Research Article

Enhanced Estimators of Population Variance with the Use of Supplementary Information in Survey Sampling

Showkat Ahmad Lone,¹ Mir Subzar,² and Ankita Sharma³

¹Department of Basic Sciences, Saudi Electronic University, Jeddah Campus, Riyadh, Saudi Arabia
²Division of Agricultural Statistics SKUAST-K, Shalimar 190025, India
³Department of Statistics, University of Jammu, Jammu, India

Correspondence should be addressed to Ankita Sharma; ankitareasi@gmail.com

Received 25 March 2021; Revised 11 April 2021; Accepted 16 April 2021; Published 26 April 2021

Academic Editor: Ishfaq Ahmad

Copyright © 2021 Showkat Ahmad Lone et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In the present study, we propose the proficient class of estimators of the finite population mean, while incorporating the nonconventional location and nonconventional measures of dispersion with coefficient of variation of the auxiliary variable. Properties associated with the suggested class of improved estimators are derived, and an efficiency comparison with the usual unbiased ratio estimator and other existing estimators under consideration in the present study is established. An empirical study has also been provided to validate the theoretical results. Finally, it is established that the proposed class of estimators of the finite population variance proves to be more efficient than the existing estimators mentioned in this study.

1. Introduction

It is very quite often that utilization of supplementary information in survey sampling which has some sort of strong positive or negative correlation with the response variable is always found to be advantageous. So for the utilization of such information, various methods in survey sampling are presently used to increase precision, with the incorporation of these supplementary information, in estimating the population parameters. Various authors have put their sincere efforts to utilize supplementary information with different sampling designs in different situations in such a way that their estimation procedure becomes more proficient; for details see [1–12]. It is very often that some of the measures are so much affected by extreme observations and can give misleading results. In case of extreme values, using classical methods of estimation provides misleading results, but authors have also put their valuable efforts to come up with solutions to this situation, so that precise results should be obtained even with the presence of outliers in the data. Authors such as Subzar et al. [13] have proposed different robust ratio type estimators in simple random sampling without replacement (SRSWOR) while utilizing the Huber-M estimation technique. Subzar et al. [14] have also proposed different robust ratio type estimators in SRSWOR by utilizing the different robust regression techniques and compared with Ordinary Least Squares (OLS) and Huber-M estimation techniques. Subzar et al. [15] also proposed different robust ratio type estimators by comparing the generalized robust regression techniques with OLS and Huber-M estimation method. Recently, Almanjahie et al. [16] have proposed the generalized class of mean estimators with known measures for outlier’s treatment. Also, Shahzad et al. [12] have given a new class of L-Moments-based calibration variance estimators. So in the present study, we made the utilization of nonconventional location parameters, nonconventional measures of dispersion, and their functions with the coefficient of variation of the auxiliary variable in order to suggest the class of estimators for estimating the population variance. The properties of the proposed class of estimators are studied under large sample approximation. It has been shown theoretically as well as empirically that the proposed class of estimators is more efficient than existing estimators mentioned in this study.
2. Notations

Consider a finite population $Z = (Z_1, Z_2, \ldots, Z_M)$ of $M$ units and let $(z, r)$ be the study and auxiliary variables defined on $Z$ taking values $(z_i, r_i)$, respectively, on $Z_i$ ($i = 1, 2, \ldots, M$). It is desired to estimate the population variance $W_z^2$ of the study variable $Z$ using the information on an auxiliary variable $r$. Let a simple random sample of size $m$ be drawn without replacement from the finite population $Z$. In this paper, we shall ignore the finite population correction (fpc) term. We denote the following.

$$Z = M^{-1} \sum_{i=1}^{M} z_i;$$
$$\bar{r} = M^{-1} \sum_{i=1}^{M} r_i;$$
$$W_z^2 = (M - 1)^{-1} \sum_{i=1}^{M} (z_i - \bar{z})^2;$$
$$W_r^2 = (M - 1)^{-1} \sum_{i=1}^{M} (r_i - \bar{r})^2;$$
$$TM = (Q_1 + 2Q_2 + Q_3)/4;$$

HWL=median $((R_j + R_k)/2, 1 \leq j < k \leq M)$: Hodges–Lehmann estimator of the auxiliary variable $r$

$G = (4/M - 1) \sum_{i=1}^{M} (2i - M - 1/2M)R_{(i)}$: Gini’s mean difference of the auxiliary variable $r$

$D = (2/\sqrt{M(M-1)}) \sum_{i=1}^{M} (-(M+1/2))R_{(j)}$: Downton’s method of the auxiliary variable $r$

$S_{pr} = (\sqrt{r}/M^2) \sum_{i=1}^{M} (2i - M - 1)R_{(j)}$: probability weighted moments of the auxiliary variable $r$

$$\mu_{40}^* = (\mu_{40} - 1),$$
$$\mu_{04}^* = (\mu_{04} - 1),$$
$$\mu_{22}^* = (\mu_{22} - 1),$$

where $\mu_{r} = \lambda_{bc}((\lambda_{bc}/2^{1/2})^{1/2})$, $\lambda_{bc} = M^{-1} \sum_{i=1}^{M} (z_i - \bar{z})b(r_i - \bar{r})^c$, and $(b, c)$ are nonnegative integers.

3. The Proposed Class of Estimators

In the present study, we have made the incorporation of nonconventional location parameters, nonconventional measures of dispersion, and their function with the coefficient of variation, whose generalized class for estimating population variance is given as

$$l = \left[ a_1 w_z^2 + a_2 (W_r^2 - w_z^2) \right] \left[ W_z^2 + \phi Q_z^2 \right] \left[ y_w^2 + \phi Q_z^2 \right],$$

where $(\gamma W_z^2 + \phi Q_z^2) > 0$, $(\gamma w_z^2 + \phi Q_z^2) > 0$, and $(\gamma, Q)$ are either real constants or functions of known parameters of an auxiliary variable $r$ and $\phi$ is a constant such that $|\phi| \leq 1$ which is a more flexible condition as given by Singh et al. [5] and Solanki et al. [17] over the constant $\phi$ in their estimators, and $(a_1, a_2)$ are constants such that the mean squared error (MSE) is minimum. Here, we note that $0 \leq \phi \leq 1$. For suitable values of $(a_1, a_2, \gamma, \phi)$, the proposed class of estimators "I" reduces to some known existing estimators based on quartiles and their functions given in Table 1.

From (2), we would like to remark the suitable values of $(a_1, a_2, \gamma, \phi)$ and one can generate different estimators. For example, we have developed some new estimators from the proposed class of estimators "I" which are listed in Table 2.

While obtaining the expression of bias and mean square error (MSE) for the suggested class of estimator "I", we write

$$w_z^2 = W_z^2(1 + \theta_0),$$
$$w_r^2 = W_r^2(1 + \theta_1),$$

such that $E(\theta_0) = E(\theta_1) = 0$, and up to the first order of approximation, while fpc term is ignored, we have

$$E(\hat{\theta}_0^*) = \left( \frac{\mu_{40}^{*}}{m} \right),$$
$$E(\hat{\theta}_1^*) = \left( \frac{\mu_{04}^{*}}{m} \right),$$
$$E(\hat{\theta}_1^{*}) = \left( \frac{\mu_{22}^{*}}{m} \right).$$

Now, in terms of $\hat{\theta}$'s, the estimator "I", given in (2), is expressed as

$$l = W_z^2[\alpha_1 (1 + \theta_0) - \alpha_2 b \theta_1] (1 + \theta^* \theta_1)^{-1},$$

where

$$b = \frac{W_z^2}{W_r^2},$$
$$\theta^* = \frac{\gamma W_z^2 + \phi Q_z^2}{W_z^2 + \phi Q_z^2}.$$
S. no. | Estimators | Values of constants | Table 1: Some existing known estimators of the class of estimators \( I \).
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( l_1 = w_2^2 ), usual unbiased estimator</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( \gamma )</td>
<td>( Q )</td>
<td>( \varphi )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( l_2 = w_2^2 (W_2^2</td>
<td>w_2^2</td>
<td>) ), Das and Tripathi [1] and Isaki [4] estimator</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( l_3 = w_2^2 (W_2^2 + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Singh et al. [5] estimator</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( Q_1 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( l_4 = w_2^2 (W_2^2 + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Singh et al. [5] estimator</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( Q_2 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( l_5 = w_2^2 (W_2^2 + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Singh et al. [5] estimator</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( Q_3 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( l_6 = w_2^2 (W_2^2 + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Singh et al. [5] estimator</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( Q_4 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>( l_7 = w_2^2 (W_2^2 + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Singh et al. [5] estimator</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( Q_5 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>( l_8 = w_2^2 (W_2^2 + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Singh et al. [5] estimator</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( Q_6 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>( l_9 = w_2^2 (C,w_2^2 + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Solanki et al. [17] estimator</td>
<td>1</td>
<td>0</td>
<td>( C_e )</td>
<td>( Q_1 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>( l_{10} = w_2^2 (\beta_2^2 \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Solanki et al. [17] estimator</td>
<td>1</td>
<td>0</td>
<td>( \beta_2^2 \varphi )</td>
<td>( Q_2 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>( l_{11} = w_2^2 (\psi R + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Solanki et al. [17] estimator</td>
<td>1</td>
<td>0</td>
<td>( \psi R )</td>
<td>( Q_3 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>( l_{12} = w_2^2 (\beta_2^2 \psi R + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Solanki et al. [17] estimator</td>
<td>1</td>
<td>0</td>
<td>( \beta_2^2 \psi )</td>
<td>( Q_4 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>( l_{13} = w_2^2 (\rho \psi R + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Solanki et al. [17] estimator</td>
<td>1</td>
<td>0</td>
<td>( \rho )</td>
<td>( Q_5 )</td>
<td>( \delta )</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>( l_{14} = w_2^2 (\rho \psi R + \varphi Q_2^2</td>
<td>w_2^2</td>
<td>+ \varphi Q_2^2) ), Solanki et al. [17] estimator</td>
<td>1</td>
<td>0</td>
<td>( \rho )</td>
<td>( Q_6 )</td>
<td>( \delta )</td>
<td></td>
</tr>
</tbody>
</table>

Squaring both sides of (7), neglecting terms of \( \theta^i \)s having the power greater than two, we have

\[
(\theta - W_2^2)^2 = W_2^4 \left[ 1 + \alpha_1^2 \left( 1 + 2\theta_0 - 2\theta^0 \theta_1 + \theta^2 + 3\theta^2 \theta_1^2 - 4\theta^2 \theta_1 \theta_3 + \alpha_2^2 \theta_1^2 \right) + 2\alpha_1 \alpha_2 b \left( 2\theta^2 \theta_1^2 - \theta^0 \theta_1 \theta_3 - 2\alpha_1 (1 + \theta_0 - \theta^0 \theta_1 - \theta^2 \theta_1 + \theta^2 \theta_3) \right) \right] + 2\alpha_1 \alpha_2 b \left( \theta^2 \theta_1^2 - \theta^0 \theta_1 \theta_3 \right) \]  

Taking expectation of both sides of (9), we get the MSE of the estimator \( I \) to the first degree of approximation (ignoring fpc term) as

\[
MSE(I) = W_2^4 \left[ 1 + \alpha_1^2 d_1 + \alpha_2^2 d_2 + 2\alpha_1 \alpha_2 - 2\alpha_1 \alpha_2 \right].
\]
\[
d_1 = \left[ 1 + \frac{1}{m} (\mu_0^* + 3\theta^2 \mu_{04}^* - 4\theta^* \mu_{22}^*) \right],
\]
\[
d_2 = \frac{b\theta^* \mu_{04}^*}{m},
\]
\[
d_3 = \left( \frac{b}{m} \right) (2\theta^* \mu_{04}^* - \mu_{22}^*),
\]
\[
d_4 = \left[ 1 + \frac{\theta^*}{m} (\theta^* \mu_{04}^* - \mu_{22}^*) \right],
\]
\[
d_5 = \frac{b\theta^*}{m} \mu_{04}^*.
\]

Differentiating MSE \((l)\) with respect to \(\alpha_1\) and \(\alpha_2\) and equating them to zero, we have

\[
\begin{bmatrix}
  d_1 \\
  d_3 \\
  d_2 \\
  d_4 \\
  d_5
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2
\end{bmatrix}
= \begin{bmatrix}
  d_4 \\
  d_5
\end{bmatrix}.
\]

(12)

Simplifying (12), we get the optimum values of \(\alpha_1\) and \(\alpha_2\) as

\[
\alpha_1 = \frac{d_4 - d_2 - d_3}{(d_1 d_2 - d_3^2)} = \alpha_{10} \text{ (say)},
\]

(13)

\[
\alpha_2 = \frac{d_4 - d_2 - d_3}{(d_1 d_2 - d_3^2)} = \alpha_{20} \text{ (say)}.
\]

Inserting (13) in (10), we get the resulting minimum MSE of \("l"\) given by

\[
\min \cdot \text{MSE}(l) = W_z^4 \left[ 1 + \frac{d_4 - d_1 - 2d_3}{d_2 - d_3^2} \right].
\]

(14)

Thus we establish the following theorem.

**Theorem 1.** To the first degree of approximation

\[
\text{MSE}(l) \geq W_z^4 \left[ 1 - \frac{(d_4 d_2^2 - 2d_3 d_1 d_2 + d_1 d_3^2)}{(d_1 d_2 - d_3^2)} \right],
\]

(15)

with equality holding if

\[
\alpha_1 = \alpha_{10},
\]

\[
\alpha_2 = \alpha_{20},
\]

where \(\alpha_{i0}^* (i = 1, 2)\) are given in (13).

**Special Case 1:** for \(\alpha_1 = 1\) in (2), we get an alternative class of estimators for the population variance \(W_z^2\) as

\[
t^* = \left[ w_z^2 + \alpha_2 (W_r^2 - w_r^2) \right] \left( \frac{\gamma W_z^2 + \varphi Q^2}{\gamma w_z^2 + \varphi Q^2} \right).
\]

(17)

Putting \(\alpha_1 = 0\) in (8) and (10), we get the bias and MSE of \(l^*\) to the first degree of approximation, given by

\[
B(l^*) = \left( \frac{W_z^2}{m} \right) \theta^* [(\theta^* + b) \mu_{04}^* - \mu_{22}^*],
\]

\[
\text{MSE}(l^*) = W_z^4 \left[ 1 + d_1 - 2d_3 + \alpha_2^* d_2 - 2\alpha_2 (d_3 - d_3) \right].
\]

(18)

The MSE \((l^*)\) is minimum when

\[
\alpha_{20}^* = \frac{(d_4 - d_1)}{(d_2)}.
\]

(19)

Thus, the resulting minimum MSE of \((l^*)\) is given by

\[
\min \cdot \text{MSE}(l^*) = W_z^4 \left[ 1 + d_1 - 2d_3 - \frac{(d_3 - d_3)^2}{d_2} \right]
\]

\[
= \left( \frac{W_z^4}{m} \right) \left( \mu_{04}^* - \mu_{22}^* \right).
\]

(20)

This equals to the minimum MSE of the difference estimator \(l_d\) envisaged by [1]

\[
l_d = w_z^2 + \alpha_2 (W_r^2 - w_r^2).
\]

(21)

**Special Case 2:** for \(\alpha_2 = 0\) in (2), we get another class of estimators for population variance \(W_z^2\) as

\[
l_1^* = \left[ \alpha_1 w_z^2 \right] \left( \frac{\gamma W_z^2 + \varphi Q^2}{\gamma w_z^2 + \varphi Q^2} \right).
\]

(22)

Putting \(\alpha_2 = 0\) in (8) and (10), we get the bias and MSE of \(l_1^*\), respectively, as

\[
B(l_1^*) = W_z^2 \left[ \alpha_1 \left( 1 + \frac{\theta^*}{m} (\theta^* \mu_{04}^* - \mu_{22}^*) \right) \right] - 1,
\]

(23)

\[
\text{MSE}(l_1^*) = W_z^4 \left[ 1 + \alpha_1^* d_1 - 2\alpha_1 d_4 \right].
\]

The MSE \((l_1^*)\) is minimum for

\[
\alpha_{10}^* = \frac{d_4}{d_1}.
\]

(24)

Thus, the resulting minimum MSE of \(l_1^*\) is given by

\[
\min \cdot \text{MSE}(l_1^*) = W_z^4 \left[ 1 - \frac{d_4^2}{d_1^2} \right].
\]

(25)

Thus, we state the following corollary.

**Corollary 1.** To the first degree of approximation

\[
\text{MSE}(l_1^*) \geq W_z^4 \left[ 1 - \frac{d_4^2}{d_1^2} \right],
\]

(26)

with equality holding if

\[
\alpha_1 = \alpha_{10}^* = \frac{d_4}{d_1}.
\]

(27)
Special Case 3: if we set \((\alpha_1, \alpha_2) = (1, 0)\) in (2), the class of estimators \(l\) reduces to the estimator
\[
I^{**}_l = w^*_z \left( \frac{yW^2 + \phi Q^2}{yu^*_z + \phi Q^2} \right),
\]
(28)
which includes Solanki et al. [17] estimators \(l_9\) to \(l_{14}\).

Putting \((\alpha_1, \alpha_2) = (1, 0)\) in (8) and (10), we get the bias and MSE of \(I^{**}_l\) to the first degree of approximation (ignoring fpc term), respectively, as
\[
B(l^{**}_1) = \left( \frac{W^2 \theta^r}{m} \right) (\theta^r \mu_{04}^* - \mu_{22}^*),
\]
(29)
\[
\text{MSE}(l^{**}_1) = W^2_z \left[ 1 + d_1 + 2d_4 \right],
\]
(30)
\[
= \left( \frac{W^2 \theta^r}{m} \right) (\mu_{40}^* + \theta^2 \mu_{04}^* - 2\theta^2 \mu_{22}^*). 
\]
(31)

Special Case 4: if we set \((\alpha_1, \alpha_2, \gamma) = (1, 0, 1)\) in (2), the class of estimators \(l\) reduces to the class of estimators of \(W^2_z\) as
\[
I^*_2 = \alpha_1 w^*_z \left( \frac{W^2 + \phi Q^2}{u^*_z + \phi Q^2} \right).
\]
(32)

Inserting \((\alpha_1, \alpha_2, \gamma) = (1, 0, 1)\) in (8) and (10), we get the bias and MSE of \(I^*_2\) to the first degree of approximation (ignoring fpc term), respectively, as
\[
B(I^*_2) = W^2_z \left[ \alpha_1 \left\{ 1 + \left( \frac{\theta}{m} \right) (\theta^r \mu_{04}^* - \mu_{22}^*) \right\} \right],
\]
(33)
\[
\text{MSE}(I^*_2) = W^2_z \left[ 1 + \alpha_1 d_1 - \alpha_1 d^*_4 \right],
\]
(34)
where
\[
d_1 = \left[ 1 + \frac{1}{m} (\mu_{40}^* + 3\theta^r \mu_{04}^* - 4\theta^2 \mu_{22}^*) \right],
\]
\[
d^*_4 = \left[ 1 + \frac{\theta}{m} (\theta \mu_{04}^* - \mu_{22}^*) \right],
\]
\[
\theta = \frac{W^2_r}{W^2_r + \phi Q^2}
\]
The MSE\((I^*_2)\) at (34) is minimized for
\[
\alpha_{10}^{**} = \frac{d_1^*}{d_1},
\]
(36)
Substitution of (36) in (34) yields the minimum MSE of \(I^*_2\) and is given by
\[
\text{min} \cdot \text{MSE}(I^*_2) = W^2_z \left[ 1 - \frac{d^*_4}{d_1} \right] .
\]
(37)
Thus, we established the following corollary.

**Corollary 2.** To the first degree of approximation
\[
\text{MSE}(l^*_2) \geq W^2_z \left[ 1 - \frac{d^*_4}{d_1} \right],
\]
(38)
with equality holdings if
\[
\alpha_1 = \alpha_{10}^{**} = \frac{d_1^*}{d_1},
\]
(39)

Special Case 5: for \((\alpha_1, \alpha_2, \gamma) = (1, 0, 1)\) in (2), the class of estimators \(l^*\) reduces to the class of estimators of \(W^2_z\) as
\[
I^*_2 = w^*_z \left( \frac{W^2 + \phi Q^2}{u^*_z + \phi Q^2} \right).
\]
(40)
which includes Singh et al. [5] estimators \(l_3\) to \(l_8\).

Inserting \((\alpha_1, \alpha_2, \gamma) = (1, 0, 1)\) in (8) and (10), we get the bias and MSE of \(I^*_2\) to the first degree of approximation (ignoring fpc term), respectively, given by
\[
I^*_2 = w^*_z \left( \frac{W^2 + \phi Q^2}{u^*_z + \phi Q^2} \right).
\]
(41)
\[
\text{MSE}(I^*_2) = W^2_z \left[ 1 + d_1 - 2d^*_4 \right],
\]
(42)

4. **Efficiency Comparison**

To the first degree of approximation, the MSE of \(w^*_z\) and \(l_2\) are, respectively, given by
\[
\text{MSE}(w^*_z) = \left( \frac{W^2}{m} \right) \mu_{04}^*,
\]
(43)
\[
\text{MSE}(l_2) = \left( \frac{W^2}{m} \right) (\mu_{04}^* + \mu_{04}^* + 2\mu_{22}^*).
\]
(44)

From (20), (43), and (44), we have
\[
\text{MSE}(w^*_z) - \min \cdot \text{MSE}(l^* \text{ or } l_2) = \left( \frac{W^2}{m} \mu_{22}^* \right) > 0,
\]
(45)
\[
\text{MSE}(l_2) - \min \cdot \text{MSE}(l^*_2) = \frac{W^2 \mu_{04}^* + \mu_{22}^*}{m \mu_{04}^*} \geq 0.
\]
(46)

It follows from (45) and (46) that the proposed class of estimators \(l^*\) or difference estimator \(l_d\) is better than the usual unbiased estimator \(w^*_z\) and the ratio type estimator \(l_2\) due to [1, 4].

From (37) and (42), we have
\[
\text{MSE}(l^*_2) - \min \cdot \text{MSE}(l^*_2) = \frac{W^2}{m} \left( \frac{d_1 - d^*_4}{d_1} \right)^2 \geq 0.
\]
(47)

This follows that the proposed class of estimators \(l^*_2\) is more efficient than the proposed class of estimators \(l^*_2\) according to Singh et al. [5]. Thus, the estimator \(l^*_2\) is an improvement over the Singh et al. [5] estimators \(l^*_2\).
is better than the class of estimators

Finally, from the above theoretical comparisons we conclude that the proposed family of estimators \( l^1 \) is better than class of estimators \( l \) respectively, and for other families of estimators (recently proposed by Singh et al. [5] and Solanki et al. [17]), the estimators \( l^1 \) are an improvement over the estimator \( l^1 \) due to Solanki et al. [17].

Furthermore, from (14) and (20), we have

$$
\text{PRE} \left( \frac{w^2}{l} \right) = \frac{\text{MSE}(l_2) - \text{MSE}(l)}{\text{MSE}(l)} \times 100
$$

and the findings are summarized in Table 4.

### Table 3: The parameters of population data set.

<table>
<thead>
<tr>
<th>( M )</th>
<th>( m )</th>
<th>( Z )</th>
<th>( R )</th>
<th>( \rho )</th>
<th>( W_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>20</td>
<td>51.8264</td>
<td>11.2646</td>
<td>0.9413</td>
<td>18.3569</td>
</tr>
</tbody>
</table>

$$
\text{PRE} \left( \frac{w^2}{l^1} \right) = \frac{\text{MSE}(w^2)}{\text{MSE}(l^1)} \times 100
$$

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( l_{15} )</th>
<th>( l_{16} )</th>
<th>( l_{17} )</th>
<th>( l_{18} )</th>
<th>( l_{19} )</th>
<th>( l_{20} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.00</td>
<td>257.0391</td>
<td>259.3591</td>
<td>291.8213</td>
<td>258.2640</td>
<td>256.4163</td>
<td>253.0021</td>
</tr>
<tr>
<td>-0.75</td>
<td>255.2481</td>
<td>257.2568</td>
<td>275.3213</td>
<td>256.0011</td>
<td>253.9286</td>
<td>252.8256</td>
</tr>
<tr>
<td>-0.50</td>
<td>254.2359</td>
<td>256.9329</td>
<td>273.3492</td>
<td>255.3651</td>
<td>253.6165</td>
<td>252.4152</td>
</tr>
<tr>
<td>-0.25</td>
<td>253.9139</td>
<td>256.1231</td>
<td>272.1029</td>
<td>254.9606</td>
<td>253.0259</td>
<td>252.0081</td>
</tr>
<tr>
<td>0.00</td>
<td>253.1029</td>
<td>255.8569</td>
<td>272.0051</td>
<td>254.0436</td>
<td>253.1851</td>
<td>252.0011</td>
</tr>
<tr>
<td>0.25</td>
<td>252.9320</td>
<td>255.0091</td>
<td>272.9321</td>
<td>253.8186</td>
<td>252.8169</td>
<td>251.3621</td>
</tr>
<tr>
<td>0.50</td>
<td>253.101</td>
<td>254.8563</td>
<td>272.3649</td>
<td>253.5646</td>
<td>251.9103</td>
<td>250.3961</td>
</tr>
<tr>
<td>0.75</td>
<td>254.2569</td>
<td>254.6561</td>
<td>272.1536</td>
<td>253.3963</td>
<td>253.3851</td>
<td>250.0011</td>
</tr>
<tr>
<td>1.00</td>
<td>254.2031</td>
<td>254.5341</td>
<td>271.9806</td>
<td>252.8646</td>
<td>251.0091</td>
<td>249.5631</td>
</tr>
</tbody>
</table>

Thus, from (48) and (50), we have the following inequality:

$$
\min \cdot \text{MSE}(l) \geq \min \cdot \text{MSE}(l^1) \geq \min \cdot \text{MSE}(l^{1*})
$$

This clearly shows that the proposed family of estimators \( l \) is better than families of estimators \( l^1 \) and \( l^{1*} \).

Finally, from the above theoretical comparisons we conclude that the proposed family of estimators \( l \) is better than the usual unbiased estimator \( l = w^2 \), [4] ratio type estimator \( l^2 \), and the estimators \( l_k, l_{16} \) and \( l_5 \) to \( l_{14} \) recently proposed by Singh et al. [5] and Solanki et al. [17], respectively, and for other families of estimators (\( l^1, l^2 \)).

### 5. Empirical Study

The performances of the proposed estimators \( l_k, l_{15}, \ldots, l_{20} \), which are members of the suggested class of estimators, \( l \), are evaluated against the usual unbiased estimators \( w^2 \) and the estimators \( l_i \) (\( i = 2, 3, \ldots, 14 \)) proposed by [1, 4, 5, 10] for the population data set (source: [18]) summarized in Table 3.

We have computed the percent relative efficiencies (PREs) of the suggested estimators \( l_k \) (\( k = 15, 16, \ldots, 20 \)) with respect to the usual unbiased estimator \( w^2 \) in the range of \( \phi \in (-1.0, 1.0) \) by using the following formula:

$$
\text{PRE} \left( \frac{w^2}{l_k} \right) = \frac{\text{MSE}(w^2)}{\text{MSE}(l_k)} \times 100
$$

and the findings are summarized in Table 4.
Table 5: PREs of the estimators \( l_i = w_{k}^2, l_2 \) and \( l_i(i = 3, \ldots, 14) \).

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Value of ( \phi )</th>
<th>PRE (( w_{k}^2 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>—</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>—</td>
<td>183.23</td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>209.67</td>
</tr>
<tr>
<td>4</td>
<td>0.50</td>
<td>230.60</td>
</tr>
<tr>
<td>5</td>
<td>0.30</td>
<td>230.28</td>
</tr>
<tr>
<td>6</td>
<td>0.70</td>
<td>230.57</td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td>200.87</td>
</tr>
<tr>
<td>8</td>
<td>0.60</td>
<td>230.80</td>
</tr>
<tr>
<td>9</td>
<td>1.00</td>
<td>270.38</td>
</tr>
<tr>
<td>10</td>
<td>1.00</td>
<td>270.61</td>
</tr>
<tr>
<td>11</td>
<td>1.00</td>
<td>263.50</td>
</tr>
<tr>
<td>12</td>
<td>0.80</td>
<td>270.58</td>
</tr>
<tr>
<td>13</td>
<td>1.00</td>
<td>270.61</td>
</tr>
<tr>
<td>14</td>
<td>0.30</td>
<td>270.58</td>
</tr>
</tbody>
</table>

For the purpose of comparison of the proposed estimators \( l_{15} \) to \( l_{20} \) with that of the usual unbiased estimator \( l_1 = w_{k}^2 \), [4] estimator \( l_2 \), [5] estimators \( l_3 \) to \( l_8 \), and [17] estimators \( l_9 \) to \( l_{14} \), in Table 5, we give the PREs of the estimators \( l_1 \) and \( l_2 \) and the PREs of the estimators \( l_3 \) to \( l_{14} \) with respect to usual unbiased estimator \( w_{k}^2 \) along with the value of 6\( d_\theta \) for which the PREs of the estimators \( l_3 \) to \( l_{14} \) are maximum as given in [17].

It is observed from Tables 4 and 5 that all the estimators \( l_k \) \((k = 15, 16, \ldots, 20)\) which are the members of the proposed class of estimators \( l \) performed better than the usual unbiased estimator \( l_1 = w_{k}^2 \), ratio type estimators \( l_2 \) due to [4, 5] estimators \( l_3 \) to \( l_8 \) and [17] estimators \( l_9 \) to \( l_{14} \), for \( \phi \in (-1.0, 1.0) \).

6. Conclusion

In the present paper, we have suggested an improved class of estimators for population variance using the auxiliary information of nonconventional location parameters, nonconventional measures of dispersion, and their function with the coefficient of variation. The bias and mean square error (MSE) expressions of the proposed class of estimators are obtained and compared with the usual unbiased estimator, estimators in [4], [5], and [17]. We have also analyzed the performance of the proposed estimators by utilizing the data set of the known population and found that in convenient cases, the proposed estimators perform better than the other existing estimators. On comparison with the usual unbiased estimator, which is shown in Table 4, it is evident that upon increasing the values of either negative or positive, the percent relative efficiency also increases. From Table 5, we observe that our estimators are more proficient than the existing ones. Hence, we strongly recommend that our proposed estimators perform better than the existing estimators for use in practical applications.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


