Research Article

Approximate Calculation Method for Noncentral t-Distribution Quantile

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In the process of structural design and structural performance evaluation, the inference of the reliable life of the structure and the representative value of the material strength is a necessary work. The determination of material strength is the presumption of the quantile of the normal distribution, and the determination of the confidence level of the quantile of the normal distribution involves the noncentral t-distribution function. However, the calculation of the quantile is very complicated and is often provided in the form of a numerical table, which often involves multiparameter interpolation calculation, so it is not convenient to apply. The existing approximate calculation methods for noncentral t-distribution quantiles have strict application conditions, and the calculation process is relatively cumbersome. It is still difficult to meet actual needs in terms of fitting accuracy, application range, and convenience. In this paper, a new calculation method for noncentral t-distribution quantiles is proposed by introducing new probability expressions and related approximate distributions, based on theoretical derivation and numerical fitting. The comparative analysis results show that the method not only is convenient for calculation but also has the advantages of higher accuracy and wider application range, and it is more in line with the actual needs of engineering.

1. Introduction

The structural reliability design covers the entire civil engineering and is the basic guarantee for the normal operation of the building structures [1, 2]. There are various reliability modeling methods. In practical engineering, uncertain failure modes can result in the system failure [3–5]. The noncentral t-distribution is mainly used for small sample inference of normal distribution quantile, and it is widely used in the estimation of structural reliability life, representative value of material strength, and so on [6, 7]. The probability density function of the noncentral t-distribution is complex, and its quantile is usually calculated by numerical methods and provided in the form of a numerical table. However, in the current numerical table, only some typical values are provided, and in other cases, interpolation calculation is required, and multiple parameters are involved, which is inconvenient to apply [8–11]. Therefore, a variety of approximate calculation methods have been

proposed, but they are still difficult to meet the actual needs in terms of fitting accuracy, scope of application, and convenience [12, 13].

In this paper, a new calculation method for noncentral t-distribution quantiles is proposed by introducing new probability expressions and related approximate distributions, based on theoretical derivation and numerical fitting. The comparative analysis results show that the method not only is convenient for calculation, but also has the advantages of higher accuracy and wider application range, and it is more in line with the actual needs of engineering.

2. Current Approximate Calculation Method

The random variable X obeys the normal distribution $N(\mu, \sigma^2)$, and the distribution parameters μ and σ^2 are unknown. Then, the upper p and $1-p$ quantile values of X can be expressed as

$$\begin{aligned} x_p &= \mu - z_{1-p}\sigma, \\ x_{1-p} &= \mu + z_{1-p}\sigma, \end{aligned} \quad (1)$$

where z_{1-p} is the upper $1-p$ quantile of the standard normal distribution; p is the assurance rate of $X \geq x_p$ or $X \leq x_{1-p}$, and its value is generally more than 0.5, and x_p and x_{1-p} are small and large quantile values, respectively.

Let $X_1, X_2, \dots, X_n$ be the n samples of the random variable X . According to the statistical method, after obtaining the sample realization value $x_1, x_2, \dots, x_n$, the lower limit estimate of x_p and the upper limit estimate of x_{1-p} are, respectively,

$$\begin{aligned} \hat{x}_p &= \bar{x} - ks, \\ \hat{x}_{1-p} &= \bar{x} + ks, \end{aligned} \quad (2)$$

where

$$\begin{aligned} \bar{x} &= \frac{1}{n} \sum_{i=1}^n x_i, \\ s &= \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}, \\ k &= \frac{t_{(n-1, z_{1-p}\sqrt{n}, \alpha)}}{\sqrt{n}}, \end{aligned} \quad (3)$$

where $t_{(n-1, z_{1-p}\sqrt{n}, \alpha)}$ is the upper α quantile value of the noncentral t-distribution with a degree of freedom of X and a parameter of $z_{1-p}\sqrt{n}$, and α is the significance level.

The coefficient k is related to the sample size n , the distribution parameter $z_{1-p}\sqrt{n}$ and the significance level α , etc., and is mainly determined by the quantile value $t_{(n-1, z_{1-p}\sqrt{n}, \alpha)}$ of the noncentral t-distribution. According to the approximate statistics of the F distribution [12], it is recommended to approximate as

$$t_{(n-1, \lambda, \alpha)} = \begin{cases} \frac{\lambda}{C_n} F_{(\gamma_1, \gamma_1, \alpha)}, & \lambda > 0, \\ \frac{\lambda}{C_n} F_{(\gamma_1, \gamma_1, 1-\alpha)}, & \lambda < 0, \end{cases} \quad (4)$$

where

$$\begin{aligned} C_n &= \sqrt{\frac{2}{n-1}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}, \\ \gamma_1 &= 2\lambda^2, \\ \gamma_2 &= \frac{2C_n^2}{1 - C_n^2}, \end{aligned} \quad (5)$$

where $t_{(n-1, \gamma, \alpha)}$ is the upper α quantile value of the noncentral t-distribution with the degree of freedom $n-1$ and the

parameter γ . $F_{(\gamma_1, \gamma_1, \alpha)}$ and $F_{(\gamma_1, \gamma_1, 1-\alpha)}$ are the upper α and $1-\alpha$ quantile values of the F distribution with parameters γ_1 and γ_2 , respectively. When γ_1 and γ_2 are nonpositive integers, the values of $F_{(\gamma_1, \gamma_1, \alpha)}$ and $F_{(\gamma_1, \gamma_1, 1-\alpha)}$ can be determined by linear interpolation method. This method is mainly applicable to the case of $n > 9$, and, in general, it is still inconvenient to check the table.

When

$$\begin{aligned} T_1 &= \frac{\bar{X} - x_p}{S/\sqrt{n}}, \\ \bar{X} &= \frac{1}{n} \sum_{i=1}^n X_i, \\ S &= \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}. \end{aligned} \quad (6)$$

Then, T_1 obeys a noncentral t-distribution with a degree of freedom of $n-1$ and a parameter of $z_{1-p}\sqrt{n}$. Jitao [14] approximates that the probability distribution of S is a normal distribution $N(\sigma, (\sigma^2/2(n-1)))$. By numerical fitting, it is recommended to approximate the noncentral t-distribution of the quantile value as

$$t_{(n-1, \gamma, \alpha)} = \gamma + z_\alpha \sqrt{1 - \left(\frac{(z_\alpha^2 - \gamma^2)/(2(n-m))}{1 - (z_\alpha^2/(2(n-m)))} z_\alpha^2 \leq 2(n-m) \right)}, \quad (7)$$

$$m = \frac{1.5}{(1 - \alpha)^2}, \quad (8)$$

where z_α is the upper α quantile value of the standard normal distribution.

This method is mainly applicable to the case of $n \geq 5$, has a wider scope of application, and is simpler than the method proposed by Huirong [12]. But it can be known from the comparison analysis of the accuracy of the following, the approximation methods proposed by Huirong [12] and Jitao and Yaokui [13] are difficult to meet the actual needs in terms of accuracy, and the scope of application is relatively limited.

3. New Approximate Calculation Method

If

$$T_2 = \frac{U + \gamma}{\sqrt{V/(n-1)}}, \quad (9)$$

where U obeys the standard normal distribution and V obeys the chi-square distribution with degrees of freedom of $n-1$, then, T_2 obeys a noncentral t-distribution with a degree of freedom of $n-1$ and a parameter of γ . When

$$\begin{aligned} P\{T_2 \leq t_{(n-1, \gamma, \alpha)}\} &= P\left\{\frac{U + \gamma}{\sqrt{V/(n-1)}} \leq t_{(n-1, \gamma, \alpha)}\right\} \\ &= 1 - \alpha, \end{aligned} \quad (10)$$

$$W = \sqrt{\frac{V}{n-1}}.$$

Then,

$$P\{U + \gamma - t_{(n-1, \gamma, \alpha)}W \leq 0\} = 1 - \alpha \quad (11)$$

The probability density function and mean value of V are

$$f_V(v) = \frac{1}{2^{(n-1)/2} \Gamma((n-1)/2)} e^{-(v/2)} v^{((n-1)/2)-1}, \quad v \geq 0,$$

$$\mu_V = n - 1. \quad (12)$$

So, the probability density function of W is

$$f_W(w) = 2(n-1)w f_V[(n-1)w^2], \quad w \geq 0. \quad (13)$$

Therefore,

$$\begin{aligned} \mu_W &= \int_0^\infty w f_W(w) dw \\ &= \int_0^\infty \sqrt{\frac{v}{n-1}} f_V(v) dv \\ &= \sqrt{\frac{2}{n-1}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}, \\ E(W^2) &= \int_0^\infty w^2 f_W(w) dw \\ &= \int_0^\infty \frac{v}{n-1} f_V(v) dv = 1, \\ \sigma_W &= \sqrt{E(W^2) - \mu_W^2} \\ &= \sqrt{1 - \frac{2}{n-1} \frac{\Gamma^2(n/2)}{\Gamma^2((n-1)/2)}}. \end{aligned} \quad (14)$$

Referring to the method proposed by Shisong et al. [15] and Jitao [14], approximating that W obeys the normal distribution $N(\mu_W, \sigma_W^2)$, then

$$\begin{aligned} P\{U + \gamma - t_{(n-1, \gamma, \alpha)}W \leq 0\} &\approx \Phi\left(\frac{\gamma - t_{(n-1, \gamma, \alpha)}\mu_W}{\sqrt{1 + t_{(n-1, \gamma, \alpha)}^2 \sigma_W^2}}\right) = 1 - \alpha, \\ \frac{t_{(n-1, \gamma, \alpha)}\mu_W - \gamma}{\sqrt{1 + t_{(n-1, \gamma, \alpha)}^2 \sigma_W^2}} &\approx z_\alpha. \end{aligned} \quad (15)$$

It can be obtained that

$$t_{(n-1, \gamma, \alpha)} = \frac{\gamma + z_\alpha \sqrt{1 - (z_\alpha^2 - \gamma^2) \delta_W^2}}{\mu_W (1 - z_\alpha^2 \delta_W^2)}, \quad (16)$$

$$\delta_W = \frac{\sigma_W}{\mu_W} = \sqrt{\frac{n-1}{2} \frac{\Gamma^2((n-1)/2)}{\Gamma^2(n/2)}} - 1 \approx 1 - \frac{0.25}{n-1}. \quad (17)$$

After the overall numerical fit, it can be found in equation (16):

$$\mu_W = 1 - \frac{0.25}{n-1} \left(p + 0.6 - \frac{0.1}{\alpha}\right), \quad (18)$$

$$p = \Phi\left(\frac{\gamma}{\sqrt{n}}\right).$$

If

$$\gamma = z_{1-p} \sqrt{n}, \quad (19)$$

then

$$k = \frac{z_{1-p} \sqrt{n} + z_\alpha \sqrt{1 - (z_\alpha^2 - z_{1-p}^2 n) \delta_W^2}}{\mu_W \sqrt{n} (1 - z_\alpha^2 \delta_W^2)}. \quad (20)$$

4. Comparative Analysis

Compare and analyze the calculation accuracy of the four methods: method 1: the method proposed by Huirong [12] calculates k according to (4); method 2: the method proposed by Jitao [13] calculates k according to (7); method 3: the method in this paper calculates k according to (20). Tables 1–5 list the corollary results of the coefficient k according to the different values of C .

In order to better research the trend of the coefficient k corollary results, the relative error scatter plots of the coefficient k corollary results can be drawn when $C=0.5$, $C=0.6$, $C=0.7$, $C=0.8$, and $C=0.9$, respectively, as shown in Figures 1–5. Since the relative error of Method 2 when $n < 5$ is too large, this part of the data is deleted when the figure is made.

From the results of comparative analysis, the error range of method 1 is $-30.59\% \sim 0.84\%$, the error range of method 2 is $-28.36\% \sim 104.4\%$, and the error range of method 3 is $-3.86\% \sim 2.75\%$. Separately, when $p=0.7$, the relative error ranges of these three methods are $-30.59\% \sim 0.84\%$, $-21.6\% \sim 76.29\%$, and $-3.86\% \sim 1.67\%$; when $p=0.8$, the relative error ranges of these three methods are $-9.9\% \sim 0.24\%$, $-28.36\% \sim 91.33\%$, and $-2.81\% \sim 2.0\%$; when $p=0.9$, the relative error ranges of these three methods are $-10.11\% \sim -0.01\%$, $-14.5\% \sim 101.86\%$, and $-2.45\% \sim 2.11\%$; and when $p=0.95$, the relative error ranges of these three methods are $-10.5\% \sim 0.85\%$, $-5.02\% \sim 104.4\%$, and $-2.35\% \sim 2.75\%$. It can be obtained that the error range calculated by method 2 is the largest, method 1 is in the middle, and method 3 is the smallest, indicating that the relative error calculated by method 3 is the smallest with the highest precision.

TABLE 1: Corollary results of coefficient k ($C=0.5$).

p 0.7							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	0.597986	0.415034	-30.59	0.524401	-12.31	0.608001	1.67
4	0.572487	0.435142	-23.99	0.524401	-8.40	0.577322	0.84
5	0.560064	0.449987	-19.65	0.524401	-6.37	0.563115	0.54
6	0.552664	0.460839	-16.61	—	—	0.554921	0.41
7	0.547827	0.469046	-14.38	0.524401	-4.28	0.54959	0.32
8	0.544407	0.475423	-12.67	0.524401	-3.68	0.545844	0.26
9	0.541868	0.480519	-11.32	0.524401	-3.22	0.543068	0.22
10	0.539877	0.484634	-10.23	0.524401	-2.87	0.585111	0.19
20	0.53166	0.503979	-5.21	0.524401	-1.37	0.562683	0.08
30	0.529147	0.510691	-3.49	0.524401	-0.90	0.554224	0.05
40	0.527907	0.514103	-2.61	0.524401	-0.66	0.549534	0.04
Error range			-30.59~-2.61		-12.31~-0.66		0.04~1.67
p 0.8							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	0.970715	0.88509	-8.82	0.841621	-13.3	0.990143	2
4	0.924547	0.859236	-7.06	0.841621	-9.0	0.935135	1.15
5	0.902606	0.849782	-5.85	—	—	0.909861	0.8
6	0.889844	0.845484	-4.99	—	—	0.895342	0.62
7	0.881479	0.843315	-4.33	0.841621	-4.52	0.885917	0.5
8	0.875564	0.842031	-3.83	0.841621	-3.88	0.879306	0.43
9	0.871202	0.841351	-3.43	0.841621	-3.4	0.874412	0.37
10	0.867798	0.840878	-3.1	0.841621	-3.02	0.870643	0.33
20	0.853853	0.840228	-1.6	0.841621	-1.43	0.855123	0.15
30	0.849585	0.840502	-1.07	0.841621	-0.94	0.850419	0.1
40	0.847534	0.84068	-0.81	0.841621	-0.7	0.848145	0.07
Error range			-8.82~-0.81		-13.3~-0.7		0.07~2
p 0.9							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	1.498545	1.481939	-1.11	1.281552	-14.5	1.530211	2.11
4	1.418883	1.403378	-1.09	—	—	1.437254	1.29
5	1.381884	1.368009	-1	—	—	1.394886	0.94
6	1.360419	1.348175	-0.9	—	—	1.370643	0.75
7	1.346566	1.335561	-0.82	1.281552	-4.83	1.354944	0.62
8	1.336729	1.326928	-0.73	1.281552	-4.13	1.343949	0.54
9	1.32965	1.320672	-0.68	1.281552	-3.62	1.335819	0.46
10	1.324126	1.315862	-0.62	1.281552	-3.22	1.329564	0.41
20	1.301272	1.29683	-0.34	1.281552	-1.52	1.303854	0.2
30	1.294449	1.291352	-0.24	1.281552	-1	1.296077	0.13
40	1.291037	1.28876	-0.18	1.281552	-0.73	1.292321	0.1
Error range			-1.11~-0.18		-14.5~-0.73		0.1~2.11
p 0.95							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	1.938445	1.954945	0.85	—	—	1.978771	2.08
4	1.829525	1.83853	0.49	—	—	1.853356	1.3
5	1.77926	1.784694	0.31	—	—	1.796427	0.96
6	1.75043	1.753874	0.20	—	—	1.763918	0.77
7	1.731816	1.734234	0.14	1.644854	-5.02	1.742891	0.64
8	1.718722	1.72031	0.09	1.644854	-4.3	1.728176	0.55
9	1.709037	1.710337	0.08	1.644854	-3.76	1.717302	0.48
10	1.701662	1.702589	0.05	1.644854	-3.34	1.708939	0.43
20	1.671163	1.67113	0	1.644854	-1.57	1.6746	0.21
30	1.661982	1.661913	0	1.644854	-1.03	1.664222	0.13
40	1.657536	1.657417	0	1.644854	-0.77	1.659212	0.1
Error range			0~0.85		-5.02~-0.77		0.1~2.08

When $C=0.9$, $p=0.7\sim 0.95$, and $n=3\sim 40$, the absolute values of the average and maximum relative errors of method 1 are 2.28%, 12.33%, the absolute values of the

average and maximum relative errors of method 2 are 10.33%, 104.4%, and the absolute values of the average and maximum relative errors of method 3 are 1.38%, 3.9%. It

TABLE 2: Corollary results of coefficient k ($C=0.6$).

0.7							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	0.799895	0.58474	-26.9	0.627209	-21.6	0.801897	0.25
4	0.733265	0.578577	-21.1	—	—	0.733647	0.05
5	0.698054	0.577128	-17.32	0.702867	0.69	0.69811	0.01
6	0.675672	0.576407	-14.69	0.659712	-2.36	0.675599	-0.01
7	0.659758	0.575744	-12.73	0.642013	-2.69	0.659767	0
8	0.647848	0.574975	-11.25	0.630945	-2.61	0.647881	0.01
9	0.63852	0.574221	-10.07	0.622911	-2.44	0.63855	0
10	0.630996	0.57343	-9.12	0.616635	-2.28	0.630981	0
20	0.594181	0.566305	-4.69	0.586898	-1.23	0.594198	0
30	0.579687	0.561364	-3.16	0.574889	-0.83	0.57971	0
40	0.571485	0.557856	-2.38	0.567906	-0.63	0.571498	0
Error range			-26.9~-2.38		-21.6~0.69		0~0.25
0.8							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	1.211267	1.111396	-8.25	0.867739	-28.36	1.216148	0.4
4	1.110931	1.038738	-6.5	—	—	1.113718	0.25
5	1.060245	1.003604	-5.34	1.082299	2.08	1.062497	0.21
6	1.02914	0.982315	-4.55	1.010636	-1.8	1.030933	0.17
7	1.007569	0.967799	-3.95	0.983455	-2.39	1.009184	0.16
8	0.991641	0.957063	-3.49	0.967535	-2.43	0.993113	0.15
9	0.97935	0.948756	-3.12	0.95652	-2.33	0.980658	0.13
10	0.969502	0.942029	-2.83	0.948208	-2.2	0.970662	0.12
20	0.922891	0.909472	-1.45	0.911562	-1.23	0.923535	0.07
30	0.905238	0.89636	-0.98	0.897651	-0.84	0.905674	0.05
40	0.895406	0.888792	-0.74	0.889712	-0.64	0.895735	0.04
Error range			-8.25~-0.74		-28.36~2.08		0.04~0.4
0.9							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	1.805399	1.77636	-1.61	—	—	1.812208	0.38
4	1.6497	1.628998	-1.25	—	—	1.655357	0.34
5	1.574205	1.557654	-1.05	1.618757	2.83	1.579205	0.32
6	1.528963	1.514778	-0.93	1.506009	-1.5	1.533232	0.28
7	1.498267	1.485975	-0.82	1.464439	-2.26	1.502045	0.25
8	1.475927	1.465039	-0.74	1.440912	-2.37	1.479285	0.23
9	1.458642	1.449051	-0.66	1.42511	-2.3	1.461825	0.22
10	1.445112	1.436289	-0.61	1.413474	-2.19	1.447934	0.2
20	1.382494	1.377987	-0.33	1.365121	-1.26	1.384065	0.11
30	1.359598	1.356631	-0.22	1.347844	-0.86	1.360658	0.08
40	1.347075	1.344737	-0.17	1.338184	-0.66	1.347853	0.06
Error range			-1.61~-0.17		-2.37~2.83		0.06~0.38
0.95							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	2.30676	2.308331	0.07	—	—	2.312866	0.26
4	2.102118	2.104901	0.13	—	—	2.108769	0.32
5	2.004726	2.006578	0.09	2.065685	3.04	2.010919	0.31
6	1.946573	1.948	0.07	1.918951	-1.42	1.952369	0.3
7	1.90781	1.9086	0.04	1.86521	-2.23	1.912919	0.27
8	1.879692	1.880188	0.03	1.835126	-2.37	1.884289	0.24
9	1.858126	1.85848	0.02	1.815139	-2.31	1.862426	0.23
10	1.841239	1.841258	0	1.800564	-2.21	1.8451	0.21
20	1.764225	1.764128	-0.01	1.741567	-1.28	1.766358	0.12
30	1.736488	1.736406	0	1.721125	-0.88	1.737963	0.08
40	1.721437	1.721317	-0.01	1.709825	-0.67	1.72256	0.07
Error range			-0.01~0.07		-2.37~3.04		0.07~0.26

can be seen that the precision of method 2 is the lowest, method 1 is in the middle, and method 3 has the highest precision.

With an absolute value of relative error not exceeding 5%, the applicable range of the method 1 is $n > 10$, and the applicable range of method 2 is $n > 7$, and method 3 satisfies

TABLE 3: Corollary results of coefficient k ($C=0.7$).

p		0.7					
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	1.056856	0.823382	-22.09	—	—	1.041203	-1.48
4	0.92804	0.770563	-16.97	0.982941	5.92	0.919138	-0.96
5	0.861097	0.742317	-13.79	0.850421	-1.24	0.855142	-0.69
6	0.818649	0.723444	-11.63	0.799625	-2.32	0.814274	-0.53
7	0.788632	0.709445	-10.04	0.769229	-2.46	0.785322	-0.42
8	0.766156	0.698382	-8.85	0.747888	-2.38	0.763447	-0.35
9	0.748413	0.689293	-7.9	0.731641	-2.24	0.74618	-0.3
10	0.73401	0.681582	-7.14	0.718652	-2.09	0.732107	-0.26
20	0.663438	0.639206	-3.65	0.655701	-1.17	0.662707	-0.11
30	0.635261	0.619653	-2.46	0.6302	-0.8	0.634827	-0.07
40	0.619144	0.607677	-1.85	0.615421	-0.6	0.61886	-0.05
Error range			-22.09~-1.85		-2.46~5.92		-1.48~-0.05
p		0.8					
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	1.528369	1.412346	-7.59	—	—	1.507871	-1.34
4	1.343264	1.267934	-5.61	1.458218	8.56	1.333216	-0.75
5	1.251379	1.1953	-4.48	1.248493	-0.23	1.245424	-0.48
6	1.19489	1.150102	-3.75	1.173717	-1.77	1.190894	-0.33
7	1.15592	1.118601	-3.23	1.131398	-2.12	1.153027	-0.25
8	1.1271	1.09512	-2.84	1.102841	-2.15	1.12485	-0.2
9	1.104595	1.076739	-2.52	1.081717	-2.07	1.102872	-0.16
10	1.0865	1.061783	-2.27	1.06519	-1.96	1.085136	-0.13
20	1.00039	0.988779	-1.16	0.988889	-1.15	0.999985	-0.04
30	0.967035	0.959519	-0.78	0.959368	-0.79	0.966858	-0.02
40	0.948261	0.942748	-0.58	0.942525	-0.6	0.948161	-0.01
Error range			-7.59~-0.58		-2.15~8.56		-1.34~-0.01
p		0.9					
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	2.219999	2.162173	-2.6	—	—	2.190339	-1.34
4	1.943907	1.912762	-1.6	2.14235	10.21	1.931962	-0.61
5	1.81224	1.790519	-1.2	1.819984	0.43	1.806181	-0.33
6	1.733163	1.716398	-0.97	1.708671	-1.41	1.729754	-0.2
7	1.679532	1.665911	-0.81	1.647691	-1.9	1.67754	-0.12
8	1.640374	1.628914	-0.7	1.60763	-2	1.63918	-0.07
9	1.610364	1.600302	-0.62	1.578625	-1.97	1.609567	-0.05
10	1.586351	1.577401	-0.56	1.556317	-1.89	1.585873	-0.03
20	1.474504	1.470441	-0.28	1.457608	-1.15	1.474811	0.02
30	1.432615	1.429924	-0.19	1.421102	-0.8	1.432884	0.02
40	1.409306	1.407292	-0.14	1.400619	-0.62	1.409556	0.02
Error range			-2.6~-0.14		-2~10.21		-1.34~0.02
p		0.95					
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	2.8089821	2.769737	-1.4	—	—	2.769095	-1.42
4	2.4530226	2.438936	-0.57	2.717347	10.8	2.437778	-0.62
5	2.2858326	2.278508	-0.32	2.300963	0.66	2.278716	-0.31
6	2.186787	2.181884	-0.22	2.158486	-1.29	2.183009	-0.17
7	2.120093	2.116571	-0.17	2.081324	-1.83	2.118108	-0.09
8	2.071657	2.069042	-0.13	2.031161	-1.95	2.070709	-0.05
9	2.034752	2.032585	-0.11	1.995165	-1.95	2.034297	-0.02
10	2.0052348	2.003434	-0.09	1.967689	-1.87	2.00528	0
20	1.8702006	1.869459	-0.04	1.848538	-1.16	1.870855	0.03
30	1.820288	1.819858	-0.02	1.805492	-0.81	1.820876	0.03
40	1.7927441	1.7925	-0.01	1.781563	-0.62	1.793272	0.03
Error range			-1.4~-0.01		-1.95~10.8		-1.42~-0.03

all the values of n in the above study. When $n < 5$, the difference of the relative error calculated by method 2 is the largest, indicating that method 2 is more suitable for the

approximate solution of the coefficient k when $n \geq 5$. It can be seen from the figure that when p is increased from 0.7 to 0.95, the absolute value of relative error calculated by these

TABLE 4: Corollary results of coefficient k ($C=0.8$).

0.7							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	1.441459	1.198475	-16.86	2.240555	55.44	1.392952	-3.37
4	1.199003	1.056408	-11.89	1.233037	2.84	1.171917	-2.26
5	1.079699	0.980532	-9.18	1.065791	-1.29	1.06182	-1.66
6	1.006262	0.931047	-7.47	0.984826	-2.13	0.993137	-1.3
7	0.955358	0.895292	-6.29	0.933649	-2.27	0.945115	-1.07
8	0.917479	0.867706	-5.42	0.897132	-2.22	0.909135	-0.91
9	0.887813	0.845589	-4.76	0.869215	-2.09	0.880892	-0.78
10	0.863964	0.827366	-4.24	0.846899	-1.98	0.857968	-0.69
20	0.748168	0.733359	-1.98	0.739788	-1.12	0.745878	-0.31
30	0.70254	0.693546	-1.28	0.697122	-0.77	0.70116	-0.2
40	0.676569	0.670218	-0.94	0.672596	-0.59	0.675594	-0.14
Error range			-16.86~-0.94		-2.27~55.44		-3.37~-0.14
0.8							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	2.015963	1.864198	-7.53	3.350253	66.19	1.959245	-2.81
4	1.67476	1.595514	-4.73	1.756294	4.87	1.644344	-1.82
5	1.513834	1.461578	-3.45	1.508846	-0.33	1.494208	-1.3
6	1.417233	1.378831	-2.71	1.395374	-1.54	1.403193	-0.99
7	1.35164	1.321513	-2.23	1.326381	-1.87	1.340847	-0.8
8	1.303523	1.278875	-1.89	1.278522	-1.92	1.294852	-0.67
9	1.266389	1.245617	-1.64	1.242699	-1.87	1.259186	-0.57
10	1.236572	1.218831	-1.43	1.214529	-1.78	1.230522	-0.49
20	1.096332	1.089234	-0.65	1.084534	-1.08	1.094046	-0.21
30	1.042614	1.038382	-0.41	1.03481	-0.75	1.041283	-0.13
40	1.012475	1.009494	-0.29	1.00666	-0.57	1.011538	-0.09
Error range			-7.53~-0.29		-1.92~66.19		-2.81~-0.09
0.9							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	2.870411	2.734096	-4.75	4.95068	72.47	2.800163	-2.45
4	2.372481	2.313091	-2.5	2.520845	6.25	2.337578	-1.47
5	2.145211	2.11023	-1.63	2.153216	0.37	2.124036	-0.99
6	2.011888	1.988188	-1.18	1.989734	-1.1	1.997447	-0.72
7	1.922926	1.905291	-0.92	1.892856	-1.56	1.912164	-0.56
8	1.858496	1.844579	-0.75	1.827017	-1.69	1.850058	-0.45
9	1.809182	1.797828	-0.63	1.778539	-1.69	1.802401	-0.37
10	1.770182	1.760544	-0.54	1.740922	-1.65	1.764428	-0.33
20	1.58978	1.586399	-0.21	1.573196	-1.04	1.587908	-0.12
30	1.522643	1.520797	-0.12	1.511473	-0.73	1.521655	-0.06
40	1.485504	1.484164	-0.09	1.477065	-0.57	1.484813	-0.05
Error range			-4.75~-0.09		-1.69~72.47		-2.45~-0.05
0.95							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	3.603836	3.451824	-4.22	6.293786	74.64	3.519022	-2.35
4	2.968097	2.908834	-2	3.169819	6.8	2.928243	-1.34
5	2.682675	2.65053	-1.2	2.700191	0.65	2.659239	-0.87
6	2.5169789	2.496323	-0.82	2.493566	-0.93	2.501342	-0.62
7	2.4069464	2.392608	-0.6	2.372342	-1.44	2.395769	-0.46
8	2.3278784	2.31704	-0.47	2.29066	-1.6	2.319351	-0.37
9	2.2677779	2.259295	-0.37	2.230948	-1.62	2.261001	-0.3
10	2.220237	2.213298	-0.31	2.184893	-1.59	2.214701	-0.25
20	2.003516	2.001622	-0.09	1.982907	-1.03	2.001987	-0.08
30	1.924118	1.92322	-0.05	1.910046	-0.73	1.923348	-0.04
40	1.880421	1.879741	-0.04	1.869768	-0.57	1.879932	-0.03
Error range			-4.22~-0.04		-1.62~74.64		-2.35~-0.03

TABLE 5: Corollary results of coefficient k ($C=0.9$).

0.7							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	2.227727	1.95304	-12.33	3.927345	76.29	2.172989	-2.46
4	1.692664	1.592182	-5.94	1.818662	7.44	1.627362	-3.86
5	1.455446	1.409432	-3.16	1.468263	0.88	1.406016	-3.4
6	1.317233	1.294781	-1.7	1.30651	-0.81	1.279076	-2.9
7	1.225071	1.214403	-0.87	1.208123	-1.38	1.19427	-2.51
8	1.15805	1.154134	-0.34	1.139942	-1.56	1.132505	-2.21
9	1.106482	1.106785	0.03	1.088974	-1.58	1.084946	-1.95
10	1.065516	1.068362	0.27	1.048945	-1.56	1.046877	-1.75
20	0.873674	0.88102	0.84	0.86496	-1	0.866203	-0.86
30	0.800592	0.806951	0.79	0.794931	-0.71	0.79604	-0.57
40	0.759473	0.764901	0.71	0.755368	-0.54	0.756278	-0.42
Error range			-12.33~0.84		-1.58~76.29		-3.86~-0.42
0.8							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	3.039117	2.738367	-9.9	5.814814	91.33	3.03893	-0.01
4	2.294540	2.186456	-4.71	2.534914	10.48	2.236538	-2.53
5	1.975691	1.92394	-2.62	2.023345	2.41	1.927019	-2.46
6	1.794452	1.766061	-1.58	1.797266	0.16	1.755281	-2.18
7	1.675329	1.658734	-0.99	1.664035	-0.67	1.643227	-1.92
8	1.589881	1.580105	-0.61	1.573872	-1.01	1.563058	-1.69
9	1.525238	1.519469	-0.38	1.507702	-1.15	1.502188	-1.51
10	1.474096	1.470974	-0.21	1.456488	-1.19	1.454012	-1.36
20	1.240591	1.243243	0.21	1.229623	-0.88	1.232126	-0.68
30	1.154152	1.15689	0.24	1.146775	-0.64	1.148946	-0.45
40	1.106253	1.108681	0.22	1.100748	-0.5	1.102528	-0.34
Error range			-9.9~0.24		-1.19~91.33		-2.53~-0.01
0.9							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	4.258037	3.827763	-10.11	8.552616	101.86	4.340567	1.94
4	3.187642	3.026395	-5.06	3.594739	12.77	3.144705	-1.35
5	2.742235	2.658306	-3.06	2.842316	3.65	2.698114	-1.61
6	2.493491	2.442339	-2.05	2.518136	0.99	2.456109	-1.5
7	2.332315	2.298158	-1.46	2.331059	-0.05	2.301023	-1.34
8	2.218358	2.193841	-1.11	2.206595	-0.53	2.191628	-1.2
9	2.132705	2.114345	-0.86	2.116515	-0.76	2.109516	-1.09
10	2.065574	2.051333	-0.69	2.047597	-0.87	2.045142	-0.99
20	1.765161	1.762608	-0.14	1.751624	-0.77	1.756306	-0.5
30	1.656991	1.656266	-0.04	1.647536	-0.57	1.651498	-0.33
40	1.597842	1.597666	-0.01	1.590631	-0.45	1.593871	-0.25
Error range			-10.11~-0.01		-0.87~101.86		-1.61~1.94
0.95							
p							
n	Exact solution	Method 1	Relative error (%)	Method 2	Relative error (%)	Method 3	Relative error (%)
3	5.311158	4.75345	-10.5	10.85606	104.4	5.457418	2.75
4	3.955887	3.743654	-5.36	4.5002	13.76	3.924381	-0.8
5	3.399631	3.286259	-3.33	3.543087	4.22	3.35901	-1.19
6	3.0916004	3.020273	-2.31	3.134426	1.39	3.055638	-1.16
7	2.8934486	2.844041	-1.71	2.90053	0.24	2.862743	-1.06
8	2.7540745	2.717418	-1.33	2.746018	-0.29	2.727541	-0.96
9	2.6498059	2.62142	-1.07	2.634866	-0.56	2.626593	-0.88
10	2.568089	2.545597	-0.88	2.550266	-0.69	2.547805	-0.79
20	2.207616	2.201785	-0.26	2.19221	-0.7	2.19873	-0.4
30	2.079723	2.077018	-0.13	2.06865	-0.53	2.074132	-0.27
40	2.010191	2.008534	-0.08	2.00167	-0.42	2.006152	-0.2
Error range			-10.5~-0.08		-0.7~104.4		-1.19~2.75

FIGURE 1: Relative error ($C=0.5$). FIGURE 2: Relative error ($C=0.6$).

three methods has a decreasing trend, indicating that the precision of these three methods is improved with the increase of p . With the increase of n , the absolute value of the relative error calculated by these three methods has a decreasing trend, indicating that the precision of these three

methods is improved with the increase of n . When $n > 10$, the relative error trend curve of the coefficient k calculated by these three methods tends to be gentle, indicating that the accuracy of these three methods is improved with the increase of n . And the relative error calculated by method 3

FIGURE 3: Relative error ($C=0.7$). FIGURE 5: Relative error ($C=0.9$). FIGURE 4: Relative error ($C=0.8$).

fluctuates around 0, and there is no large variability. In summary, method 3 has a wider range of applications than the first two methods.

5. Concluding Remarks

Through the study of the inference method of the noncentral t-distribution value, the following main conclusions can be obtained:

- (1) By comparing and analyzing the error range of coefficient k under different conditions, it is shown that the quantile value corollary method proposed in

this paper is more accurate than the current approximation method.

- (2) By comparing the influence of different number of samples n on the coefficient k , it indicates that the proposed method in this paper has a wider scope of application.
- (3) The method of inferring the quantile value proposed in this paper does not need to repeatedly check the value table and does not need to perform linear interpolation calculation, which simplifies the calculation process and has better convenience.

Data Availability

Some or all data, models, or codes that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Jun Gao and Jitao Yao contributed the central idea, and Jun Gao analysed most of the data and wrote the initial draft of the paper. The remaining authors contributed to refining the ideas, carrying out additional analyses, and finalizing this paper.

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