Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 9920240, 8 pages
https://doi.org/10.1155/2021/9920240

*Research Article*

# Bond and Option Prices under Skew Vasicek Model with Transaction Cost

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Received 17 March 2021; Accepted 8 May 2021; Published 21 May 2021

Academic Editor: Eric Campos



This paper studies the European option pricing on the zero-coupon bond in which the Skew Vasicek model uses to predict the interest rate amount. To do this, we apply the skew Brownian motion as the random part of the model and show that results of the model predictions are better than other types of the model. Besides, we obtain an analytical formula for pricing the zero-coupon bond and find the European option price by constructing a portfolio that contains the option and a share of the bond. Since the skew Brownian motion is not a martingale, thus we add transaction costs to the portfolio, where the time between trades follows the exponential distribution. Finally, some numerical results are presented to show the efficiency of the proposed model.

## 1. Introduction

The interest rate is widely used in the financial system. It influences the cost of borrowing and the return on savings and provides insight into future economic and financial market activity. It is one of the debt instruments that is issued by countries and corporations that need to increase capital. Indeed, sometimes they cover their budget deficit by selling bonds. According to importance of subject, researchers try to introduce different models to forecast the behavior of the interest rate paths [1–3].

Classical models are popular financial models to forecast interest rate, where Brownian motion is used to display the random part of these models [2,4,5]. Since Brownian motion is a martingale, thus these models find a fair price or amount of the financial instruments. Use of these models sometimes is associated with some problems. For instance, when the interest rate amount increases or decreases in a specific time interval, models that find the fair amounts cannot obtain an appropriate amount. Because the parameters of the models are calibrated based on the past market data. Even if we know that the interest rate will increase or decrease in the future, there is no parameter in the models so that we can adapt the model with the future data. Therefore, in such cases, the skew financial models are a better choice [6, 7].

The skew-normal process is a class of asymmetric probability distributions that includes the normal distribution. In general, the process can be displayed as follows:

$$X_t = \sqrt{1-\delta^2}B_t + \delta|W_t|, \quad \delta \in (-1,1),\ t \geq 0, \tag{1}$$

where $\{B_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ are two independent Brownian motion processes. By considering different values of $\delta$ parameter $\delta < 0$, $\delta = 0$, and $\delta > 0$, the process has different properties. If $\delta < 0$, the mathematical expectation of the process is a descending function and used in the financial models when interest rate amounts decrease. If $\delta = 0$, then the process is the standard Brownian motion. Moreover, if $\delta > 0$, the mathematical expectation of the process is an ascending function. Therefore, the process is a suitable noise for financial models, when interest rate amounts increase. Besides, to make a more effective model, researchers use different methods to calibrate models' parameters by considering the real data market [8–12]. The maximum likelihood estimation (MLE) is a famous method to calibrate the model's parameters. However, when the model's structure is

complex, this method is not an appropriate choice. The Newton–Raphson algorithm, known as "Newton's Method" after Sir Isaac Newton, was developed in 1669. The idea of the method is as follows: "one starts with an initial guess which is reasonably close to the true root. The function of interest is approximated using its tangent line. Next, one computes the $x$-intercept of this tangent line. This value will typically be a better approximation to the function's root than the original guess. The method is repeated until the prespecified convergence criteria are satisfied. In this study, we use this algorithm to calibrate the model's parameters" [12].

The bond contract is a debt instruments in the financial market. The value of zero-coupon bond is related to interest rate amounts. Recently, researchers are showing interest in trading options on the zero-coupon bond [1, 13–15]. An option is a contract that gives the right to the buyer (the owner or holder of the option) without any obligation so that the buyer buys or sells an underlying asset or instrument at a specified strike price prior to or on a specified date, depending on the form of the option. However, when fractional or skew models are used to predict the interest rate amount, there may be arbitrage opportunities. To overcome this problem, researchers applied the Leland and Kabanov's strategies [16, 17]. Indeed, they constructed a portfolio which contains a European option and share of the zero coupon bond. Then, they used the delta hedging strategy and added transaction cost to the portfolio. Researchers often divide the time interval into $n > 0$ equal parts such that the length of these parts is the same. Then, they calculate the value of the transaction costs at these times [18, 19]. However, in the reality of the market, it cannot be assumed that the duration between transactions is equal because the timing of investors' transactions is usually unknown. In this study, we remove this limitation and assume the duration between trades which follow the exponential distribution.

The rest of the paper is structured as follows. In Section 2, we discuss the skew Brownian motion process and the skew Vasicek model. Also, we calibrate model's parameters by the Newton–Raphson method. In Section 3, we derive an analytical approximation formula to find the value of the zero-coupon bond. In Section 4, we remove arbitrage opportunities by Leland and Kabanov's strategies and find a formula for calculating the transaction cost when the time interval between trades follows the exponential distribution. Furthermore, we consider the effect of changes in the model's parameters on the option and zero-coupon bond prices.

## 2. Skew Version of the Vasicek Model

In this section, we present a skew version of the Vasicek model and calibrate parameters of the model based on the real data market.

**Lemma 1.** *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a probability space and $\{X_t\}_{t\geq 0}$ be the skew Brownian motion that satisfies equation (1). For $t \geq s$, we obtain the following:*

*(1) If $\delta \in (0,1)$, then $\mathbb{E}(X_t) \geq \mathbb{E}(X_s)$*

*(2) If $\delta = 0$, then the skew Brownian motion $\{X_t\}_{t\leq 0}$ is the standard Brownian motion*

*(3) If $\delta \in (-1,0)$, then $\mathbb{E}(X_t) \leq \mathbb{E}(X_s)$*

*Proof.* The proof is straightforward by using the following equation:

$$\begin{aligned}\mathbb{E}[X_t] &= \mathbb{E}\left[\sqrt{1-\delta^2}\,B_t + \delta|W_t|\right] = \delta\mathbb{E}\left[|W_t|\right] \\ &= \delta\int_{-\infty}^{+\infty}\frac{|x|}{\sqrt{2\pi t}}e^{-(x^2/2t)}\mathrm{d}x \\ &= \delta\sqrt{\frac{2}{\pi t}}\int_0^{+\infty} xe^{-(x^2/2t)}\mathrm{d}x = \delta\sqrt{\frac{2t}{\pi}}.\end{aligned} \tag{2}$$

□

Recently, researchers used different stochastic models to predict the interest rate amount. The Vasicek model is one of the popular financial interest rate models and is introduced by Vasicek [4]. He used the mean reverting process to deal with the interest rate movement. The structure of the Vasicek model is as follows:

$$dR_t = \kappa(\theta - R_t)dt + \sigma\, dB_t, \tag{3}$$

where $\kappa$ is the mean reversion rate, $\theta$ is the mean interest rate, $\sigma$ is the volatility, and $B$ is the Brownian motion. The skew Vasicek model can be obtained by using the skew process instead of the standard Brownian motion in equation (3):

$$dR_t = \kappa(\theta - R_t)dt + \sigma\, dX_t, \tag{4}$$

where $X_t$ denotes the skew Brownian motion. In the following, we calibrate the mentioned model's parameter by the Newton–Raphson method.

*2.1. Calibration the Skew Vasicek Model's Parameters by Newton–Raphson Method.* The Newton–Raphson method is a technique for solving equations numerically. Like so much of differential calculus, it is based on the simple idea of linear approximation. Suppose $r$ be a root of function $g(x)$. The method begins with an estimate $r_0$ of $r$. Using $r_0$, it produces another assessment of $r$ as $r_1$. By applying $r_1$, the process has a new estimate of $r$ as $r_2$, and thus, a sequence of calculation $r_0, r_1, r_2, \ldots, r_n$ is generated such that $r_n$ is "close enough" to $r$. Then, we use $r_n$ instead of $r$ as the root of the equation $g(x) = 0$.

To use the method, we need the maximum likelihood function. The maximum likelihood method selects the set of values of the model's parameters that maximizes the likelihood function. The log-likelihood function of a set of observation $R_0, R_1, \ldots, R_n$ from the skew Vasicek model is as follows:

$$\begin{aligned}\mathscr{L}(\kappa,\theta,\delta,\sigma) &= \ln\left(\prod_{i=1}^{n} f\left(R_{i+1}|R_i,\kappa,\theta,\delta,\sigma\right)\right)\\ &= \sum_{i=1}^{n}\ln\ f\left(R_{i+1}|R_i,\kappa,\theta,\delta,\sigma\right)\\ &= \sum_{i=1}^{n}\ln\left[\frac{2}{e^{-(\kappa h/2)}\sqrt{t}}\phi\left(\frac{R_{i+1}-\left(e^{-\kappa h}R_i+\theta\left(1-e^{-(\kappa h/2)}\right)\right)}{e^{-\kappa h}\sqrt{t}}\right)\times\Phi\left(\lambda\left(\frac{R_{i+1}-\left(e^{-\kappa h}R_i+\theta\left(1-e^{-\kappa h}\right)\right)}{e^{-(\kappa h/2)}\sqrt{t}}\right)\right)\right].\end{aligned} \tag{5}$$

The steps of the Newton–Raphson algorithm to calibrate the model's parameters are as follows:

*Step 1*. Identify initial values $\kappa^0$, $\theta^0$, $\sigma^0$, and $\delta^0$ for $\kappa$, $\theta$, $\sigma$, and $\delta$.

*Step 2*. Compute the vector $U(\kappa^m,\theta^m,\sigma^m,\delta^m)$ and gradient matrix $V(\kappa^m,\theta^m,\sigma^m,\delta^m)$, where $m$ is the number of iterations, by taking values $m = 0, 1, \ldots$:

$$U = \left(\frac{\partial \ln\ L}{\partial\kappa},\frac{\partial \ln\ L}{\partial\theta},\frac{\partial \ln\ L}{\partial\sigma},\frac{\partial \ln\ L}{\partial\delta}\right),$$

$$V = \begin{pmatrix} \frac{\partial^2 \ln L}{\partial\kappa^2} & \frac{\partial^2 \ln L}{\partial\kappa\,\partial\theta} & \frac{\partial^2 \ln L}{\partial\kappa\,\partial\sigma} & \frac{\partial^2 \ln L}{\partial\kappa\,\partial\delta}\\ \frac{\partial^2 \ln L}{\partial\theta\,\partial\kappa} & \frac{\partial^2 \ln L}{\partial\theta^2} & \frac{\partial^2 \ln L}{\partial\theta\,\partial\sigma} & \frac{\partial^2 \ln L}{\partial\theta\,\partial\delta}\\ \frac{\partial^2 \ln L}{\partial\sigma\,\partial\kappa} & \frac{\partial^2 \ln L}{\partial\sigma\,\partial\theta} & \frac{\partial^2 \ln L}{\partial\sigma^2} & \frac{\partial^2 \ln L}{\partial\sigma\,\partial\delta}\\ \frac{\partial^2 \ln L}{\partial\delta\,\partial\kappa} & \frac{\partial^2 \ln L}{\partial\delta\,\partial\theta} & \frac{\partial^2 \ln L}{\partial\delta\,\partial\sigma} & \frac{\partial^2 \ln L}{\partial\delta^2} \end{pmatrix}. \tag{6}$$

*Step 3*. Calculate the values of the $\kappa$, $\theta$, $\sigma$, and $\delta$ in $(m+1)^{\text{th}}$ iteration by using the following equality:

$$\begin{bmatrix}\kappa^{m+1}\\ \theta^{m+1}\\ \sigma^{m+1}\\ \delta^{m+1}\end{bmatrix} = \begin{bmatrix}\kappa^{m}\\ \theta^{m}\\ \sigma^{m}\\ \delta^{m}\end{bmatrix} - V^{-1}\left(\kappa^m,\theta^m,\sigma^m,\delta^m\right)U\left(\kappa^m,\theta^m,\sigma^m,\delta^m\right). \tag{7}$$

*Step 4*. Suppose $A^m = (\kappa^m,\theta^m,\sigma^m,\delta^m)$. Then, the iterations $A^m$ are stopped as soon as $\|A^{m+1} - A^m\| < 10^{-6}$, where $\|\cdot\|$ denotes the Euclidean norm.

In the following, we compare the mentioned model with standard and fractional versions of the Vasicek model [20] by considering Ireland's interest rate data. The first column of Table 1 indicates the amounts of parameters of models from 12/2010 to 12/2011. The first column of Table 2 shows the value of Ireland's interest rate forecast in 12/2012 according to different versions of the model and obtains parameters of the first column of Table 1. The other columns of Table 1 show the calibrate amounts of model's parameters according to Ireland's interest rate data from 12/2011 to 12/2012 (the second column), from 12/2012 to 12/2013 (the third column), and from 12/2013 to 12/2014 (the fourth column), respectively. Besides, the other columns of Table 2 present the interest rate forecast data of the country in 12/2013 (the second column), in 12/2014 (the third column), and in 12/2015 (the fourth column), respectively.

## 3. Zero-Coupon Bond Formula

In this section, we present a formula to estimate the value of the zero-coupon bond and compare this formula with the Monte Carlo method as a famous simulation method. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space. The value of the zero-coupon bond with maturity time $T$ at time $0 \le t \le T$ is denoted by $P_{t,T}$ and

$$P_{t,T} = \mathbb{E}_t^{\mathbb{P}}\left[\exp\left(-\int_t^T R_s\mathrm{d}s\right)\right], \tag{8}$$

where $R_t, t \ge 0$ satisfies the skew Vasicek model (4). Since the integral existing in equation (8) is not solvable, thus we use the Trapezoidal method to estimate this integral. Here, an approximate solution to estimate the value of the zero-coupon bond is presented.

**Theorem 1.** *The zero-coupon bond price with maturity time $T$ at $t = 0$ can be estimated with*

$$\begin{aligned}P_{0,T} &= \mathbb{E}\left[\exp\left(-\int_0^T R_s\mathrm{d}s\right)\right]\\ &= \exp\Bigg(-\theta(T-B(T)) - R_0B(T) + \frac{\sigma^2\Phi(\delta\sigma)h}{4\kappa^2}\Bigg[(\Psi(1,1,0)-\Psi(1,1,-n)) - 2\left(1-\frac{\kappa h}{2}\right)[\Psi((1-\kappa h),1,0)\\ &\quad -(1-\kappa h)^n\Psi((1-\kappa h),1,-n)] + \left(1-\frac{\kappa h}{2}\right)^2\left[\Psi\left((1-\kappa h)^2,1,0\right)-(1-\kappa h)^{2n}\Psi\left((1-\kappa h)^2,1,-n\right)\right]\Bigg]\Bigg),\end{aligned} \tag{9}$$

Table 1: Calibrate model parameters by considering Ireland's interest rate data under the Newton–Raphson method at different times.

| Time | 12/2011 | 12/2012 | 12/2013 | 12/2014 |
|---|---|---|---|---|
| Newton–Raphson method | $\kappa = 1.2$<br>$\theta = 0.77$<br>$\sigma = 0.5565$<br>$H = 0.74$<br>$\delta = 0.15$ | $\kappa = 0.9$<br>$\theta = 0.675$<br>$\sigma = 0.5095$<br>$H = 0.81$<br>$\delta = 0.13$ | $\kappa = 0.88$<br>$\theta = 0.606$<br>$\sigma = 0.4738$<br>$H = 0.76$<br>$\delta = -0.05$ | $\kappa = 6$<br>$\theta = 0.553$<br>$\sigma = 0.4447$<br>$H = 0.59$<br>$\delta = 0.06$ |

Table 2: The comparison of the skew and other versions of the Vasicek model by Ireland interest rate data.

| Time | 12/2012 | 12/2013 | 12/2014 | 12/2015 |
|---|---|---|---|---|
| Real data | 0.467 | 0.348 | 0.131 | 0.112 |
| Standard Vasicek model | 0.3283 | 0.2243 | 0.1740 | 0.1000 |
| Fractional Vasicek model | 0.3609 | 0.2436 | 0.1732 | 0.0960 |
| Skew Vasicek model | 0.4216 | 0.3628 | 0.1268 | 0.1191 |

*where* $B(T) = B(0,T) = (1/\kappa)(1 - e^{-\kappa T})$.

*Proof.* Let $0 = t_0 < t_1 \cdots < t_n = T$ be a partition for $[0,T]$ such that $h = T/n$ and $t_k = kh$. By the Trapezoidal method, we can write

$$\int_0^T R_s \mathrm{d}s \approx h\left[\frac{R_0}{2} + R_1 + \cdots + R_{n-1} + \frac{R_n}{2}\right]. \tag{10}$$

The conditional expectation is based on the information until $t_{n-1}$ is obtained by

$$\begin{aligned}
&\mathbb{E}_{n-1}\left[\exp\left(-\int_0^T R_s \mathrm{d}s\right)\right] \\
&= \exp\left(-h\left[\frac{R_0}{2} + R_1 + \cdots + R_{n-2}\right] - hR_{n-1}\right) \times \mathbb{E}_{n-1}\left[\exp\left(-\frac{h}{2}\left[R_{n-1} + \kappa(\theta - R_{n-1})h + \sigma\Delta X_{n-1}(h)\right]\right)\right] \\
&= \exp\left(-h\left[\frac{R_0}{2} + R_1 + \cdots + R_{n-2}\right] - \frac{\kappa\theta h^2}{2} - \frac{h}{2}\left[(3 - \kappa h)R_{n-1}\right] + AF\right),
\end{aligned} \tag{11}$$

where $AF_u = \exp(((\sigma^2 h^3(\sqrt{(n-u+1)} - \sqrt{(n-u)})^2)/8)a_u^2\,\Phi(\delta\sigma)) \approx \exp((\sigma^2 h^3/16(n-u+1))\,a_u^2\,\Phi(\delta\sigma))$, $u \geq 2$ and $a_u$ satisfies the recursive relation:

$$\begin{aligned}
a_1 &= 1, \\
a_u &= 2 + (1 - \kappa h)a_{u-1},
\end{aligned} \tag{12}$$

and it can be explicitly rewritten as

$$a_u = -\frac{2}{\kappa h}\left[(1 - \kappa h)^{u-1}\left(1 - \frac{\kappa h}{2}\right) - 1\right]. \tag{13}$$

By considering $a_n$ frequently, the general formula for $\mathbb{E}_{n-u}$ can be written as follows:

$$\begin{aligned}
&\mathbb{E}_{n-u}\left[\frac{\exp\left(-\int_0^T R_s \mathrm{d}s\right)}{\prod_{i=1}^{u-1} AF_i}\right] \\
&= \exp\left(-h\left[\frac{R_0}{2} + R_1 + \cdots + R_{n-u-1}\right] - \frac{\kappa\theta h^2}{2}\sum_{i=1}^{u} a_i - \frac{h}{2}R_{n-u}a_{u+1} + AF_u\right).
\end{aligned} \tag{14}$$

Therefore,

$$\mathbb{E}_{n-u}\left[\frac{\exp\left(-\int_0^T R_s ds\right)}{\prod_{i=1}^{u-1} AF_i}\right] = \exp\left(-\frac{\kappa\theta h^2}{2}\sum_{i=1}^{n} a_i - \frac{h}{2}R_0\left[1 + a_n(1-\kappa h)\right] + AF_n\right). \tag{15}$$

In equation (15), by using equations (12) and (13), one can conclude that

$$\frac{h}{2}\left[1 + a_n(1-\kappa h)\right] = \frac{h}{2} + \frac{1}{\kappa}\left[1 - (1-\kappa h)^{n-1}\right](1-\kappa h) + \frac{h}{2}(1-\kappa h)^{n+1}. \tag{16}$$

If $n \longrightarrow \infty$, then

$$\frac{h}{2}\left[1 + a_n(1-\kappa h)\right] = \frac{1}{\kappa}\left(1 - e^{-\kappa T}\right). \tag{17}$$

Also, $\sum_{i=1}^{n} a_i$ can be written as follows:

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \frac{2}{\kappa h}\sum_{i=1}^{n}\left[1 - \left(1 - \frac{\kappa h}{2}\right)(1-\kappa h)^{i-1}\right] \\ &= \frac{2T}{\kappa h^2} - \frac{2}{\kappa h}\left(1 - \frac{\kappa h}{2}\right)\frac{\left(1 - (1-\kappa h)^n\right)}{\kappa h}.\end{aligned} \tag{18}$$

Thus,

$$\frac{\kappa\theta h^2}{2}\sum_{i=1}^{n} a_i = \theta(T - B(T)). \tag{19}$$

Finally, $\ln\left(\prod_{i=1}^{n} AF_i\right)$ can be estimated by

$$\begin{aligned}\ln\left(\prod_{i=1}^{n} AF_i\right) &= \frac{\sigma^2\,\Phi(\delta\sigma)h^3}{16}\frac{4}{\kappa^2 h^2}\sum_{i=1}^{n}\frac{1}{(n-i+1)}\left[1 - \left(1 - \frac{\kappa h}{2}\right)(1-\kappa h)^{i-1}\right]^2 \\ &= \frac{\sigma^2\,\Phi(\delta\sigma)h}{4\kappa^2}\left[\sum_{i=1}^{n}\frac{1}{(n-i+1)} - 2\left(1 - \frac{\kappa h}{2}\right)\sum_{i=1}^{n}\frac{(1-\kappa h)^{i-1}}{(n-(i-1))} + \left(1 - \frac{\kappa h}{2}\right)^2\sum_{i=1}^{n}\frac{(1-\kappa h)^{2i-2}}{(n-i+1)}\right] \\ &= \frac{\sigma^2\,\Phi(\delta\sigma)h}{4\kappa^2}\left[(\Psi(1,1,0) - \Psi(1,1,-n)) - 2\left(1 - \frac{\kappa h}{2}\right)[\Psi((1-\kappa h),1,0)\right. \\ &\quad -(1-\kappa h)^n\Psi((1-\kappa h),1,-n)] + \left(1 - \frac{\kappa h}{2}\right)^2\left[\Psi\left((1-\kappa h)^2,1,0\right)\right. \\ &\quad \left.\left.-(1-\kappa h)^{2n}\Psi\left((1-\kappa h)^2,1,-n\right)\right]\right],\end{aligned} \tag{20}$$

where $\Psi$ is the Lerch transcendent. Therefore, we can write

$$\begin{aligned}P_{0,T} &= \mathbb{E}\left[\exp\left(-\int_0^T R_s \mathrm{d}s\right)\right] \\ &= \exp\left(-\theta(T - B(T)) - R_0 B(T) + \frac{\sigma^2\Phi(\delta\sigma)h}{4\kappa^2}\left[(\Psi(1,1,0) - \Psi(1,1,-n))\right.\right. \\ &\quad - 2\left(1 - \frac{\kappa h}{2}\right)\left[\Psi((1-\kappa h),1,0) - (1-\kappa h)^n\Psi((1-\kappa h),1,-n)\right] \\ &\quad \left.\left.+\left(1 - \frac{\kappa h}{2}\right)^2\left[\Psi\left((1-\kappa h)^2,1,0\right) - (1-\kappa h)^{2n}\Psi\left((1-\kappa h)^2,1,-n\right)\right]\right]\right).\end{aligned} \tag{21}$$

□

In Tables 3 and 4, we compare the presented method with the Monte Carlo method. When the number of repetitions is low, the Monte Carlo method has low accuracy. As the number of repetitions increases, the accuracy of the method increases, but the duration of the program also increases. In fact, when the number of iterations is $n < 100000$, the mentioned method has more speed and accuracy than the Monte Carlo method. For a high number of iterations ($n > 100000$), both methods have approximately the same results, but the proposed formula is much faster.

## 4. Option Pricing with Transaction Cost

An option is a derivative instrument that gives its holder a right to buy or sell an asset at a set price (the strike price) on a set date (contract expiration). A European option can be defined as a type of options' contract (call or put option) that restricts its execution until the expiration date. A European call option is an option for the right to buy an asset at a specified time and price. The call option price formula on the zero coupon bond $P$ is as follows:

$$\text{Call} = \mathbb{E}\left[P_{0,T} \max\left(P_T - K_{\text{Call}}\right)^+\right]. \tag{22}$$

Also, a European put option is an option for the right to sell an asset at a specified time and price. The put option formula on the zero coupon bond $P$ can be written as

$$\text{Put} = \mathbb{E}\left[P_{0,T} \max\left(K_{\text{Put}} - P_T\right)^+\right]. \tag{23}$$

In the reality of securities market, investors pay money for each transaction called the transaction cost. Leland used this fact and suggested that the no-arbitrage assumption can be replaced by the delta hedging strategy under the condition of discrete time occasions and transaction costs. After that, according to Leland's idea, Yuri M. Kabanov and Mher M. Safarian showed that, in order to eliminate arbitrage, the transaction cost coefficient should be as $k = k_0 n^{\alpha - (1/2)}$, where $k_0 > 0$ and $\alpha \in [0, (1/2)]$ are constant and $n$ is the number of revisions.

In the following, we calculate transaction costs formula by constructing a portfolio containing a European option and share of the zero coupon bond. Let $(\Omega, F, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ be a complete probability space, where $(\mathscr{F}_t)_{0\le t\le T}$ denotes the $\sigma$-algebra generated by $P_{t,T}$. Then, the value of the bond at time $t$ by maturity time $T$ is as follows:

$$P_{t,T} = \mathbb{E}\left[\exp\left(-\int_t^T R(u)\mathrm{d}u\right)|\mathscr{F}_t\right], \tag{24}$$

where $R(t)$ is the value of the interest rate at time $t$ under the skew Vasicek model. Here, we put some limitations on the number of transactions and suppose the time between trades follows the exponential distribution. The exponential distribution is one of the widely used continuous distributions. It is often used to model the time elapsed between events.

**Theorem 2.** *Suppose the underlying bond price $P_{t,T}$ satisfied in equation (24). Then, the value of the transaction cost can be written as follows:*

$$k\sum_{i=1}^{n}\left|\Delta_{(i)\delta t} - \Delta_{(i-1)\delta t}\right|P_{i\delta t,T}, \tag{25}$$

*where $\Delta_t = (\partial C/\partial P_{t,T})$, $\delta t = T/n$, $k = (d/n^{(1/2)-\alpha})$, $d$ is contant, $\alpha \in [0, (1/2)]$, and $n$ is the number of revision intervals.*

*Proof.* In the constant volatility model, the random part of the option value is dependent only on the asset that exists in the option, and since the asset is tradable, therefore the option can be hedged and market can be completed. In this paper, the following basic assumptions were needed:

(i) The portfolio is revised every $\delta t$, where $\delta t$ is a finite, fixed, and small time step

(ii) Underlying asset price, $P_{t,T}$, satisfied equation (24)

Since the trading occurs in a discrete time in a real financial market and there exist transaction costs in a real financial market, we will consider option pricing in a discrete time setting with transaction costs. FBM is not a semimartingale for $H \neq 1/2$, we cannot use Ito's formula. Therefore, we use the fractional Taylor's theorem for calculating.

Let $C$ be a European option. We construct a replicating portfolio $X_t$ with a long position of the put option $C$ by maturity time $T_{\text{option}}$ and buy $\Delta$ share of Bond $B(t,T)$ by maturity time $T_{\text{bond}}$, where $T_1 \le T$. By considering Leland strategy, we find the number of the zero-coupon bond and the value of the transaction cost until time $t$ needed to eliminate arbitrage from replication portfolio $X(t)$. To do this, let $X(t)$ be a replication portfolio as follows:

$$X(t) = C\left(t, P_{t,T}\right) - \Delta_t P_{t,T}. \tag{26}$$

After the time interval $\delta t = T/n$, the change in the value of the portfolio $X$ is as follows:

$$\delta X(t) = \delta C\left(t, P_{t,T}\right) - \Delta_t \delta P_{t,T} - k|\vartheta_t|B(t+\delta t, T), \tag{27}$$

where $\Delta_t \delta P_{t,T}$ is the change of the value of the bond, $\Delta_t$ unit of the bond hold in the portfolio, and $k|\vartheta_t|P_{t,T}$ represents the value of transaction costs associated with trading $\vartheta$ of the bond $B(t,T)$. Since the time step is very small, from Taylor's theorem, we have

$$\begin{aligned}\delta X(t) = {} & \frac{\partial C\left(t, P_{t,T}\right)}{\partial t}\delta t + \frac{\partial^2 C\left(t, P_{t,T}\right)}{\partial P_{t,T}^2}(\delta B(t,T))^2 \\ & + \left(\Delta_t - \frac{\partial C\left(t, P_{t,T}\right)}{\partial P_{t,T}}\right)\delta P_{t,T} - k|\vartheta_t|p_{(t+\delta t),T},\end{aligned} \tag{28}$$

where $\delta P_{t,T}$ denotes the change in the bond price and $\vartheta_t = \Delta_{(t+\delta t)} - \Delta_t$ is the change of the bond price share in $[t, t+\delta t]$, and we have

$$\Delta_t = \frac{\partial C\left(t, P_{t,T}\right)}{\partial P_{t,T}}. \tag{29}$$

All the transaction costs in the interval $[0, T]$ are obtained as follows:

Table 3: Comparison of the presented formula and Monte Carlo method with regard to the skew Vasicek model and $R_0 = 0.1, T = 1, \kappa = 0.4$, $\theta = 0.1$, $\sigma = 0.1447$, $\delta = 0.1$, and $K = 0.15$.

| Time expiration | $n = 10$ | $n = 100$ | $n = 1000$ |
|---|---|---|---|
| The Monte Carlo method | Price = 0.6654/time = 0.060516 | Price = 0.6829/time = 0.063422 | Price = 0.6704/time = 0.547654 |
| The mentioned method | Price = 0.6762/time = 0.019768 | Price = 0.6762/time = 0.022893 | Price = 0.6762/time = 0.024370 |

Table 4: Comparison of the presented formula and Monte Carlo method with regard to the skew Vasicek model and $R_0 = 0.1, T = 1, \kappa = 0.4$, $\theta = 0.1$, $\sigma = 0.1447$, $\delta = 0.1$, and $K = 0.15$.

| Time expiration | $n = 10000$ | $n = 100000$ | $n = 1000000$ |
|---|---|---|---|
| The Monte Carlo method | Price = 0.6752/time = 4.045152 | Price = 0.6761/time = 44.419408 | Price = 0.6763/time = 1198.8371 |
| The mentioned method | Price = 0.6762/time = 0.030113 | Price = 0.6762/time = 0.102343 | Price = 0.6762/time = 0.802953 |

Table 5: The amount of transaction costs under different amounts of trades with $\mu = 0.2$, $\kappa = 0.4$, $H = 0.6$, $T = 1$, $R_0 = 0.1\, E = 0.95$, and $\sigma = 0.14$.

| $n$ | 5 | 10 | 20 | 50 | 100 | 200 | 1000 |
|---|---|---|---|---|---|---|---|
| The exponential distribution | 0.0467 | 0.0656 | 0.0691 | 0.0712 | 0.0728 | 0.0753 | 0.0863 |
| The same trading time's interval | 0.0317 | 0.0526 | 0.0553 | 0.0598 | 0.0601 | 0.0622 | 0.0645 |

$$k \sum_{i=1}^{n} \left| \Delta_{(i)\delta t} - \Delta_{(i-1)\delta t} \right| P_{i\delta t, T}. \tag{30}$$

□

Theorem 2 is a suitable tool for investigating bond pricing market efficiency under transaction cost. Previous research has assumed that the time between transactions is the same. However, this limitation on trading time is not correct. The investor can trade whenever he wants, and his trades cannot be limited to specific times. In the paper, we assume that the time between trades follows the exponential distribution. Table 5 uses the assumption and calculates the transaction cost for the portfolio according to the number of investor transactions. The results are also compared with a situation in which the time between trades is equal.

The value of the European call option is related to the value of the strike price and Expiration time. When the value of the strike price increases or the time expiration decreases; then, the values of the European call option and zero coupon bond decrease. However, put option price increases. These results are outlined in Tables 6 and 7.

In the following, we obtain the option price by using equations (22) and (23), the analytic approximation formula (9), and the transaction cost's formula (25). When the expiration time amount increases, by equation (24), one can deduce that the value of the zero coupon bond decreases. Thus, by equations (22) and (23), it is obvious the call option price decreases and the put option price increases. These results can be seen in Table 6.

Strike price has an important role in the price of the option. Here, we investigate the value of the call and put option under zero coupon bond when the bond price is obtained by equation (9). The put (call) option with a high strike price is more (less) expensive than the put option with a low strike price. These effects are illustrated in Table 7.

Table 6: The value of the European call and put options and zero coupon bond under the different amounts of the expiration time $\kappa = 0.4$, $\delta = 0.1$, $\theta = 0.1$, $R_0 = 0.1$, $\sigma = 0.1447$, $K_{\text{call}} = 0.85$, $K_{\text{put}} = 0.92$, and $T_{\text{option}} = 24$ months.

| Time expiration (months) | 1 | 6 | 12 | 18 | 24 |
|---|---|---|---|---|---|
| Bond | 0.9650 | 0.9473 | 0.8911 | 0.8407 | 0.8187 |
| Option call | 0.0949 | 0.0837 | 0.0372 | 0 | 0 |
| Option put | 0 | 0 | 0.0261 | 0.0683 | 0.0836 |

Table 7: The value of the European call and put options under the different amounts of the strike price, where $\kappa = 0.4$, $\delta = 0.1$, $\theta = 0.1$, $\sigma = 0.1447$, $T_{\text{bond}} = 12$ months, and $T_{\text{option}} = 12$ months.

| Strike price | 0.6 | 0.7 | 0.8 | 0.9 | 0.99 |
|---|---|---|---|---|---|
| Option call | 0.2634 | 0.1729 | 0.0824 | 0 | 0 |
| Option put | 0 | 0 | 0 | 0.0081 | 0.0895 |

Now, we study the effect of the skew parameter on the European call and put options prices. Investors are always interested in adopting call and put options with more profit and consequently; they should pay more for them. Indeed they can make a good profit if they have the right information about future interest rate changes. Using fair pricing models are not appropriate when the interest rate amount may fall or rise over a period of time due to events. In these cases, we must apply general models. In the mentioned model (4), if $\delta = 0$, then the model is the Vasicek model. Under different amounts of the skew parameter $\delta \neq 0$, the mathematical expectation of the skew process $\{X_t\}_{t \geq 0}$ is in

Table 8: The value of the European call and put options and zero coupon bond under the different amounts of the skew parameters, where $\kappa = 0.4$, $K_{call} = 0.65$, $K_{put} = 0.85$, $\theta = 0.1$, $\sigma = 0.1447$, and $T = 12$ months.

| Skew parameter ($\delta$) | −0.05 | 0 | 0.05 | 0.1 | 0.15 | 0.2 |
|---|---|---|---|---|---|---|
| Bond | 0.9919 | 0.9083 | 0.7625 | 0.5409 | 0.4023 | 0.2433 |
| Option call | 0.3094 | 0.2337 | 0.1018 | 0 | 0 | 0 |
| Option put | 0 | 0 | 0.0792 | 0.2797 | 0.4051 | 0.5490 |

ascending or descending; thus, we can apply the related financial models in different states. When the skew parameter amount increases, the amount of the interest rate increases. Thus, by equations (22), (23), and (24), one can conclude that the values of the zero coupon bond and call option decrease but the put option price increases. These results can be seen in Table 8.

## 5. Conclusions

In this study, we used the skew Brownian motion as the random part of the Vasicek model and derived the analytical formula for pricing the zero-coupon bond. We showed that the formula is much faster than the Monte Carlo method. In addition, to make the model more consistent with real data market, we assumed that time between transactions has the exponential distribution and calibrated the parameters by the Newton–Raphson method. Then, we calculated the price of the European option on the bond by the obtained results and formulas and discussed on the value of the option by changing the exist parameters in the model. We will seek the further extensions and applications of the model in the future works.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon reasonable request.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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