Research Article

Analytic Functions Related with Starlikeness

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The aim of present investigation is to study a new class of analytic function related with the Sokol-Nunokawa class. We derived relationships of this class with strongly starlike functions and obtained many interesting results.

1. Introduction

Let $\mathcal{A}$ be the class of functions λ having the series representation,

$$\lambda(\omega) = \omega + \sum_{n=2}^{\infty} a_n \omega^n, \quad (1)$$

which are analytic in the open unit disk $E = \{\omega \in \mathbb{C}: |\omega| < 1\}$ and normalized by $\lambda(0) = 0$ and $\lambda'(0) = 1$.

A function $\lambda \in \mathcal{A}$ is in class $\mathcal{S}$ of univalent function if and only if $\lambda(\omega_1) = \lambda(\omega_2)$ implies $\omega_1 = \omega_2$, for all $\omega_1, \omega_2 \in E$. An analytic function $\lambda \in \mathcal{A}$ subordinate to an analytic function δ denoted as $\lambda < \delta$ if and only if there exists an analytic function ς in E with $\varsigma(0) = 0$ and $|\varsigma(\omega)| < 1$, $\omega \in E$, such that $\lambda(\omega) = \delta(\varsigma(\omega))$, $\omega \in E$. In particular, if δ is

univalent in E , then $\lambda < \delta$ if and only if $\lambda(0) = \delta(0)$ and $\lambda(E) \subset \delta(E)$.

Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ such that $\Re(p(\omega)) > 0$ in E . A function $\lambda \in \mathcal{S}$ is in class $\mathcal{C}$ of normalized convex function if and only if $\Re(1 + (z\lambda''(\omega)/\lambda'(\omega))) > 0$. Also, a function $\lambda \in \mathcal{S}$ is in class $\mathcal{S}^*$ if and only if $\Re(\omega\lambda'(\omega)/\lambda(\omega)) > 0$.

In 1991, Goodman [1] introduced the class of uniformly convex function $\mathcal{UCV}$. A function $\lambda \in \mathcal{C}$ is in class $\mathcal{UCV}$ if, for every circular arc $\varphi \subset E$ with center in E , the arc $\lambda(\varphi)$ is convex. Also, a function $\lambda \in \mathcal{S}$ is in class $\mathcal{UST}$ if, for every circular arc γ contained in E with center in E , the arc $\lambda(\gamma)$ is starlike.

The classes $\mathcal{UCV}$ and $\mathcal{UST}$ were studied by Ma and Minda [2, 3] and Ronning [4, 5]. They provided the analytic characterization of class $\mathcal{UCV}$ and $\mathcal{UST}$ as follows:

$$\lambda \in \mathcal{UCV} \text{ if and only if } \Re \left\{ 1 + \frac{\lambda''(\omega)}{\lambda'(\omega)} \right\} > \left| \frac{z\lambda''(\omega)}{\lambda'(\omega)} \right|, \quad \omega \in E, \quad (2)$$

$$\lambda \in \mathcal{UST} \text{ if and only if } \Re \left\{ \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right\} > \left| \frac{\omega \lambda'(\omega)}{\lambda(\omega)} - 1 \right|, \quad \omega \in E. \quad (3)$$

In 1999, Kanas et al. [6] introduced the class $k - \mathcal{UCV}$, $k \geq 0$, of k -uniformly convex functions as follows:

$$\lambda \in k - \mathcal{UCV} \text{ if and only if } \lambda \in \mathcal{A} \text{ and } \Re \left\{ 1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)} \right\} > k \left| \frac{\omega \lambda''(\omega)}{\lambda'(\omega)} \right|, \quad \omega \in E. \quad (4)$$

Furthermore, the class of k -uniformly starlike functions is defined as follows:

$$\lambda \in k - \mathcal{UST} \text{ if and only if } \lambda \in \mathcal{A} \text{ and } \Re \left\{ \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right\} > k \left| \frac{\omega \lambda'(\omega)}{\lambda(\omega)} - 1 \right|, \quad \omega \in E. \quad (5)$$

Many authors had investigated the interesting properties of the abovementioned classes. For details, see [6–13]. The class $\mathcal{SS}^*(\beta)$ [14, 15] denotes the class of strongly starlike functions of order $\beta \in (0, 1)$, which is defined as follows:

$$\mathcal{SS}^*(\beta) = \left\{ \lambda \in \mathcal{A} : \left| \arg \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right| < \frac{\beta \pi}{2} \right\}. \quad (6)$$

In 2015, Sokol et al. [16] introduced the class $\mathcal{MN}$ as follows:

$$\lambda \in \mathcal{MN} \text{ if and only if } \lambda \in \mathcal{A} \text{ and } \Re \left\{ \frac{(\omega \lambda'(\omega))'}{\lambda'(\omega)} \right\} > \left| \frac{\omega \lambda'(\omega)}{\lambda(\omega)} - 1 \right|, \quad \omega \in E. \quad (7)$$

For $k \geq 0$, $0 \leq \alpha \leq 1$, in a recent paper, Darus et al. [17] defined a new class $k - \mathcal{UST}(\alpha)$ as follows:

$$k - \mathcal{UST}(\alpha) = \left\{ \lambda \in \mathcal{A} : \Re \left(\frac{(\omega \lambda'(\omega))'}{\lambda'(\omega)} \right) > k \left| (1 - \alpha) \frac{\omega \lambda'(\omega)}{\lambda(\omega)} + \alpha \frac{(\omega \lambda'(\omega))'}{\lambda'(\omega)} - 1 \right|, \quad \omega \in E \right\}. \quad (8)$$

The class $k - \mathcal{UST}(\alpha)$ has some interesting relationship with other classes of univalent functions. Motivated from [17–21], we introduced a new class of analytic functions and found its relation with some other existing classes of analytic functions.

Definition 1. Let $\lambda \in \mathcal{A}$, $0 \leq k < 1$. Then, λ is in class $k - \mathcal{EST}$ if and only if

$$\Re \left(\frac{(\omega \lambda'(\omega))'}{\lambda'(\omega)} \right) > k \left| \frac{(\omega \lambda'(\omega))'}{\lambda'(\omega)} - \frac{\omega \lambda'(\omega)}{\lambda(\omega)} - 1 \right|. \quad (9)$$

For special values of parameter k , we obtained many known classes of analytic functions. For example, if we take $k = 0$, then $k - \mathcal{EST} = \mathcal{C}$, the class of convex functions.

2. Set of Lemmas

To prove our main results, we need the following lemmas.

Lemma 1. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u, v): D \subset \mathbb{C}^2 \rightarrow \mathbb{C}$ be a complex-valued function satisfying the conditions:

- (i) $\Psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$.
- (ii) $(1, 0) \in D$ and $\Psi(1, 0) > 0$.
- (iii) $\Re \Psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -(1/2)(1 + u_2^2)$. If $h(\omega) = 1 + c_1\omega + c_2\omega^2 + \dots$ is a function that is analytic in E such that $(h(\omega), \omega h'(\omega)) \in D$ and $\Re(\Psi(h(\omega), \omega h'(\omega))) > 0$ for $\omega \in E$, then $\Re(h(\omega)) > 0$. This result is due to Miller [22].

Lemma 2. Let $\beta > 0$ and $p(\omega) + \beta\omega(p'(\omega)/p(\omega)) < (1 + \omega/1 - \omega)$. Then, $p(\omega) < (1 + (1 - 2\xi)\omega/1 - \omega)$, where

$$\xi = \frac{(2k - \beta) + \sqrt{(2k - \beta)^2 + 8\beta}}{4}. \quad (10)$$

Proof. Let $p(\omega) = (1 - \xi)p_1(\omega) + \xi$. Then, $p(\omega) + (\beta\omega p'(\omega)/p(\omega)) = (1 - \xi)p_1(\omega) + \xi + \beta\omega((1 - \xi)p_1'(\omega)/(1 - \xi)p_1(\omega) + \xi)$. Since $p(\omega) + \beta\omega(p'(\omega)/p(\omega)) < (1 + \omega/1 - \omega)$.

Therefore, $\{(1 - \xi)p_1(\omega) + \xi + ((1 - \xi)\beta\omega p_1'(\omega)/(1 - \xi)p_1(\omega) + \xi) - k\} \in P$, where $0 \leq k < 1$.

We formulate a functional $\phi(u, v)$ by taking $u = p_1$, $v = \omega p_1'$, thus $\phi(u, v) = (1 - \xi)u + (\xi - k) + (\beta(1 - \xi)v/(1 - \xi)u + \xi)$. Clearly,

(i) $\phi(u, v)$ is continuous in domain $D \subset \mathbb{C} \setminus \{-\xi/1 - \xi\} \times \mathbb{C}$

(ii) $\phi(1, 0) = 1 - k > 0$

Consider $\phi(iu_2, v_1) = (1 - \xi)iu_2 + (\xi - k) + \beta(1 - \xi)v_1/(1 - \xi)iu_2 + \xi$. Then, $\Re(\phi(iu_2, v_1)) \leq (\xi - k) - (\beta\xi(1 - \delta)(1 + u_2^2)/2(\xi^2 + (1 - \xi)^2 u_2^2))$, where we have used the fact that $v_1 \leq -((1 + u_2^2)/2)$. This implies that $\Re(\phi(iu_2, v_1)) \leq (A + Bu_2^2/2C)$, where $A = \xi[2\xi(\xi - k) - \beta(1 - \xi)]$, $B = (1 - \xi)[2(\xi - k)(1 - \xi) - \beta\xi]$, and $C = (1 - \xi)^2 u_2^2 + \xi^2 > 0$.

Clearly, $\Re(\phi(iu_2, v_1)) \leq 0$, if and only if $A \leq 0$ and $B \leq 0$.

From $A \leq 0$, $\xi = ((2k - \beta) + \sqrt{(2k - \beta)^2 + 8\beta})/4$.

From $B \leq 0$, we have $0 \leq \xi < 1$. Now, using Lemma 1, we have the required result. $\square$

Lemma 3 (see [23]). If $-1 \leq B < A \leq 1$, $\lambda > 0$ and the complex number γ satisfies $\Re\{\gamma\} \geq -\lambda(1 - A)/(1 - B)$, then the differential equation $q(\omega) + (\omega q'(\omega)/\lambda q(\omega) + \gamma) = (1 + A\omega/1 + B\omega)$, $\omega \in E$, has a univalent solution in E given by

$$q(\omega) = \begin{cases} \frac{\omega^{\lambda+\gamma}(1 + B\omega)^{\lambda(A-B)/B}}{\lambda_0 t^{\lambda+\gamma-1}(1 + Bt)^{\lambda(A-B)/B}} - \frac{\gamma}{\lambda}, & \text{for } B \neq 0, \\ \frac{z^{\lambda+\gamma} e^{\lambda A \omega}}{\lambda_0 t^{\lambda+\gamma-1} e^{\lambda A t}} - \frac{\gamma}{\lambda}, & \text{for } B = 0. \end{cases} \quad (11)$$

If $h(\omega) = 1 + c_1\omega + c_2\omega^2 + \dots$ is analytic in E and satisfies $h(\omega) + (\omega h'(\omega)/\lambda h(\omega) + \gamma) < (1 + A\omega/1 + B\omega)$, $\omega \in E$, then $h(\omega) < q(\omega) < (1 + A\omega/1 + B\omega)$, and $q(\omega)$ is the best dominant.

Lemma 4 (see [24]). Let a, b and $c \neq 0, -1, -2, \dots$ be complex numbers. Then, for $\Re(c) > \Re(b) > 0$,

- (i) ${}_2\chi_1(a, b, c; \omega) (\Gamma(c)/\Gamma(c - b)\Gamma(b))_0^{1/b-1} (1 - t)^{c-b-1} (1 - t\omega)^{-a} dt$,
- (ii) ${}_2\chi_1(a, b, c; \omega) {}_2\chi_1(b, a, c; \omega)$,
- (iii) ${}_2\chi_1(a, b, c; \omega) (1 - \omega)_2^{-a} \chi_1(a, c - b, c; \omega/\omega - 1)$.

Lemma 5. Let p be analytic in E of the form

$p(\omega) = 1 + \sum_{n=m}^{\infty} c_n \omega^n$, $c_m \neq 0$, with $p(\omega) \neq 0$ in E . If there exists a point ω_0 , $|\omega_0| < 1$ such that $|\arg p(\omega)| < (\pi\varphi/2)$ ($|\omega| < |\omega_0|$) and $|\arg p(\omega_0)| = (\pi\varphi/2)$ for some

$\varphi > 0$, then we have $(\omega_0 p'(\omega_0)/p(\omega_0)) = i\ell\varphi$, where $\begin{cases} \ell \geq (m/2)(a + (1/a)), & \text{when } \arg p(\omega_0) = (\pi\varphi/2), \\ \ell \leq -(m/2)(a + (1/a)), & \text{when } \arg p(\omega_0) = -(\pi\varphi/2), \end{cases}$ where $(p(\omega_0))^{1/\varphi} = \pm ia$ ($a > 0$). This result is the generalization of Nunokawa's lemma [25, 26].

3. Main Results

In this section, we derived our main results.

Theorem 1. The function $\lambda(\omega) = (\omega/1 - A\omega)$ is in the class $k - \mathcal{ES}\mathcal{T}$ if and only if $|A| \leq ((1 - k)/(1 + 2k))$.

Proof. Let $\lambda(\omega) = (\omega/1 - A\omega)$, then some simple calculation gives us

$$\Re\left(\frac{(\omega\lambda'(\omega))'}{\lambda'(\omega)}\right) = \Re\left(\frac{1 + A\omega}{1 - A\omega}\right). \quad (12)$$

By using (9) and (12), we obtain

$$k \left| \frac{(\omega\lambda'(\omega))'}{\lambda'(\omega)} - \frac{z\lambda'(\omega)}{\lambda(\omega)} - 1 \right| = k \left| \frac{2A\omega - 1}{1 - A\omega} \right|. \quad (13)$$

Using (12) and (13), it is clear that $\lambda \in k - \mathcal{ES}\mathcal{T}$ if and only if

$$\Re\left(\frac{1 + A\omega}{1 - A\omega}\right) > k \left| \frac{2A\omega - 1}{1 - A\omega} \right|. \quad (14)$$

It suffices to study $|\omega| = 1$, $|A| = r$, and $A\omega = re^{i\theta}$ in (14).

$$\Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{|1 - re^{i\theta}|^2}, \quad (15)$$

$$= \frac{1 - r^2}{|1 - re^{i\theta}|^2} \geq k \left| \frac{2re^{i\theta} - 1}{1 - re^{i\theta}} \right|.$$

Then, equation (15) leads to a simple computation which shows that

$$(1 - r^2/|1 - re^{i\theta}|) \geq k\sqrt{4r^2 - 4r \cos \theta + 1} \text{ or}$$

$$k \leq \frac{1 - r^2}{|1 - re^{i\theta}| \sqrt{4r^2 - 4r \cos \theta + 1}}. \quad (16)$$

The right-hand side of the above inequality (16), is seen to have minimum for $\theta = \pi$ and minimum value is $(1 - r/2r + 1)$. Hence, a necessary and sufficient conditions for (16) is $k \leq (1 - r/2r + 1)$ or $|A| = r \leq (1 - k/2k + 1)$. This completes the proof. $\square$

Corollary 1. The function $\lambda(\omega) = (\omega/1 - \omega)$ belongs to a class $k - \mathcal{ES}\mathcal{T}$ if and only if $k = 0$.

Remark 1. It is important to note that if $k_1 \geq k_2$, then $k_1 - \mathcal{ES}\mathcal{T} \subseteq k_2 - \mathcal{ES}\mathcal{T}$.

Theorem 2. Let $\lambda \in k - \mathcal{CSST}$. Then, $\lambda \in \mathcal{S}^*(\xi)$, where $\mathcal{S}^*$ is the class of starlike functions of order ξ , where ξ is defined in (10) and $0 \leq k < 1$ and $\beta = 1 + k$.

Proof. Let $(\omega\lambda'(\omega)/\lambda(\omega)) = p(\omega)$, where p is analytic in unit disk and $p(0) = 1$. Now, using (9), we obtain $\Re\{p(\omega) + (\omega p'(\omega)/p(\omega))\} > k|(\omega p'(\omega)/p(\omega)) - 1| = k|1 - (\omega p'(\omega)/p(\omega))|$. This implies that $\Re\{p(\omega) + \beta(\omega p'(\omega)/p(\omega))\} > k\Re\{1 - (\omega p'(\omega)/p(\omega))\} > k$, where $\beta = 1 + k, 0 \leq k < 1$. The above relation can be written in the following Briot–Bouquet differential subordination $p(\omega) + (\omega p'(\omega)/(1/\beta)p(\omega)) < (1 + \omega/1 - \omega)$. Now, using Lemma 3, we have $p \in P(\xi)$ this complete the proof.

Special case: for $k = 0$ and $\beta = 1$, we have $\xi = 1/2$, that is, $\lambda(\omega) \in 0 - \mathcal{CSST}$ implies $\lambda \in P^*(1/2)$. $\square$

Theorem 3. Let $\lambda \in k - \mathcal{CSST}$ of the form $\lambda(\omega) = \omega + \sum_{n=m+1}^{\infty} a_n \omega^n$, $a_{m+1} \neq 0$; then, λ is strongly starlike of order β_0 , where $\beta_0 = \min\{k^2 - (a^\beta \cos \beta\pi/2)^2 + k^2\beta^2(m(a^2 + 1)/2a)^2\}$.

Proof. Let $p(\omega) = (\omega\lambda'(\omega)/\lambda(\omega))$; then, p is of the form $p(\omega) = 1 + \sum_{n=m}^{\infty} c_n \omega^n$, $c_m \neq 0$. Now using the definition of class $k - \mathcal{CSST}$, we have $\Re\{p(\omega) + (\omega p'(\omega)/p(\omega))\} > k|(\omega p'(\omega)/p(\omega)) - 1|$.

If there exists a point $\omega_0 \in E$, we have $|\arg p(\omega)| < \beta\pi/2$, $|\omega| < |\omega_0|$, and $|\arg p(\omega_0)| = \beta\pi/2$. Then, from Lemma 5, we have $(\omega_0 p'(\omega_0)/p(\omega_0)) = i\ell\beta$, where $(p(\omega_0))^{(1/\beta)} = \pm ia$ ($a > 0$) and

$$l: \begin{cases} l \geq \frac{m}{2} \left(a + \frac{1}{a}\right), & \text{when } \arg p(\omega_0) = \frac{\pi\varphi}{2}, \\ l \leq -\frac{m}{2} \left(a + \frac{1}{a}\right), & \text{when } \arg p(\omega_0) = -\frac{\pi\varphi}{2}. \end{cases} \quad (17)$$

For the case $\arg(p(\omega_0)) = (\beta\pi/2)$, we obtain

$$\Re\left\{p(\omega_0) + \frac{\omega_0 p'(\omega_0)}{p(\omega_0)}\right\} = \Re\{(ia)^\beta + i\ell\beta\} = a^\beta \cos \frac{\beta\pi}{2}. \quad (18)$$

Also, we have $k|(\omega_0 p'(\omega_0)/p(\omega_0)) - 1| = k|il\beta - 1| = k\sqrt{1 + l^2\beta^2}$.

Then, from $l \geq (m(a^2 + 1)/2a)$ for $\beta \geq \beta_0$, we have

$$0 \leq k^2 - \left(a \cos \frac{\beta\pi}{2}\right)^2 + k^2 \left\{\left(\frac{m(a^2 + 1)}{2a}\right)^2 \beta^2\right\}. \quad (19)$$

Therefore, $0 \leq k^2 - (a \cos(\beta\pi/2))^2 + k^2\{(m(a^2 + 1)/2a)\beta^2\}$.

By using (18) and (19), we have $\Re\{p(\omega_0) + (\omega_0 p'(\omega_0)/p(\omega_0))\} < k|(\omega_0 p'(\omega_0)/p(\omega_0)) - 1|$, which is a contradiction, therefore, $|\arg p(\omega)| < (\beta\pi/2)$ for $|\omega| < 1$. Similarly, we can prove the case $\arg(p(\omega_0)) = -(\beta\pi/2)$, by using the same method as the above and will get a contradiction. This proves that λ is strongly starlike of order β_0 . $\square$

Theorem 4. If $\lambda \in k - \mathcal{CSST}$, then $(\omega\lambda'/\lambda) < q(\omega) = (1/\delta(\omega))$, where $q(\omega) = [\chi_1((2/\beta); 1; (1/\beta) + 1; (\omega/\omega - 1))]$, with $\beta = 1 + k$ and $0 \leq k < 1$.

Proof. Let $(\omega\lambda'/\lambda) = p(\omega)$, where $p(\omega)$ is analytic in E and $p(0) = 1$. $\Re\{p(\omega) + (\omega p'(\omega)/p(\omega))\} > k|(\omega p'(\omega)/p(\omega)) - 1| = k|1 - (\omega p'(\omega)/p(\omega))| > \Re\{1 - (\omega p'(\omega)/p(\omega))\}$.

This implies that $\Re\{p(\omega) + \beta(\omega p'(\omega)/p(\omega))\} > k$, where $0 \leq k < 1, \beta = 1 + k$. The above relation can be written in the following Briot–Bouquet differential subordination, $p(\omega) + \beta(\omega p'(\omega)/p(\omega)) < (1 + \omega/1 - \omega)$. Using Lemma 4, for $\lambda = (1/\beta)$ and $\gamma = 0$, we have $p(\omega) < q(\omega) = 1/\delta(\omega) = 1/(1/\beta) \int_0^1 t^{(1/\beta-1)} (1 - t\omega/1 - \omega)^{(-2/\beta)} dt = \{\chi_1(2/\beta, 1; 1/\beta + 1; (\omega/\omega - 1))\} - 1$, which is required. $\square$

Remark 2. If $\lambda \in k - \mathcal{CSST}$, then $|a_2| \leq 2(1 - \xi)$ and $|a_3| \leq 2(1 - \xi)$, where ξ is defined in (10).

Theorem 5. Let $0 \leq k < 1$ and $\lambda(\omega) \in k - \mathcal{CSST}$; then, $\lambda(E)$ contains an open disk of radius $|1/\omega_0| \leq 2 + 2(1 - \xi)$.

Proof. Let $\omega_0 \neq 0$ be a complex number such that $\lambda(\omega) \neq 0$, for $\omega \in E$, then $\lambda_1(\omega) = (\omega_0\lambda(\omega)/\omega_0 - \lambda(\omega)) = \{\omega + (a_2 + 1/\omega_0)\omega^2 + \dots\}$.

Since $\lambda_1(\omega)$ is univalent in E so that

$$\begin{aligned} \left|a_2 + \frac{1}{\omega_0}\right| &\leq 2 = \left|\frac{1}{\omega_0}\right| - |a_2| \leq 2, \\ &= \left|\frac{1}{\omega_0}\right| \leq 2 + |a_2|. \end{aligned} \quad (20)$$

Since $\lambda(\omega) \in k - \mathcal{CSST}$, then

$$|a_2| \leq 2(1 - \xi). \quad (21)$$

Using (21) in (20), $|1/\omega_0| \leq 2 + 2(1 - \xi)$, where ξ is defined in (10). $\square$

Theorem 6. Let $\lambda \in k - \mathcal{CSST}$ such that $\chi(\omega) = \lambda^{-1}(\omega) = \omega + d_2\omega^2 + d_3\omega^3 + \dots$. Then,

$$|d_2| \leq 2(1 - \xi), \quad (22)$$

$$|d_3| \leq (1 - \xi)(5 - 8\xi), \quad (23)$$

where ξ is defined in (10).

Proof. As $\chi(\omega) = \lambda^{-1}(\omega)$, $\omega = \lambda(\omega)$ and $\chi(\lambda(\omega)) = \omega$. Now, $\chi(\omega) = \omega + d_2\omega^2 + d_3\omega^3 + \dots$, and $\lambda(\omega) = \omega + a_2\omega^2 + a_3\omega^3 + \dots$. Then, $\chi(\lambda(\omega)) = \omega + (a_2 + d_2)\omega^2 + (a_3 + 2a_2d_2 + d_3)\omega^3 + \dots$, this implies $\chi(\lambda(\omega)) = \omega + (a_2 + d_2)\omega^2 + (a_3 + 2a_2d_2 + d_3)\omega^3 + \dots$, where $d_2 = -a_2, d_3 = 2a_2^3 - a_3$, and $|d_3| \leq (1 - \xi)(5 - 8\xi)$, in which ξ is defined in (10). $\square$

4. Conclusions

In this article, we studied a new class of analytic functions. We derived certain necessary and sufficient condition, order of strongly starlike. We also pointed out some existing results in literature.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

SGAS and SH came with the main thoughts and helped to draft the manuscript, AT wrote, reviewed, and edited the manuscript, AR proved the main theorems, and MD and SN are revised the paper. All authors read and approved the final manuscript.

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