be viewed as an extension of those provided in [22].

This paper is divided into five sections. In Section 2, we provide some notations and basic known results. In Section 3, we study the new existence results to problem (1) under some weak conditions, and after that, we show the existence and uniqueness using Banach's contraction principle. In Section 4, we give two examples to illustrate the results. We end the paper with a conclusion.

2. Preliminaries and Notations

In this section, we state some notations, definitions, and lemmas which are used in this paper.

Definition 1 (see [5]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds. \quad (3)$$

Definition 2 (see [5]). The Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad (4)$$

where $n \in \mathbb{N}$, $0 \leq n-1 < \alpha < n$, $t > 0$.

Theorem 1 (see [29]). Let M be a bounded, closed, convex, and nonempty subset of a Banach space X . Let A and B be operators such that

- (i) $Ax + By \in M$ whenever $x, y \in M$.
- (ii) A is compact and continuous.
- (iii) B is a contraction mapping.

Then, there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2 (see [30]). (Arzelà–Ascoli theorem).

A set of functions in $C[a, b]$ is relatively compact if and only it is uniformly bounded and equicontinuous on $[a, b]$.

Theorem 3 (see [30]). If a set is closed and relatively compact, then it is compact.

Lemma 1 (see [5]). Let $\alpha, \beta \geq 0$; then, the following relation holds:

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \quad (5)$$

Lemma 2 (see [5]). Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have

$$I^\alpha ({}^c D^\alpha f(t)) = f(t) + a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1}. \quad (6)$$

Lemma 3. Let $y \in C([0, 1], \mathbb{R})$. Then, a unique solution of the following boundary value problem:

$$\begin{cases} {}^c D^\alpha ({}^c D^\beta)x(t) = y(t), & t \in [0, 1], \\ x(0) = \lambda_1 x(1), \\ x'(0) = \lambda_2 x'(1), \\ {}^c D^\beta x(0) = \lambda_3 {}^c D^\beta x(1), \\ {}^c D^{\beta+1} x(0) = \lambda_4 {}^c D^{\beta+1} x(1), \end{cases} \quad (7)$$

is given by

$$\begin{aligned}x(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} y(s)ds + \frac{A_1(t)}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-1} y(s)ds \\ &\quad + \frac{A_2(t)}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s)ds + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha+\beta)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha+\beta-1} y(s)ds \\ &\quad + \frac{A_3(t)}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} y(s)ds,\end{aligned}\quad (8)$$

where

$$\begin{aligned}
A_1(t) &= \frac{\lambda_3}{1-\lambda_3} \left(\frac{t^\beta}{\Gamma(\beta+1)} + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\beta+1)} + \frac{\lambda_2\lambda_1}{(1-\lambda_2)(1-\lambda_1)\Gamma(\beta)} + \frac{t\lambda_2}{(1-\lambda_2)\Gamma(\beta)} \right), \\
A_2(t) &= \frac{\lambda_4}{1-\lambda_4} \left(\frac{t^\beta\lambda_3}{\Gamma(\beta+1)(1-\lambda_3)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{\lambda_2\lambda_1}{(1-\lambda_2)(1-\lambda_1)\Gamma(\beta+1)} \right. \\
&\quad + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\beta+2)} + \frac{\lambda_3\lambda_2\lambda_1}{(1-\lambda_3)(1-\lambda_2)(1-\lambda_1)\Gamma(\beta)} + \frac{\lambda_3\lambda_1}{(1-\lambda_3)(1-\lambda_1)\Gamma(\beta+1)} \\
&\quad \left. + \frac{\lambda_3\lambda_2t}{(1-\lambda_3)(1-\lambda_2)\Gamma(\beta)} + \frac{t\lambda_2}{(1-\lambda_2)\Gamma(\beta+1)} \right), \\
A_3(t) &= \frac{\lambda_2}{1-\lambda_2} \left(t + \frac{\lambda_1}{1-\lambda_1} \right).
\end{aligned} \tag{9}$$

Proof. By Lemma 2, we have

$$\begin{aligned}
{}^c D^\beta x(t) &= I^\alpha y(t) + c_0 + c_1 t, \\
{}^c D^{\beta+1} x(t) &= I^{\alpha-1} y(t) + c_1, \\
x(t) &= I^{\alpha+\beta} y(t) + I^\beta c_0 + I^\beta c_1 t + c_2 + c_3 t,
\end{aligned} \tag{10}$$

where $c_0, c_1, c_2, c_3 \in \mathbb{R}$. Using the condition ${}^c D^{\beta+1} x(0) = \lambda_4 {}^c D^{\beta+1} x(1)$, we get

$$c_1 = \frac{\lambda_4}{(1-\lambda_4)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds. \tag{11}$$

By applying ${}^c D^\beta x(0) = \lambda_3 {}^c D^\beta x(1)$, we have

$$\begin{aligned}
c_0 &= \frac{\lambda_3}{(1-\lambda_3)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
&\quad + \frac{\lambda_3\lambda_4}{(1-\lambda_4)(1-\lambda_3)\Gamma(\alpha-1)} \\
&\quad \times \int_0^1 (1-s)^{\alpha-2} y(s) ds.
\end{aligned} \tag{12}$$

Now we use $x'(0) = \lambda_2 x'(1)$ to get

$$\begin{aligned}
c_3 &= \frac{\lambda_2}{(1-\lambda_2)\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} y(s) ds \\
&\quad + \frac{\lambda_2\lambda_4}{(1-\lambda_4)(1-\lambda_2)\Gamma(\beta+1)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\
&\quad + \frac{\lambda_2\lambda_3\lambda_4}{(1-\lambda_4)(1-\lambda_3)(1-\lambda_2)\Gamma(\beta)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\
&\quad + \frac{\lambda_2\lambda_3}{(1-\lambda_3)(1-\lambda_2)\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} y(s) ds.
\end{aligned} \tag{13}$$

In view of $x(0) = \lambda_1 x(1)$, we have

$$\begin{aligned}
 c_2 = & \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} y(s) ds + \frac{\lambda_1 \lambda_2}{(1-\lambda_1)(1-\lambda_2)\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} y(s) ds \\
 & + \left(\frac{\lambda_1 \lambda_3}{(1-\lambda_1)(1-\lambda_3)\Gamma(\beta+1)} + \frac{\lambda_1 \lambda_2 \lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)\Gamma(\beta)} \right) \\
 & \times \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
 & + \left(\frac{\lambda_1 \lambda_3 \lambda_4}{(1-\lambda_1)(1-\lambda_3)(1-\lambda_4)\Gamma(\beta+1)} + \frac{\lambda_1 \lambda_4}{(1-\lambda_1)(1-\lambda_4)\Gamma(\beta+2)} \right. \\
 & \left. + \frac{\lambda_1 \lambda_2 \lambda_4}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_4)\Gamma(\beta+1)} \right. \\
 & \left. + \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)(1-\lambda_4)\Gamma(\beta)} \right) \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds.
 \end{aligned} \tag{14}$$

Substituting the value of c_0 , c_1 , and c_2 , we get the desired results. Directly computing, one can prove the converse of the lemma. $\square$

3. Main Results

Denote by X the Banach space of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ endowed with the norms

$\|x\| = \sup\{|x(t)|: t \in [0, 1]\}$ and $\|y\|_v = \sup_{t \in [0, 1]} (|y(t)|/e^{vt})$, where $v > (1 + \phi^* + \phi^*/\Gamma(\alpha + \beta))\|\sigma\|$, and σ will be defined later.

By Lemma 3, we transform problem (1) into a fixed point problem as $x = Px$, where $P: X \rightarrow X$ is given by

$$\begin{aligned}
 Px(t) = & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s), \phi x(s), \psi x(s)) ds \\
 & + \frac{A_1(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi x(s), \psi x(s)) ds \\
 & + \frac{A_2(t)}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, x(s), \phi x(s), \psi x(s)) ds \\
 & + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} f(s, x(s), \phi x(s), \psi x(s)) ds \\
 & + \frac{A_3(t)}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} f(s, x(s), \phi x(s), \psi x(s)) ds.
 \end{aligned} \tag{15}$$

For computational convenience, we set

$$\begin{aligned}
 A_1 &= \frac{|\lambda_3|}{|1-\lambda_3|} \left(\frac{1}{\Gamma(\beta+1)} + \frac{|\lambda_1|}{|(1-\lambda_1)\Gamma(\beta+1)|} + \frac{|\lambda_2\lambda_1|}{|(1-\lambda_2)(1-\lambda_1)\Gamma(\beta)|} + \frac{|\lambda_2|}{|(1-\lambda_2)\Gamma(\beta)|} \right), \\
 A_2 &= \frac{|\lambda_4|}{|1-\lambda_4|} \left(\frac{|\lambda_3|}{\Gamma(\beta+1)|1-\lambda_3|} + \frac{1}{\Gamma(\beta+2)} + \frac{|\lambda_2\lambda_1|}{|(1-\lambda_2)(1-\lambda_1)\Gamma(\beta+1)|} \right. \\
 &\quad \left. + \frac{|\lambda_1|}{|(1-\lambda_1)\Gamma(\beta+2)|} + \frac{|\lambda_3\lambda_2\lambda_1|}{|(1-\lambda_3)(1-\lambda_2)(1-\lambda_1)\Gamma(\beta)|} \right. \\
 &\quad \left. + \frac{|\lambda_3\lambda_1|}{|(1-\lambda_3)(1-\lambda_1)\Gamma(\beta+1)|} + \frac{|\lambda_3\lambda_2|}{|(1-\lambda_3)(1-\lambda_2)\Gamma(\beta)|} \right. \\
 &\quad \left. + \frac{|\lambda_2|}{|(1-\lambda_2)\Gamma(\beta+1)|} \right), \\
 A_3 &= \frac{|\lambda_2|}{|1-\lambda_2|} \left(1 + \frac{|\lambda_1|}{|1-\lambda_1|} \right).
 \end{aligned} \tag{16}$$

Theorem 4. Suppose that

(H_1) For all $t \in [0, 1]$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$, we have $|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq \sigma(t)(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|)$ with $\sigma \in C([0, 1]; [t0, \infty))$.

(H_2) $|f(t, x, y, z)| \leq \theta(t)$, $\forall (t, x, y, z) \in [0, 1] \times \mathbb{R}^3$ with $\theta \in C([0, 1]; \mathbb{R}^+)$.

Then, problem (1) has at least one solution.

Proof. Define $B_{r'} = \{x \in X: \|x\|_v \leq r'\}$ with

$$r' \geq \frac{\|\theta\|_v (e^v - 1)}{v} \left(\frac{1}{\Gamma(\alpha + \beta)} + \frac{A_1}{\Gamma(\alpha)} + \frac{A_2}{\Gamma(\alpha - 1)} + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha + \beta)} + \frac{A_3}{\Gamma(\alpha + \beta - 1)} \right). \tag{17}$$

We introduce the decomposition $P = P_1 + P_2$, where

$$\begin{aligned}
 P_1 x(t) &= \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s), \phi x(s), \psi x(s)) ds, \\
 P_2 x(t) &= \frac{A_1(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi x(s), \psi x(s)) ds \\
 &\quad + \frac{A_2(t)}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, x(s), \phi x(s), \psi x(s)) ds \\
 &\quad + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha+\beta-1} f(s, x(s), \phi x(s), \psi x(s)) ds \\
 &\quad + \frac{A_3(t)}{\Gamma(\alpha + \beta - 1)} \int_0^1 (1-s)^{\alpha+\beta-2} f(s, x(s), \phi x(s), \psi x(s)) ds.
 \end{aligned} \tag{18}$$

For $x, y \in B_{r'}$, we have

$$\begin{aligned}
\|P_1 x(t)\|_v &\leq \sup_{t \in [0,1]} \frac{1}{e^{vt}} \left| \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s), \phi x(s), \psi x(s)) ds \right| \\
&\leq \sup_{t \in [0,1]} \frac{1}{e^{vt}} \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} |\theta(s)| ds \\
&\leq \sup_{t \in [0,1]} \frac{1}{e^{vt}} \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} \frac{|\theta(s)| e^{vs}}{e^{vs}} ds \\
&\leq \sup_{t \in [0,1]} \frac{\|\theta\|_v}{e^{vt} \Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} e^{vs} ds \\
&\leq \sup_{t \in [0,1]} \frac{\|\theta\|_v}{e^{vt} \Gamma(\alpha + \beta)} \int_0^t e^{vs} ds \\
&\leq \sup_{t \in [0,1]} \frac{\|\theta\|_v}{v \Gamma(\alpha + \beta)} \frac{e^{vt} - 1}{e^{vt}} \\
&\leq \frac{\|\theta\|_v (e^v - 1)}{v \Gamma(\alpha + \beta)}, \\
\|P_2 y(t)\|_v &\leq \sup_{t \in [0,1]} \frac{1}{e^{vt}} \left| \frac{A_1(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s), \phi y(s), \psi y(s)) ds \right. \\
&\quad + \frac{A_2(t)}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, y(s), \phi y(s), \psi y(s)) ds \\
&\quad + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} f(s, y(s), \phi y(s), \psi y(s)) ds \\
&\quad \left. + \frac{A_3(t)}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} f(s, y(s), \phi y(s), \psi y(s)) ds \right| \\
&\leq \sup_{t \in [0,1]} \frac{1}{e^{vt}} \left[\frac{A_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \frac{|\theta(s)| e^{vs}}{e^{vs}} ds + \frac{A_2}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \times \frac{|\theta(s)| e^{vs}}{e^{vs}} ds \right. \\
&\quad \left. + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \frac{|\theta(s)| e^{vs}}{e^{vs}} ds + \frac{A_3}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} \frac{|\theta(s)| e^{vs}}{e^{vs}} ds \right] \\
&\leq \sup_{t \in [0,1]} \frac{\|\theta\|_v}{e^{vt}} \left[\frac{A_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} e^{vs} ds + \frac{A_2}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} e^{vs} ds \right. \\
&\quad \left. + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} e^{vs} ds + \frac{A_3}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} e^{vs} ds \right] \\
&\leq \sup_{t \in [0,1]} \frac{\|\theta\|_v}{e^{vt}} \left[\frac{A_1}{\Gamma(\alpha)} \int_0^1 e^{vs} ds + \frac{A_2}{\Gamma(\alpha-1)} \int_0^1 e^{vs} ds + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} \times \int_0^1 e^{vs} ds + \frac{A_3}{\Gamma(\alpha+\beta-1)} \int_0^1 e^{vs} ds \right] \\
&\leq \sup_{t \in [0,1]} \frac{e^v - 1}{v e^{vt}} \|\theta\|_v \left(\frac{A_1}{\Gamma(\alpha)} + \frac{A_2}{\Gamma(\alpha-1)} + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} + \frac{A_3}{\Gamma(\alpha+\beta-1)} \right) \\
&\leq \frac{e^v - 1}{v} \|\theta\|_v \left(\frac{A_1}{\Gamma(\alpha)} + \frac{A_2}{\Gamma(\alpha-1)} + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} + \frac{A_3}{\Gamma(\alpha+\beta-1)} \right).
\end{aligned} \tag{19}$$

Thus,

$$\|P_1x + P_2y\|_v \leq \frac{\|\theta\|_v(e^v - 1)}{v} \left(\frac{1}{\Gamma(\alpha + \beta)} + \frac{A_1}{\Gamma(\alpha)} + \frac{A_2}{\Gamma(\alpha - 1)} + \frac{|\lambda_1|}{|1 - \lambda_1|\Gamma(\alpha + \beta)} + \frac{A_3}{\Gamma(\alpha + \beta - 1)} \right). \quad (20)$$

Hence, $P_1x + P_2y \in B_{r'}$.

For $x, y \in B_{r'}$, consider

$$\begin{aligned} \|P_1y(t) - P_1x(t)\|_v &\leq \sup_{t \in [0, 1]} \frac{1}{\Gamma(\alpha + \beta)e^{vt}} \int_0^t (t-s)^{\alpha+\beta-1} \\ &\quad \times |f(s, y(s), \phi y(s), \psi y(s)) - f(s, x(s), \phi x(s), \psi x(s))| ds \\ &\leq \sup_{t \in [0, 1]} \frac{1}{\Gamma(\alpha + \beta)e^{vt}} \int_0^t (t-s)^{\alpha+\beta-1} \sigma(s) (|y(s) - x(s)| + |\phi y(s) - \phi x(s)| + |\psi y(s) - \psi x(s)|) ds \\ &\leq \sup_{t \in [0, 1]} \frac{\|\sigma\|}{\Gamma(\alpha + \beta)e^{vt}} \int_0^t e^{vs} (\|y - x\|_v + \phi^* \|y - x\|_v + \psi^* \|y - x\|_v) ds \\ &\leq \sup_{t \in [0, 1]} \frac{(1 + \phi^* + \psi^*) \|\sigma\|}{v\Gamma(\alpha + \beta)} \frac{e^{vt} - 1}{e^{vt}} \|y - x\|_v \\ &\leq \frac{(1 + \phi^* + \psi^*) \|\sigma\| \|y - x\|_v}{\Gamma(\alpha + \beta)v}. \end{aligned} \quad (21)$$

By using the condition of the new norm, we have that P_1 is a contraction.

Next, we will show that P_2 is compact and continuous.

Continuity of f implies that the operator P_2 is continuous. Also, P_2 is uniformly bounded on $B_{r'}$ as

$$\|P_2y\|_v \leq \frac{e^v - 1}{v} \|\theta\|_v \left(\frac{A_1}{\Gamma(\alpha)} + \frac{A_2}{\Gamma(\alpha - 1)} + \frac{|\lambda_1|}{|1 - \lambda_1|\Gamma(\alpha + \beta)} + \frac{A_3}{\Gamma(\alpha + \beta - 1)} \right). \quad (22)$$

Suppose that $0 \leq t_1 < t_2 \leq 1$. We have

$$\begin{aligned} |P_2y(t_2) - P_2y(t_1)| &\leq \frac{|A_1(t_2) - A_1(t_1)|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, y(s), \phi y(s), \psi y(s))| \\ &\quad \times ds + \frac{|A_1(t_2) - A_1(t_1)|}{\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} \\ &\quad \times |f(s, y(s), \phi y(s), \psi y(s))| ds + \frac{|A_3(t_2) - A_3(t_1)|}{\Gamma(\alpha + \beta - 1)} \\ &\quad \times \int_0^1 (1-s)^{\alpha+\beta-2} |f(s, y(s), \phi y(s), \psi y(s))| ds. \end{aligned} \quad (23)$$

Thus,

$$|P_2 y(t_2) - P_2 y(t_1)| \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2 \text{ independently from } y \in B_{r'}.$$
(24)

This proves that the operator P_2 is relatively compact on $B_{r'}$. Then, by the Arzelà–Ascoli theorem, we have that P_2 is compact on $B_{r'}$.

Therefore, problem (1) has at least one solution on $B_{r'}$. $\square$

Theorem 5. Assume that $f: [0, 1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous function satisfying

(H₁) For all $t \in [0, 1]$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$, we have $|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq \Sigma(t) (|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|)$ with $\Sigma \in L^1([0, 1]; [t_0, \infty))$ and $\Sigma^* = \int_0^1 \Sigma(s) ds$.

Then, there exists a unique solution for the boundary value problem (1) provided that

$$r_1 = (1 + \phi^* + \psi^*) \Sigma^* \left(\frac{1}{\Gamma(\alpha + \beta)} + \frac{A_1}{\Gamma(\alpha)} + \frac{A_2}{\Gamma(\alpha - 1)} + \frac{|\lambda_1|}{|(1 - \lambda_1)|\Gamma(\alpha + \beta)} + \frac{A_3}{\Gamma(\alpha + \beta - 1)} \right) < 1. \quad (25)$$

Proof. Set $\sup_{0 \leq t \leq 1} |f(t, 0, 0, 0)| = M$.

Let $B_r = \{x \in X: \|x\| \leq r\}$ be the ball with radius r , where $r \geq (r_2/1 - r_1)$, with

$$r_2 = \frac{M}{\Gamma(\alpha + \beta)} + \frac{A_1 M}{\Gamma(\alpha)} + \frac{A_2 M}{\Gamma(\alpha - 1)} + \frac{|\lambda_1| M}{|(1 - \lambda_1)|\Gamma(\alpha + \beta)} + \frac{A_3 M}{\Gamma(\alpha + \beta - 1)}.$$
(26)

Then, B_r is a closed, convex, and nonempty subset of the Banach space X .

Our objective is to show that the operator P has a unique fixed point on B_r .

We prove that $P B_r \subseteq B_r$.

For $x \in B_r$, $t \in [0, 1]$, we have

$$\begin{aligned} |Px(t)| &\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s, x(s), \phi x(s), \psi x(s))| ds \\ &\quad + \frac{|A_1(t)|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, x(s), \phi x(s), \psi x(s))| ds \\ &\quad + \frac{|A_2(t)|}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x(s), \phi x(s), \psi x(s))| ds \\ &\quad + \frac{|\lambda_1|}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} |f(s, x(s), \phi x(s), \psi x(s))| ds \\ &\quad + \frac{|A_3(t)|}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} |f(s, x(s), \phi x(s), \psi x(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} (|f(s, x(s), \phi x(s), \psi x(s)) - f(s, 0, 0, 0)| + |f(s, 0, 0, 0)|) ds \\ &\quad + \frac{|A_1(t)|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (|f(s, x(s), \phi x(s), \psi x(s)) - f(s, 0, 0, 0)| + |f(s, 0, 0, 0)|) ds \\ &\quad + \frac{|A_2(t)|}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \times (|f(s, x(s), \phi x(s), \psi x(s)) - f(s, 0, 0, 0)| + |f(s, 0, 0, 0)|) ds \\ &\quad + \frac{|\lambda_1|}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} (|f(s, x(s), \phi x(s), \psi x(s)) - f(s, 0, 0, 0)| + |f(s, 0, 0, 0)|) ds \\ &\quad + \frac{|A_3(t)|}{\Gamma(\alpha+\beta-1)} \\ &\quad \times \int_0^1 (1-s)^{\alpha+\beta-2} |f(s, x(s), \phi x(s), \psi x(s)) - f(s, 0, 0, 0)| + |f(s, 0, 0, 0)| ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} (\Sigma(s)(|x(s)|+|\phi x(s)|+|\psi x(s)|)+M)ds \\
&\quad + \frac{A_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (\Sigma(s)(|x(s)|+|\phi x(s)|+|\psi x(s)|)+M)ds \\
&\quad + \frac{A_2}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} (\Sigma(s)(|x(s)|+|\phi x(s)|+|\psi x(s)|)+M)ds \\
&\quad + \frac{|\lambda_1|}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} (\Sigma(s)(|x(s)|+|\phi x(s)|+|\psi x(s)|)+M)ds \\
&\quad + \frac{A_3}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} (\Sigma(s) \times (|x(s)|+|\phi x(s)|+|\psi x(s)|)+M)ds \\
&\leq \frac{(1+\phi^*+\psi^*)\|x\|}{\Gamma(\alpha+\beta)} \int_0^1 \Sigma(s)ds + \frac{M}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} ds + \frac{(1+\phi^*+\psi^*)A_1\|x\|}{\Gamma(\alpha)} \int_0^1 \Sigma(s)ds \\
&\quad + \frac{A_1M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{(1+\phi^*+\psi^*)A_2\|x\|}{\Gamma(\alpha-1)} \int_0^1 \Sigma(s)ds + \frac{A_2M}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \\
&\quad + \frac{|\lambda_1|(1+\phi^*+\psi^*)\|x\|}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} \int_0^1 \Sigma(s)ds + \frac{|\lambda_1|M}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} ds \\
&\quad + \frac{A_3(1+\phi^*+\psi^*)\|x\|}{\Gamma(\alpha+\beta-1)} \int_0^1 \Sigma(s)ds + \frac{A_3M}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} ds,
\end{aligned} \tag{27}$$

which implies that

$$\begin{aligned}
\|Px\| &\leq \frac{(1+\phi^*+\psi^*)\Sigma^*\|x\|}{\Gamma(\alpha+\beta)} + \frac{M}{\Gamma(\alpha+\beta)} \\
&\quad + \frac{(1+\phi^*+\psi^*)\Sigma^*A_1\|x\|}{\Gamma(\alpha)} + \frac{A_1M}{\Gamma(\alpha)} \\
&\quad + \frac{(1+\phi^*+\psi^*)\Sigma^*A_2\|x\|}{\Gamma(\alpha-1)} + \frac{A_2M}{\Gamma(\alpha-1)} \\
&\quad + \frac{|\lambda_1|(1+\phi^*+\psi^*)\Sigma^*\|x\|}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} + \frac{|\lambda_1|M}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} \\
&\quad + \frac{A_3(1+\phi^*+\psi^*)\Sigma^*\|x\|}{\Gamma(\alpha+\beta-1)} + \frac{A_3M}{\Gamma(\alpha+\beta-1)} \\
&\leq \left[\frac{(1+\phi^*+\psi^*)\Sigma^*}{\Gamma(\alpha+\beta)} + \frac{(1+\phi^*+\psi^*)\Sigma^*A_1}{\Gamma(\alpha)} + \frac{(1+\phi^*+\psi^*)\Sigma^*A_2}{\Gamma(\alpha-1)} + \frac{|\lambda_1|(1+\phi^*+\psi^*)\Sigma^*}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} + \frac{A_3(1+\phi^*+\psi^*)\Sigma^*}{\Gamma(\alpha+\beta-1)} \right] \\
&\quad \|x\| + \frac{M}{\Gamma(\alpha+\beta)} + \frac{A_1M}{\Gamma(\alpha)} + \frac{A_2M}{\Gamma(\alpha-1)} + \frac{|\lambda_1|M}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} + \frac{A_3M}{\Gamma(\alpha+\beta-1)} \\
&\leq r_1r + r_2 \\
&\leq r.
\end{aligned} \tag{28}$$

Now, for $x, y \in B_r$ and for $t \in [0, 1]$,

$$\begin{aligned}
 |Px(t) - Py(t)| &\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s, x(s), \phi x(s), \psi x(s)) - f(s, y(s), \phi y(s), \psi y(s))| ds \\
 &\quad + \frac{|A_1(t)|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, x(s), \phi x(s), \psi x(s)) - f(s, y(s), \phi y(s), \psi y(s))| ds \\
 &\quad + \frac{|A_2(t)|}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x(s), \phi x(s), \psi x(s)) - f(s, y(s), \phi y(s), \psi y(s))| ds \\
 &\quad + \frac{|\lambda_1|}{|(1-\lambda_1)|\Gamma(\alpha+\beta)} \\
 &\quad \times \int_0^1 (1-s)^{\alpha+\beta-1} |f(s, x(s), \phi x(s), \psi x(s)) - f(s, y(s), \phi y(s), \psi y(s))| ds \\
 &\quad + \frac{|A_3(t)|}{\Gamma(\alpha+\beta-1)} \\
 &\quad \times \int_0^1 (1-s)^{\alpha+\beta-2} |f(s, x(s), \phi x(s), \psi x(s)) - f(s, y(s), \phi y(s), \psi y(s))| ds,
 \end{aligned} \tag{29}$$

and then

$$\|Px - Py\| \leq r_1 \|x - y\|. \tag{30}$$

Since $r_1 < 1$, then the operator P is a contraction mapping. Thus, problem (1) has a unique solution. $\square$

4. Examples

Example 1. Consider the following fractional problem:

$$\left\{ \begin{aligned}
 {}^c D^{(17/11)} ({}^c D^{(15/11)}) x(t) &= \frac{t^4}{800} \left(\frac{|x(t)|e^{-t}}{1+|x(t)|} + \int_0^t \frac{(t+s)^4 |x(s)| (e^{-2s} + e^{-3s})}{800(1+|x(s)|)} ds \right), \quad t \in [0, 1], \\
 x(0) &= \frac{1}{200} x(1), \\
 x'(0) &= \frac{1}{200} x'(1), \\
 {}^c D^{(15/11)} x(0) &= \frac{1}{200} {}^c D^{(15/11)} x(1), \\
 {}^c D^{(26/11)} x(0) &= \frac{1}{200} {}^c D^{(26/11)} x(1).
 \end{aligned} \right. \tag{31}$$

We choose

$$\beta = \frac{15}{11},$$

$$\alpha = \frac{17}{11},$$

$$f(t, x, y, z) = \frac{t^4}{800} \left(\frac{|x(t)|e^{-t}}{1+|x(t)|} + \frac{|y(t)|e^{-2t}}{1+|y(t)|} + \frac{|z(t)|e^{-3t}}{1+|z(t)|} \right),$$

$$\lambda(t, s) = \delta(t, s) = \frac{(t+s)^4}{800}.$$

(32)

We have

$$|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq \frac{t^4}{800} (|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|), \quad \text{for all } t \in [0, 1] \text{ and } x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R},$$

$$|f(t, x_1, x_2, x_3)| \leq \frac{3t^4}{800}.$$

(33)

Clearly, $\phi^* = \psi^* = (31/4000)$, $\|\sigma\| = (1/800)$.

Hence, by Theorem 4, problem (31) has a least one solution.

Example 2. Consider the following boundary value problem:

$$\left\{ \begin{array}{l} {}^c D^{(10/6)} ({}^c D^{(8/6)}) x(t) = \frac{t^3}{3000} \left(\frac{1}{1+|x(t)|} + \frac{1}{1000} \int_0^t t^5 s^4 x(s) ds \right), \quad t \in [0, 1], \\ x(0) = \frac{1}{201} x(1), \\ x'(0) = \frac{1}{201} x'(1), \\ {}^c D^{(8/6)} x(0) = \frac{1}{201} {}^c D^{(8/6)} x(1), \\ {}^c D^{(14/6)} x(0) = \frac{1}{201} {}^c D^{(14/6)} x(1). \end{array} \right. \quad (34)$$

Here $\beta = (8/6)$, $\alpha = (10/6)$, $f(t, x, y, z) = (t^3/3000) ((1/1+|x(t)|) + y(t) + z(t))$, $\lambda(t, s) = \delta(t, s) = (t^5 s^4/2000)$.

We get

$$|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq \frac{t^3}{3000} (|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|), \quad \text{for all } t \in [0, 1] \text{ and } x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}. \quad (35)$$

Clearly, $\phi^* = \psi^* = (1/10000)$, $\Sigma^* = (1/12000)$.
Hence, we have $r_1 \approx 0.000043 < 1$.
Then, problem (34) has a unique solution.

5. Conclusion

In this paper, new existence and uniqueness results have been studied for nonlinear fractional integro-differential equations equipped with nonseparated conditions. By using a new norm, we have established the new existence of solution for (1) under some weak conditions. The uniqueness of solutions has been given by using the sup norm and applying the Banach fixed point theorem. We have ended the article with some examples to illustrate the results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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