Research Article

Some Hardy-Type Inequalities for Superquadratic Functions via Delta Fractional Integrals

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In this paper, Jensen and Hardy inequalities, including Pólya–Knopp type inequalities for superquadratic functions, are extended using Riemann–Liouville delta fractional integrals. Furthermore, some inequalities are proved by using special kernels. Particular cases of obtained inequalities give us the results on time scales calculus, fractional calculus, discrete fractional calculus, and quantum fractional calculus.

1. Introduction

The study of Hardy inequalities has gained huge attention in the literature, and now it became a major field in applied and pure mathematics. The Hardy inequality has a long history and many variants. Together with the Sobolev inequalities, it is one of the most frequently used inequalities in the analysis. Firstly, Hardy inequality was discovered to simplify the proof of another inequality. It was then studied in its own right and acquired several useful variants, and it eventually turned out to be extremely useful in the theory of partial differential equations. Except for the direct application of Hardy's inequality to the Schrodinger operator, other useful variants have been successfully developed for applications in other areas of physics [1].

Hardy-type inequalities have crucial importance in the study of function spaces, especially of fractional regularity [2]. Another fundamental consequence is the trace theory of weighted Sobolev spaces; in turn, weighted Sobolev spaces

are useful in the regularity theory of the superposition operators [2]. The familiar Hardy inequality (both in the discrete and continuous settings) as presented in [3] has been enormously studied and used as a model for the inquisition of more general integral inequalities [4, 5].

In [6], Hardy proved the discrete inequality,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n a(i) \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a^p(n), \quad p > 1, \quad (1)$$

where $a(n) \geq 0$ for $n \geq 1$. Also, in [7], Hardy proved the continuous inequality, using the calculus of variation, which states that for nonnegative, integrable function over any finite interval $(0, x)$ and f^p is integrable for $p > 1$,

$$\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x) dx, \quad (2)$$

where the constant $(p/(p-1))^p$ is best possible.

Hardy inequality (2) has been generalized by G. H. Hardy himself in [8]. There he showed that for any integral function $f(x) > 0$ on $(0, \infty)$, $p > 1$,

$$\int_0^\infty \frac{1}{x^m} \left(\int_x^\infty f(t) dt \right)^p dx \leq \left(\frac{p}{1-m} \right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x) dx, \quad m < 1, \quad (3)$$

$$\int_0^\infty \frac{1}{x^m} \left(\int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{m-1} \right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x) dx, \quad m > 1. \quad (4)$$

Littlewood and Hardy [9] established the discrete version of (3) and (4).

Since the discovery of these two inequalities, various papers which deal with new proofs, generalizations, and extensions have appeared in the literature, [6, 7, 10–12] are referred to readers. During the last decades, these inequalities were extended, a time scale version of weighted Hardy-type inequality is proved in [11], and some preliminary dynamic inequalities [13] are proved in time scales calculus. In [14, 15], some classical inequalities are proved for isotonic linear functionals and superquadratic functions.

Recently, many new Hardy inequalities have been proved, including Hardy and Rellich inequalities for Bessel pairs [16], Hardy–Sobolev inequalities on hypersurfaces for Euclidean space [17], improved Hardy inequalities with exact remainder terms [18], Hardy inequalities with double singular weights [19], and Hardy inequalities for class functions in one-dimensional fractional Orlicz–Sobolev spaces [20]. A new approach for the fractional integral operator in time scales with variable exponent Lebesgue spaces is presented in [21] and n -dimensional integral-type inequalities via time scale calculus is studied in [22].

2. Preliminaries

2.1. Some Basics on Time Scales. A time scale is a closed subset of the real line $\mathbb{R}$, and its common notation is $\mathbb{T}$. A time scale $\mathbb{T}$ may or may not be connected. Forward jump operator and backward jump operator are respectively defined as

$$\begin{aligned} \sigma(\omega) &= \inf\{s \in \mathbb{T} : s > \omega\}, \\ \rho(\omega) &= \sup\{s \in \mathbb{T} : s < \omega\}. \end{aligned} \quad (5)$$

In general, $\sigma(\omega) \geq \omega$ and $\rho(\omega) \leq \omega$. The forward and backward graininess functions $\mu, \nu: \mathbb{T} \rightarrow [0, +\infty)$ are, respectively, defined by

$$\begin{aligned} \mu(\omega) &= \sigma(\omega) - \omega, \\ \nu(\omega) &= \omega - \rho(\omega). \end{aligned} \quad (6)$$

If $\sigma(\omega) > \omega$, then ω is said to be right scattered. If $\rho(\omega) < \omega$, then ω is said to be left scattered. Points that are left-scattered and right scattered at the same time are called isolated. If $\omega < \sup \mathbb{T}$ and $\sigma(\omega) = \omega$, then ω is called the right-dense and if $\omega > \inf \mathbb{T}$ and $\rho(\omega) = \omega$, then ω is called

left-dense. Points that are left-dense and right-dense at the same time are called dense.

Rd-continuous function: a function $h: \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$.

Notation: if $\mathbb{T}$ is a time scale, then the set $\mathbb{T}^k$ is defined as

$$\mathbb{T}^k = \begin{cases} \frac{\mathbb{T}}{(\rho \sup \mathbb{T}, \sup \mathbb{T}]}, & \sup \mathbb{T} < \infty; \\ \mathbb{T}, & \sup \mathbb{T} = \infty. \end{cases} \quad (7)$$

Delta derivatives: take a function $h: \mathbb{T} \rightarrow \mathbb{R}$. The delta derivative (Hilger derivative) $h^\Delta(\xi)$ exists iff, for every $\mathfrak{F} > 0$, there exists a neighborhood $\mathfrak{N}$ of ξ such that

$$|h(\sigma(\xi)) - h(s) - h^\Delta(\xi)(\sigma(\xi) - s)| \leq \mathfrak{F}|\sigma(\xi) - s|, \quad (8)$$

for all $s \in \mathfrak{N}$, $\xi \in \mathbb{T}$.

2.2. Some Principles of Fractional Calculus

2.2.1. Riemann–Liouville Derivative. Riemann–Liouville derivative is the generalization of the usual derivative. It is based on Cauchy's formula for calculating iterated integrals, if the first integral of a function, which must be equal to deriving it to -1 , is as follows:

$$D^{-1}h(\vartheta) = \int_0^\vartheta h(\omega) d\omega. \quad (9)$$

The calculation of the second can be simplified by interchanging the integration order,

$$\begin{aligned} D^{-2}h(\vartheta) &= \int_0^\vartheta \int_0^{\omega_2} h(\omega_1) d\omega_1 d\omega_2 \\ &= \int_0^\vartheta \int_{\omega_1}^\vartheta h(\omega_1) d\omega_2 d\omega_1 \\ &= \int_0^\vartheta h(\omega_1) \int_{\omega_1}^\vartheta d\omega_2 d\omega_1 \\ &= \int_0^\vartheta h(\omega) (\vartheta - \omega) d\omega. \end{aligned} \quad (10)$$

This method can be applied repeatedly, resulting in the following formula for calculating iterated integrals,

$$D^{-n}h(\vartheta) = \frac{1}{(n-1)!} \int_0^\vartheta h(\omega) (\vartheta - \omega)^{n-1} d\omega. \quad (11)$$

Now, this can be easily generalized to noninteger values, in what is the Riemann–Liouville derivative [23],

$$D^\gamma h(\vartheta) = \frac{1}{\Gamma(-\gamma)} \int_0^\vartheta \frac{h(\omega)}{(\vartheta - \omega)^{\gamma+1}} d\omega. \quad (12)$$

The Riemann–Liouville derivative with the lower integration limit a would be [23]

$${}_a D_{\vartheta}^{\gamma} h(\vartheta) = \frac{1}{\Gamma(-\gamma)} \int_a^{\vartheta} \frac{h(\omega)}{(\vartheta - \omega)^{\gamma+1}} d\omega. \quad (13)$$

2.2.2. Riemann–Liouville Integral. The Riemann–Liouville integral relates with a real function $h: \mathbb{R} \rightarrow \mathbb{R}$, another same kind of function $I^{\gamma}h$ for each value of the parameter $\gamma > 0$. The integral is a generalization of the repeated anti-derivative of h of order real number γ .

Let $J = [a, b]$ ($-\infty < a < b < \infty$) be a finite interval of $\mathbb{R}$. The left and right Riemann–Liouville fractional integrals ${}_a D_{\omega}^{-\gamma} h$ and ${}_b D_b^{-\gamma} h$ of order $\gamma \in \mathbb{R}^+$ are defined in [23] by

$${}_a D_{\omega}^{-\gamma} h(\omega) = \frac{1}{\Gamma(\gamma)} \int_a^{\omega} (\omega - s)^{\gamma-1} h(s) ds, \quad \omega > a, \gamma > 0, \quad (14)$$

$${}_b D_b^{-\gamma} h(\omega) = \frac{1}{\Gamma(\gamma)} \int_{\omega}^b (s - \omega)^{\gamma-1} h(s) ds, \quad \omega < b, \gamma > 0, \quad (15)$$

respectively, provided the right-hand sides are pointwise defined on $[a, b]$. When $\gamma = n \in \mathbb{N}^+$, definitions (14) and (15) coincide with the n th integrals of the form,

$$\begin{aligned} {}_a D_{\omega}^{-n} h(\omega) &= \frac{1}{(n-1)!} \int_a^{\omega} (\omega - s)^{n-1} h(s) ds, \\ {}_b D_b^{-n} h(\omega) &= \frac{1}{(n-1)!} \int_{\omega}^b (s - \omega)^{n-1} h(s) ds. \end{aligned} \quad (16)$$

2.3. Some Principles of Delta Fractional Calculus. For $\mu, \nu > 0$, one has that (see [24], p. 256)

$$\int_u^{\xi} \left(\frac{(\xi - r)^{\ell-1}}{\Gamma(\ell)} \right) \left(\frac{(r - u)^{\mathcal{J}-1}}{\Gamma(\mathcal{J})} \right) dr = \frac{(\xi - u)^{\ell+\mathcal{J}-1}}{\Gamma(\ell + \mathcal{J})}, \quad (17)$$

where Γ is the Gamma function. If time scale $\mathbb{T}$ is considered to be $\mathbb{T} = \mathbb{T}^k$, then the coordinate-wise rd-continuous functions are given by

$$h_{\gamma}: \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}, \quad \gamma \geq 0, \quad (18)$$

such that $h_0(\xi, r) = 1, \forall \xi, r \in \mathbb{T}$ and

$$h_{\gamma+1}(\xi, r) = \int_r^{\xi} h_{\gamma}(y, r) \Delta y, \quad \forall \xi, r, y \in \mathbb{T}. \quad (19)$$

Furthermore, for $\gamma, \beta > 1$, it is assumed that

$$\int_{\sigma(r)}^{\xi} h_{\gamma-1}(\xi, \sigma(s)) h_{\beta-1}(s, \sigma(r)) \Delta s = h_{\gamma+\beta-1}(\xi, \sigma(r)), \quad \forall r < \xi; r, \xi \in \mathbb{T}. \quad (20)$$

In the case $\mathbb{T} = \mathbb{R}$ and for $\sigma(r) = r$, one obtains

$$\begin{aligned} h_{\kappa}(r, \xi) &= \frac{(r - \xi)^{\kappa}}{\kappa!}, \quad \kappa \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \\ h_{\gamma}(r, \xi) &= \frac{(r - \xi)^{\gamma}}{\Gamma(\gamma + 1)}, \quad \gamma \geq 0, \end{aligned} \quad (21)$$

and also

$$\int_r^{\xi} \frac{(\xi - r)^{\gamma}}{\Gamma(\gamma + 1)} d\xi = \frac{(\xi - r)^{\gamma+1}}{\Gamma(\gamma + 2)} = h_{\gamma+1}(\xi, r), \quad (22)$$

satisfying (19). Furthermore, it is observed that for $\gamma, \beta > 1$ and by using (17), one gets

$$\begin{aligned} &\int_r^{\varsigma} h_{\gamma-1}(\varsigma, \xi) h_{\beta-1}(\xi, \omega) d\xi \\ &= \int_r^{\varsigma} \frac{(\varsigma - \xi)^{\gamma-1}}{\Gamma(\gamma)} \frac{(\xi - \omega)^{\beta-1}}{\Gamma(\beta)} d\xi \\ &= \frac{(\varsigma - \omega)^{\gamma+\beta-1}}{\Gamma(\gamma + \beta)} \\ &= h_{\gamma+\beta-1}(\varsigma, \omega), \end{aligned} \quad (23)$$

which is satisfying (20).

If $g \in C_{rd}$ and $x \in \mathbb{T}^k$ ([25], Chap: 1), then

$$\int_x^{\sigma(x)} g(\varsigma) \Delta \varsigma = \mu(x) g(x). \quad (24)$$

2.3.1. Riemann–Liouville Delta Fractional Integral. For $\gamma \geq 1$, the Δ -Riemann–Liouville type fractional integral is defined in [26] as

$$K_a^{\gamma} \phi(\omega) = \int_a^{\omega} h_{\gamma-1}(\omega, \sigma(\nu)) \phi(\nu) \Delta \nu, \quad (25)$$

for $[a, b] \in \mathbb{T}$ and $\omega \in [a, b] \cap \mathbb{T}$. Notice that $K_a^0 \phi = \phi$, where $\phi \in L_1([a, b] \cap \mathbb{T})$ (Lebesgue Δ -integrable functions on $[a, b] \cap \mathbb{T}$, [27]).

2.3.2. Fubini's Theorem on Time Scale. Consider $r, l \in \mathbb{T}$ and $\phi \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$. Then,

$$\int_r^l \int_{\eta}^l \phi(\eta, \zeta) \Delta \zeta \Delta \eta = \int_r^l \int_r^{\sigma(\zeta)} \phi(\eta, \zeta) \Delta \eta \Delta \zeta. \quad (26)$$

It can be found in [28].

2.4. Superquadratic Function

Definition 1. A function $\Theta: [0, \infty) \rightarrow \mathbb{R}$ is called superquadratic 29 provided that, for all $\omega \geq 0$, there exists a constant $C(\omega) \in \mathbb{R}$ such that

$$\Theta(\nu) - \Theta(\omega) - \Theta(|\nu - \omega|) \geq C(\omega)(\nu - \omega), \quad (27)$$

for all $\nu \geq 0$. We say that ϕ is subquadratic if $-\phi$ is a superquadratic function.

The following lemma (see [29]) shows that every positive superquadratic function is also a convex function.

Lemma 1. *Let Θ be a superquadratic function with $C(\omega)$ as in Definition 1. Then,*

- (i) $\Theta(0) \leq 0$.
- (ii) If $\Theta(0) = \Theta'(0) = 0$, then $C(\omega) = \Theta'(\omega)$, where Θ is differentiable at $\omega > 0$.
- (iii) If $\Theta \geq 0$, then Θ is convex and $\Theta(0) = \Theta'(0) = 0$.

3. Jensen Inequality for Superquadratic Functions via Delta Fractional Integrals

Theorem 1. *Let $h_{\gamma-1}: \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ where $\gamma \geq 1$ and let $\vartheta_1 \in \mathbb{T}$. If $Y_1 = [\lambda_1, \lambda_2] \cap \mathbb{T}$ with $\lambda_1, \lambda_2 \in T$ and $\phi: Y_1 \rightarrow [0, \infty)$ is rd-continuous, then for every continuous superquadratic function $\Theta: [0, \infty) \rightarrow \mathbb{R}$,*

$$\Theta\left(\frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}\right) \leq \frac{1}{F(\vartheta_1)} \int_{\lambda_1}^{\vartheta_1} \left(\Theta(\phi(s)) - \Theta\left(\left|\phi(s) - \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(s)}\right|\right) \right) \Delta s, \quad (28)$$

where

Proof. Let $\Theta: [0, \infty) \rightarrow \mathbb{R}$ be a superquadratic function and $\vartheta_0 \in [0, \infty)$. According to (27), there is a constant $C(\vartheta_0)$ such that

$$\Theta(\vartheta_2) \geq \Theta(\vartheta_0) + C(\vartheta_0)(\vartheta_2 - \vartheta_0) + \Theta(|\vartheta_2 - \vartheta_0|). \quad (30)$$

Since ϕ is rd-continuous,

$$\vartheta_0 = \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}, \quad (31)$$

is well defined. The function $\Theta \circ \phi$ is also rd-continuous, so we may apply (30) with $\vartheta_2 = \phi(s)$ and (31) to obtain

$$\begin{aligned} \Theta(\phi(s)) &\geq \Theta\left(\frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}\right) + C(\vartheta_0)\left(\phi(s) - \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}\right) \\ &\quad + \Theta\left(\left|\phi(s) - \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}\right|\right). \end{aligned} \quad (32)$$

Integrating from λ_1 to ϑ_1 , we get

$$\begin{aligned} &\int_{\lambda_1}^{\vartheta_1} h_{\gamma-1}(\vartheta_1, \sigma(s)) \Theta(\phi(s)) \Delta s - \int_{\lambda_1}^{\vartheta_1} h_{\gamma-1}(\vartheta_1, \sigma(s)) \Theta\left(\left|\phi(s) - \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}\right|\right) \Delta s \\ &\quad - \int_{\lambda_1}^{\vartheta_1} h_{\gamma-1}(\vartheta_1, \sigma(s)) \Theta\left(\frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}\right) \Delta s \\ &\geq C(\vartheta_0) \int_{\lambda_1}^{\vartheta_1} h_{\gamma-1}(\vartheta_1, \sigma(s)) \left(\phi(s) - \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)}\right) \Delta s \\ &= C(\vartheta_0) \left(K_{\lambda_1}^\gamma \phi(\vartheta_1) - \int_{\lambda_1}^{\vartheta_1} h_{\gamma-1}(\vartheta_1, \sigma(s)) \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)} \Delta s \right) \\ &= C(\vartheta_0) \left(K_{\lambda_1}^\gamma \phi(\vartheta_1) - F(\vartheta_1) \cdot \frac{K_{\lambda_1}^\gamma \phi(\vartheta_1)}{F(\vartheta_1)} \right) \\ &= C(\vartheta_0) (K_{\lambda_1}^\gamma \phi(\vartheta_1) - K_{\lambda_1}^\gamma \phi(\vartheta_1)) = 0. \end{aligned} \quad (33)$$

□

Remark 1. Since Θ is nonnegative superquadratic function, therefore (by Lemma 1) a convex one too, the result of (28) refines the result given in [30] (Theorem 1) and by assuming $\gamma = 1$, (28) becomes [31] (Theorem 2.5), and

result of [31] (Theorem 2.5) refines the result given in [31] (Theorem 2.4). Furthermore, respective inequality is obtained in fractional calculus ([29], Theorem 2.3) by assuming $\gamma = 1$ and $\mathbb{T} = \mathbb{R}$.

Example 1. If $\mathbb{T} = \mathbb{N}$, then $\sigma(\vartheta_2) = \vartheta_2 + 1$, $\forall \vartheta_2 \in \mathbb{N}$, and

$$h_{\gamma-1}(\vartheta_1, \sigma(\vartheta_2)) = \frac{(\vartheta_1 - \sigma(\vartheta_2))^{\gamma-1}}{(\gamma-1)!} = \frac{(\vartheta_1 - \vartheta_2 - 1)^{\gamma-1}}{(\gamma-1)!}. \quad (34)$$

By substituting (28) and (34), it becomes

$$\Theta\left(\frac{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - 1)^{\gamma-1} \phi(\vartheta_2)}{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - 1)^{\gamma-1}}\right) \leq \frac{(\gamma-1)!}{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - 1)^{\gamma-1}} \sum_{s=\lambda_1}^{\vartheta_1-1} \left(\Theta(\phi(s)) - \Theta\left(\left|\phi(s) - \frac{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - 1)^{\gamma-1} \phi(\vartheta_2)}{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - 1)^{\gamma-1}}\right|\right) \right). \quad (35)$$

Example 2. If $\mathbb{T} = q^{\mathbb{Z}}$, then $\sigma(\vartheta_2) = q\vartheta_2$. In this case,

$$h_{\gamma-1}(\vartheta_1, \sigma(\vartheta_2)) = \prod_{v=0}^{\gamma-2} \frac{\vartheta_1 - q^v \sigma(\vartheta_2)}{\sum_{\mu=0}^v q^\mu} = \prod_{v=0}^{\gamma-2} \frac{\vartheta_1 - q^{v+1} \vartheta_2}{\sum_{\mu=0}^v q^\mu}. \quad (36)$$

By substituting (36) in (28), one gets

$$\Theta\left(\frac{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} \left(\prod_{v=0}^{\gamma-2} (\vartheta_1 - q^{v+1} \vartheta_2 / \sum_{\mu=0}^v q^\mu)\right) \phi(\vartheta_2)}{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} \left(\prod_{v=0}^{\gamma-2} (\vartheta_1 - q^{v+1} \vartheta_2 / \sum_{\mu=0}^v q^\mu)\right)}\right) \leq \frac{1}{\prod_{v=0}^{\gamma-2} (\vartheta_1 - q^{v+1} \vartheta_2 / \sum_{\mu=0}^v q^\mu)} \sum_{s=\lambda_1}^{\vartheta_1-1} \left(\Theta(\phi(s)) - \Theta\left(\left|\phi(s) - \frac{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} \left(\prod_{v=0}^{\gamma-2} (\vartheta_1 - q^{v+1} \vartheta_2 / \sum_{\mu=0}^v q^\mu)\right) \phi(\vartheta_2)}{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} \left(\prod_{v=0}^{\gamma-2} (\vartheta_1 - q^{v+1} \vartheta_2 / \sum_{\mu=0}^v q^\mu)\right)}\right|\right) \right). \quad (37)$$

4. Hardy Inequalities for Superquadratic Functions with General Kernels

$\lambda_1, \lambda_2, \eta_1, \eta_2 \in \mathbb{T}$, and $\vartheta_1 \in \mathbb{T}^\kappa, s_1, s_2 \in \mathbb{T}$ such that $\lambda_1 < s_1 < \lambda_2$ and $\eta_1 < s_2 < \eta_2$. Define

Theorem 2. Assume $h_{\gamma-1}: \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ where $\gamma \geq 1, \beta \geq 1$, and $Y_1 = [\lambda_1, \lambda_2] \cap \mathbb{T}$ and $Y_2 = [\eta_1, \eta_2] \cap \mathbb{T}$ where

$$A_\kappa(\phi(\vartheta_1)) = \frac{1}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2 - \int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2 \right), \quad (38)$$

where $\phi: Y_1 \rightarrow \mathbb{R}$ is a λ_Δ -measurable function, $\kappa: Y_2 \times Y_1 \rightarrow \mathbb{R}$ is a nonnegative kernel, and

$$\chi(\vartheta_1) = \left(\int_{\lambda_1}^{\vartheta_1} h_{\gamma-1}(\vartheta_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Delta \vartheta_2 - \int_{\lambda_2}^{\vartheta_1} h_{\gamma-1}(\vartheta_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Delta \vartheta_2 \right) < \infty, \quad \vartheta_1 \in Y_2. \quad (39)$$

Let $\zeta: Y_2 \rightarrow \mathbb{R}_+$ be μ_Δ -integrable function. Denote

$$w(\vartheta_2) = \int_{\eta_1}^{\eta_2} \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1, \quad \vartheta_2 \in Y_1. \quad (40)$$

Further assume $I = [a, c], 0 \leq a < c \leq \infty$, and $\Theta: I \rightarrow \mathbb{R}$. If $\Theta \in C(I, \mathbb{R})$ is superquadratic, then

$$\begin{aligned}
& \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \Theta(A_\kappa(\phi(\vartheta_1))) \zeta(\vartheta_1) \Delta \vartheta_1 \\
& + \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \frac{\zeta(\vartheta_1) \kappa(\vartheta_1, \vartheta_2)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
& \leq \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \Theta(\phi(\vartheta_1)) w(\vartheta_2) \Delta \vartheta_2,
\end{aligned} \tag{41}$$

holds for all λ_Δ -integrable $\phi: Y_1 \longrightarrow \mathbb{R}$ such that $\phi(Y_1) \subset I$. *Proof.* Consider
 If Θ is subquadratic, then inequality (41) is reversed.

$$\begin{aligned}
& \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \Theta(A_\kappa(\phi(\vartheta_1))) \zeta(\vartheta_1) \Delta \vartheta_1 \\
& = \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \zeta(\vartheta_1) \times \Theta \left(\frac{1}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2 - \int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2 \right) \right) \Delta \vartheta_1 \\
& = \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \zeta(\vartheta_1) \Theta \left(\frac{1}{\chi(\vartheta_1)} \int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \quad - \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \zeta(\vartheta_1) \Theta \left(\frac{1}{\chi(\vartheta_1)} \int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2 \right) \Delta \vartheta_1.
\end{aligned} \tag{42}$$

By using Jensen's inequality (28),

$$\begin{aligned}
& \leq \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(\phi(\vartheta_2)) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \quad - \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \quad - \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(\phi(\vartheta_2)) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \quad + \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \leq \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(\phi(\vartheta_2)) \Delta \vartheta_2 - \int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(\phi(\vartheta_2)) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \quad - \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \Delta \vartheta_2 \right. \\
& \quad \left. - \int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \leq \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(\phi(\vartheta_2)) \Delta \vartheta_2 \right) \Delta \vartheta_1 \\
& \quad - \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \frac{\zeta(\vartheta_1)}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \Delta \vartheta_2 \right) \Delta \vartheta_1.
\end{aligned} \tag{43}$$

By using Fubini's theorem,

$$\begin{aligned}
&= \int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \Theta(\phi(\vartheta_2)) \left(\int_{\eta_1}^{\eta_2} \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
&\quad - \int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
&\quad - \int_{\lambda_1}^{s_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \Theta(\phi(\vartheta_2)) \left(\int_{\eta_1}^{\eta_2} \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
&\quad + \int_{\lambda_1}^{s_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
&= \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \Theta(\phi(\vartheta_2)) \left(\int_{\eta_1}^{\eta_2} \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
&\quad - \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
&= \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \Theta(\phi(\vartheta_2)) w(\vartheta_2) \Delta \vartheta_2 \\
&\quad - \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} \Theta(|\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|) \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2.
\end{aligned} \tag{44}$$

Remark 2. If the function Θ is subquadratic, then Jensen's inequality for superquadratic functions on delta fractional integrals is reversed, which implies, according to the conclusions made above, that inequality sign in (41) is reversed.

In [30], authors prove the analogous results for convex functions; now, it is easy to see that when Θ in (41) is nonnegative, then inequality (41) gives the refinement of [30] (Theorem 2). By assuming $\gamma, \beta = 1$, we obtain the

inequality (41) in time scales ([32], Theorem 2.1), when ϕ is nonnegative in [32] (Theorem 2.1), then [32] (Theorem 2.1) gives the refinement of [27] (Theorem 3.2). Moreover, the inequality (41) is obtained in fractional calculus [33] by assuming $\mathbb{T} = \mathbb{R}$ and $\gamma, \beta = 1$. □

Corollary 1. Assume (38)–(40). If $p \geq 2$, then

$$\begin{aligned}
&\int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) A_\kappa^p(\phi(\vartheta_2)) \zeta(\vartheta_1) \Delta \vartheta_1 \\
&\quad + \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} |\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|^p \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
&\leq \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \phi^p(\vartheta_2) w(\vartheta_2) \Delta \vartheta_2.
\end{aligned} \tag{45}$$

holds for all λ_Δ -integrable $\phi: Y_1 \longrightarrow \mathbb{R}$ such that $\phi(Y_1) \subset I$.

If $0 \leq p \leq 2$, then (45) holds in reverse direction.

Proof. Use $\Theta(r) = r^p$ in Theorem 2. □

Remark 3. We obtain the respective results in time scales ([32], Corollary 2.3) by assuming $\gamma, \beta = 1$ in (45).

Remark 4. In particular, if $p = 2$ in Corollary 4.2, we get the following identity:

$$\begin{aligned}
& \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) A_\kappa^2(\phi(\vartheta_2)) \zeta(\vartheta_1) \Delta \vartheta_1 \\
& + \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} |\phi(\vartheta_2) - A_\kappa(\phi(\vartheta_1))|^2 \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\
& = \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \phi^2(\vartheta_2) w(\vartheta_2) \Delta \vartheta_2.
\end{aligned} \tag{46}$$

Corollary 2. Assume (38)–(40), then

$$\begin{aligned}
& \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) e^{A_\kappa(h(\vartheta_1))} \zeta(\vartheta_1) \Delta \vartheta_1 + I \\
& \leq \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) h(\vartheta_2) w(\vartheta_2) \Delta \vartheta_2,
\end{aligned} \tag{47}$$

holds for all λ_Δ -integrable $h: Y_1 \longrightarrow \mathbb{R}$ such that $\phi(Y_1) \subset I$ with

$$\begin{aligned}
A_\kappa(h(\vartheta_1)) &= \frac{1}{\chi(\vartheta_1)} \left(\int_{\lambda_1}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \ln h(\vartheta_2) \Delta \vartheta_2 - \int_{\lambda_2}^{s_1} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) \kappa(\vartheta_1, \vartheta_2) \ln h(\vartheta_2) \Delta \vartheta_2 \right), \\
I &= \int_{\lambda_1}^{\lambda_2} \left[h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \times \left(\int_{\eta_1}^{\eta_2} \left(e^{|\ln h(\vartheta_2) - A_\kappa(h(\vartheta_1))|} - |\ln h(\vartheta_2) - A_\kappa(h(\vartheta_1))| \right) \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\chi(\vartheta_1)} \Delta \vartheta_1 \right) \right] \\
& \Delta \vartheta_2 + \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \ln h(\vartheta_2) w(\vartheta_2) \Delta \vartheta_2 - \int_{\eta_1}^{\eta_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) (1 + A_\kappa(h(\vartheta_1))) \zeta(\vartheta_1) \Delta \vartheta_1.
\end{aligned} \tag{48}$$

Proof. Use $\Theta(r) = e^r - r - 1$ and $\phi(r) = \ln h(r)$ in Theorem 2. $\square$

Remark 5. By choosing $\gamma, \beta = 1$ in (47), the respective result is obtained in [32] (Corollary 2.5).

Corollary 3. Assume (38)–(40) and further if $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned}
& \int_{\eta_1}^{\eta_2} \frac{(s_2 - \vartheta_1)^{\beta-1}}{(\beta-1)!} \Theta \left(\frac{\int_{\lambda_1}^{\lambda_2} (s_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} \right) \zeta(\vartheta_1) d\vartheta_1 \\
& + \int_{\lambda_1}^{\lambda_2} \left(\frac{(s_1 - \vartheta_2)^{\gamma-1} (s_2 - \vartheta_1)^{\beta-1}}{(\beta-1)!} \times \left(\int_{\eta_1}^{\eta_2} \Theta \left(\left| \phi(\vartheta_2) - \frac{\int_{\lambda_1}^{\lambda_2} (s_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} \right| \right) \right) \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} d\vartheta_1 \right) d\vartheta_2 \\
& \leq \int_{\lambda_1}^{\lambda_2} \frac{(s_1 - \vartheta_2)^{\gamma-1} (s_2 - \vartheta_1)^{\beta-1}}{(\beta-1)!} \Theta(\phi(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} d\vartheta_1 \right) d\vartheta_2.
\end{aligned} \tag{49}$$

Proof. If $\mathbb{T} = \mathbb{R}$, then $\sigma(\vartheta_2) = \vartheta_2, \forall \vartheta_2 \in \mathbb{R}$. In this case,

$$h_{\gamma-1}(s_1, \sigma(\vartheta_2)) = \frac{(s_1 - \sigma(\vartheta_2))^{\gamma-1}}{(\gamma-1)!} = \frac{(s_1 - \vartheta_2)^{\gamma-1}}{(\gamma-1)!}, \quad (50)$$

$$h_{\beta-1}(s_2, \sigma(\vartheta_1)) = \frac{(s_2 - \sigma(\vartheta_1))^{\beta-1}}{(\beta-1)!} = \frac{(s_2 - \vartheta_1)^{\beta-1}}{(\beta-1)!}. \quad (51)$$

Substituting (50) and (51) in (41), one gets

$$\begin{aligned} & \int_{\eta_1}^{\eta_2} \frac{(s_2 - \vartheta_1)^{\beta-1}}{(\beta-1)!} \Theta \left(\frac{\int_{\lambda_1}^{\lambda_2} (s_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} \right) \zeta(\vartheta_1) d\vartheta_1 \\ & + \int_{\lambda_1}^{\lambda_2} \frac{(s_1 - \vartheta_2)^{\gamma-1} (s_2 - \vartheta_1)^{\beta-1}}{(\beta-1)!} \times \left(\int_{\eta_1}^{\eta_2} \Theta \left(\left| \phi(\vartheta_2) - \frac{\int_{\lambda_1}^{\lambda_2} (s_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2) \Delta \vartheta_2}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} \right| \right) \right. \\ & \quad \left. \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} d\vartheta_1 \right) d\vartheta_2 \\ & \leq \int_{\lambda_1}^{\lambda_2} \frac{(s_1 - \vartheta_2)^{\gamma-1} (s_2 - \vartheta_1)^{\beta-1}}{(\beta-1)!} \Theta(\phi(\vartheta_1)) \left(\int_{\eta_1}^{\eta_2} \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\int_{\lambda_1}^{\lambda_2} (\vartheta_1 - \vartheta_2)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) d\vartheta_2} d\vartheta_1 \right) d\vartheta_2. \end{aligned} \quad (52)$$

5. Inequalities with Special Kernels

$$\kappa(\vartheta_1, \vartheta_2) = \begin{cases} 1, & \text{if } \lambda_1 \leq \vartheta_2 < \sigma(\vartheta_1) \leq \lambda_2, \\ 0, & \text{otherwise,} \end{cases} \quad \square \quad (53)$$

Corollary 4. Assume (38)–(40) with kernel κ such that *then*

$$\begin{aligned} & \int_{\lambda_1}^{\lambda_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\frac{K_{\lambda_1}^{\gamma} \phi(\sigma(\vartheta_1))}{F(\sigma(\vartheta_1))} \right)^p \zeta(\vartheta_1) \Delta \vartheta_1 \\ & + \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\vartheta_2}^{\lambda_2} \left| \phi(\vartheta_2) - \frac{K_{\lambda_1}^{\gamma} \phi(\sigma(\vartheta_1))}{F(\sigma(\vartheta_1))} \right|^p \frac{\zeta(\vartheta_1)}{F(\sigma(\vartheta_1))} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\ & \leq \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \phi^p(\vartheta_2) \left(\int_{\vartheta_2}^{\lambda_2} \frac{\zeta(\vartheta_1)}{F(\sigma(\vartheta_1))} \Delta \vartheta_1 \right) \Delta \vartheta_2, \end{aligned} \quad (54)$$

holds for all λ_{Δ} -integrable $h: Y_1 \longrightarrow \mathbb{R}$ such that $\phi(Y_1) \subset I$.

$$\chi(\vartheta_1) = \int_{\lambda_1}^{\sigma(\vartheta_1)} h_{\gamma-1}(\sigma(\vartheta_1), \sigma(\vartheta_2)) \Delta \vartheta_2 = F(\sigma(\vartheta_1)), \quad (55)$$

Proof. By using (53) in (38)–(40), respectively,

$$w(\vartheta_2) = \int_{\vartheta_2}^{\lambda_2} \frac{\zeta(\vartheta_1)}{F(\sigma(\vartheta_1))} \Delta \vartheta_1, \quad (56)$$

$$A_{\kappa}(\phi(\vartheta_1)) = \frac{\int_{\lambda_1}^{\sigma(\vartheta_1)} h_{\gamma-1}(\sigma(\vartheta_1), \sigma(\vartheta_2)) \phi(\vartheta_2) \Delta \vartheta_2}{\int_{\lambda_1}^{\sigma(\vartheta_1)} h_{\gamma-1}(\sigma(\vartheta_1), \sigma(\vartheta_2)) \Delta \vartheta_2} = \frac{K_{\lambda_1}^{\gamma} \phi(\sigma(\vartheta_1))}{F(\sigma(\vartheta_1))}. \quad (57)$$

We use $Y_1 = Y_2 = [\lambda_1, \lambda_2]$ and (55)–(57) in Corollary 4.2 to get

$$\begin{aligned} & \int_{\lambda_1}^{\lambda_2} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\frac{K_{\lambda_1}^{\gamma} \phi(\sigma(\vartheta_1))}{F(\sigma(\vartheta_1))} \right)^p \zeta(\vartheta_1) \Delta \vartheta_1 \\ & + \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\vartheta_2}^{\lambda_2} \left| \phi(\vartheta_2) - \frac{K_{\lambda_1}^{\gamma} \phi(\sigma(\vartheta_1))}{F(\sigma(\vartheta_1))} \right|^p \frac{\zeta(\vartheta_1)}{F(\sigma(\vartheta_1))} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\ & \leq \int_{\lambda_1}^{\lambda_2} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \phi^p(\vartheta_2) \left(\int_{\vartheta_2}^{\lambda_2} \frac{\zeta(\vartheta_1)}{F(\sigma(\vartheta_1))} \Delta \vartheta_1 \right) \Delta \vartheta_2. \end{aligned} \quad (58)$$

Corollary 5. Assume $Y_1 = Y_2 = [\lambda_1, \lambda_2]$ with $\lambda_2 = \infty$ and (55)–(57) hold, then

$$\begin{aligned} & \int_{\lambda_1}^{\infty} h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\frac{K_{\lambda_1}^{\gamma} \phi(\sigma(\vartheta_1))}{F(\sigma(\vartheta_1))} \right)^p \zeta(\vartheta_1) \Delta \vartheta_1 \\ & + \int_{\lambda_1}^{\infty} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \left(\int_{\vartheta_2}^{\infty} \left| \phi(\vartheta_2) - \frac{K_{\lambda_1}^{\gamma} \phi(\sigma(\vartheta_1))}{F(\sigma(\vartheta_1))} \right|^p \frac{\zeta(\vartheta_1)}{F(\sigma(\vartheta_1))} \Delta \vartheta_1 \right) \Delta \vartheta_2 \\ & \leq \int_{\lambda_1}^{\infty} h_{\gamma-1}(s_1, \sigma(\vartheta_2)) h_{\beta-1}(s_2, \sigma(\vartheta_1)) \phi^p(\vartheta_2) \left(\int_{\vartheta_2}^{\infty} \frac{\zeta(\vartheta_1)}{F(\sigma(\vartheta_1))} \Delta \vartheta_1 \right) \Delta \vartheta_2, \end{aligned} \quad (59)$$

holds for all λ_{Δ} -integrable $h: Y_1 \rightarrow \mathbb{R}$ such that $\phi(Y_1) \subset I$.

Proof. The statement follows from Corollary 4 by using $\lambda_2 = \infty$. $\square$

Remark 6. By assuming $\gamma, \beta = 1$ in Corollaries 4 and 5, respectively, we obtain the results given in [32] (Corollaries 3.2 and 3.4) (for $n = 1$).

6. Pólya–Knopp Type Inequalities

Example 3. For $\mathbb{T} = h\mathbb{Z}$, $\sigma(\vartheta_2) = \vartheta_2 + h$, $\forall \vartheta_2 \in \mathbb{Z}$. In this case,

$$h_{\gamma-1}(s_1, \sigma(\vartheta_2)) = \frac{(s_1 - \sigma(\vartheta_2))^{\gamma-1}}{(\gamma-1)!} = \frac{(s_1 - \vartheta_2 - h)^{\gamma-1}}{(\gamma-1)!}, \quad (60)$$

$$h_{\beta-1}(s_2, \sigma(\vartheta_2)) = \frac{(s_2 - \sigma(\vartheta_1))^{\beta-1}}{(\beta-1)!} = \frac{(s_2 - \vartheta_1 - h)^{\beta-1}}{(\beta-1)!}. \quad (61)$$

By using (60) and (61) in (41), one gets

$$\begin{aligned}
 & \sum_{\vartheta_1=\eta_1}^{\eta_2-1} \frac{(s_2 - \vartheta_1 - h)^{\beta-1}}{(\beta-1)!} \Theta \left(\frac{\sum_{\vartheta_2=\lambda_1}^{\lambda_2-1} (s_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2)}{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) - \sum_{\vartheta_2=\lambda_2}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2)} \right) \zeta(\vartheta_1) \\
 & + \sum_{\vartheta_2=\lambda_1}^{\lambda_2-1} \left[\left(\sum_{\vartheta_1=\eta_1}^{\eta_2-1} \Theta \left(\left| \phi(\vartheta_2) - \frac{\sum_{\vartheta_2=\lambda_1}^{\lambda_2-1} (s_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \phi(\vartheta_2)}{\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) - \sum_{\vartheta_2=\lambda_2}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2)} \right| \right) \right) \right. \\
 & \quad \times \frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\left(\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) - \sum_{\vartheta_2=\lambda_2}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \right)} \left. \right] \\
 & \leq \sum_{\vartheta_2=\lambda_1}^{\lambda_2-1} \left[\frac{(s_1 - \vartheta_2 - h)^{\gamma-1} (s_2 - \vartheta_1 - h)^{\beta-1}}{(\beta-1)!} \Theta(\phi(\vartheta_1)) \right. \\
 & \quad \times \sum_{\vartheta_1=\eta_1}^{\eta_2-1} \left(\frac{\kappa(\vartheta_1, \vartheta_2) \zeta(\vartheta_1)}{\left(\sum_{\vartheta_2=\lambda_1}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) - \sum_{\vartheta_2=\lambda_2}^{\vartheta_1-1} (\vartheta_1 - \vartheta_2 - h)^{\gamma-1} \kappa(\vartheta_1, \vartheta_2) \right)} \right) \left. \right].
 \end{aligned} \tag{62}$$

Corollary 6. Suppose time scale $\mathbb{T}$ consists of isolated points, $Y_1 = Y_2 = [\lambda_1, \infty) \cap \mathbb{T}$, $\lambda_1 \geq 0$, and $\zeta: Y_2 \rightarrow \mathbb{R}_+$ is a μ_Δ -integrable function. For $p \geq 2$ inequality,

$$\begin{aligned}
 & \sum_{\vartheta_1 \in [\lambda_1, \infty)_\mathbb{T}} \frac{(s_2 - \vartheta_1 - 1)^{\beta-1} \zeta(\vartheta_1)}{(\beta-1)!} \left(\frac{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_\mathbb{T}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \phi(\vartheta_2) \mu(\vartheta_2)}{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_\mathbb{T}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \mu(\vartheta_2)} \right)^p \mu(\vartheta_1) \\
 & + \sum_{\vartheta_2 \in [\lambda_1, \infty)_\mathbb{T}} \left[\frac{(s_1 - \vartheta_2 - 1)^{\gamma-1} (s_2 - \vartheta_1 - 1)^{\beta-1}}{(\beta-1)!} \times \left(\sum_{\vartheta_1 \in [\vartheta_2, \infty)_\mathbb{T}} \left| \phi(\vartheta_2) - \frac{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_\mathbb{T}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \phi(\vartheta_2) \mu(\vartheta_2)}{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_\mathbb{T}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \mu(\vartheta_2)} \right|^p \right) \right. \\
 & \quad \left. \frac{\zeta(\vartheta_1)}{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_\mathbb{T}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \mu(\vartheta_2)} \right] \mu(\vartheta_1) \mu(\vartheta_2) \\
 & \leq \sum_{\vartheta_2 \in [\lambda_1, \infty)_\mathbb{T}} \frac{(s_1 - \vartheta_2 - 1)^{\gamma-1} (s_2 - \vartheta_1 - 1)^{\beta-1} \phi^p(\vartheta_2)}{(\beta-1)!} \left(\sum_{\vartheta_1 \in [\vartheta_2, \infty)_\mathbb{T}} \frac{\zeta(\vartheta_1)}{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_\mathbb{T}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \mu(\vartheta_2)} \mu(\vartheta_1) \right) \mu(\vartheta_2),
 \end{aligned} \tag{63}$$

holds for all λ_Δ -integrable $\phi: Y_1 \rightarrow \mathbb{R}$ such that $\phi(Y_1) \subset I$, where $\mu(r) = \sigma(r) - r$.

Proof. If $\mathbb{T}$ consists of isolated points, then

$$h_{\gamma-1}(s_1, \sigma(\vartheta_2)) = \frac{(s_1 - \sigma(\vartheta_2))^{\gamma-1}}{(\gamma-1)!} = \frac{(s_1 - \vartheta_2 - 1)^{\gamma-1}}{(\gamma-1)!}, \quad (64)$$

$$h_{\gamma-1}(\sigma(\vartheta_1), \sigma(\vartheta_2)) = \frac{(\sigma(\vartheta_1) - \sigma(\vartheta_2))^{\gamma-1}}{(\gamma-1)!} = \frac{(\vartheta_1 - \vartheta_2)^{\gamma-1}}{(\gamma-1)!}, \quad (65)$$

$$h_{\beta-1}(s_2, \sigma(\vartheta_1)) = \frac{(s_2 - \sigma(\vartheta_1))^{\beta-1}}{(\beta-1)!} = \frac{(s_2 - \vartheta_1 - 1)^{\beta-1}}{(\beta-1)!}. \quad (66)$$

By using (64)–(66) in Corollary 5, one gets

$$\begin{aligned} & \sum_{\vartheta_1 \in [\lambda_1, \infty)_{\mathbb{T}}} \frac{(s_2 - \vartheta_1 - 1)^{\beta-1} \zeta(\vartheta_1)}{\beta-1!} \left(\frac{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_{\mathbb{T}}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \phi(\vartheta_2) \mu(\vartheta_2)}{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_{\mathbb{T}}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \mu(\vartheta_2)} \right) \mu(\vartheta_1) \\ & + \sum_{\vartheta_2 \in [\lambda_1, \infty)_{\mathbb{T}}} \left[\frac{(s_1 - \vartheta_2 - 1)^{\gamma-1} (s_2 - \vartheta_1 - 1)^{\beta-1}}{(\beta-1)!} \times \left(\sum_{\vartheta_1 \in [\vartheta_2, \infty)_{\mathbb{T}}} \left| \phi(\vartheta_2) - \frac{\sum_{\vartheta_1 \in [\lambda_1, \sigma(\vartheta_1))_{\mathbb{T}}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \phi(\vartheta_2) \mu(\vartheta_2)}{\sum_{\vartheta_1 \in [\lambda_1, \sigma(\vartheta_1))_{\mathbb{T}}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \mu(\vartheta_2)} \right|^p \right) \mu(\vartheta_1) \right] \mu(\vartheta_2) \\ & \leq \sum_{\vartheta_2 \in [\lambda_1, \infty)_{\mathbb{T}}} \frac{(s_1 - \vartheta_2 - 1)^{\gamma-1} (s_2 - \vartheta_1 - 1)^{\beta-1} \phi^p(\vartheta_2)}{(\beta-1)!} \left(\sum_{\vartheta_1 \in [\vartheta_2, \infty)_{\mathbb{T}}} \frac{\zeta(\vartheta_1)}{\sum_{\vartheta_2 \in [\lambda_1, \sigma(\vartheta_1))_{\mathbb{T}}} (\vartheta_1 - \vartheta_2)^{\gamma-1} \mu(\vartheta_2)} \mu(\vartheta_1) \right) \mu(\vartheta_2). \end{aligned} \quad (67)$$

Remark 7. By choosing $\gamma, \beta = 1$ in (63), the respective Polya–Knopp type inequality is obtained in time scales ([32], Corollary 4.1).

Example 4. If we consider $\mathbb{T} = h\mathbb{N} = \{h\iota : \iota \in \mathbb{N}\}$ with fixed $h > 0$, $\lambda_1 = h$ and taking $\zeta(\vartheta_1) = (1/\sigma(\vartheta_1))$, (63) takes the form,

$$\begin{aligned} & \sum_{\iota=1}^{\infty} \frac{(\mathcal{J} - \iota - 1)^{\beta-1}}{(\beta-1)! (\iota+1)} \left(\frac{\sum_{\kappa=1}^{\iota} (\iota - \kappa)^{\gamma-1} \phi(\kappa(h))}{\sum_{\kappa=1}^{\iota} (\iota - \kappa)^{\gamma-1}} \right)^p \\ & + \sum_{\kappa=1}^{\infty} \left[\frac{(\iota h - \kappa(h) - h)^{\gamma-1} (\mathcal{J} h - \iota h - h)^{\beta-1}}{(\beta-1)!} \times \left(\sum_{\iota=1}^{\infty} \left| \phi(\vartheta_2) - \frac{\sum_{\kappa=1}^{\iota} (\iota - \kappa)^{\gamma-1} \phi(\vartheta_2)}{\sum_{\kappa=1}^{\iota} (\iota - \kappa)^{\gamma-1}} \right|^p \frac{1}{\iota+1 \sum_{\kappa=1}^{\iota} (\iota h - \kappa(h))^{\gamma-1}} \right) \right] \\ & \leq \sum_{\kappa=1}^{\infty} \frac{(\iota h - \kappa(h) - h)^{\gamma-1} (\mathcal{J} - \iota - 1)^{\beta-1} \phi(\kappa(h))}{(\beta-1)!} \left(\sum_{\iota=\kappa}^{\infty} \frac{1}{\iota+1 \sum_{\kappa=1}^{\iota} (\iota h - \kappa(h))^{\gamma-1}} \right). \end{aligned} \quad (68)$$

Remark 8. Notice that (68) becomes [32] Example 4.3 by choosing $\gamma, \beta = 1$.

Example 5. If we consider time scale $\mathbb{T} = \mathbb{N}^2 = \{\iota^2 : \iota \in \mathbb{N}\}$, $\lambda_1 = 1$ and

$$\zeta(\vartheta_1) = \frac{2(\sigma(\vartheta_1) - 1)}{(\sigma(\vartheta_1) - \vartheta_1)(2\sqrt{\vartheta_1} + 3)} = \frac{2((\iota+1)^2 - 1)}{((\iota+1)^2 - \iota^2)(2\iota + 3)} = \frac{2\iota(\iota+2)}{(2\iota+1)(2\iota+3)}. \quad (69)$$

(63) takes the form,

$$\begin{aligned} & \sum_{\iota=1}^{\infty} \frac{(\mathcal{J}^2 - (\iota+1)^2)^{\beta-1} 2\iota(\iota+2)}{(\beta-1)!(2\iota+3)} \left(\frac{\sum_{\kappa=1}^{\iota} ((\iota+1)^2 - (\kappa+1)^2)^{\gamma-1} \phi(\kappa^2) (2\kappa+1)}{\sum_{\kappa=1}^{\iota} ((\iota+1)^2 - (\kappa+1)^2)^{\gamma-1} (2\kappa+1)} \right)^p \\ & + \sum_{\kappa=1}^{\infty} \left[\frac{(\iota^2 - (\kappa+1)^2)^{\gamma-1} (\mathcal{J}^2 - (\iota+1)^2)^{\beta-1}}{(\beta-1)!} \times \left(\sum_{\iota=\kappa}^{\infty} \left| \phi(\kappa^2) - \frac{\sum_{\kappa=1}^{\iota} ((\iota+1)^2 - (\kappa+1)^2)^{\gamma-1} \phi(\kappa^2) (2\kappa+1)}{\sum_{\kappa=1}^{\iota} ((\iota+1)^2 - (\kappa+1)^2)^{\gamma-1} (2\kappa+1)} \right|^p \right. \right. \\ & \quad \left. \left. \times \frac{2\iota(\iota+2)}{(2\iota+3) \sum_{\kappa=1}^{\iota} ((\iota+1)^2 - (\kappa+1)^2)^{\gamma-1} (2\kappa+1)} \right) \right] (2\kappa+1) \\ & \leq \sum_{\kappa=1}^{\infty} \left[\frac{(\iota^2 - kh - h)^{\gamma-1} (\mathcal{J}^2 - nh - h)^{\beta-1} \phi(\kappa^2)}{(\beta-1)!} \times \left(\sum_{\iota=\kappa}^{\infty} \frac{2\iota(\iota+2)}{(2\iota+3) \sum_{\kappa=1}^{\iota} ((\iota+1)^2 - (\kappa+1)^2)^{\gamma-1} (2\kappa+1)} \right) \right] (2\kappa+1). \end{aligned} \quad (70)$$

Example 6. If $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ with fixed $q > 1$, then $\sigma(\vartheta) = q\vartheta = q^{n+1}$, and assume

$$\zeta(\vartheta) = \frac{2(\sigma(\vartheta) - 1)}{(\sigma(\vartheta) - \vartheta)(2\vartheta + 3)} = \frac{2(q^{n+1} - 1)}{q^n(q - 1)(2q^n + 3)}. \quad (71)$$

In this case,

$$h_{\gamma-1}(s_1, \sigma(\vartheta_2)) = \prod_{v=0}^{\gamma-2} \frac{s_1 - q^v \sigma(\vartheta_2)}{\sum_{\mu=0}^v q^\mu} = \prod_{v=0}^{\gamma-2} \frac{s_1 - q^{v+k+1}}{\sum_{\mu=0}^v q^\mu}, \quad (72)$$

$$h_{\beta-1}(s_2, \sigma(\vartheta_1)) = \prod_{v=0}^{\beta-2} \frac{s_2 - q^v \sigma(\vartheta_1)}{\sum_{\mu=0}^v q^\mu} = \prod_{v=0}^{\beta-2} \frac{s_2 - q^{v+n+1}}{\sum_{\mu=0}^v q^\mu}. \quad (73)$$

By using (63), (71)–(73) take the form,

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\left(\prod_{v=0}^{\beta-2} \frac{s_2 - q^{v+n+1}}{\sum_{\mu=0}^v q^\mu} \right) \frac{2(q^{n+1} - 1)}{q^n(2q^n + 3)} \left(\frac{\sum_{k=1}^n (\prod_{v=0}^{\gamma-2} (s_1 - q^{v+k+1} / \sum_{\mu=0}^v q^\mu)) \phi(q^k)}{\sum_{k=1}^n (\prod_{v=0}^{\gamma-2} (q^n - q^{v+k+1} / \sum_{\mu=0}^v q^\mu)) \phi(q^k)} \right)^p \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \left(\prod_{v=0}^{\gamma-2} \frac{s_1 - q^{v+k+1}}{\sum_{\mu=0}^v q^\mu} \cdot \prod_{v=0}^{\beta-2} \frac{s_2 - q^{v+n+1}}{\sum_{\mu=0}^v q^\mu} \sum_{n=k}^{\infty} \left(\left| \phi(q^k) - \frac{\sum_{k=1}^n (\prod_{v=0}^{\gamma-2} (s_1 - q^{v+k+1} / \sum_{\mu=0}^v q^\mu)) \phi(q^k)}{\sum_{k=1}^n (\prod_{v=0}^{\gamma-2} (q^n - q^{v+k+1} / \sum_{\mu=0}^v q^\mu)) \phi(q^k)} \right|^p \frac{(2(q^{n+1} - 1)/q^n(2q^n + 3))}{\sum_{k=1}^n \prod_{v=0}^{\gamma-2} (q^n - q^{v+k+1} / \sum_{\mu=0}^v q^\mu)} \right) \right) \right) \\ & \leq \sum_{k=1}^{\infty} \left(\prod_{v=0}^{\gamma-2} \frac{s_1 - q^{v+k+1}}{\sum_{\mu=0}^v q^\mu} \cdot \prod_{v=0}^{\beta-2} \frac{s_2 - q^{v+n+1}}{\sum_{\mu=0}^v q^\mu} (\phi(q^k))^p \sum_{n=k}^{\infty} \left(\frac{(2(q^{n+1} - 1)/q^n(2q^n + 3))}{\sum_{k=1}^n (\prod_{v=0}^{\gamma-2} (q^n - q^{v+k+1} / \sum_{\mu=0}^v q^\mu))} \right) \right). \end{aligned} \quad (74)$$

Remark 9. By choosing $\gamma, \beta = 1$ in (70), the respective Polya–Knopp type inequality is obtained in time scales ([32], Example 4.4).

7. Conclusion

In the paper, some dynamic inequalities of Jensen and Hardy for superquadratic functions with Riemann–Liouville delta fractional integrals are generalized. Particular cases of obtained inequalities have been given in time scales [32] by assuming $\gamma, \beta = 1$ and respective inequalities in fractional calculus [33] by choosing $\mathbb{T} = \mathbb{R}$ are also obtained. Furthermore, some discrete fractional inequalities for superquadratic functions are

obtained by choosing $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T} = \mathbb{N}^2$. Inequalities in quantum discrete calculus are also obtained by choosing $\mathbb{T} = q^{\mathbb{Z}}$, which are new, up to the knowledge of authors.

Data Availability

All data generated or analyzed during this study are included in this published article. There are no experimental data in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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