Research Article

Enhanced Estimators of Population Variance with the Use of Supplementary Information in Survey Sampling

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In the present study, we propose the proficient class of estimators of the finite population mean, while incorporating the nonconventional location and nonconventional measures of dispersion with coefficient of variation of the auxiliary variable. Properties associated with the suggested class of improved estimators are derived, and an efficiency comparison with the usual unbiased ratio estimator and other existing estimators under consideration in the present study is established. An empirical study has also been provided to validate the theoretical results. Finally, it is established that the proposed class of estimators of the finite population variance proves to be more efficient than the existing estimators mentioned in this study.

1. Introduction

It is very quite often that utilization of supplementary information in survey sampling which has some sort of strong positive or negative correlation with the response variable is always found to be advantageous. So for the utilization of such information, various methods in survey sampling are presently used to increase precision, with the incorporation of these supplementary information, in estimating the population parameters. Various authors have put their sincere efforts to utilize supplementary information with different sampling designs in different situations in such a way that their estimation procedure becomes more proficient; for details see [1–12]. It is very often that some of the measures are so much affected by extreme observations and can give misleading results. In case of extreme values, using classical methods of estimation provides misleading results, but authors have also put their valuable efforts to come up with solutions to this situation, so that precise results should be obtained even with the presence of outliers in the data. Authors such as Subzar et al. [13] have proposed different robust ratio type estimators in simple random sampling

without replacement (SRSWOR) while utilizing the Huber-M estimation technique. Subzar et al. [14] have also proposed different robust ratio type estimators in SRSWOR by utilizing the different robust regression techniques and compared with Ordinary Least Squares (OLS) and Huber-M estimation techniques. Subzar et al. [15] also proposed different robust ratio type estimators by comparing the generalized robust regression techniques with OLS and Huber-M estimation method. Recently, Almanjahie et al. [16] have proposed the generalized class of mean estimators with known measures for outlier's treatment. Also, Shahzad et al. [12] have given a new class of L-Moments-based calibration variance estimators. So in the present study, we made the utilization of nonconventional location parameters, nonconventional measures of dispersion, and their functions with the coefficient of variation of the auxiliary variable in order to suggest the class of estimators for estimating the population variance. The properties of the proposed class of estimators are studied under large sample approximation. It has been shown theoretically as well as empirically that the proposed class of estimators is more efficient than existing estimators mentioned in this study.

2. Notations

Consider a finite population $Z = (Z_1, Z_2, \dots, Z_M)$ of M units and let (z, r) be the study and auxiliary variables defined on Z taking values (z_i, r_i) , respectively, on Z_i ($i = 1, 2, \dots, M$). It is desired to estimate the population variance W_z^2 of the study variable z using the information on an auxiliary variable r . Let a simple random sample of size m be drawn without replacement from the finite population Z . In this paper, we shall ignore the finite population correction (fpc) term. We denote the following.

$\bar{Z} = M^{-1} \sum_{i=1}^M z_i$: the population mean of the study variable z

$\bar{R} = M^{-1} \sum_{i=1}^M r_i$: the population mean of the auxiliary variable r

$W_z^2 = (M-1)^{-1} \sum_{i=1}^M (z_i - \bar{Z})^2$: the population mean square/variance of the study variable z

$W_r^2 = (M-1)^{-1} \sum_{i=1}^M (r_i - \bar{R})^2$: the population mean square/variance of the auxiliary variable r

TM = $(Q_1 + 2Q_2 + Q_3/4)$: tri-mean of the auxiliary variable r

HL = median $((R_j + R_k)/2, 1 \leq j \leq k \leq M)$: Hodges-Lehmann estimator of the auxiliary variable r

MR = $(R_{(1)} + R_{(M)})/2$: mid-range of the auxiliary variable r

$G = (4/M-1) \sum_{i=1}^M (2i - M - 1/2M)R_{(i)}$: Gini's mean difference of the auxiliary variable r

$D = (2\sqrt{\pi}/M(M-1)) \sum_{i=1}^M (i - (M+1/2))R_{(i)}$: Downton's method of the auxiliary variable r

$S_{pw} = (\sqrt{\pi}/M^2) \sum_{i=1}^M (2i - M - 1)R_{(i)}$: probability weighted moments of the auxiliary variable r :

$$\begin{aligned}\mu_{40}^* &= (\mu_{40} - 1), \\ \mu_{04}^* &= (\mu_{04} - 1), \\ \mu_{22}^* &= (\mu_{22} - 1),\end{aligned}\quad (1)$$

where $\mu_{rs} = \lambda_{bc}/(\lambda_{02}^{c/2} \lambda_{20}^{b/s})$, $\lambda_{rs} = M^{-1} \sum_{i=1}^M (z_i - \bar{Z})^b (r_i - \bar{R})^c$, and (b, c) are nonnegative integers.

3. The Proposed Class of Estimators

In the present study, we have made the incorporation of nonconventional location parameters, nonconventional measures of dispersion, and their function with the coefficient of variation, whose generalized class for estimating population variance is given as

$$l = [\alpha_1 w_z^2 + \alpha_2 (W_r^2 - w_r^2)] \left(\frac{\gamma W_r^2 + \varphi Q^2}{\gamma w_r^2 + \varphi Q^2} \right), \quad (2)$$

where $(\gamma W_r^2 + \varphi Q^2) > 0$, $(\gamma w_r^2 + \varphi Q^2) > 0$, and (γ, Q) are either real constants or functions of known parameters of an auxiliary variable r and φ is a constant such that $|\varphi| \leq 1$ which is a more flexible condition as given by Singh et al. [5] and

Solanki et al. [17] over the constant φ in their estimators, and (α_1, α_2) are constants such that the mean squared error (MSE) is minimum. Here, we note that $0 \leq \varphi \leq 1$. For suitable values of $(\alpha_1, \alpha_2, \gamma, \varphi)$, the proposed class of estimators "I" reduces to some known existing estimators based on quartiles and their functions given in Table 1.

From (2), we would like to remark the suitable values of $(\alpha_1, \alpha_2, \gamma, \varphi)$ and one can generate different estimators. For example, we have developed some new estimators from the proposed class of estimators "I" which are listed in Table 2.

While obtaining the expression of bias and mean square error (MSE) for the suggested class of estimator "I", we write

$$\begin{aligned}w_z^2 &= W_z^2 (1 + \vartheta_0), \\ w_r^2 &= W_r^2 (1 + \vartheta_1),\end{aligned}\quad (3)$$

such that $E(\vartheta_0) = E(\vartheta_1) = 0$, and up to the first order of approximation, while fpc term is ignored, we have

$$\begin{aligned}E(\vartheta_0^2) &= \left(\frac{\mu_{40}^*}{m} \right), \\ E(\vartheta_1^2) &= \left(\frac{\mu_{04}^*}{m} \right), \\ E(\vartheta_0 \vartheta_1) &= \left(\frac{\mu_{22}^*}{m} \right).\end{aligned}\quad (4)$$

Now, in terms of ϑ 's, the estimator "I", given in (2), is expressed as

$$l = W_z^2 [\alpha_1 (1 + \vartheta_0) - \alpha_2 b \vartheta_1] (1 + \theta^* \vartheta_1)^{-1}, \quad (5)$$

where

$$\begin{aligned}b &= \frac{W_r^2}{W_z^2}, \\ \theta^* &= \frac{\gamma W_r^2}{(\gamma W_r^2 + \varphi Q^2)}.\end{aligned}\quad (6)$$

We assume that $|\theta^* \vartheta_1| < 1$, so that $(1 + \theta^* \vartheta_1)^{-1}$ is expandable. Expanding the right-hand side of (5) and multiplying out, we have

$$\begin{aligned}\theta &\cong W_z^2 [\alpha_1 (1 + \vartheta_0 - \theta^* \vartheta_1 - \theta^* \vartheta_0 \vartheta_1 + \theta^{*2} \vartheta_1^2) \\ &\quad + \alpha_2 b (\theta^* \vartheta_1^2 - \vartheta_1)] \\ \text{or } (\theta - W_z^2) &\cong W_z^2 [\alpha_1 (1 + \vartheta_0 - \theta^* \vartheta_1 - \theta^* \vartheta_0 \vartheta_1 + \theta^{*2} \vartheta_1^2) \\ &\quad + \alpha_2 b (\theta^* \vartheta_1^2 - \vartheta_1) - 1].\end{aligned}\quad (7)$$

Taking expectation of both sides of (7), we get the bias of the estimators l to the first degree of approximation as

$$B(l) = W_z^2 \left[\alpha_1 \left\{ 1 + \left(\frac{\theta^*}{m} \right) (\theta^* \mu_{04}^* - \mu_{22}^*) \right\} + \alpha_2 \left(\frac{b \theta^*}{m} \right) \mu_{04}^* - 1 \right]. \quad (8)$$

TABLE 1: Some existing known estimators of the class of estimators l .

S. no.	Estimators	Values of constants				
		α_1	α_2	γ	Q	φ
1	$l_1 = w_z^2$, usual unbiased estimator	1	0	0	—	—
2	$l_2 = w_z^2 (W_r^2/w_r^2)$, Das and Tripathi [1] and Isaki [4] estimator	1	0	1	—	0
3	$l_3 = w_z^2 (W_r^2 + \phi Q_1^2/w_r^2 + \phi Q_2^2)$, Singh et al. [5] estimator	1	0	1	Q_1	δ
4	$l_4 = w_z^2 (W_r^2 + \phi Q_2^2/w_r^2 + \phi Q_3^2)$, Singh et al. [5] estimator	1	0	1	Q_2	δ
5	$l_5 = w_z^2 (W_r^2 + \phi Q_3^2/w_r^2 + \phi Q_4^2)$, Singh et al. [5] estimator	1	0	1	Q_3	δ
6	$l_6 = w_z^2 (W_r^2 + \phi Q_4^2/w_r^2 + \phi Q_r^2)$, Singh et al. [5] estimator	1	0	1	Q_r	δ
7	$l_7 = w_z^2 (W_r^2 + \phi Q_d^2/w_r^2 + \phi Q_a^2)$, Singh et al. [5] estimator	1	0	1	Q_d	δ
8	$l_8 = w_z^2 (W_r^2 + \phi Q_a^2/w_r^2 + \phi Q_a^2)$, Singh et al. [5] estimator	1	0	1	Q_a	δ
9	$l_9 = w_z^2 (C_r W_r^2 + \phi Q_1^2/C_r w_r^2 + \phi Q_1^2)$, Solanki et al. [17] estimator	1	0	C_r	Q_1	δ
10	$l_{10} = w_z^2 (\beta_{2(r)} W_r^2 + \phi Q_2^2/\beta_{2(r)} s_r^2 + \phi Q_2^2)$, Solanki et al. [17] estimator	1	0	$\beta_{2(r)}$	Q_2	δ
11	$l_{11} = w_z^2 (R W_r^2 + \phi Q_3^2/R w_r^2 + \phi Q_3^2)$, Solanki et al. [17] estimator	1	0	$\bar{R}$	Q_3	δ
12	$l_{12} = w_z^2 (\beta_{2(r)} W_r^2 + \phi Q_r^2/\beta_{2(r)} w_r^2 + \phi Q_r^2)$, Solanki et al. [17] estimator	1	0	$\beta_{2(r)}$	Q_r	δ
13	$l_{13} = w_z^2 (\rho W_r^2 + \phi Q_d^2/\rho w_r^2 + \phi Q_d^2)$, Solanki et al. [17] estimator	1	0	ρ	Q_d	δ
14	$l_{14} = w_z^2 (\rho W_r^2 + \phi Q_a^2/\rho w_r^2 + \phi Q_a^2)$, Solanki et al. [17] estimator	1	0	ρ	Q_a	δ

TABLE 2: Some new members of the class of estimators l .

S. no.	Estimators	Values of constant				
		w_1	w_2	γ	Q	φ
15	$l_{15} = [\alpha_1 w_z^2 + \alpha_2 (W_r^2 - w_r^2)] (\gamma W_r^2 + \phi T M^2 / \gamma w_r^2 + \phi T M^2)$	α_1	α_2	C_r	TM	φ
16	$l_{16} = [\alpha_1 w_z^2 + \alpha_2 (W_r^2 - w_r^2)] (\gamma W_r^2 + \phi M R^2 / \gamma w_r^2 + \phi M R^2)$	α_1	α_2	C_r	MR	φ
17	$l_{17} = [\alpha_1 w_z^2 + \alpha_2 (W_r^2 - w_r^2)] (\gamma W_r^2 + \phi H L^2 / \gamma w_r^2 + \phi H L^2)$	α_1	α_2	C_r	HL	φ
18	$l_{18} = [\alpha_1 w_z^2 + \alpha_2 (W_r^2 - w_r^2)] (\gamma W_r^2 + \phi G^2 / \gamma w_r^2 + \phi G^2)$	α_1	α_2	C_r	G	φ
19	$l_{19} = [\alpha_1 w_z^2 + \alpha_2 (W_r^2 - w_r^2)] (\gamma W_r^2 + \phi D^2 / \gamma w_r^2 + \phi D^2)$	α_1	α_2	C_r	D	φ
20	$l_{20} = [\alpha_1 w_z^2 + \alpha_2 (W_r^2 - w_r^2)] (\gamma W_r^2 + \phi S_{pw}^2 / \gamma w_r^2 + \phi S_{pw}^2)$	α_1	α_2	C_r	S_{pw}	φ

Squaring both sides of (7), neglecting terms of ϑ' 's having the power greater than two, we have

$$\begin{aligned}
(\theta - W_z^2)^2 &\cong W_z^4 [1 + \alpha_1^2 (1 + 2\vartheta_0 - 2\theta^* \vartheta_1 + \vartheta_0^2 + 3\theta^{*2} \vartheta_1^2 - 4\theta^* \vartheta_0 \vartheta_1) + \alpha_2^2 b^2 \vartheta_1^2 \\
&\quad + 2\alpha_1 \alpha_2 b (2\theta^* \vartheta_1^2 - \vartheta_0 \vartheta_1 - \vartheta_1) - 2\alpha_1 (1 + \vartheta_0 - \theta^* \vartheta_1 - \theta^* \vartheta_0 \vartheta_1 + \theta^{*2} \vartheta_1) \\
&\quad - 2\alpha_2 b (\theta^* \vartheta_1^2 - \vartheta_1)].
\end{aligned} \tag{9}$$

Taking expectation of both sides of (9), we get the MSE of the estimator l to the first degree of approximation (ignoring fpc term) as

$$MSE(l) = W_z^4 [1 + \alpha_1^2 d_1 + \alpha_2^2 d_2 + 2\alpha_1 \alpha_2 - 2\alpha_1 d_4 - 2\alpha_2 d_5], \tag{10}$$

where

$$\begin{aligned}
d_1 &= \left[1 + \frac{1}{m} (\mu_{40}^* + 3\theta^{*2} \mu_{04}^* - 4\theta^* \mu_{22}^*) \right], \\
d_2 &= \frac{b^2 \mu_{04}^*}{m}, \\
d_3 &= \left(\frac{b}{m} \right) (2\theta^* \mu_{04}^* - \mu_{22}^*), \\
d_4 &= \left[1 + \frac{\theta^*}{m} (\theta^* \mu_{04}^* - \mu_{22}^*) \right], \\
d_5 &= \frac{b\theta^*}{m} \mu_{04}^*.
\end{aligned} \tag{11}$$

Differentiating MSE (l) with respect to α_1 and α_2 and equating them to zero, we have

$$\begin{bmatrix} d_1 & d_3 \\ d_3 & d_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} d_4 \\ d_5 \end{bmatrix}. \tag{12}$$

Simplifying (12), we get the optimum values of α_1 and α_2 as

$$\begin{aligned}
\alpha_1 &= \frac{(d_2 d_4 - d_3 d_5)}{(d_1 d_2 - d_3^2)} = \alpha_{10} \text{ (say)}, \\
\alpha_2 &= \frac{(d_1 d_5 - d_3 d_4)}{(d_1 d_2 - d_3^2)} = \alpha_{20} \text{ (say)}.
\end{aligned} \tag{13}$$

Inserting (13) in (10), we get the resulting minimum MSE of " l " given by

$$\min \cdot \text{MSE}(l) = W_z^4 \left[1 - \frac{(d_2 d_4^2 - 2d_3 d_4 d_5 + d_1 d_5^2)}{(d_1 d_2 - d_3^2)} \right]. \tag{14}$$

Thus we establish the following theorem.

Theorem 1. *To the first degree of approximation*

$$\text{MSE}(l) \geq W_z^4 \left[1 - \frac{(d_2 d_4^2 - 2d_3 d_4 d_5 + d_1 d_5^2)}{(d_1 d_2 - d_3^2)} \right], \tag{15}$$

with equality holding if

$$\begin{aligned}
\alpha_1 &= \alpha_{10}, \\
\alpha_2 &= \alpha_{20},
\end{aligned} \tag{16}$$

where α'_{i0} ($i = 1, 2$) are given in (13).

Special Case 1: for $\alpha_1 = 1$ in (2), we get an alternative class of estimators for the population variance W_z^2 as

$$t^* = \left[w_z^2 + \alpha_2 (W_r^2 - w_r^2) \right] \left(\frac{\gamma W_r^2 + \varphi Q^2}{\gamma w_r^2 + \varphi Q^2} \right). \tag{17}$$

Putting $\alpha_1 = 0$ in (8) and (10), we get the bias and MSE of l^* to the first degree of approximation, given by

$$B(l^*) = \left(\frac{W_z^2}{m} \right) \theta^* [(\theta^* + b) \mu_{04}^* - \mu_{22}^*],$$

$$\text{MSE}(l^*) = W_z^4 \left[1 + d_1 - 2d_4 + \alpha_2^2 d_2 - 2\alpha_2 (d_5 - d_3) \right]. \tag{18}$$

The MSE (l^*) is minimum when

$$\alpha_{20}^* = \frac{(d_5 - d_3)}{(d_2)}. \tag{19}$$

Thus, the resulting minimum MSE of (l^*) is given by

$$\begin{aligned}
\min \cdot \text{MSE}(l^*) &= W_z^4 \left[1 + d_1 - 2d_4 - \frac{(d_5 - d_3)^2}{d_2} \right] \\
&= \left(\frac{W_z^4}{m} \right) \left[\mu_{40}^* - \frac{\mu_{22}^{*2}}{\mu_{04}^*} \right].
\end{aligned} \tag{20}$$

This equals to the minimum MSE of the difference estimator l_d envisaged by [1]

$$l_d = w_z^2 + \alpha_2 (W_r^2 - w_r^2). \tag{21}$$

Special Case 2: for $\alpha_2 = 0$ in (2), we get another class of estimators for population variance W_z^2 as

$$l_1^* = \left[\alpha_1 w_z^2 \right] \left(\frac{\gamma W_r^2 + \varphi Q^2}{\gamma w_r^2 + \varphi Q^2} \right). \tag{22}$$

Putting $\alpha_2 = 0$ in (8) and (10), we get the bias and MSE of l_1^* , respectively, as

$$B(l^*) = W_z^2 \left[\alpha_1 \left\{ 1 + \frac{\theta^*}{m} (\theta^* \mu_{04}^* - \mu_{22}^*) \right\} - 1 \right], \tag{23}$$

$$\text{MSE}(l^*) = W_z^4 \left[1 + \alpha_1^2 d_1 - 2\alpha_1 d_4 \right].$$

The MSE (l^*) is minimum for

$$\alpha_{10}^* = \frac{d_4}{d_1}. \tag{24}$$

Thus, the resulting minimum MSE of l^* is given by

$$\min \cdot \text{MSE}(l^*) = W_z^4 \left[1 - \frac{d_4^2}{d_1} \right]. \tag{25}$$

Thus, we state the following corollary.

Corollary 1. *To the first degree of approximation*

$$\text{MSE}(l_1^*) \geq W_z^4 \left[1 - \frac{d_4^2}{d_1} \right], \tag{26}$$

with equality holding if

$$\alpha_1 = \alpha_{10}^* = \frac{d_4}{d_1}. \tag{27}$$

Special Case 3: if we set $(\alpha_1, \alpha_2) = (1, 0)$ in (2), the class of estimators l reduces to the estimator

$$l_1^{**} = w_z^2 \left(\frac{\gamma W_r^2 + \phi Q^2}{\gamma w_r^2 + \phi Q^2} \right), \quad (28)$$

which includes Solanki et al. [17] estimators l_9 to l_{14} .

Putting $(\alpha_1, \alpha_2) = (1, 0)$ in (8) and (10), we get the bias and MSE of l_1^{**} to the first degree of approximation (ignoring fpc term), respectively, as

$$B(l_1^{**}) = \left(\frac{W_z^2 \theta^*}{m} \right) (\theta^* \mu_{04}^* - \mu_{22}^*), \quad (29)$$

$$\text{MSE}(l_1^{**}) = W_z^4 [1 + d_1 - 2d_4], \quad (30)$$

$$= \left(\frac{W_z^4}{m} \right) (\mu_{40}^* + \theta^{*2} \mu_{04}^* - 2\theta^* \mu_{22}^*). \quad (31)$$

Special Case 4: if we set $(\alpha_1, \alpha_2, \gamma) = (\alpha_1, 0, 1)$ in (2), the class of estimators l reduces to the class of estimators of W_z^2 as

$$l_2^* = \alpha_1 w_z^2 \left(\frac{W_r^2 + \phi Q^2}{w_r^2 + \phi Q^2} \right). \quad (32)$$

Inserting $(\alpha_1, \alpha_2, \gamma) = (\alpha_1, 0, 1)$ in (8) and (10), we get the bias and MSE of l_2^* to the first degree of approximation (ignoring fpc term), respectively, as

$$B(l_2^*) = W_z^2 \left[\alpha_1 \left\{ 1 + \left(\frac{\theta}{m} \right) (\theta \mu_{04}^* - \mu_{22}^*) \right\} - 1 \right], \quad (33)$$

$$\text{MSE}(l_2^*) = W_z^4 [1 + \alpha_1^2 d_1^* - \alpha_1 d_4^*], \quad (34)$$

where

$$\begin{aligned} d_1^* &= \left[1 + \frac{1}{m} (\mu_{40}^* + 3\theta^2 \mu_{04}^* - 4\theta \mu_{22}^*) \right], \\ d_4^* &= \left[1 + \frac{\theta}{m} (\theta \mu_{04}^* - \mu_{22}^*) \right], \\ \theta &= \frac{W_r^2}{W_r^2 + \phi Q^2}. \end{aligned} \quad (35)$$

The MSE(l_2^*) at (34) is minimized for

$$\alpha_{10}^{**} = \frac{d_4^*}{d_1^*}. \quad (36)$$

Substitution of (36) in (34) yields the minimum MSE of l_2^* and is given by

$$\min \cdot \text{MSE}(l_2^*) = W_z^4 \left[1 - \frac{d_4^{*2}}{d_1^*} \right]. \quad (37)$$

Thus, we established the following corollary.

Corollary 2. To the first degree of approximation

$$\text{MSE}(l_2^*) \geq W_z^4 \left[1 - \frac{d_4^{*2}}{d_1^*} \right], \quad (38)$$

with equality holdings if

$$\alpha_1 = \alpha_{10}^{**} = \frac{d_4^*}{d_1^*}. \quad (39)$$

Special Case 5: for $(\alpha_1, \alpha_2, \gamma) = (1, 0, 1)$ in (2), the class of estimators “ l ” reduces to the class of estimators of W_z^2 as

$$l_2^{**} = w_z^2 \left(\frac{W_r^2 + \phi Q^2}{w_r^2 + \phi Q^2} \right), \quad (40)$$

which includes Singh et al. [5] estimators l_3 to l_8 .

Inserting $(\alpha_1, \alpha_2, \gamma) = (1, 0, 1)$ in (8) and (10), we get the bias and MSE of l_2^{**} to the first degree of approximation (ignoring fpc term), respectively, given by

$$l_2^{**} = \left(\frac{W_z^2 \theta^*}{m} \right) (\theta^* \mu_{04}^* - \mu_{22}^*), \quad (41)$$

$$\begin{aligned} \text{MSE}(l_2^{**}) &= W_z^4 [1 + d_1^* - 2d_4^*] \\ &= \left(\frac{W_z^4}{m} \right) (\mu_{40}^* + \theta^{*2} \mu_{04}^* - 2\theta \mu_{22}^*). \end{aligned} \quad (42)$$

4. Efficiency Comparison

To the first degree of approximation, the MSE of w_z^2 and l_2 are, respectively, given by

$$\text{MSE}(w_z^2) = \left(\frac{W_z^4}{m} \right) \mu_{40}^*, \quad (43)$$

$$\text{MSE}(l_2) = \left(\frac{W_z^4}{m} \right) (\mu_{40}^* + \mu_{04}^* + 2\mu_{22}^*). \quad (44)$$

From (20), (43), and (44), we have

$$\text{MSE}(w_z^2) - \min \cdot \text{MSE}(l^* \text{ or } l_d) = \left(\frac{W_z^4 \mu_{22}^{*2}}{m \mu_{04}^*} \right) > 0, \quad (45)$$

$$\text{MSE}(l_2) - \min \cdot \text{MSE}(l_2^*) = \frac{W_z^4 (\mu_{04}^* - \mu_{22}^*)^2}{m \mu_{04}^*} \geq 0. \quad (46)$$

It follows from (45) and (46) that the proposed class of estimators l^* or difference estimator l_d is better than the usual unbiased estimator w_z^2 and the ratio type estimator l_2 due to [1, 4].

From (37) and (42), we have

$$\text{MSE}(l_2^{**}) - \min \cdot \text{MSE}(l_2^*) = \frac{W_z^4 (d_1^* - d_4^*)^2}{d_1^*} \geq 0. \quad (47)$$

This follows that the proposed class of estimators l_2^* is more efficient than the proposed class of estimators l_2^{**} according to Singh et al. [5]. Thus, the estimator l_2^* is an improvement over the Singh et al. [5] estimators l_2^{**} .

TABLE 3: The parameters of population data set.

M	80	C_z	0.3542	Q_1	5.1500	TM	9.318
m	20	W_r	8.4563	Q_2	10.300	MR	10.405
$\bar{Z}$	51.8264	C_r	0.7507	Q_3	16.975	HL	17.955
$\bar{R}$	11.2646	μ_{04}	2.8664	Q_r	11.825	G	9.0408
ρ	0.9413	μ_{40}	2.2667	Q_d	5.9125	D	8.0138
W_z	18.3569	μ_{22}	2.2209	Q_a	11.0625	S_{pw}	7.9136

TABLE 4: PRE of estimators l_k ($k = 15, 16, \dots, 20$) with respect to $l_1 = w_z^2$.

ϕ	PRE (w_z^2)					
	l_{15}	l_{16}	l_{17}	l_{18}	l_{19}	l_{20}
-1.00	257.0391	259.3591	291.8213	258.2640	256.4163	253.0021
-0.75	255.2481	257.2658	275.3213	256.0011	253.9286	252.8256
-0.50	254.2359	256.9329	273.3492	255.3651	253.6165	252.4152
-0.25	253.9139	256.1231	273.1029	254.9606	253.0259	252.0081
0.00	253.1029	255.8569	273.0051	254.0436	252.8169	251.5621
0.25	252.9320	255.0091	272.9321	253.8186	252.0411	251.0561
0.50	253.1001	254.8563	272.3649	253.5646	251.9103	250.3961
0.75	254.2569	254.6561	272.1536	253.3963	251.3851	250.0011
1.00	254.2031	254.5341	271.9806	252.8646	251.0091	249.5651

From (25) and (30), we have

$$\text{MSE}(l_1^{**}) - \min \cdot \text{MSE}(l_1^*) = \frac{W_z^4 (d_1 - d_4)^2}{d_1} \geq 0. \quad (48)$$

Expression (48) clearly indicates that the proposed class of estimators l_1^* is better than class of estimator l_1^{**} recently

proposed by Solanki et al. [17]. Thus, the estimators l_1^* are an improvement over the estimator l_1^{**} due to Solanki et al. [17].

Furthermore, from (14) and (20), we have

$$\min \cdot \text{MSE}(l^*) - \min \cdot \text{MSE}(l) = \frac{W_z^4 [d_2 (d_1 - d_4) + d_3 (d_5 - d_3)]^2}{d_2 (d_1 d_2 - d_3^2)} > 0. \quad (49)$$

It follows from (49) that the proposed class of estimators " l^* " is better than the class of estimators l^* and the difference estimator l_d .

Next, from (14) and (25), we have

$$\min \cdot \text{MSE}(l_1^*) - \min \cdot \text{MSE}(l) = \frac{W_z^4 (d_1 d_5 - d_3 d_4)^2}{d_1 (d_1 d_2 - d_3^2)} \geq 0. \quad (50)$$

Thus, from (48) and (50), we have the following inequality:

$$\min \cdot \text{MSE}(l) \leq \min \cdot \text{MSE}(l_1^*) \leq \min \cdot \text{MSE}(l_1^{**}). \quad (51)$$

This clearly shows that the proposed family of estimators l is better than families of estimators l_1^* and l_1^{**} .

Finally, from the above theoretical comparisons we conclude that the proposed family of estimators l is better than the usual unbiased estimator $l_1 = w_z^2$, [4] ratio type estimator l_2 , and the estimators (l_3 to l_8) and (l_9 to l_{14}) recently proposed by Singh et al. [5] and Solanki et al. [17], respectively, and for other families of estimators (l_1^* , l_2^*).

5. Empirical Study

The performances of the proposed estimators l_k ($k = 15, 16, \dots, 20$), which are members of the suggested class of estimators, l , are evaluated against the usual unbiased estimators w_z^2 and the estimators l_i ($i = 2, 3, \dots, 14$) proposed by [1, 4, 5, 10] for the population data set (source: [18]) summarized in Table 3.

We have computed the percent relative efficiencies (PREs) of the suggested estimators l_k ($k = 15, 16, \dots, 20$) with respect to the usual unbiased estimator w_z^2 in the range of $\phi \in (-1.0, 1.0)$ by using the following formula:

$$\begin{aligned} \text{PRE}(l_k, w_z^2) &= \frac{\text{MSE}(w_z^2)}{\text{MSE}(l_k)} \times 100 \\ &= \frac{\mu_{40}^*}{m[1 - (d_1 d_4^2 - 2d_3 d_4 d_5 + (d_1 d_3^2 / d_1 d_2) - d_3^2)]} \times 100, \end{aligned} \quad (52)$$

and the findings are summarized in Table 4.

TABLE 5: PREs of the estimators $l_1 = w_z^2$, l_2 and l_i ($i = 3, \dots, 14$).

Estimator	Value of ϕ	PRE (w_z^2)
l_1	—	100.00
l_2	—	183.23
l_3	1.00	209.67
l_4	0.50	230.60
l_5	0.30	230.28
l_6	0.70	230.57
l_7	1.00	200.87
l_8	0.60	230.80
l_9	1.00	270.38
l_{10}	1.00	270.61
l_{11}	1.00	263.50
l_{12}	0.80	270.58
l_{13}	1.00	270.61
l_{14}	0.30	270.58

For the purpose of comparison of the proposed estimators l_{15} to l_{20} with that of the usual unbiased estimator $l_1 = w_z^2$, [4] estimator l_2 , [5] estimators (l_3 to l_8), and [17] estimators (l_9 to l_{14}), in Table 5, we give the PREs of the estimators l_1 , l_2 and the PREs of the estimators l_3 to l_{14} with respect to usual unbiased estimator w_z^2 along with the value of δ for which the PREs of the estimators l_3 to l_{14} are maximum as given in [17].

It is observed from Tables 4 and 5 that all the estimators l_k ($k = 15, 16, \dots, 20$) which are the members of the proposed class of estimators l performed better than the usual unbiased estimator $l_1 = w_z^2$, ratio type estimators l_2 due to [4, 5] estimators (l_3 to l_8) and [17] estimators (l_9 to l_{14}), for $\phi \in (-1.0, 1.0)$.

6. Conclusion

In the present paper, we have suggested an improved class of estimators for population variance using the auxiliary information of nonconventional location parameters, nonconventional measures of dispersion, and their function with the coefficient of variation. The bias and mean square error (MSE) expressions of the proposed class of estimators are obtained and compared with the usual unbiased estimator, estimators in [4], [5], and [17]. We have also analyzed the performance of the proposed estimators by utilizing the data set of the known population and found that in convenient cases, the proposed estimators perform better than the other existing estimators. On comparison with the usual unbiased estimator, which is shown in Table 4, it is evident that upon increasing the values of ϕ either negative or positive, the percent relative efficiency also increases. From Table 5, we observe that our estimators are more proficient than the existing ones. Hence, we strongly recommend that our proposed estimators perform better than the existing estimators for use in practical applications.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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