Research Article

A Hybrid Method for Complex Pythagorean Fuzzy Decision Making

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This article takes advantage of advancements in two different fields in order to produce a novel decision-making framework. First, we contribute to the theory of aggregation operators, which are mappings that combine large amounts of data into more advantageous forms. They are extensively used in different settings from classical to fuzzy set theory alike. Secondly, we expand the literature on complex Pythagorean fuzzy model, which has an edge over other models due to its ability to handle uncertain data of periodic nature. We propose some aggregation operators for complex Pythagorean fuzzy numbers that depend on the Hamacher t -norm and t -conorm, namely, the complex Pythagorean fuzzy Hamacher weighted average operator, the complex Pythagorean fuzzy Hamacher ordered weighted average operator, and the complex Pythagorean fuzzy Hamacher hybrid average operator. We explore some properties of these operators inclusive of idempotency, monotonicity, and boundedness. Then, the operators are applied to multicriteria decision-making problems under the complex Pythagorean fuzzy environment. Furthermore, we present an algorithm along with a flow chart, and we demonstrate their applicability with the assistance of two numerical examples (selection of most favorable country for immigrants and selection of the best programming language). We investigate the adequacy of this algorithm by conducting a comparative study with the case of complex intuitionistic fuzzy aggregation operators.

1. Introduction

Aggregation operators (AOs) are the tools to convert an n -tuple crisp information into a single beneficial form. The AOs find extensive applications in decision-making. Multicriteria decision-making (MCDM) refers to a procedure for choosing the most worthwhile alternative with respect to some crucial factors. In real world MCDM problems, the human decisions are usually unclear and inexact which lead us to take help from the fuzzy set theory to handle the uncertain data.

The crisp set theory was developed on the basis of crisp logic in which logical connectors conjunction and disjunction are employed to define operations. On the other hand, triangular norm (t -norm) and triangular conorm (t -conorm) are used to define the operations in fuzzy set theory as a generalization of Boolean logical connectives. The concept of t -norm and t -conorm was originally initiated by

Menger [1] in the context of probabilistic metric spaces in 1942. Later, Schweizer and Sklar [2] worked for the development of t -norm and t -conorms. The notion of t -norm and t -conorm gained the attention of researchers. Many researchers contributed in this field and proposed the multipurpose t -norms, including drastic product, algebraic product, Lukasiewicz t -norm, Yager t -norm, Schweizer and Sklar t -norm and Frank t -norm, and the corresponding t -conorms. Later on, Zimmermann and Zysno [3] noticed that t -norm and t -conorm exhibit extreme behaviors. This led to the development of compensation and average operators [4] that provide the results inside the interval. In 1978, Hamacher [5] introduced a parameterized t -norm and its dual t -conorm as a generalization of Einstein product and Einstein sum, respectively.

Many AOs, including arithmetic mean, median, minimum, maximum, weighted maximum, weighted minimum,

geometric mean, harmonic mean, and quasiarithmetic mean, are used to aggregate the crisp data in classical set theory. Yager [6] proposed the notion of ordered weighted average operators which was further extended by many quality researchers in different models. The fundamental properties of AOs are monotonicity, continuity, associativity, symmetry, bisymmetry, idempotency, and invariance.

In 1965, Zadeh [7] proposed the fuzzy set (FS) as an extension of classical set and introduced several operations for FS. Zadeh's idea made a way to solve those decision-making and MCDM problems, comprising imprecise and ambiguous data that cannot be solved by the notion of crisp set. Song et al. [8] discussed fuzzy operators and studied their properties. Merigó et al. [9] utilized the generalized AOs for decision-making in fuzzy environment. Atanassov [10] associated the nonmembership degree with the membership degree along with the condition that the sum of membership and nonmembership degree should not exceed 1 and named the new structure as intuitionistic fuzzy set (IFS). Xu [11] extended weighted averaging operators in the intuitionistic fuzzy (IF) environment. Wei [12] and Xu and Wang [13] presented induced AOs and induced generalized AOs for IF as well as interval-valued IF information and illustrate their application in group decision-making. Zhao et al. [14] introduced IF generalized weighted average operators. Wei and Zhao [15] proposed a decision-making approach on the basis of induced IF correlated averaging operators and induced IF correlated geometric operators. Huang [16] and Yang et al. [17] examined IF Hamacher AOs and dynamic intuitionistic normal fuzzy AOs, respectively. Later, Yager [18, 19] presented the notion of Pythagorean fuzzy set (PFS) which has more generalized structure and accommodates more uncertainty than IFS. Zhang [20] investigated some AOs for Pythagorean fuzzy (PF) model and presented a decision-making approach. Akram et al. [21] and Jana et al. [22] worked on the Pythagorean Dombi fuzzy AOs and utilized these operators for decision-making in textile industry and enterprise resource planning system, respectively. Wu and Wei [23] presented the PF Hamacher AOs with their properties and apply these operators to opt the most talented enterpriser. Wei et al. [24, 25] considered the Pythagorean hesitant fuzzy AOs and Pythagorean fuzzy power AOs. Shahzadi et al. [26] presented a decision-making approach on the basis of PF Yager operators. Aydin et al. [27] established harmonic AOs for trapezoidal PF numbers.

To overcome the limitations of FS, IFS, and PFS, Ramot introduced the complex fuzzy set (CFS) [28] as well as complex fuzzy logic [29]. The membership function of a CFS, restricted to complex unit circle, consists of two real valued terms, i.e., amplitude term and phase term which make it distinct and superior to all existing models. The novelty of CFS is due to the phase term associated with membership which enables it to handle periodic data. Bi et al. developed the complex fuzzy arithmetic AOs [30] and complex fuzzy geometric AOs [31]. Alkouri et al. [32, 33] extended the CFS to a superior model, namely, complex intuitionistic fuzzy set (CIFS) and introduced some basic operations along with complex intuitionistic fuzzy (CIF) relations. The nonmembership function in the CIFS

distinguishes it from CFS and provides more accurate results than CFS. Garg and Rani [34] presented decision-making approaches on the basis of CIF weighted average operator and CIF weighted geometric operator. Further, Garg et al. [35, 36] put forward some new complex intuitionistic fuzzy aggregation operators (CIFAOs) on the basis of Archimedean t -norm and t -conorm. Rani and Garg [37] introduced CIF power AOs and explained their application in practical decision-making. Akram et al. [38] presented the CIF Hamacher aggregation operators with impactful applications in the decision-making scenarios.

Ullah et al. [39] proposed the complex Pythagorean fuzzy set (CPFS), having relatively relaxed conditions for amplitude and phase terms, to overcome the deficiencies of CIFS. Phase term of CPFS is of vital importance and makes it dominant to all other models due to its tendency to tackle two-dimensional vague information efficiently. Akram et al. [40–42] worked for the development of aggregation operators on the basis of Yager and Dombi operations for complex Pythagorean fuzzy (CPF) model. Tan et al. [43], Wei et al. [44], and Waseem et al. [45] contributed to literature by proposing the decision-making models based on hesitant fuzzy Hamacher AOs, bipolar fuzzy Hamacher AOs, and m -polar fuzzy Hamacher AOs, respectively. Recently, Akram et al. [46] proposed an innovative extension of the existing models, namely, complex spherical fuzzy sets.

The motivation of this article can be described as follows:

- (i) The models of fuzzy set theory with complex membership and nonmembership have an edge over the other existing models as they are proficient enough to deal with two-dimensional obscure data and information due to phase term. The traditional decision-making approaches of fuzzy set theory cannot be applied to periodic data because this may cause loss of important information.
- (ii) The constraints on the amplitude and phase terms of CPFS allow it to capture more imprecision and vagueness. Therefore, CPFS works effectively when the sum of amplitude or phase term does not fall under the conditions of CIFS.
- (iii) Hamacher AOs represent a generalized family of operators which provide more accurate results in decision-making. The purpose of this study is to define a MCDM approach for CPF model using the foundations of Hamacher operations to exploit the strengths of both models for more precise decisions.
- (iv) The complex Pythagorean fuzzy Hamacher arithmetic aggregation operators (CPFHAAOs) also overcome the deficiencies of existing operators and MCDM approaches for two-dimensional information, including complex fuzzy model and CIF model.

In this research article, we propose complex Pythagorean fuzzy Hamacher weighted average (CPFHWA) operator, complex Pythagorean fuzzy Hamacher ordered weighted average (CPFHOWA) operator, and complex Pythagorean fuzzy Hamacher hybrid average (CPFHHA) operator to

aggregate the CPF data for decision-making purpose. We also present a MCDM strategy based on these operators under CPF environment. We illustrate the implementation of the proposed algorithm by two explanatory numerical examples: one for the selection of best country for immigrants and other for the selection of best programming language. We verify the results of the proposed strategy by conducting a comparative study with existing MCDM techniques using complex intuitionistic fuzzy weighted average (CIFWA) operator, complex intuitionistic fuzzy Einstein weighted average (CIFEWA) operator and complex intuitionistic fuzzy Hamacher weighted average (CIFHWA) operator, complex intuitionistic fuzzy weighted geometric (CIFWG) operator, complex intuitionistic fuzzy Einstein weighted geometric (CIFEWG) operator, and complex intuitionistic fuzzy Hamacher weighted geometric (CIFHWG) operator. The main contributions of this research article can be summarized as follows:

- (i) The considerable contribution of this study is to make capital of the parametric and flexible framework of Hamacher operations under the competitive and innovative model of CPFs to accumulate the obscure periodic data for decision-making.
- (ii) The novelty of the proposed operators is due to their flexible structure and authentic outputs as they compile the CPF data deploying the brilliance of Hamacher operations, whereas the existing operators, developed on the basis of Hamacher norms, are not applicable for CPF data due to nonavailability or strict condition of phase terms.
- (iii) The main goal of this article is to employ the competency and potential of the proposed operators for the development of a MCDM approach to enhance the accuracy of decision-making results for two-dimensional information.
- (iv) The presented methodology is supported with the help of two rational numerical examples, one for the selection of best country for immigrants and other for the selection of best programming language.
- (v) The calibre of the proposed methodology is demonstrated via comparative study to prove the dominance of the proposed operators over existing operators.

The rest of this research article is organized as follows: Section 2 comprises the basic concepts and Hamacher operations of CPFNs which have been employed to define the CPFHAAOs and their properties. Section 3 presents a mathematical approach to address the MCDM problem under CPF environment along with two illustrative numerical examples. Section 4 describes the comparison of the proposed operators with well-known complex intuitionistic fuzzy AOs. Section 5 highlights the eminence and excellence of the proposed operators. Section 6 summarizes the article with concluding remarks and future directions.

2. Complex Pythagorean Fuzzy Hamacher Aggregation Operators

Definition 1 (see [39]). Let P be a universe of discourse. A complex Pythagorean fuzzy set $\mathfrak{B}$ over the universe P is an object of the following form:

$$\mathfrak{B} = \{ \langle p, \mu_{\mathfrak{B}}(p), \nu_{\mathfrak{B}}(p) \rangle \mid p \in P \}, \quad (1)$$

where the membership function $\mu_{\mathfrak{B}}$ and nonmembership function $\nu_{\mathfrak{B}}$ are defined by the mapping $\mu_{\mathfrak{B}}, \nu_{\mathfrak{B}}: P \longrightarrow \{z \mid z \in \mathbb{C}: |z| \leq 1\}$. For every $p \in P$, the membership grade is of the form $\mu_{\mathfrak{B}}(p) = u_{\mathfrak{B}}(p)e^{i\alpha_{\mathfrak{B}}(p)}$ and nonmembership grade has the form $\nu_{\mathfrak{B}}(p) = v_{\mathfrak{B}}(p)e^{i\beta_{\mathfrak{B}}(p)}$, where $i = \sqrt{-1}$, $u_{\mathfrak{B}}, v_{\mathfrak{B}} \in [0, 1]$, $\alpha_{\mathfrak{B}}, \beta_{\mathfrak{B}} \in [0, 2\pi]$, $0 \leq u_{\mathfrak{B}}^2(p) + v_{\mathfrak{B}}^2(p) \leq 1$, and $0 \leq (\alpha_{\mathfrak{B}}(p)/2\pi)^2 + (\beta_{\mathfrak{B}}(p)/2\pi)^2 \leq 1$. The pair of membership and nonmembership $(\mu_{\mathfrak{B}}, \nu_{\mathfrak{B}})$ is called a complex Pythagorean fuzzy number (CPFN).

Definition 2 (see [42]). The score function s of a CPFN $\mathfrak{R} = (\mu_{\mathfrak{R}}, \nu_{\mathfrak{R}})$ is defined as follows:

$$s(\mathfrak{R}) = (u^2 - v^2) + \frac{1}{4\pi^2}(\alpha^2 - \beta^2), \quad (2)$$

where $s(\mathfrak{R}) \in [-2, 2]$.

Definition 3 (see [42]). The accuracy function h of a CPFN $\mathfrak{R} = (\mu_{\mathfrak{R}}, \nu_{\mathfrak{R}})$ is defined as

$$h(\mathfrak{R}) = (u^2 + v^2) + \frac{1}{4\pi^2}(\alpha^2 + \beta^2), \quad (3)$$

where $h(\mathfrak{R}) \in [0, 2]$.

Definition 4 (see [42]). For the comparison of any two CPFNs $\mathfrak{R}_1 = (u_1e^{i\alpha_1}, v_1e^{i\beta_1})$ and $\mathfrak{R}_2 = (u_2e^{i\alpha_2}, v_2e^{i\beta_2})$,

- (1) If $s(\mathfrak{R}_1) > s(\mathfrak{R}_2)$, then $\mathfrak{R}_1 > \mathfrak{R}_2$ ($\mathfrak{R}_1$ is superior to $\mathfrak{R}_2$);
- (2) If $s(\mathfrak{R}_1) = s(\mathfrak{R}_2)$, then
 - (i) If $h(\mathfrak{R}_1) > h(\mathfrak{R}_2)$, then $\mathfrak{R}_1 > \mathfrak{R}_2$ ($\mathfrak{R}_1$ is superior to $\mathfrak{R}_2$);
 - (ii) If $h(\mathfrak{R}_1) = h(\mathfrak{R}_2)$, then $\mathfrak{R}_1 \sim \mathfrak{R}_2$ ($\mathfrak{R}_1$ is equivalent to $\mathfrak{R}_2$).

Definition 5 (see [47]). Let $\mathfrak{R} = (u_{\mathfrak{R}}e^{i\alpha_{\mathfrak{R}}}, v_{\mathfrak{R}}e^{i\beta_{\mathfrak{R}}})$, $\mathfrak{R}_1 = (u_{\mathfrak{R}_1}e^{i\alpha_{\mathfrak{R}_1}}, v_{\mathfrak{R}_1}e^{i\beta_{\mathfrak{R}_1}})$, and $\mathfrak{R}_2 = (u_{\mathfrak{R}_2}e^{i\alpha_{\mathfrak{R}_2}}, v_{\mathfrak{R}_2}e^{i\beta_{\mathfrak{R}_2}})$ be three CPFNs. The operations corresponding to these three CPFNs can be defined as follows:

- (1) $\sigma\mathfrak{R} = (\sqrt[1-(1-u_{\mathfrak{R}}^2)^{\sigma}]{u_{\mathfrak{R}}^{\sigma}e^{i2\pi(\beta_{\mathfrak{R}}/2\pi)^{\sigma}}}, (\sigma > 0); e^{i2\pi\sqrt{1-(1-(\alpha_{\mathfrak{R}}/2\pi)^2)^{\sigma}}}$,
- (2) $\mathfrak{R}^{\sigma} = (\sqrt[1-(1-v_{\mathfrak{R}}^2)^{\sigma}]{u_{\mathfrak{R}}^{\sigma}e^{i2\pi(\alpha_{\mathfrak{R}}/2\pi)^{\sigma}}}, \sqrt{1-(1-v_{\mathfrak{R}}^2)^{\sigma}} e^{i2\pi\sqrt{1-(1-(\beta_{\mathfrak{R}}/2\pi)^2)^{\sigma}}}, (\sigma > 0);$
- (3) $\mathfrak{R}_1 \oplus \mathfrak{R}_2 = (\sqrt{u_{\mathfrak{R}_1}^2 + u_{\mathfrak{R}_2}^2 - u_{\mathfrak{R}_1}^2 u_{\mathfrak{R}_2}^2} e^{i2\pi\sqrt{(\alpha_{\mathfrak{R}_1}/2\pi)^2 + (\alpha_{\mathfrak{R}_2}/2\pi)^2 - (\alpha_{\mathfrak{R}_1}/2\pi)^2 (\alpha_{\mathfrak{R}_2}/2\pi)^2}}, v_{\mathfrak{R}_1} v_{\mathfrak{R}_2} e^{i2\pi(\beta_{\mathfrak{R}_1}/2\pi)(\beta_{\mathfrak{R}_2}/2\pi)});$

$$(4) \mathfrak{R}_1 \otimes \mathfrak{R}_2 = (u_{\mathfrak{R}_1} u_{\mathfrak{R}_2} e^{i2\pi(\alpha_{\mathfrak{R}_1}/2\pi)(\alpha_{\mathfrak{R}_2}/2\pi)}, \\ \sqrt{v_{\mathfrak{R}_1}^2 + v_{\mathfrak{R}_2}^2 - v_{\mathfrak{R}_1}^2 v_{\mathfrak{R}_2}^2} e^{i2\pi\sqrt{(\beta_{\mathfrak{R}_1}/2\pi)^2 + (\beta_{\mathfrak{R}_2}/2\pi)^2}} \\ - (\beta_{\mathfrak{R}_1}/2\pi)^2 (\beta_{\mathfrak{R}_2}/2\pi)^2).$$

2.1. Hamacher t-Norm and Hamacher t-Conorm. The notions of t-norm and t-conorm are the elementary tools to define operations in fuzzy set theory. Later, Hamacher [5] presented the more general t-norm and t-conorm, namely, Hamacher product and Hamacher sum, respectively. For all $g, h \in [0, 1]$, Hamacher operations including product and sum are defined as follows:

$$\mathfrak{H}(g, h) = g \otimes h = \frac{gh}{\gamma + (1 - \gamma)(g + h - gh)}, \quad \gamma > 0, \\ \mathfrak{H}^*(g, h) = g \oplus h = \frac{g + h - gh - (1 - \gamma)gh}{1 - (1 - \gamma)gh}, \quad \gamma > 0. \quad (4)$$

Special cases:

(i) For $\gamma = 1$, Hamacher t-norm and t-conorm give algebraic t-norm and t-conorm, respectively.

$$\mathfrak{H}(g, h) = g \otimes h = gh, \\ \mathfrak{H}^*(g, h) = g \oplus h = g + h - gh. \quad (5)$$

(ii) For $\gamma = 2$, Hamacher product and Hamacher sum reduce to Einstein product and Einstein sum, respectively.

$$\mathfrak{H}(g, h) = g \otimes h = \frac{gh}{1 + (1 - g)(1 - h)}, \\ \mathfrak{H}^*(g, h) = g \oplus h = \frac{g + h - gh}{1 + gh}. \quad (6)$$

2.2. Hamacher Operations of Complex Pythagorean Fuzzy Numbers. For three CPFNs $\mathfrak{R} = (u_{\mathfrak{R}} e^{i\alpha_{\mathfrak{R}}}, v_{\mathfrak{R}} e^{i\beta_{\mathfrak{R}}})$, $\mathfrak{R}_1 = (u_{\mathfrak{R}_1} e^{i\alpha_{\mathfrak{R}_1}}, v_{\mathfrak{R}_1} e^{i\beta_{\mathfrak{R}_1}})$ and $\mathfrak{R}_2 = (u_{\mathfrak{R}_2} e^{i\alpha_{\mathfrak{R}_2}}, v_{\mathfrak{R}_2} e^{i\beta_{\mathfrak{R}_2}})$, Hamacher operations are defined as

$$\mathfrak{R}_1 \otimes \mathfrak{R}_2 = \left(\frac{u_1 u_2}{\sqrt{\gamma + (1 - \gamma)(u_1^2 + u_2^2 - u_1^2 u_2^2)}} e^{i2\pi \left((\alpha_1/2\pi)(\alpha_2/2\pi) / \sqrt{\gamma + 1 - \gamma \left((\alpha_1/2\pi)^2 + (\alpha_2/2\pi)^2 - (\alpha_1/2\pi)^2 (\alpha_2/2\pi)^2 \right)} \right)}, \right. \\ \left. \sqrt{\frac{v_1^2 + v_2^2 - v_1^2 v_2^2 - (1 - \gamma)v_1^2 v_2^2}{1 - (1 - \gamma)v_1^2 v_2^2}} e^{i2\pi \left(\sqrt{(\beta_1/2\pi)^2 + (\beta_2/2\pi)^2 - (\beta_1/2\pi)^2 (\beta_2/2\pi)^2} - 1 - \gamma (\beta_1/2\pi)^2 (\beta_2/2\pi)^2 / 1 - (1 - \gamma) (\beta_1/2\pi)^2 (\beta_2/2\pi)^2 \right)} \right), \\ \mathfrak{R}_1 \oplus \mathfrak{R}_2 = \left(\sqrt{\frac{u_1^2 + u_2^2 - u_1^2 u_2^2 - (1 - \gamma)u_1^2 u_2^2}{1 - (1 - \gamma)u_1^2 u_2^2}} e^{i2\pi \left(\sqrt{(\alpha_1/2\pi)^2 + (\alpha_2/2\pi)^2 - (\alpha_1/2\pi)^2 (\alpha_2/2\pi)^2} - 1 - \gamma (\alpha_1/2\pi)^2 (\alpha_2/2\pi)^2 / 1 - (1 - \gamma) (\alpha_1/2\pi)^2 (\alpha_2/2\pi)^2 \right)}, \right. \\ \left. \frac{v_1 v_2}{\sqrt{\gamma + (1 - \gamma)(v_1^2 + v_2^2 - v_1^2 v_2^2)}} e^{i2\pi \left((\beta_1/2\pi)(\beta_2/2\pi) / \sqrt{\gamma + 1 - \gamma \left((\beta_1/2\pi)^2 + (\beta_2/2\pi)^2 - (\beta_1/2\pi)^2 (\beta_2/2\pi)^2 \right)} \right)}, \right. \\ \left. \sigma \mathfrak{R} = \left(\sqrt{\frac{(1 + (\gamma - 1)u^2)^\sigma - (1 - u^2)^\sigma}{(1 + (\gamma - 1)u^2)^\sigma + (\gamma - 1)(1 - u^2)^\sigma}} e^{i2\pi \left(\sqrt{(1 + (\gamma - 1)(\alpha/2\pi)^2)^\sigma - (1 - (\alpha/2\pi)^2)^\sigma} / (1 + (\gamma - 1)(\alpha/2\pi)^2)^\sigma + (\gamma - 1)(1 - (\alpha/2\pi)^2)^\sigma \right)} \right. \right. \\ \left. \frac{\sqrt{\gamma} (v)^\sigma}{\sqrt{(1 + (\gamma - 1)(1 - v^2))^\sigma + (\gamma - 1)v^{2\sigma}}} e^{i2\pi \left(\sqrt{\gamma} (\beta/2\pi)^\sigma / \sqrt{(1 + (\gamma - 1)(1 - (\beta/2\pi)^2)^\sigma + (\gamma - 1)(\beta/2\pi)^{2\sigma}} \right)} \right), \\ \mathfrak{R}^\sigma = \left(\frac{\sqrt{\gamma} (u)^\sigma}{\sqrt{(1 + (\gamma - 1)(1 - u^2))^\sigma + (\gamma - 1)(u^2)^\sigma}} e^{i2\pi \left(\sqrt{\gamma} (\alpha/2\pi)^\sigma / \sqrt{(1 + (\gamma - 1)(1 - (\alpha/2\pi)^2)^\sigma + (\gamma - 1)(\alpha/2\pi)^{2\sigma}} \right)} \right. \\ \left. \sqrt{\frac{(1 + (\gamma - 1)v^2)^\sigma - (1 - v^2)^\sigma}{(1 + (\gamma - 1)v^2)^\sigma + (\gamma - 1)(1 - v^2)^\sigma}} e^{i2\pi \left(\sqrt{(1 + (\gamma - 1)(\beta/2\pi)^2)^\sigma - (1 - (\beta/2\pi)^2)^\sigma} / (1 + (\gamma - 1)(\beta/2\pi)^2)^\sigma + (\gamma - 1)(1 - (\beta/2\pi)^2)^\sigma \right)} \right). \quad (7)$$

2.3. Complex Pythagorean Fuzzy Hamacher Arithmetic Aggregation Operators. In this subsection, we present a few CPFHAAOs based on the Hamacher operations of CPFNs.

Definition 6. For any collection $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) of CPFNs and the weight vector $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$, the complex Pythagorean fuzzy Hamacher weighted average (CPFHWA) operator is defined as follows:

$$\text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \bigoplus_{t=1}^n \phi_t \mathfrak{R}_t, \quad (8)$$

where ϕ_t , representing the weight of $\mathfrak{R}_t$, belongs to $[0, 1]$ and satisfies the condition $\sum_{t=1}^n \phi_t = 1$.

Theorem 1. For any collection $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) of CPFNs and the weight vector $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$, the accumulated value by deploying the CPFHWA operator is also a CPFN which is given as

$$\begin{aligned} \text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \bigoplus_{t=1}^n \phi_t \mathfrak{R}_t \\ &= \left(\sqrt{\frac{\prod_{t=1}^n (1 + (\gamma - 1)u_t^2)^{\phi_t} - \prod_{t=1}^n (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1)u_t^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^n (1 - u_t^2)^{\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1)(\alpha_t/2\pi)^2) - \prod_{t=1}^n (1 - (\alpha_t/2\pi)^2)^{\phi_t}}}{\prod_{t=1}^n (1 + (\gamma - 1)(\alpha_t/2\pi)^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^n (1 - (\alpha_t/2\pi)^2)^{\phi_t}} \right)}, \right. \\ &\quad \left. \frac{\sqrt{\gamma} \prod_{t=1}^n (v_t)^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1)(1 - v_t^2))^{\phi_t} + (\gamma - 1)\prod_{t=1}^n v_t^{2\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\gamma} \prod_{t=1}^n (\beta_t/2\pi)^{\phi_t} / \sqrt{\prod_{t=1}^n (1 + (\gamma - 1)(1 - (\beta_t/2\pi)^2))^{\phi_t}}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1)(1 - v_t^2))^{\phi_t} + (\gamma - 1)\prod_{t=1}^n v_t^{2\phi_t}}} \right)} \right), \end{aligned} \quad (9)$$

where $\gamma > 0$, ϕ_t , representing the weight of $\mathfrak{R}_t$, belongs to $[0, 1]$ and satisfies the condition $\sum_{t=1}^n \phi_t = 1$.

Case 1. When $n = 1$, the CPFHWA operator given in equation (9) gives

Proof. We prove the theorem with the help of mathematical induction.

$$\text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \phi_1 \mathfrak{R}_1 = \mathfrak{R}_1 \text{ (Here } \phi_1 = 1)$$

$$\begin{aligned} &= \left(\sqrt{\frac{(1 + (\gamma - 1)u_1^2) - (1 - u_1^2)}{(1 + (\gamma - 1)u_1^2) + (\gamma - 1)(1 - u_1^2)}} e^{i2\pi \left(\frac{\sqrt{(1 + (\gamma - 1)(\alpha_1/2\pi)^2) - (1 - (\alpha_1/2\pi)^2)}}{(1 + (\gamma - 1)(\alpha_1/2\pi)^2) + (\gamma - 1)(1 - (\alpha_1/2\pi)^2)} \right)}, \right. \\ &\quad \left. \frac{\sqrt{\gamma} (v_1)}{\sqrt{(1 + (\gamma - 1)(1 - v_1^2)) + (\gamma - 1)v_1^2}} e^{i2\pi \left(\frac{\sqrt{\gamma} (\beta_1/2\pi)}{\sqrt{(1 + (\gamma - 1)(1 - (\beta_1/2\pi)^2)) + (\gamma - 1)(\beta_1/2\pi)^2}} \right)} \right) \\ &= (u_1 e^{i\alpha_1}, v_1 e^{i\beta_1}). \end{aligned} \quad (10)$$

Thus, aggregated value is the same CPFN and equation (9) holds for $n = 1$.

Case 2. We now assume that equation (9) is true for $n = r$, where r denotes a natural number. Then, equation (9) becomes

$$\begin{aligned} \text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_r) &= \bigoplus_{t=1}^r \phi_t \mathfrak{R}_t \\ &= \left(\sqrt{\frac{\prod_{t=1}^r (1 + (\gamma - 1)u_t^2)^{\phi_t} - \prod_{t=1}^r (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^r (1 + (\gamma - 1)u_t^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^r (1 - u_t^2)^{\phi_t}}} \cdot e^{i2\pi \left(\sqrt{\prod_{t=1}^r (1 + (\gamma - 1)(\alpha_t/2\pi)^2)^{\phi_t} - \prod_{t=1}^r (1 - (\alpha_t/2\pi)^2)^{\phi_t}} / \prod_{t=1}^r (1 + (\gamma - 1)(\alpha_t/2\pi)^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^r (1 - (\alpha_t/2\pi)^2)^{\phi_t}} \right)}, \right. \\ &\quad \left. \frac{\sqrt{\gamma} \prod_{t=1}^r (v_t)^{\phi_t}}{\sqrt{\prod_{t=1}^r (1 + (\gamma - 1)(1 - v_t^2))^{\phi_t} + (\gamma - 1)\prod_{t=1}^r v_t^{2\phi_t}}} \cdot e^{i2\pi \left(\sqrt{\gamma} \prod_{t=1}^r (\beta_t/2\pi)^{\phi_t} / \sqrt{\prod_{t=1}^r (1 + (\gamma - 1)(1 - (\beta_t/2\pi))^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^r (\beta_t/2\pi)^{2\phi_t}} \right)} \right). \end{aligned} \quad (11)$$

Now, for $n = r + 1$,

$$\begin{aligned} \text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_r, \mathfrak{R}_{r+1}) &= \bigoplus_{t=1}^r \phi_t \mathfrak{R}_t \oplus \phi_{r+1} \mathfrak{R}_{r+1} \\ &= \left(\sqrt{\frac{\prod_{t=1}^r (1 + (\gamma - 1)u_t^2)^{\phi_t} - \prod_{t=1}^r (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^r (1 + (\gamma - 1)u_t^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^r (1 - u_t^2)^{\phi_t}}} \cdot e^{i2\pi \left(\sqrt{\prod_{t=1}^r (1 + (\gamma - 1)(\alpha_t/2\pi)^2)^{\phi_t} - \prod_{t=1}^r (1 - (\alpha_t/2\pi)^2)^{\phi_t}} / \prod_{t=1}^r (1 + (\gamma - 1)(\alpha_t/2\pi)^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^r (1 - (\alpha_t/2\pi)^2)^{\phi_t}} \right)}, \right. \\ &\quad \left. \frac{\sqrt{\gamma} \prod_{t=1}^r (v_t)^{\phi_t}}{\sqrt{\prod_{t=1}^r (1 + (\gamma - 1)(1 - v_t^2))^{\phi_t} + (\gamma - 1)\prod_{t=1}^r v_t^{2\phi_t}}} \cdot e^{i2\pi \left(\sqrt{\gamma} \prod_{t=1}^r (\beta_t/2\pi)^{\phi_t} / \sqrt{\prod_{t=1}^r (1 + (\gamma - 1)(1 - (\beta_t/2\pi))^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^r (\beta_t/2\pi)^{2\phi_t}} \right)} \right) \\ &\quad \oplus \left(\sqrt{\frac{(1 + (\gamma - 1)u_{r+1}^2)^{\phi_{r+1}} - (1 - u_{r+1}^2)^{\phi_{r+1}}}{(1 + (\gamma - 1)u_{r+1}^2)^{\phi_{r+1}} + (\gamma - 1)(1 - u_{r+1}^2)^{\phi_{r+1}}}} \cdot e^{i2\pi \left(\sqrt{(1 + (\gamma - 1)(\alpha_{r+1}/2\pi)^2)^{\phi_{r+1}} - (1 - (\alpha_{r+1}/2\pi)^2)^{\phi_{r+1}}} / (1 + (\gamma - 1)(\alpha_{r+1}/2\pi)^2)^{\phi_{r+1}} + (\gamma - 1)(1 - (\alpha_{r+1}/2\pi)^2)^{\phi_{r+1}} \right)}, \right. \\ &\quad \left. \frac{\sqrt{\gamma} (v_{r+1})^{\phi_{r+1}}}{\sqrt{(1 + (\gamma - 1)(1 - v_{r+1}^2))^{\phi_{r+1}} + (\gamma - 1)v_{r+1}^{2\phi_{r+1}}}} \cdot e^{i2\pi \left(\sqrt{\gamma} (\beta_{r+1}/2\pi)^{\phi_{r+1}} / \sqrt{(1 + (\gamma - 1)(1 - (\beta_{r+1}/2\pi))^2)^{\phi_{r+1}} + (\gamma - 1)(\beta_{r+1}/2\pi)^{2\phi_{r+1}}} \right)} \right) \\ &= \left(\sqrt{\frac{\prod_{t=1}^{r+1} (1 + (\gamma - 1)u_t^2)^{\phi_t} - \prod_{t=1}^{r+1} (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^{r+1} (1 + (\gamma - 1)u_t^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^{r+1} (1 - u_t^2)^{\phi_t}}} \cdot e^{i2\pi \left(\sqrt{\prod_{t=1}^{r+1} (1 + (\gamma - 1)(\alpha_t/2\pi)^2)^{\phi_t} - \prod_{t=1}^{r+1} (1 - (\alpha_t/2\pi)^2)^{\phi_t}} / \prod_{t=1}^{r+1} (1 + (\gamma - 1)(\alpha_t/2\pi)^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^{r+1} (1 - (\alpha_t/2\pi)^2)^{\phi_t}} \right)}, \right. \\ &\quad \left. \frac{\sqrt{\gamma} \prod_{t=1}^{r+1} (v_t)^{\phi_t}}{\sqrt{\prod_{t=1}^{r+1} (1 + (\gamma - 1)(1 - v_t^2))^{\phi_t} + (\gamma - 1)\prod_{t=1}^{r+1} v_t^{2\phi_t}}} \cdot e^{i2\pi \left(\sqrt{\gamma} \prod_{t=1}^{r+1} (\beta_t/2\pi)^{\phi_t} / \sqrt{\prod_{t=1}^{r+1} (1 + (\gamma - 1)(1 - (\beta_t/2\pi))^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^{r+1} (\beta_t/2\pi)^{2\phi_t}} \right)} \right). \end{aligned} \quad (12)$$

Thus, equation (9) holds for $n = r + 1$. Hence, it is proved that equation (9) is true for all n (natural numbers). $\square$

Special cases:

(i) For $\gamma = 1$, CPFHWA operator becomes

$$\text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \left(\sqrt{1 - \prod_{t=1}^n (1 - u_t^2)^{\phi_t}} \cdot e^{i2\pi \left(\sqrt{1 - \prod_{t=1}^n (1 - (\alpha_t/2\pi)^2)^{\phi_t}} \right)}, \prod_{t=1}^n (v_t)^{\phi_t} e^{i2\pi \left(\prod_{t=1}^n (\beta_t/2\pi)^{\phi_t} \right)} \right), \quad (13)$$

which represents the complex Pythagorean fuzzy weighted averaging (CPFWA) operator.

(ii) For $\gamma = 2$, CPFWA operator becomes

$$\text{CPFWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \left(\sqrt[n]{\frac{\prod_{t=1}^n (1 + u_t^2)^{\phi_t} - \prod_{t=1}^n (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^n (1 + u_t^2)^{\phi_t} + \prod_{t=1}^n (1 - u_t^2)^{\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (1 + (\alpha_t/2\pi)^2)^{\phi_t} - \prod_{t=1}^n (1 - (\alpha_t/2\pi)^2)^{\phi_t}} / \sqrt{\prod_{t=1}^n (1 + (\alpha_t/2\pi)^2)^{\phi_t} + \prod_{t=1}^n (1 - (\alpha_t/2\pi)^2)^{\phi_t}}} \right)} \right. \\ \left. \frac{\sqrt[n]{\prod_{t=1}^n (v_t)^{\phi_t}}}{\sqrt{\prod_{t=1}^n (2 - v_t^2)^{\phi_t} + \prod_{t=1}^n v_t^{2\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (\beta_t/2\pi)^{\phi_t}} / \sqrt{\prod_{t=1}^n (2 - (\beta_t/2\pi)^{\phi_t} + \prod_{t=1}^n (\beta_t/2\pi)^{\phi_t}}} \right)} \right), \quad (14)$$

which represents the complex Pythagorean fuzzy Einstein weighted averaging (CPFEWA) operator.

(iii) When $\alpha_t = \beta_t = 0, \forall t$, CPFWA operator becomes

$$\text{CPFWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) \\ = \left(\sqrt[n]{\frac{\prod_{t=1}^n (1 + (\gamma - 1)u_t^2)^{\phi_t} - \prod_{t=1}^n (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1)u_t^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^n (1 - u_t^2)^{\phi_t}}}, \frac{\sqrt[n]{\prod_{t=1}^n (v_t)^{\phi_t}}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1)(1 - v_t^2))^{\phi_t} + (\gamma - 1)\prod_{t=1}^n v_t^{2\phi_t}}} \right), \quad (15)$$

which represents the Pythagorean fuzzy Hamacher weighted averaging (PFHWA) operator.

Example 1. Let $\mathfrak{R}_1 = (0.8e^{i2\pi(0.73)}, 0.34e^{i2\pi(0.45)})$, $\mathfrak{R}_2 = (0.65e^{i2\pi(0.83)}, 0.43e^{i2\pi(0.29)})$, and $\mathfrak{R}_3 = (0.92e^{i2\pi(0.85)}, 0.18e^{i2\pi(0.25)})$ be three CPFNs. Let $\phi = (0.5, 0.2, 0.3)^T$ be the associated weight vector. Then, for $\gamma = 4$,

$$\text{CPFWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3) = \prod_{t=1}^3 \phi_t \mathfrak{R}_t \\ = \left(\sqrt[n]{\frac{\prod_{t=1}^3 (1 + 3u_t^2)^{\phi_t} - \prod_{t=1}^3 (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^3 (1 + 3u_t^2)^{\phi_t} + 3\prod_{t=1}^3 (1 - u_t^2)^{\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^3 (1 + 3(\alpha_t/2\pi)^2)^{\phi_t} - \prod_{t=1}^3 (1 - (\alpha_t/2\pi)^2)^{\phi_t}} / \sqrt{\prod_{t=1}^3 (1 + 3(\alpha_t/2\pi)^2)^{\phi_t} + 3\prod_{t=1}^3 (1 - (\alpha_t/2\pi)^2)^{\phi_t}}} \right)} \right. \\ \left. \frac{2\prod_{t=1}^3 (v_t)^{\phi_t}}{\sqrt{\prod_{t=1}^3 (1 + 3(1 - v_t^2))^{\phi_t} + 3\prod_{t=1}^3 \prod_{t=1}^3 (v_t)^{\phi_t}}} e^{i2\pi \left(\frac{2\prod_{t=1}^3 (\beta_t/2\pi)^{\phi_t}}{\sqrt{\prod_{t=1}^3 (1 + 3(1 - (\beta_t/2\pi)^2))^{\phi_t} + 3\prod_{t=1}^3 (\beta_t/2\pi)^{2\phi_t}}} \right)} \right) \\ = \left(\sqrt[n]{\frac{[1.7089 \times 1.1779 \times 1.4611] - [0.6 \times 0.896 \times 0.5701]}{[1.7089 \times 1.1779 \times 1.4611] + 3[0.6 \times 0.896 \times 0.5701]}} \right. \\ \left. \cdot e^{i2\pi \left(\frac{[1.6120 \times 1.2512 \times 1.4132] - [0.6834 \times 0.7917 \times 0.6807]}{[1.6120 \times 1.2512 \times 1.4132] + 3[0.6834 \times 0.7917 \times 0.6807]} \right)} \right. \\ \left. \frac{2(0.5831 \times 0.8447 \times 0.5978)}{\sqrt{[1.9113 \times 1.2807 \times 1.3130] + 3[0.34 \times 0.7135 \times 0.3574]}} e^{i2\pi \left(\frac{2(0.6708 \times 0.7807 \times 0.6598)}{\sqrt{[1.8419 \times 1.3024 \times 1.4940] + 3[0.45 \times 0.6095 \times 0.4353]}} \right)} \right) \\ = \left(\sqrt[n]{\frac{2.9411 - 0.3065}{2.9411 + 0.9195}} e^{i2\pi \left(\frac{\sqrt{2.8503 - 0.3683/2.8503 + 1.1049}}{\sqrt{3.2140 + 0.2601}} \right)}, \frac{0.5889}{\sqrt{3.2140 + 0.2601}} e^{i2\pi \left(\frac{0.6910}{\sqrt{3.5839 + 0.3582}} \right)} \right) \\ = (0.8216e^{i2\pi(0.7922)}, 0.3160e^{i2\pi(0.3480)}). \quad (16)$$

Theorem 2 (idempotency property). Let $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) be a family of CPFNs. If $\mathfrak{R}_t = \mathfrak{R} \forall t$, then

$$\text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \mathfrak{R}. \quad (17)$$

Proof. Since $\mathfrak{R}_t = \mathfrak{R} = (u e^{i\alpha}, v e^{i\beta}) \forall t$, then equation (9) becomes

$$\begin{aligned} \text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \bigoplus_{t=1}^n \phi_t \mathfrak{R}_t \\ &= \left(\sqrt{\frac{\prod_{t=1}^n (1 + (\gamma - 1) u_t^2)^{\phi_t} - \prod_{t=1}^n (1 - u_t^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) u_t^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - u_t^2)^{\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (1 + (\alpha_t/2\pi)^2)^{\phi_t} - \prod_{t=1}^n (1 - (\alpha_t/2\pi)^2)^{\phi_t}}}{\prod_{t=1}^n (1 + (\alpha_t/2\pi)^2)^{\phi_t} + \prod_{t=1}^n (1 - (\alpha_t/2\pi)^2)^{\phi_t}} \right)} \right. \\ &\quad \left. \frac{\sqrt{\gamma} \prod_{t=1}^n (v_t)^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - v_t^2))^{\phi_t} + (\gamma - 1) \prod_{t=1}^n v_t^{2\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\gamma} \prod_{t=1}^n (\beta_t/2\pi)^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - (\beta_t/2\pi)^2))^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (\beta_t/2\pi)^{2\phi_t}}} \right)} \right) \\ &= \left(\sqrt{\frac{(1 + (\gamma - 1) u^2) - (1 - u^2)}{(1 + (\gamma - 1) u^2) + (\gamma - 1) (1 - u^2)}} e^{i2\pi \left(\frac{\sqrt{(1 + (\gamma - 1) (\alpha/2\pi)^2) - (1 - (\alpha/2\pi)^2)}}{(1 + (\gamma - 1) (\alpha/2\pi)^2) + (\gamma - 1) (1 - (\alpha/2\pi)^2)} \right)} \right. \\ &\quad \left. \frac{\sqrt{\gamma} v}{\sqrt{(1 + (\gamma - 1) (1 - v^2)) + (\gamma - 1) v^2}} e^{i2\pi \left(\frac{\sqrt{\gamma} (\beta/2\pi)}{\sqrt{(1 + (\gamma - 1) (1 - (\beta/2\pi)^2)) + (\gamma - 1) (\beta/2\pi)^2}} \right)} \right) \\ &= (u e^{i\alpha}, v e^{i\beta}) = \mathfrak{R}. \end{aligned} \quad (18)$$

Hence, it is proved that $\text{CPFHWA}_\phi(\mathfrak{R}, \mathfrak{R}, \dots, \mathfrak{R}) = \mathfrak{R}$. $\square$

Theorem 3 (boundedness property). Let $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) be a family of CPFNs. Let

$$\begin{aligned} \mathfrak{R}^+ &= \left(\max_t u_t e^{i \max_t \alpha_t}, \min_t v_t e^{i \min_t \beta_t} \right), \\ \mathfrak{R}^- &= \left(\min_t u_t e^{i \min_t \alpha_t}, \max_t v_t e^{i \max_t \beta_t} \right), \end{aligned} \quad (19)$$

then

$$\mathfrak{R}^- \leq \text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) \leq \mathfrak{R}^+. \quad (20)$$

Theorem 4 (monotonicity property). Let $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ and $\mathfrak{R}'_t = (\mu'_t, \nu'_t) = (u'_t e^{i\alpha'_t}, v'_t e^{i\beta'_t})$ ($t = 1, 2, \dots, n$) be two families of CPFNs. If $u_t \leq u'_t$, $\alpha_t \leq \alpha'_t$, $v_t \geq v'_t$ and $\beta_t \geq \beta'_t$, then

$$\text{CPFHWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) \leq \text{CPFHWA}_\phi(\mathfrak{R}'_1, \mathfrak{R}'_2, \dots, \mathfrak{R}'_n). \quad (21)$$

Definition 7. For any collection $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) of CPFNs and the weight vector $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$, the complex Pythagorean fuzzy Hamacher ordered weighted average (CPFHOWA) operator is defined as follows:

$$\text{CPFHOWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \bigoplus_{t=1}^n \phi_t \mathfrak{R}_{h(t)}, \quad (22)$$

where $h(1), h(2), \dots, h(n)$ is a permutation of $(1, 2, \dots, n)$, such that $\mathfrak{R}_{h(t)} \geq \mathfrak{R}_{h(t+1)}, \forall t = 1, 2, \dots, n-1$. ϕ_t , representing the weight of $\mathfrak{R}_t$, belongs to $[0, 1]$ and satisfies the condition $\sum_{t=1}^n \phi_t = 1$.

Theorem 5. For any collection $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) of CPFNs and the weight vector $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$, the accumulated value by deploying the CPFHOWA operator is also a CPFN which is given as

$$\begin{aligned} \text{CPFHOWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \bigoplus_{t=1}^n \phi_t \mathfrak{R}_{h(t)} \\ &= \left(\sqrt[n]{\frac{\prod_{t=1}^n (1 + (\gamma - 1) u_{h(t)}^2)^{\phi_t} - \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) u_{h(t)}^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}}, \right. \\ &\quad \cdot e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (\alpha_{h(t)}/2\pi)^2}^{\phi_t} - \prod_{t=1}^n (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) (\alpha_{h(t)}/2\pi)^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}} \right)}, \\ &\quad \left. \frac{\sqrt{\gamma} \prod_{t=1}^n (v_{h(t)})^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - v_{h(t)}^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n v_{h(t)}^{2\phi_t}}} \cdot e^{i2\pi \left(\frac{\sqrt{\gamma} \prod_{t=1}^n (\beta_{h(t)}/2\pi)^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - (\beta_{h(t)}/2\pi)^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (\beta_{h(t)}/2\pi)^{2\phi_t}}} \right)} \right), \end{aligned} \quad (23)$$

where $\gamma > 0$ and $h(1), h(2), \dots, h(n)$ is a permutation of $(1, 2, \dots, n)$, such that $\mathfrak{R}_{h(t)} \geq \mathfrak{R}_{h(t+1)}, \forall t = 1, 2, \dots, n-1$. ϕ_t , representing the weight of $\mathfrak{R}_t$, belongs to $[0, 1]$ and satisfies the condition $\sum_{t=1}^n \phi_t = 1$.

Example 2. Let $\mathfrak{R}_1 = (0.8e^{i2\pi(0.73)}, 0.34e^{i2\pi(0.45)})$, $\mathfrak{R}_2 = (0.65e^{i2\pi(0.83)}, 0.43e^{i2\pi(0.29)})$, and $\mathfrak{R}_3 = (0.92e^{i2\pi(0.85)}, 0.18e^{i2\pi(0.25)})$ be three CPFNs. Let $\phi = (0.5, 0.2, 0.3)^T$ be the associated weight vector. The scores of these CPFNs can be evaluated by equation (2).

$$\begin{aligned} s(\mathfrak{R}_1) &= 0.80^2 - 0.34^2 + 0.73^2 - 0.45^2 = 0.8548, \\ s(\mathfrak{R}_2) &= 0.65^2 - 0.43^2 + 0.83^2 - 0.29^2 = 0.8424, \\ s(\mathfrak{R}_3) &= 0.92^2 - 0.18^2 + 0.85^2 - 0.25^2 = 1.4740. \end{aligned} \quad (24)$$

Since, $s(\mathfrak{R}_3) > s(\mathfrak{R}_1) > s(\mathfrak{R}_2)$, therefore,

$$\begin{aligned} \mathfrak{R}_{h(1)} &= \mathfrak{R}_3 = (0.92e^{i2\pi(0.85)}, 0.18e^{i2\pi(0.25)}), \\ \mathfrak{R}_{h(2)} &= \mathfrak{R}_1 = (0.80e^{i2\pi(0.73)}, 0.34e^{i2\pi(0.45)}), \\ \mathfrak{R}_{h(3)} &= \mathfrak{R}_2 = (0.65e^{i2\pi(0.83)}, 0.43e^{i2\pi(0.29)}). \end{aligned} \quad (25)$$

Then, for $\gamma = 4$,

$$\begin{aligned} \text{CPFHOWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3) &= \bigoplus_{t=1}^3 \phi_t \mathfrak{R}_{h(t)} \\ &= \left(\sqrt[n]{\frac{\prod_{t=1}^3 (1 + 3u_{h(t)}^2)^{\phi_t} - \prod_{t=1}^3 (1 - u_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^3 (1 + 3u_{h(t)}^2)^{\phi_t} + 3 \prod_{t=1}^3 (1 - u_{h(t)}^2)^{\phi_t}}}, \right. \\ &\quad \cdot e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^3 (1 + 3(\alpha_{h(t)}/2\pi)^2}^{\phi_t} - \prod_{t=1}^3 (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}}{\prod_{t=1}^3 (1 + 3(\alpha_{h(t)}/2\pi)^2)^{\phi_t} + 3 \prod_{t=1}^3 (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}} \right)}, \\ &\quad \left. \frac{2 \prod_{t=1}^3 (v_{h(t)})^{\phi_t}}{\sqrt{\prod_{t=1}^3 (1 + 3(1 - v_{h(t)}^2)^{\phi_t} + 3 \prod_{t=1}^3 v_{h(t)}^{2\phi_t}}} \cdot e^{i2\pi \left(\frac{2 \prod_{t=1}^3 (\beta_{h(t)}/2\pi)^{\phi_t}}{\sqrt{\prod_{t=1}^3 (1 + 3(1 - (\beta_{h(t)}/2\pi)^2)^{\phi_t} + 3 \prod_{t=1}^3 (\beta_{h(t)}/2\pi)^{2\phi_t}}} \right)} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\sqrt{\frac{[1.8813 \times 1.2390 \times 1.2784] - [0.3919 \times 0.8152 \times 0.8481]}{[1.8813 \times 1.2390 \times 1.2784] + 3[0.3919 \times 0.8152 \times 0.8481]}} \right. \\
&\quad \cdot e^{i2\pi(\sqrt{[1.7797 \times 1.2105 \times 1.3996] - [0.5268 \times 0.8588 \times 0.7045]} / [1.7797 \times 1.2105 \times 1.3996] + 3[0.5268 \times 0.8588 \times 0.7045]})} \\
&\quad \left. \frac{2(0.4243 \times 0.8059 \times 0.7763)}{\sqrt{[1.9756 \times 1.2958 \times 1.4493] + 3[0.18 \times 0.6495 \times 0.6027]}} e^{i2\pi(2(0.5 \times 0.8524 \times 0.6898) / \sqrt{[1.9526 \times 1.2767 \times 1.4864] + 3[0.25 \times 0.7266 \times 0.4758]})} \right) \\
&= \left(\sqrt{\frac{2.9799 - 0.2709}{2.9799 + 0.8128}} e^{i2\pi(\sqrt{3.0152 - 0.3187/3.0152 + 0.9562})}, \frac{0.5309}{\sqrt{3.7102 + 0.2115}} e^{i2\pi(0.5880 / \sqrt{3.7054 + 0.2592})} \right) \\
&= (0.8451 e^{i2\pi(0.8240)}, 0.2681 e^{i2\pi(0.2953)}).
\end{aligned} \tag{26}$$

Special cases:

(i) For $\gamma = 1$, CPFHOWA operator becomes

$$\text{CPFHOWA}_\phi \mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n = \left(\sqrt{1 - \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}} e^{i2\pi \sqrt{1 - \prod_{t=1}^n (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}}}, \prod_{t=1}^n (v_{h(t)})^{\phi_t} e^{i2\pi \left(\prod_{t=1}^n (\beta_{h(t)}/2\pi)^{\phi_t} \right)} \right), \tag{27}$$

which represents the complex Pythagorean fuzzy ordered weighted averaging (CPFOWA) operator.

(ii) For $\gamma = 2$, CPFHOWA operator becomes

$$\begin{aligned}
\text{CPFHOWA}_\phi (\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \sqrt{\frac{\prod_{t=1}^n (1 + u_{h(t)}^2)^{\phi_t} - \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^n (1 + u_{h(t)}^2)^{\phi_t} + \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (1 + (\alpha_{h(t)}/2\pi)^2)^{\phi_t}} - \sqrt{\prod_{t=1}^n (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}}}{\sqrt{\prod_{t=1}^n (1 + (\alpha_{h(t)}/2\pi)^2)^{\phi_t}} + \sqrt{\prod_{t=1}^n (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}}} \right)}, \\
&\quad \frac{\sqrt{\prod_{t=1}^n (v_{h(t)})^{\phi_t}}}{\sqrt{\prod_{t=1}^n (2 - v_{h(t)}^2)^{\phi_t} + \prod_{t=1}^n v_{h(t)}^{2\phi_t}}} e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (\beta_{h(t)}/2\pi)^{\phi_t}}}{\sqrt{\prod_{t=1}^n (2 - (\beta_{h(t)}/2\pi)^{\phi_t} + \prod_{t=1}^n (\beta_{h(t)}/2\pi)^{2\phi_t}}} \right)},
\end{aligned} \tag{28}$$

which represents the complex Pythagorean fuzzy Einstein ordered weighted averaging (CPFEOWA) operator.

(iii) When $\alpha_t = \beta_t = 0, \forall t$, CPFHOWA operator becomes

$$\begin{aligned}
\text{CPFHOWA}_\phi (\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \left(\sqrt{\frac{\prod_{t=1}^n (1 + (\gamma - 1)u_{h(t)}^2)^{\phi_t} - \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1)u_{h(t)}^2)^{\phi_t} + (\gamma - 1)\prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}}, \right. \\
&\quad \left. \frac{\sqrt{\prod_{t=1}^n (v_{h(t)})^{\phi_t}}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1)(1 - v_{h(t)}^2))^{\phi_t} + (\gamma - 1)\prod_{t=1}^n v_{h(t)}^{2\phi_t}}} \right),
\end{aligned} \tag{29}$$

which represents the Pythagorean fuzzy Hamacher ordered weighted averaging (PFHOWA) operator.

Theorem 6 (idempotency property). *Let $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) be a family of CPFNs. If $\mathfrak{R}_t = \mathfrak{R} \forall t$, then*

$$\text{CPFHOWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \mathfrak{R}. \quad (30)$$

Proof. Since $\mathfrak{R}_t = \mathfrak{R} = (u e^{i\alpha}, v e^{i\beta})$, $\forall t$, then equation (9) becomes

$$\begin{aligned} \text{CPFHOWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \bigoplus_{t=1}^n \phi_t \mathfrak{R}_{h(t)} \\ &= \sqrt{\frac{\prod_{t=1}^n (1 + (\gamma - 1) u_{h(t)}^2)^{\phi_t} - \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) u_{h(t)}^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - u_{h(t)}^2)^{\phi_t}}} \\ &\quad \cdot e^{i2\pi \left(\frac{\prod_{t=1}^n (1 + (\gamma - 1) (\alpha_{h(t)}/2\pi)^2)^{\phi_t} - \prod_{t=1}^n (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) (\alpha_{h(t)}/2\pi)^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - (\alpha_{h(t)}/2\pi)^2)^{\phi_t}} \right)} \\ &\quad \cdot \frac{\sqrt{\gamma} \prod_{t=1}^n (v_{h(t)})^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - v_{h(t)}^2))^{\phi_t} + (\gamma - 1) \prod_{t=1}^n v_{h(t)}^{2\phi_t}}} \\ &\quad \cdot e^{i2\pi \left(\frac{\sqrt{\gamma} \prod_{t=1}^n (\beta_{h(t)}/2\pi)^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - (\beta_{h(t)}/2\pi)^2))^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (\beta_{h(t)}/2\pi)^{2\phi_t}}} \right)} \\ &= \sqrt{\frac{(1 + (\gamma - 1) u^2) - (1 - u^2)}{(1 + (\gamma - 1) u^2) + (\gamma - 1) (1 - u^2)}} e^{i2\pi \left(\frac{\sqrt{(1 + (\gamma - 1) (\alpha/2\pi)^2) - (1 - (\alpha/2\pi)^2)}}{(1 + (\gamma - 1) (\alpha/2\pi)^2) + (\gamma - 1) (1 - (\alpha/2\pi)^2)} \right)} \\ &\quad \cdot \frac{\sqrt{\gamma} v}{\sqrt{(1 + (\gamma - 1) (1 - v^2)) + (\gamma - 1) v^2}} e^{i2\pi \left(\frac{\sqrt{\gamma} (\beta/2\pi)}{\sqrt{(1 + (\gamma - 1) (1 - (\beta/2\pi)^2)) + (\gamma - 1) (\beta/2\pi)^2}} \right)} \\ &= (u e^{i\alpha}, v e^{i\beta}) = \mathfrak{R}. \end{aligned} \quad (31)$$

Theorem 7 (boundedness property). *Let $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) be a family of CPFNs. Let*

$$\begin{aligned} \mathfrak{R}^+ &= \left(\max_t u_t e^{i \max_t \alpha_t}, \min_t v_t e^{i \min_t \beta_t} \right), \\ \mathfrak{R}^- &= \left(\min_t u_t e^{i \min_t \alpha_t}, \max_t v_t e^{i \max_t \beta_t} \right), \end{aligned} \quad (32)$$

then

$$\mathfrak{R}^- \leq \text{CPFHOWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) \leq \mathfrak{R}^+. \quad (33)$$

Theorem 8 (monotonicity property). *Let $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ and $\mathfrak{R}'_t = (\mu'_t, \nu'_t) = (u'_t e^{i\alpha'_t}, v'_t e^{i\beta'_t})$ ($t = 1, 2, \dots, n$) be two families of CPFNs. If $u_t \leq u'_t$, $\alpha_t \leq \alpha'_t$, $v_t \geq v'_t$ and $\beta_t \geq \beta'_t$, then*

$$\text{CPFHOWA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) \leq \text{CPFHOWA}_\phi(\mathfrak{R}'_1, \mathfrak{R}'_2, \dots, \mathfrak{R}'_n). \quad (34)$$

The CPFHWA operator weights the CPFNs itself. On the other hand, CPFHOWA operator weights the ordered arrangements of CPFNs. To combine both of these properties in a single operator, we propose complex Pythagorean fuzzy Hamacher hybrid average (CPFHHA) operator.

Definition 8. For any collection $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) of CPFNs, the complex Pythagorean fuzzy Hamacher hybrid average (CPFHHA) operator is defined as follows:

$$\text{CPFHHA}_{\phi, \wp}(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \bigoplus_{t=1}^n \phi_t \tilde{\mathfrak{R}}_{h(t)}, \quad (35)$$

where $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$ denotes the associated weight vector with $\phi_t \in [0, 1]$ and $\sum_{t=1}^n \phi_t = 1$. $h(1), h(2), \dots, h(n)$ is a permutation of $(1, 2, \dots, n)$ such that $\tilde{\mathfrak{R}}_{h(t)} \geq \tilde{\mathfrak{R}}_{h(t+1)}$, $\forall t = 1, 2, \dots, n-1$. $\tilde{\mathfrak{R}}_{h(t)} = n \wp \mathfrak{R}_t$, ($t = 1, 2, \dots, n$), where n is the balancing coefficient. $\wp = (\wp_1, \wp_2, \dots, \wp_n)$ denotes the weight vector such that $\wp_t \in [0, 1]$ and $\sum_{t=1}^n \wp_t = 1$.

Theorem 9. For any collection $\mathfrak{R}_t = (\mu_t, \nu_t) = (u_t e^{i\alpha_t}, v_t e^{i\beta_t})$ ($t = 1, 2, \dots, n$) of CPFNs, the accumulated value by

deploying the CPFHHA operator is also a CPFN which is given as

$$\begin{aligned} \text{CPFHHA}_{\phi, \wp}(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \bigoplus_{t=1}^n \phi_t \tilde{\mathfrak{R}}_{h(t)} \\ &= \left(\sqrt{\frac{\prod_{t=1}^n (1 + (\gamma - 1) \tilde{u}_{h(t)}^2)^{\phi_t} - \prod_{t=1}^n (1 - \tilde{u}_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) \tilde{u}_{h(t)}^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - \tilde{u}_{h(t)}^2)^{\phi_t}}} \right. \\ &\quad \cdot e^{i2\pi \left(\frac{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}} - \prod_{t=1}^n (1 - (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}} \right)} \\ &\quad \frac{\sqrt{\gamma} \prod_{t=1}^n (\tilde{v}_{h(t)})^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - \tilde{v}_{h(t)}^2))^{\phi_t} + (\gamma - 1) \prod_{t=1}^n \tilde{v}_{h(t)}^{2\phi_t}}} \\ &\quad \cdot e^{i2\pi \left(\frac{\sqrt{\gamma} \prod_{t=1}^n (\tilde{\beta}_{h(t)}/2\pi)^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - (\tilde{\beta}_{h(t)}/2\pi)^2))^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (\tilde{\beta}_{h(t)}/2\pi)^{2\phi_t}}} \right)} \Bigg), \end{aligned} \quad (36)$$

where $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$ denotes the associated weight vector with $\phi_t \in [0, 1]$ and $\sum_{t=1}^n \phi_t = 1$. $h(1), h(2), \dots, h(n)$ is a permutation of $(1, 2, \dots, n)$ such that $\tilde{\mathfrak{R}}_{h(t)} \geq \tilde{\mathfrak{R}}_{h(t+1)}, \forall t = 1, 2, \dots, n-1$ and $\tilde{\mathfrak{R}}_{h(t)} = n\wp \mathfrak{R}_t$, ($t = 1, 2, \dots, n$), where n is the balancing coefficient. $\wp = (\wp_1, \wp_2, \dots, \wp_n)$ denotes the weight vector such that $\wp_t \in [0, 1]$ and $\sum_{t=1}^n \wp_t = 1$.

Special cases:

- (i) When $\phi = (1/n), (1/n), \dots, (1/n)^T$, CPFHHA operator reduces to CPFHWA operator.
- (ii) When $\wp = (1/n), (1/n), \dots, (1/n)$, CPFHHA operator reduces to CPFHOWA operator.
- (iii) When $\alpha_t = \beta_t = 0, \forall t$, the CPFHHA operator becomes

$$\begin{aligned} \text{CPFHHA}_{\phi, \wp}(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) &= \left(\sqrt{\frac{\prod_{t=1}^n (1 + (\gamma - 1) \tilde{u}_{h(t)}^2)^{\phi_t} - \prod_{t=1}^n (1 - \tilde{u}_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^n (1 + (\gamma - 1) \tilde{u}_{h(t)}^2)^{\phi_t} + (\gamma - 1) \prod_{t=1}^n (1 - \tilde{u}_{h(t)}^2)^{\phi_t}}} \right. \\ &\quad \frac{\sqrt{\gamma} \prod_{t=1}^n (\tilde{v}_{h(t)})^{\phi_t}}{\sqrt{\prod_{t=1}^n (1 + (\gamma - 1) (1 - \tilde{v}_{h(t)}^2))^{\phi_t} + (\gamma - 1) \prod_{t=1}^n \tilde{v}_{h(t)}^{2\phi_t}}} \Bigg), \end{aligned} \quad (37)$$

which represents the Pythagorean fuzzy Hamacher hybrid average (PFHHA) operator.

- (iv) For $\gamma = 1$, CPFHHA operator becomes

$$\text{CPFHHA}_{\phi, \wp}(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = \left(\sqrt{1 - \prod_{t=1}^n (1 - \tilde{u}_{h(t)}^2)^{\phi_t}} e^{i2\pi \left(\sqrt{1 - \prod_{t=1}^n (1 - (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}} \right)}, \prod_{t=1}^n (\tilde{v}_{h(t)})^{\phi_t} e^{i2\pi \left(\prod_{t=1}^n (\tilde{\beta}_{h(t)}/2\pi)^{\phi_t} \right)} \right), \quad (38)$$

which represents the complex Pythagorean fuzzy hybrid average (CPFHA) operator.

- (v) For $\gamma = 2$, CPFHHA operator becomes

$$\begin{aligned} \text{CPFHHA}_{\phi, \wp}(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n) = & \left(\sqrt{\frac{\prod_{t=1}^n (1 + \tilde{u}_{h(t)}^2)^{\phi_t} - \prod_{t=1}^n (1 - \tilde{u}_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^n (1 + \tilde{u}_{h(t)}^2)^{\phi_t} + \prod_{t=1}^n (1 - \tilde{u}_{h(t)}^2)^{\phi_t}}} \right. \\ & \cdot e^{i2\pi \left(\sqrt{\prod_{t=1}^n (1 + (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}} - \prod_{t=1}^n (1 - (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t} / \prod_{t=1}^n (1 + (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t} + \prod_{t=1}^n (1 - (\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}} \right)}, \\ & \frac{\sqrt{\gamma} \prod_{t=1}^n (\tilde{v}_{h(t)})^{\phi_t}}{\sqrt{\prod_{t=1}^n (2 - \tilde{v}_{h(t)}^2)^{\phi_t} + \prod_{t=1}^n \tilde{v}_{h(t)}^{2\phi_t}}} \\ & \left. e^{i2\pi \left(\sqrt{\gamma} \prod_{t=1}^n (\tilde{\beta}_{h(t)}/2\pi)^{\phi_t} / \sqrt{\prod_{t=1}^n (2 - (\tilde{\beta}_{h(t)}/2\pi)^2)^{\phi_t} + \prod_{t=1}^n (\tilde{\beta}_{h(t)}/2\pi)^{2\phi_t}} \right)} \right), \end{aligned} \quad (39)$$

which represents the complex Pythagorean fuzzy Einstein hybrid average (CPFEHA) operator.

$0.18e^{i2\pi(0.25)}$ be three CPFNs. Let $\phi = (0.5, 0.2, 0.3)^T$ and $\wp = (0.3, 0.45, 0.25)$ be the weight vectors. Then, for $\gamma = 4$,

Example 3. Let $\mathfrak{R}_1 = (0.8e^{i2\pi(0.73)}, 0.34e^{i2\pi(0.45)})$, $\mathfrak{R}_2 = (0.65e^{i2\pi(0.83)}, 0.43e^{i2\pi(0.29)})$, and $\mathfrak{R}_3 = (0.92e^{i2\pi(0.85)}, 0.18e^{i2\pi(0.25)})$

$$\begin{aligned} \tilde{\mathfrak{R}}_1 = & \left(\sqrt{\frac{(1 + 3u_1^2)^{\eta_{\wp_1}} - (1 - u_1^2)^{\eta_{\wp_1}}}{(1 + 3u_1^2)^{\eta_{\wp_1}} + 3(1 - u_1^2)^{\eta_{\wp_1}}}} e^{i2\pi \left(\sqrt{(1 + 3(\alpha_1/2\pi)^2)^{\eta_{\wp_1}} - (1 - (\alpha_1/2\pi)^2)^{\eta_{\wp_1}} / (1 + 3(\alpha_1/2\pi)^2)^{\eta_{\wp_1}} + 3(1 - (\alpha_1/2\pi)^2)^{\eta_{\wp_1}}} \right)}, \right. \\ & \left. \frac{\sqrt{\gamma} (v_1)^{\eta_{\wp_1}}}{\sqrt{(1 + 3(1 - v_1^2))^{\eta_{\wp_1}} + 3v_1^{2\eta_{\wp_1}}}} e^{i2\pi \left(\sqrt{\gamma} (\beta_1/2\pi)^{\eta_{\wp_1}} / \sqrt{(1 + 3(1 - (\beta_1/2\pi)^2))^{\eta_{\wp_1}} + 3(\beta_1/2\pi)^{2\eta_{\wp_1}}} \right)} \right) \\ = & \left(\sqrt{\frac{2.92^{3(0.3)} - 0.36^{3(0.3)}}{2.92^{3(0.3)} + 3(0.36^{3(0.3)})}} e^{i2\pi \left(\sqrt{2.5987^{3(0.3)} - 0.4671^{3(0.3)} / 2.5987^{3(0.3)} + 3(0.4671^{3(0.3)})} \right)}, \right. \\ & \left. \frac{2(0.34^{3(0.3)})}{\sqrt{(1 + 3(0.8844))^{3(0.3)} + 3((0.34)^{2(3)(0.3)}}} e^{i2\pi \left(2(0.45^{3(0.3)}) / \sqrt{(1 + 3(0.7975))^{3(0.3)} + 3((0.45)^{2(3)(0.3)})} \right)} \right) \\ = & (0.7632e^{i2\pi(0.6925)}, 0.3970e^{i2\pi(0.5057)}), \\ \tilde{\mathfrak{R}}_2 = & \left(\sqrt{\frac{(1 + 3u_2^2)^{\eta_{\wp_2}} - (1 - u_2^2)^{\eta_{\wp_2}}}{(1 + 3u_2^2)^{\eta_{\wp_2}} + 3(1 - u_2^2)^{\eta_{\wp_2}}}} e^{i2\pi \left(\sqrt{(1 + 3(\alpha_2/2\pi)^2)^{\eta_{\wp_2}} - (1 - (\alpha_2/2\pi)^2)^{\eta_{\wp_2}} / (1 + 3(\alpha_2/2\pi)^2)^{\eta_{\wp_2}} + 3(1 - (\alpha_2/2\pi)^2)^{\eta_{\wp_2}}} \right)}, \right. \\ & \left. \frac{\sqrt{\gamma} (v_2)^{\eta_{\wp_2}}}{\sqrt{(1 + 3(1 - v_2^2))^{\eta_{\wp_2}} + 3v_2^{2\eta_{\wp_2}}}} e^{i2\pi \left(\sqrt{\gamma} (\beta_2/2\pi)^{\eta_{\wp_2}} / \sqrt{(1 + 3(1 - (\beta_2/2\pi)^2))^{\eta_{\wp_2}} + 3(\beta_2/2\pi)^{2\eta_{\wp_2}}} \right)} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\sqrt{\frac{2.2675^{3(0.45)} - 0.5775^{3(0.45)}}{2.2675^{3(0.45)} + 3(0.5775^{3(0.45)})}} e^{i2\pi \left(\sqrt{3.0667^{3(0.45)} - 0.3111^{3(0.45)}/3.0667^{3(0.45)} + 3(0.3111^{3(0.45)})} \right)}, \right. \\
&\quad \left. \frac{2(0.43^{3(0.45)})}{\sqrt{(1+3(0.8151))^{3(0.45)} + 3((0.43)^{2(3)(0.45)})}} e^{i2\pi \left(2(0.29^{3(0.45)})/\sqrt{(1+3(0.9159))^{3(0.45)} + 3((0.29)^{2(3)(0.45)})} \right)} \right) \\
&= (0.7560e^{i2\pi(0.9164)}, 0.2700e^{i2\pi(0.1528)}), \\
\tilde{\mathfrak{R}}_3 &= \left(\sqrt{\frac{(1+3u_3^2)^{\eta_{\mathcal{Q}_3}} - (1-u_3^2)^{\eta_{\mathcal{Q}_3}}}{(1+3u_3^2)^{\eta_{\mathcal{Q}_3}} + 3(1-u_3^2)^{\eta_{\mathcal{Q}_3}}}} e^{i2\pi \left(\sqrt{(1+3(\alpha_3/2\pi)^2)^{\eta_{\mathcal{Q}_3}} - (1-(\alpha_3/2\pi)^2)^{\eta_{\mathcal{Q}_3}} / (1+3(\alpha_3/2\pi)^2)^{\eta_{\mathcal{Q}_3}} + 3(1-(\alpha_3/2\pi)^2)^{\eta_{\mathcal{Q}_3}}} \right)}, \right. \\
&\quad \left. \frac{\sqrt{\gamma} (v_3)^{\eta_{\mathcal{Q}_3}}}{\sqrt{(1+3(1-v_3^2))^{\eta_{\mathcal{Q}_3}} + 3v_3^{2\eta_{\mathcal{Q}_3}}}} e^{i2\pi \left(\sqrt{\gamma} (\beta_3/2\pi)^{\eta_{\mathcal{Q}_3}} / \sqrt{(1+3(1-(\beta_3/2\pi)^2)^{\eta_{\mathcal{Q}_3}} + 3(\beta_3/2\pi)^{2\eta_{\mathcal{Q}_3}}} \right)} \right) \\
&= \left(\sqrt{\frac{3.5392^{3(0.25)} - 0.1536^{3(0.25)}}{3.5392^{3(0.25)} + 3(0.1536^{3(0.25)})}} e^{i2\pi \left(\sqrt{3.1675^{3(0.25)} - 0.2775^{3(0.25)}/3.1675^{3(0.25)} + 3(0.2775^{3(0.25)})} \right)}, \right. \\
&\quad \left. \frac{2(0.18^{3(0.25)})}{\sqrt{(1+3(0.9676))^{3(0.25)} + 3((0.18)^{2(3)(0.25)})}} e^{i2\pi \left(2(0.25^{3(0.25)})/\sqrt{(1+3(0.9375))^{3(0.25)} + 3((0.25)^{2(3)(0.25)})} \right)} \right) \\
&= (0.8391e^{i2\pi(0.7521)}, 0.3187e^{i2\pi(0.4014)}).
\end{aligned} \tag{40}$$

The score degrees are computed as follows:

$$\begin{aligned}
s(\tilde{\mathfrak{R}}_1) &= 0.7632^2 - 0.3970^2 + 0.6925^2 - 0.5057^2 = 0.6487, \\
s(\tilde{\mathfrak{R}}_2) &= 0.7560^2 - 0.2700^2 + 0.9164^2 - 0.1528^2 = 1.3151, \\
s(\tilde{\mathfrak{R}}_2) &= 0.8391^2 - 0.3187^2 + 0.7521^2 - 0.4014^2 = 1.0071.
\end{aligned} \tag{41}$$

Since $s(\tilde{\mathfrak{R}}_2) > s(\tilde{\mathfrak{R}}_3) > s(\tilde{\mathfrak{R}}_1)$, therefore,

$$\tilde{\mathfrak{R}}_{h(1)} = \tilde{\mathfrak{R}}_2 = (0.7560e^{i2\pi(0.9164)}, 0.2700e^{i2\pi(0.1528)}),$$

$$\tilde{\mathfrak{R}}_{h(2)} = \tilde{\mathfrak{R}}_3 = (0.8391e^{i2\pi(0.7521)}, 0.3187e^{i2\pi(0.4014)}),$$

$$\tilde{\mathfrak{R}}_{h(3)} = \tilde{\mathfrak{R}}_1 = (0.7632e^{i2\pi(0.6925)}, 0.3970e^{i2\pi(0.5057)}),$$

$$\text{CPFHHA}_\phi(\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3) = \bigoplus_{t=1}^3 \phi_t \tilde{\mathfrak{R}}_{h(t)}$$

$$\begin{aligned}
&= \left(\sqrt{\frac{\prod_{t=1}^3 (1+3\tilde{u}_{h(t)}^2)^{\phi_t} - \prod_{t=1}^3 (1-\tilde{u}_{h(t)}^2)^{\phi_t}}{\prod_{t=1}^3 (1+3\tilde{u}_{h(t)}^2)^{\phi_t} + 3\prod_{t=1}^3 (1-\tilde{u}_{h(t)}^2)^{\phi_t}}}, \right. \\
&\quad \left. e^{i2\pi \left(\sqrt{\prod_{t=1}^3 (1+3(\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t} - \prod_{t=1}^3 (1-(\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}} / \sqrt{\prod_{t=1}^3 (1+3(\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t} + 3\prod_{t=1}^3 (1-(\tilde{\alpha}_{h(t)}/2\pi)^2)^{\phi_t}} \right)} \right),
\end{aligned}$$

$$\begin{aligned}
& \frac{\sqrt{\gamma} \prod_{t=1}^3 (\tilde{v}_{h(t)})^{\phi_t}}{\sqrt{\prod_{t=1}^3 (1 + 3(1 - \tilde{v}_{h(t)}^2))^{\phi_t} + 3 \prod_{t=1}^3 \prod_{t=1}^3 (\tilde{v}_{h(t)})^{\phi_t}}} \\
& \cdot e^{i2\pi \left(\sqrt{\gamma} \prod_{t=1}^3 (\tilde{\beta}_{h(t)/2\pi})^{\phi_t} / \sqrt{\prod_{t=1}^3 (1 + 3(1 - (\tilde{\beta}_{h(t)/2\pi})^2))^{\phi_t} + 3 \prod_{t=1}^3 (\tilde{\beta}_{h(t)/2\pi})^{2\phi_t}} \right)} \\
& = \left(\sqrt{\frac{[1.6476 \times 1.2549 \times 1.3542] - [0.6546 \times 0.7838 \times 0.7695]}{[1.6476 \times 1.2549 \times 1.3542] + 3[0.6546 \times 0.7838 \times 0.7695]}} \right. \\
& \cdot e^{i2\pi \left(\sqrt{[1.876 \times 1.2195 \times 1.3066] - [0.4002 \times 0.8464 \times 0.8221]} / \sqrt{[1.876 \times 1.2195 \times 1.3066] + 3[0.4002 \times 0.8464 \times 0.8221]} \right)} \Bigg), \quad (42) \\
& \frac{2(0.5196 \times 0.7956 \times 0.7579)}{\sqrt{[1.9446 \times 1.2988 \times 1.4596] + 3[0.27 \times 0.6329 \times 0.5745]}} \\
& \cdot e^{i2\pi (2(0.3909 \times 0.8331 \times 0.815) / \sqrt{[1.9824 \times 1.286 \times 1.4219] + 3[0.1528 \times 0.6941 \times 0.6643]})} \Bigg) \\
& = \left(\sqrt{\frac{2.7999 - 0.3948}{2.7999 + 1.1844}} e^{i2\pi (\sqrt{2.9892 - 0.2785/2.9892 + 0.8355})}, \frac{0.6266}{\sqrt{3.6864 + 0.8835}} e^{i2\pi (0.5308 / \sqrt{3.6249 + 0.2115})} \right) \\
& = (0.7769 e^{i2\pi (0.8419)}, 0.2931 e^{i2\pi (0.2710)}).
\end{aligned}$$

3. Mathematical Model for Multicriteria Decision-Making Using Complex Pythagorean Fuzzy Information

In this section, we describe the application of CPFHWA operator and CPFHOWA operator in MCDM problems to identify the best alternative. The structure of a MCDM problem is described as follows:

Let $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_p\}$ be the set of alternatives from which the best one is to be chosen with respect to a number of decision criteria $c_1, c_2, \dots, c_g$. A decision maker is appointed to analyze the needs of the MCDM problem. The weight vector $\phi = (\phi_1, \phi_2, \dots, \phi_g)^T$ denotes the normalized weights of decision criteria. The expert investigates the capabilities of an alternative $\mathfrak{B}_m$ relative to criterion c_n and assigns it a CPFN $Q_{mn} = (\mu_{mn}, \nu_{mn}) = (\mu_{mn} e^{i\alpha_{mn}}, \nu_{mn} e^{i\beta_{mn}})$ according to its performance. These CPFNs are tabulated to form complex Pythagorean fuzzy decision matrix (CPFDM) $\mathfrak{Q}$ which is given by

$$\mathfrak{Q} = \begin{pmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \dots & (\mu_{1g}, \nu_{1g}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \dots & (\mu_{2g}, \nu_{2g}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{p1}, \nu_{p1}) & (\mu_{p2}, \nu_{p2}) & \dots & (\mu_{pg}, \nu_{pg}) \end{pmatrix}. \quad (43)$$

We propose Algorithm 1 to solve MCDM problems with the help of CPFHWA operator or CPFHOWA operators.

The flow chart of the proposed algorithm is represented in Figure 1.

3.1. Selection of the Best Country for Immigrants. Immigration refers to an international movement of the people from their birth place to any other developed and modern country without having the nationality of that

country. Few factors behind immigration are attractive job, establishment of business, higher education from a well-known institution, better medical treatment, permanent residence to enhance the quality of life, natural disasters, and moving to partner's place after marriage. Immigration is a key factor that strengthens the economy of any country and helps to improve gross domestic product of a country. Immigrants consider various factors before moving to a country involving legal framework, laws, citizenship, culture, climate, religious freedom, gender equality, literacy rate, food, political environment, and economic condition of that country. The aim of this study is to select the most suitable country for immigrants relative to their desires (criteria) under the framework of MCDM for complex Pythagorean fuzzy model. For this purpose, the data have been collected from usnews.com regarding each country. The following countries have been treated as alternatives to opt the most favorable country for immigrants by deploying CPFHWA operator and CPFHOWA operator:

- $\mathfrak{B}_1$: Switzerland,
- $\mathfrak{B}_2$: Germany,
- $\mathfrak{B}_3$: Australia,
- $\mathfrak{B}_4$: Canada,
- $\mathfrak{B}_5$: Sweden.

The decision-making expert decides the following factors to be used as criteria for this MCDM problem:

- c_1 : citizenship (human rights preservation, environment, gender equality, progressive, religious freedom, property rights, trustworthy, and political environment),
- c_2 : entrepreneurship (foreign policy, literacy rate, legal framework, infrastructure, business friendly policies, and hardworking labor),

c_3 : quality of life (attractive job opportunities, strong economy, high wages and income, safe, politically stable, education system, and health system),

c_4 : cultural influence (moral values, strong cultural influence, fashionable, modern, and beautiful).

First, we solve this MCDM problem by following the proposed algorithm and using CPFHWA operator

Step 1: the decision maker assigns a CPFN to each alternative relative to every criteria which are assembled in CPFDM, as shown in Table 1.

Step 2: the preference value $\mathcal{U}_m$ regarding each alternative $\mathfrak{B}_m$ can be computed with the help of CPFHWA operator by taking $\gamma = 4$ and the weight vector $\phi = (0.23, 0.32, 0.25, 0.20)^T$ of criteria:

$$\begin{aligned}\mathcal{U}_1 &= [0.8531e^{i2\pi(0.8019)}, 0.3083e^{i2\pi(0.3030)}], \\ \mathcal{U}_2 &= [0.7888e^{i2\pi(0.7426)}, 0.3826e^{i2\pi(0.4230)}], \\ \mathcal{U}_3 &= [0.7592e^{i2\pi(0.7158)}, 0.4512e^{i2\pi(0.4570)}], \\ \mathcal{U}_4 &= [0.8661e^{i2\pi(0.8168)}, 0.2395e^{i2\pi(0.2648)}], \\ \mathcal{U}_5 &= [0.8432e^{i2\pi(0.7895)}, 0.2897e^{i2\pi(0.3146)}].\end{aligned}\quad (44)$$

Step 3: the score degree of preference value for each alternative $\mathfrak{B}_m$ is given by

$$\begin{aligned}s(\mathcal{U}_1) &= 1.1841, \\ s(\mathcal{U}_2) &= 0.8484, \\ s(\mathcal{U}_3) &= 0.6763, \\ s(\mathcal{U}_4) &= 1.2898, \\ s(\mathcal{U}_5) &= 1.1515.\end{aligned}\quad (45)$$

Step 4: from the scores of preference values, we infer the following ranking for the alternatives:

$$\mathfrak{B}_4 > \mathfrak{B}_1 > \mathfrak{B}_5 > \mathfrak{B}_2 > \mathfrak{B}_3. \quad (46)$$

Thus, Canada is the best country for immigrants.

Now, we solve the same MCDM problem by following the proposed algorithm and using CPFHOWA operator for $\gamma = 4$.

Step 1: the CPFDM $\mathfrak{Q}$, comprising CPFNs, is given by Table 1.

Step 2: the preference value $\mathcal{U}_m$ of each alternative $\mathfrak{B}_m$ can be computed with the help of CPFHOWA operator and the weight vector $\phi = (0.23, 0.32, 0.25, 0.20)^T$ of criteria:

$$\begin{aligned}\mathcal{U}_1 &= [0.8531e^{i2\pi(0.8019)}, 0.3083e^{i2\pi(0.3030)}], \\ \mathcal{U}_2 &= [0.7831e^{i2\pi(0.7372)}, 0.3939e^{i2\pi(0.4340)}], \\ \mathcal{U}_3 &= [0.7779e^{i2\pi(0.7313)}, 0.4287e^{i2\pi(0.4312)}], \\ \mathcal{U}_4 &= [0.8735e^{i2\pi(0.8253)}, 0.2227e^{i2\pi(0.2512)}], \\ \mathcal{U}_5 &= [0.8532e^{i2\pi(0.7995)}, 0.2698e^{i2\pi(0.2945)}].\end{aligned}\quad (47)$$

Step 3: the score degree of the preference value for each alternative $\mathfrak{B}_m$ is given by

$$\begin{aligned}s(\mathcal{U}_1) &= 1.1841, \\ s(\mathcal{U}_2) &= 0.8131, \\ s(\mathcal{U}_3) &= 0.7701, \\ s(\mathcal{U}_4) &= 1.3314, \\ s(\mathcal{U}_5) &= 1.2075.\end{aligned}\quad (48)$$

Step 4: from the scores of preference values, we infer the following ranking for the alternatives:

$$\mathfrak{B}_4 > \mathfrak{B}_5 > \mathfrak{B}_1 > \mathfrak{B}_2 > \mathfrak{B}_3. \quad (49)$$

Thus, Canada is the best country for immigrants.

To observe the worth of parameter γ , we solve this numerical example by CPFHWA operator and CPFHOWA operator for different values of γ and analyze the results. Tables 2 and 3 represent the outcomes of the CPHWA operator and CPFHOWA operator, including the score of preference values and final ranking, for the different values of parameter γ , respectively.

It is clear from Table 2 that the score of preference values differs slightly corresponding to different values of parameter γ but the final ranking of alternatives relative to any value of parameter ($\gamma > 0$) is the same. Hence, the optimal alternative is also the same corresponding to any value of parameter γ for CPFHWA operator. Similarly, Table 3 shows the consistency of the CPHOWA operator corresponding to any value of parameter γ . So, the parameter γ in CPFHWA operator and CPFHOWA operator makes them more generalized and authentic.

3.2. Selection of the Best Programming Language. A programming language is basically a collection of instructions, having a unique syntax, to design algorithms and computer programs which allows the computer to perform that specific task. A programming language is a tool that enables the programmer to communicate with the computer in a human readable form. With the advancement of technology, the computer programs have become an essential way to deal a large amount of data with accuracy. Computer programming is used in web designing, document formatting, video games, mobile applications, distance-learning, architecture, medical instruments, and business. A programmer is a person who is completely familiar with the syntax and coding of programming languages and is proficient in program designing for different purposes. Due to change in popularity and demand of the programming languages, the programmers need to learn the modern and popular languages to meet the standards of the modern world. A programmer wants to learn a programming language from the popular and trending languages with latest features which is beneficial for his professional projects. The following programming languages are considered as alternatives:

- $\mathfrak{B}_1$: Java,
 $\mathfrak{B}_2$: Python,
 $\mathfrak{B}_3$: PHP,
 $\mathfrak{B}_4$: JavaScript,
 $\mathfrak{B}_5$: C++.

The following properties and features of the programming languages are considered as criteria for this MCDM problem:

- c_1 : latest programming features,
 c_2 : reliability,
 c_3 : demand in job sector,
 c_4 : popularity.

First, we solve this MCDM problem by the proposed algorithm and CPFHWA operator.

Step 1: the opinions of decision maker about each alternative regarding all considered criteria are summarized in Table 4.

Step 2: the preference value of each alternative, evaluated by CPFHWA operator using $\gamma = 4$ and weight vector $\phi = (0.2, 0.25, 0.25, 0.3)^T$, is

$$\begin{aligned}
 \mathfrak{U}_1 &= [0.6788e^{i2\pi(0.6377)}, 0.2256e^{i2\pi(0.2312)}], \\
 \mathfrak{U}_2 &= [0.7512e^{i2\pi(0.7751)}, 0.1563e^{i2\pi(0.1476)}], \\
 \mathfrak{U}_3 &= [0.6753e^{i2\pi(0.5982)}, 0.2570e^{i2\pi(0.2556)}], \\
 \mathfrak{U}_4 &= [0.6704e^{i2\pi(0.5892)}, 0.2316e^{i2\pi(0.2067)}], \\
 \mathfrak{U}_5 &= [0.5350e^{i2\pi(0.4914)}, 0.2418e^{i2\pi(0.2451)}].
 \end{aligned} \quad (50)$$

Step 3: the score of the preference value for each alternative is given by

$$\begin{aligned}
 s(\mathfrak{U}_1) &= 0.7632, \\
 s(\mathfrak{U}_2) &= 1.1190, \\
 s(\mathfrak{U}_3) &= 0.6825, \\
 s(\mathfrak{U}_4) &= 0.7002, \\
 s(\mathfrak{U}_5) &= 0.4092.
 \end{aligned} \quad (51)$$

Step 4: from the scores of preference values, we deduce the following ranking for the alternatives:

$$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5. \quad (52)$$

Thus, Python is the best programming language to learn.

Now, we solve the same MCDM problem by following the proposed algorithm and using CPFHOWA operator for $\gamma = 4$.

Step 1: the CPFDM, comprising the CPFNs, is shown in Table 4.

Step 2: the preference value $\mathfrak{U}_m$ of each alternative $\mathfrak{B}_m$ can be computed with the help of CPFHOWA and the weight vector $\phi = (0.20, 0.25, 0.25, 0.30)^T$ of criteria:

$$\begin{aligned}
 \mathfrak{U}_1 &= [0.6788e^{i2\pi(0.6377)}, 0.2256e^{i2\pi(0.2312)}], \\
 \mathfrak{U}_2 &= [0.7260e^{i2\pi(0.7519)}, 0.1732e^{i2\pi(0.1604)}], \\
 \mathfrak{U}_3 &= [0.6661e^{i2\pi(0.5888)}, 0.2609e^{i2\pi(0.2595)}], \\
 \mathfrak{U}_4 &= [0.6534e^{i2\pi(0.5677)}, 0.2401e^{i2\pi(0.2165)}], \\
 \mathfrak{U}_5 &= [0.5119e^{i2\pi(0.4680)}, 0.2480e^{i2\pi(0.2517)}].
 \end{aligned} \quad (53)$$

Step 3: the score degree of the preference value for each alternative is given by

$$\begin{aligned}
 s(\mathfrak{U}_1) &= 0.7632, \\
 s(\mathfrak{U}_2) &= 1.0367, \\
 s(\mathfrak{U}_3) &= 0.6550, \\
 s(\mathfrak{U}_4) &= 0.6446, \\
 s(\mathfrak{U}_5) &= 0.3561.
 \end{aligned} \quad (54)$$

Step 4: from the scores of preference values, we conclude the following ranking for the alternatives:

$$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_3 > \mathfrak{B}_4 > \mathfrak{B}_5. \quad (55)$$

Thus, Python is the most beneficial language for the programmer.

4. Comparative Analysis and Discussion

In this section, we provide the comparative analysis of our proposed CPFHAAOs with different CIFAOs, proposed by Garg et al. [35, 36], to demonstrate their competence and check the accuracy of results. For this purpose, we apply the different CIFAOs, based on Archimedean t-norm and t-conorm, to our numerical example for the “selection of best programming language”.

- (i) We apply well-established CIFAOs [35], namely, CIFWA operator, CIFEWA operator, and CIFHWA operator as well as complex intuitionistic fuzzy Archimedean weighted geometric (CIFAWG) operators [36], namely, CIFWG operator, CIFEWG operator, and CIFHWG operator to select the most worthwhile programming language and investigate their results. The outcomes of CIFAOs, including the scores of preference values and final ranking, are assembled in Table 5 to inspect the veracity of our proposed operators.

It is clear from Table 5 that score of the preference values for each alternative may differ slightly but the best alternative obtained from the proposed operators and existing operators is the same which

FIGURE 1: Flow chart of mathematical approach for MCDM.

demonstrate the consistency of the proposed operators.

- (ii) The comparative analysis of the proposed CPFHAAOs with CIFAOs is graphically represented in Figure 2 which shows the consistency of CPFHAAOs.
- (iii) The CPFHAAOs work efficiently in decision-making environment and overcome the limitations of all existing AOs which are unable to deal with periodic data due to the absence of phase term.
- (iv) The proposed CPFHAAOs not only deal with CPF data proficiently rather they can be successfully applied to Pythagorean fuzzy numbers and intuitionistic fuzzy numbers by taking phase term equal to zero to tackle imprecise information of one dimension.
- (v) The AOs in CIF model are competent to deal two-dimensional vague information but they are bound by some conditions. The CIFAOs lost their worth if

the sum of amplitude or phase terms exceeds 1. In such situations, our proposed CPFHAAOs work efficiently as CPFs possess more relaxed conditions than CIFs. Therefore, CPFHWA operator, CPFHOWA operator, and CPFHHA operator have an edge over the existing CIFAOs because they own the advantageous properties of CPFs, AOs, Hamacher t-norm, and Hamacher t-conorm.

5. Merits of the Proposed Operators

- (i) The multipurpose proposed operators present a worthy blend of the theory of AOs with advantageous characteristics of the CPFs to cumulate the two-dimensional inexact data. The existing Hamacher operators inclusive of intuitionistic fuzzy hamacher AOs and Pythagorean fuzzy hamacher AOs are inept to address the two dimensional uncertain information.
- (ii) Another edge of the proposed operators lies in their parametric construction as they secure the

Input: $\mathfrak{B}$, the set of p feasible alternatives.

c , the set of g criteria.

$\phi = (\phi_1, \phi_2, \dots, \phi_g)^T$, the weight vector of decision criteria.

Step 1: construct the CPFDM by arranging the CPFNs assigned by decision-making expert.

Step 2: determine the preference value $\mathfrak{U}_m = (\mu_m, \nu_m) = (u_m e^{i\alpha_m}, v_m e^{i\beta_m})$ of each alternative $\mathfrak{B}_m$ using the CPFHWA operator as follows:

$$\begin{aligned} \mathfrak{U}_m &= \text{CPFHWA}_\phi(Q_{m1}, Q_{m2}, \dots, Q_{mg}) = \oplus_{n=1}^g \phi_n Q_{mn} \\ &= \sqrt{\prod_{n=1}^g (1 + (\gamma - 1)u_{mn}^2)^{\phi_n} - \prod_{n=1}^g (1 - u_{mn}^2)^{\phi_n} / \prod_{n=1}^g (1 + (\gamma - 1)u_{mn}^2)^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (1 - u_{mn}^2)^{\phi_n}} \\ &\quad e^{i2\pi(\sqrt{\prod_{n=1}^g (1 + (\gamma - 1)(\alpha_{mn}/2\pi)^2)^{\phi_n} - \prod_{n=1}^g (1 - (\alpha_{mn}/2\pi)^2)^{\phi_n} / \prod_{n=1}^g (1 + (\gamma - 1)(\alpha_{mn}/2\pi)^2)^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (1 - (\alpha_{mn}/2\pi)^2)^{\phi_n}}}, \\ &\quad \sqrt{\gamma} \prod_{n=1}^g (v_{mn})^{\phi_n} / \sqrt{\prod_{n=1}^g (1 + (\gamma - 1)(1 - v_{mn}^2))^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (v_{mn})^{2\phi_n}} \\ &\quad e^{i2\pi(\sqrt{\gamma} \prod_{n=1}^g (\beta_{mn}/2\pi)^{\phi_n} / \sqrt{\prod_{n=1}^g (1 + (\gamma - 1)(1 - (\beta_{mn}/2\pi)^2))^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (\beta_{mn}/2\pi)^{2\phi_n}})}, \end{aligned}$$

or by using CPFHOWA operator as follows:

$$\begin{aligned} \mathfrak{U}_m &= \text{CPFHOWA}_\phi(Q_{m1}, Q_{m2}, \dots, Q_{mg}) = \oplus_{n=1}^g \phi_n Q_{mh(n)} \\ &= \sqrt{\prod_{n=1}^g (1 + (\gamma - 1)u_{mh(n)}^2)^{\phi_n} - \prod_{n=1}^g (1 - u_{mh(n)}^2)^{\phi_n} / \prod_{n=1}^g (1 + (\gamma - 1)u_{mh(n)}^2)^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (1 - u_{mh(n)}^2)^{\phi_n}} \\ &\quad e^{i2\pi(\sqrt{\prod_{n=1}^g (1 + (\gamma - 1)(\alpha_{mh(n)}/2\pi)^2)^{\phi_n} - \prod_{n=1}^g (1 - (\alpha_{mh(n)}/2\pi)^2)^{\phi_n} / \prod_{n=1}^g (1 + (\gamma - 1)(\alpha_{mh(n)}/2\pi)^2)^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (1 - (\alpha_{mh(n)}/2\pi)^2)^{\phi_n}}}, \\ &\quad \sqrt{\gamma} \prod_{n=1}^g (v_{mh(n)})^{\phi_n} / \sqrt{\prod_{n=1}^g (1 + (\gamma - 1)(1 - v_{mh(n)}^2))^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (v_{mh(n)})^{2\phi_n}} \\ &\quad e^{i2\pi(\sqrt{\gamma} \prod_{n=1}^g (\beta_{mh(n)}/2\pi)^{\phi_n} / \sqrt{\prod_{n=1}^g (1 + (\gamma - 1)(1 - (\beta_{mh(n)}/2\pi)^2))^{\phi_n} + (\gamma - 1) \prod_{n=1}^g (\beta_{mh(n)}/2\pi)^{2\phi_n}})}, \end{aligned}$$

where $h(1), h(2), \dots, h(g)$ is a permutation of $(1, 2, \dots, g)$ such that $Q_{mh(n)} \geq Q_{mh(n+1)} \forall n = 1, 2, \dots, g - 1$.

Step 3: calculate the score degree $s(\mathfrak{U}_m)$ of preference value for each alternative $\mathfrak{B}_m$ as follows:

$$s(\mathfrak{U}_m) = u_m^2 - v_m^2 + (\alpha_m/2\pi)^2 - (\beta_m/2\pi)^2$$

In case of the same score degree of two alternatives, we seek help from the accuracy degree which can be evaluated using the formula:

$$h(\mathfrak{U}_m) = u_m^2 + v_m^2 + (\alpha_m/2\pi)^2 + (\beta_m/2\pi)^2$$

Step 4: rank all the alternatives according to Definition 4 using their score degree and accuracy degree.

Output: identify the optimal alternative, having the highest score degree, as the solution of MCDM problem.

ALGORITHM 1: The stepwise procedure to solve MCDM problem using CPFHWA (or CPFHOWA) operator.

generalized and parametric structure of Hamacher t-norm and t-conorm. The parameter γ empowers the proposed operators to cover the existing operators, including CPFWA, CPFOWA, CPFOWA, and CPFOWA operators and emphasizes the accuracy of results.

(iii) Furthermore, the competency of the proposed operators is not only limited to address the two-dimensional obscure information but these operators operate excellently for some existing and traditional environments inclusive of PF and IF models.

(iv) In a nutshell, the proposed decision-making approach, based on the proposed operators, is a beneficial addition to the literature that presents an appropriate and befitted framework to capture the vagueness of human decisions with the target to point out the optimal alternative.

(v) Besides the proposed decision-making strategy, the proposed operators find extensive applications in different well-known decision-making procedures for the sake of aggregation of individual data to evaluate a group satisfactory solution.

TABLE 1: Complex Pythagorean fuzzy decision matrix $\mathfrak{Q}$.

$\mathfrak{Q}$	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_3$	$\mathfrak{c}_4$
$\mathfrak{B}_1$	$[0.91e^{i2\pi(0.86)}, 0.16e^{i2\pi(0.18)}]$	$[0.86e^{i2\pi(0.81)}, 0.32e^{i2\pi(0.29)}]$	$[0.86e^{i2\pi(0.81)}, 0.32e^{i2\pi(0.29)}]$	$[0.72e^{i2\pi(0.68)}, 0.55e^{i2\pi(0.58)}]$
$\mathfrak{B}_2$	$[0.71e^{i2\pi(0.67)}, 0.55e^{i2\pi(0.58)}]$	$[0.76e^{i2\pi(0.71)}, 0.48e^{i2\pi(0.51)}]$	$[0.93e^{i2\pi(0.89)}, 0.12e^{i2\pi(0.15)}]$	$[0.60e^{i2\pi(0.59)}, 0.62e^{i2\pi(0.66)}]$
$\mathfrak{B}_3$	$[0.86e^{i2\pi(0.81)}, 0.32e^{i2\pi(0.29)}]$	$[0.60e^{i2\pi(0.59)}, 0.62e^{i2\pi(0.66)}]$	$[0.80e^{i2\pi(0.75)}, 0.40e^{i2\pi(0.40)}]$	$[0.77e^{i2\pi(0.72)}, 0.44e^{i2\pi(0.46)}]$
$\mathfrak{B}_4$	$[0.93e^{i2\pi(0.89)}, 0.12e^{i2\pi(0.15)}]$	$[0.84e^{i2\pi(0.78)}, 0.34e^{i2\pi(0.32)}]$	$[0.93e^{i2\pi(0.89)}, 0.12e^{i2\pi(0.15)}]$	$[0.60e^{i2\pi(0.59)}, 0.62e^{i2\pi(0.66)}]$
$\mathfrak{B}_5$	$[0.91e^{i2\pi(0.86)}, 0.16e^{i2\pi(0.18)}]$	$[0.77e^{i2\pi(0.72)}, 0.44e^{i2\pi(0.46)}]$	$[0.91e^{i2\pi(0.86)}, 0.16e^{i2\pi(0.18)}]$	$[0.71e^{i2\pi(0.67)}, 0.55e^{i2\pi(0.58)}]$

TABLE 2: Significance of parameter γ in CPFHWA operator.

Parameter (γ)	Operator	$s(\mathfrak{U}_1)$	$s(\mathfrak{U}_2)$	$s(\mathfrak{U}_3)$	$s(\mathfrak{U}_4)$	$s(\mathfrak{U}_5)$	Ranking
1	CPFWA operator	1.1980	0.9082	0.7093	1.3232	1.1772	$\mathfrak{B}_4 > \mathfrak{B}_1 > \mathfrak{B}_5 > \mathfrak{B}_2 > \mathfrak{B}_3$
2	CPFEWA operator	1.1896	0.8723	0.6903	1.3039	1.1617	$\mathfrak{B}_4 > \mathfrak{B}_1 > \mathfrak{B}_5 > \mathfrak{B}_2 > \mathfrak{B}_3$
4	CPFHWA operator	1.1841	0.8484	0.6763	1.2898	1.1515	$\mathfrak{B}_4 > \mathfrak{B}_1 > \mathfrak{B}_5 > \mathfrak{B}_2 > \mathfrak{B}_3$
10	CPFHWA operator	1.1801	0.8310	0.6653	1.2788	1.1442	$\mathfrak{B}_4 > \mathfrak{B}_1 > \mathfrak{B}_5 > \mathfrak{B}_2 > \mathfrak{B}_3$
50	CPFHWA operator	1.1776	0.8205	0.6582	1.2718	1.1398	$\mathfrak{B}_4 > \mathfrak{B}_1 > \mathfrak{B}_5 > \mathfrak{B}_2 > \mathfrak{B}_3$
100	CPFHWA operator	1.1773	0.8191	0.6572	1.2708	1.1393	$\mathfrak{B}_4 > \mathfrak{B}_1 > \mathfrak{B}_5 > \mathfrak{B}_2 > \mathfrak{B}_3$

TABLE 3: Significance of parameter γ in CPFHOWA operator.

Parameter (γ)	Operator	$s(\mathfrak{U}_1)$	$s(\mathfrak{U}_2)$	$s(\mathfrak{U}_3)$	$s(\mathfrak{U}_4)$	$s(\mathfrak{U}_5)$	Ranking
1	CPFOWA operator	1.1980	0.8725	0.7929	1.3637	1.2314	$\mathfrak{B}_4 > \mathfrak{B}_5 > \mathfrak{B}_1 > \mathfrak{B}_2 > \mathfrak{B}_3$
2	CPFEOWA operator	1.1896	0.8367	0.7798	1.3450	1.2170	$\mathfrak{B}_4 > \mathfrak{B}_5 > \mathfrak{B}_1 > \mathfrak{B}_2 > \mathfrak{B}_3$
4	CPFHOWA operator	1.1841	0.8131	0.7701	1.3314	1.2075	$\mathfrak{B}_4 > \mathfrak{B}_5 > \mathfrak{B}_1 > \mathfrak{B}_2 > \mathfrak{B}_3$
10	CPFHOWA operator	1.1801	0.7960	0.7625	1.3207	1.2007	$\mathfrak{B}_4 > \mathfrak{B}_5 > \mathfrak{B}_1 > \mathfrak{B}_2 > \mathfrak{B}_3$
50	CPFHOWA operator	1.1776	0.7856	0.7575	1.3138	1.1966	$\mathfrak{B}_4 > \mathfrak{B}_5 > \mathfrak{B}_1 > \mathfrak{B}_2 > \mathfrak{B}_3$
100	CPFHOWA operator	1.1773	0.7842	0.7569	1.3128	1.1960	$\mathfrak{B}_4 > \mathfrak{B}_5 > \mathfrak{B}_1 > \mathfrak{B}_2 > \mathfrak{B}_3$

TABLE 4: Complex Pythagorean fuzzy decision matrix $\mathfrak{Q}$.

$\mathfrak{Q}$	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_3$	$\mathfrak{c}_4$
$\mathfrak{B}_1$	$[0.87e^{i2\pi(0.82)}, 0.11e^{i2\pi(0.15)}]$	$[0.65e^{i2\pi(0.60)}, 0.30e^{i2\pi(0.22)}]$	$[0.64e^{i2\pi(0.52)}, 0.21e^{i2\pi(0.23)}]$	$[0.54e^{i2\pi(0.60)}, 0.30e^{i2\pi(0.32)}]$
$\mathfrak{B}_2$	$[0.49e^{i2\pi(0.56)}, 0.34e^{i2\pi(0.26)}]$	$[0.84e^{i2\pi(0.88)}, 0.14e^{i2\pi(0.10)}]$	$[0.81e^{i2\pi(0.79)}, 0.14e^{i2\pi(0.16)}]$	$[0.74e^{i2\pi(0.76)}, 0.11e^{i2\pi(0.13)}]$
$\mathfrak{B}_3$	$[0.78e^{i2\pi(0.73)}, 0.16e^{i2\pi(0.21)}]$	$[0.60e^{i2\pi(0.54)}, 0.35e^{i2\pi(0.41)}]$	$[0.55e^{i2\pi(0.45)}, 0.31e^{i2\pi(0.26)}]$	$[0.74e^{i2\pi(0.65)}, 0.23e^{i2\pi(0.19)}]$
$\mathfrak{B}_4$	$[0.71e^{i2\pi(0.61)}, 0.24e^{i2\pi(0.18)}]$	$[0.46e^{i2\pi(0.35)}, 0.25e^{i2\pi(0.24)}]$	$[0.69e^{i2\pi(0.62)}, 0.30e^{i2\pi(0.34)}]$	$[0.76e^{i2\pi(0.70)}, 0.17e^{i2\pi(0.13)}]$
$\mathfrak{B}_5$	$[0.39e^{i2\pi(0.36)}, 0.16e^{i2\pi(0.18)}]$	$[0.76e^{i2\pi(0.72)}, 0.14e^{i2\pi(0.17)}]$	$[0.42e^{i2\pi(0.36)}, 0.43e^{i2\pi(0.41)}]$	$[0.47e^{i2\pi(0.43)}, 0.30e^{i2\pi(0.26)}]$

TABLE 5: Comparative study.

Additive generator	Operator	$s(\mathfrak{U}_1)$	$s(\mathfrak{U}_2)$	$s(\mathfrak{U}_3)$	$s(\mathfrak{U}_4)$	$s(\mathfrak{U}_5)$	Ranking
$t(y) = -\log(y)$	CIFWA [35]	0.8742	1.2333	0.7703	0.8308	0.5572	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5$
	CIFWG [36]	0.7815	1.1344	0.6968	0.7386	0.4086	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5$
$t(y) = \log(2 - y/y)$	CIFEWA [35]	0.8623	1.2247	0.7610	0.8198	0.5348	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5$
	CIFEWG [36]	0.7936	1.1509	0.7089	0.7548	0.4301	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5$
$t(y) = (\gamma + (1 - \gamma)y/y)$	CIFHWA [35] ($\gamma = 4$)	0.8547	1.2192	0.7547	0.8116	0.5187	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5$
	CIFHWG [36] ($\gamma = 4$)	0.8076	1.1686	0.7210	0.7696	0.4515	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5$
	CPFHWA (proposed) ($\gamma = 4$)	0.7632	1.1190	0.6825	0.7002	0.4092	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_4 > \mathfrak{B}_3 > \mathfrak{B}_5$
	CPFHOWA (proposed) ($\gamma = 4$)	0.7632	1.0367	0.6550	0.6446	0.3561	$\mathfrak{B}_2 > \mathfrak{B}_1 > \mathfrak{B}_3 > \mathfrak{B}_4 > \mathfrak{B}_5$

FIGURE 2: Comparative analysis.

6. Conclusion

The traditional models of fuzzy set theory can handle one-dimensional uncertain information. But the realization that real data often have other characteristics has boosted the research on extended models of vague knowledge. In particular, many problems in decision-making concern periodic phenomena, so they cannot be completely addressed by models such as FS, IFS, and PFS. This limitation was overcome by Ramot with the development of CFSs. Even this model has been further extended to avoid issues with nonmemberships, thus producing, e.g., CIFs and CPFs. CPFs are better suited for the faithful expression of two-dimensional data, as they can incorporate more amount of vagueness.

In this article, we have combined the advantages of the Hamacher operator and the versatility of CPFs to develop some AOs with an eye to their applicability in decision-making. We have proposed the CPFHWA, CPFHOWA, and CPFHHA operators. We have analyzed their fundamental properties which include idempotency, monotonicity, and boundedness. In addition, we have highlighted some special cases to describe the generalization of these parameterized operators.

Our new tools have enabled us to propose an MCDM approach for CPF data. We have illustrated the proposed algorithm with two numerical examples and a detailed comparative study. The latter goal has been achieved with a comparison of the outcomes of CPFHAAOs and the cases of other well-established CFAOs in decision-making environment. We have emphasized the superiorities and strengths of the selected operators along with the limitations of existing operators.

The CPFHWA, CPFHOWA, and CPFHHA operators are more versatile due to the parameterized nature of the Hamacher t-norm and t-conorm. The proposed operators can be effectively applied to any CPFNs in order to aggregate the data. They ultimately led us to deal with very general

MCDM problems that allow us to find the best alternative in the CPF as well as the CIF environment.

In short, Hamacher operators under different modern extensions of fuzzy sets serve as major contributions to decision-making scenarios. In the future, our aim is to extend our study to (1) complex Pythagorean fuzzy Hamacher geometric AOs and (2) complex Pythagorean fuzzy Hamacher power AOs. Besides the highlighted efficiencies of the proposed operators, there is a notable deficiency in these operators owing to the restrictions of the CPFs. Therefore, it will also be a beneficial task to exploit the competency of the Hamacher norms to construct the more practical and flexible operators by employing the theoretical background of the boarder and generalized model, namely, complex q -rung orthopair fuzzy set.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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