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Research Article

On Estimation of Three-Component Mixture of Distributions via Bayesian and Classical Approaches

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In this study, we model a heterogeneous population assuming the three-component mixture of the Pareto distributions assuming type I censored data. In particular, we study some statistical properties (such as various entropies, different inequality indices, and order statistics) of the three-component mixture distribution. The ML estimation and the Bayesian estimation of the mixture parameters have been performed in this study. For the ML estimation, we used the Newton Raphson method. To derive the posterior distributions, different noninformative priors are assumed to derive the Bayes estimators. Furthermore, we also discussed the Bayesian predictive intervals. We presented a detailed simulation study to compare the ML estimates and Bayes estimates. Moreover, we evaluated the performance of different estimates assuming various sample sizes, mixing weights and test termination times (a fixed point of time after which all other tests are dismissed). The real-life data application is also a part of this study.

1. Introduction

In the last decade, finite mixture models have emerged as flexible models due to their applications in applied sciences, engineering, and physical sciences. As explained by Mendenhall and Hader [1], for real-life purposes, an engineer split the failures of a structure into more than one kind of causes. For example, to know the proportion of failures due to a definite reason and to recover the engineering system, Acheson and McElwee [2] separated electronic tube flops into three different faults such as mechanical faults, gaseous faults, and normal decline of the cathode.

Moreover, the mixture models can also be used in a situation when the data are presented in the form of the overall mixture models. The overall mixture models are also called the direct application of the mixture models, and their applications can be seen in medicine, botany, zoology, agriculture, economics, life testing, reliability, and survival analysis. The various aspects of mixture models were discussed by Li and Sedransk [3]. The interested readers can refer the work of Harris [4], Kanji [5], and Jones and McLachlan [6] on the application of mixture models for real-life problems.

The mixture models have been extensively used for heterogeneous nature of the process in comparison to the simple models. Most of the researchers have comprehensively applied mixture distributions in various real-life situations and estimated parameters using the Bayesian and classical methods. For a detailed appraisal of classical techniques for estimation and applications of mixture distributions, we refer to studies by Sultan et al. [7], Abu-Zinadah [8], and Kamaruzzaman et al. [9] among others. On the other hand, the estimation of parameters in Bayesian framework for a mixture of two distributions has been considered by many researchers [10–21]. Contrary to the two-component mixture modeling, some authors have discussed situations where data are assumed to follow a three-component mixture of suitable probability distributions [22–28].

Censoring is a significant characteristic of the real data application. Due to time and cost problem, it is very difficult to continue the lifetime testing experiment till observing the last failure. Although there are many censoring schemes, the type I right censoring is commonly used in life testing experiments. In this scheme, we consider a fixed censoring time, and the values larger than the specific t (life test termination time) are observed as censored observations. Romeu [29] and Kalbfleisch and Prentice [30] explained various censoring schemes.

To motivate the readers about the mixture modeling, consider a sample of sand which is based on the mixture of some minerals. With the application of mixture modeling, estimates of the proportions of various minerals in the sand can be obtained. Similarly, the grain size distributions for the different minerals can also be estimated. It is worth mentioning that mixture models can be classified into type I (if component densities of the various components belong to the same family) and type II (if the component densities belong to different families) mixtures.

It has been noticed that from the recent literature that the Pareto distribution can be applied efficiently in various situations rather than other distributions to model data. The significance of Pareto distribution in forming various real phenomena is patent from the following revealed research and references mentioned therein, Abdel-All et al. [31], Ismail [32], Sankaran and Nair [33], and Nadarajah and Kotz [34]. Inspired by the wide real-life applications of mixture distributions, the main objective of this study is to develop a new three-component mixture of Pareto distributions (TCMPD) for lifetime data modeling under type I mixture. Furthermore, we also compare the maximum likelihood (ML) estimates, ML variances (MLVs), Bayes estimates, and their posterior variances (PVs) assuming type I right censored data.

2. The Population and the Three-Component Mixture Distribution

The finite k-components mixture density function can be defined for random variable Y as $f(y) = \sum_{d=1}^{k} w_d f_d(y)$, where $f_d(y)$ is the d^{th} component density function, w_d, ($d = 1, 2, \ldots, h$) is the d^{th} mixing proportion, and $\sum_{d=1}^{k} w_d = 1$. A finite TCMPD is defined as

$$f(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2) = w_1 f_1(y;\lambda_1) + w_2 f_2(y;\lambda_2) + (1-w_1-w_2) f_3(y;\lambda_3)$$
$$w_1, w_2 \geq 0, w_1 + w_2 \leq 1, \tag{1}$$

where $f_d(y;\lambda_d) = \lambda_d e^{-(\lambda_d+1)\ln y}$, $1 < y < \infty, \lambda_d > 0$, $d = 1, 2, 3.$

The cdf $F(y)$ of a TCMPD is

$$F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2) = w_1 F_1(y;\lambda_1) + w_2 F_2(y;\lambda_2) + (1-w_1-w_2) F_3(y;\lambda_3), \tag{2}$$

where $F_d(y;\lambda_d) = 1 - e^{-\lambda_d \ln y}$.

3. Properties of the TCMPD

The statistical properties such as moments, mean, variance, and mode of the TCMPD are derived in this section.

r^{th} moment about zero: the r^{th} moment about zero of a TCMPD is derived as

$$E(Y^r) = w_1\lambda_1(\lambda_1 - r)^{-1} + w_2\lambda_2(\lambda_2 - r)^{-1} + (1-w_1-w_2)\lambda_3(\lambda_3 - r)^{-1}. \tag{3}$$

k^{th} order negative moment: the k^{th} order negative moment is derived as

$$E\left(Y^{-k}\right) = w_1\lambda_1(\lambda_1 + k)^{-1} + w_2\lambda_2(\lambda_2 + k)^{-1} + (1-w_1-w_2)\lambda_3(\lambda_3 + k)^{-1}. \tag{4}$$

Factorial moments: the factorial moments can be determined as

$$E(Y(Y-1)(Y-2),\ldots,(Y-\delta+1)) = \sum_{\omega=0}^{\delta-1}\xi_{\omega}(-1)^{\omega}E\left(Y^{\delta-\omega}\right), \tag{5}$$

Where $\xi_{\omega}'s$ is the real number. The $E(Y^{\delta-\omega})$ can be obtained ash

$$E\left(Y^{\delta-\omega}\right) = w_1\lambda_1(\lambda_1-\delta+\omega)^{-1} + w_2\lambda_2(\lambda_2-\delta+\omega)^{-1} + (1-w_1-w_2)\lambda_3(\lambda_3-\delta+\omega)^{-1}. \tag{6}$$

Mean: the mean of a TCMPD is evaluated ash

$$E(Y) = w_1\lambda_1(\lambda_1-1)^{-1} + w_2\lambda_2(\lambda_2-1)^{-1} + (1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}. \tag{7}$$

Variance: the variance of a TCMPD is derived as

$$\operatorname{Var}(Y) = \sum_{d=1}^{3} w_d\lambda_d(\lambda_d-2)^{-1} - \left\{\sum_{d=1}^{3} w_d\lambda_d(\lambda_d-1)^{-1}\right\}^2. \tag{8}$$

Median: the median (y) is determined by evaluating the following nonlinear equation for y.

$$w_1 e^{-\lambda_1 \ln y} + w_2 e^{-\lambda_2 \ln y} + (1-w_1-w_2)e^{-\lambda_3 \ln y} = 0.5. \tag{9}$$

Mode: the mode (y) is obtained by solving the following nonlinear equation for y.

$$w_1\lambda_1(\lambda_1+1)e^{-(\lambda_1+2)\ln y} + w_2\lambda_2(\lambda_2+1)e^{-(\lambda_2+2)\ln y} + (1-w_1-w_2)\lambda_3(\lambda_3+1)e^{-(\lambda_3+2)\ln y} = 0. \tag{10}$$

Using the above expressions, mean, median, variance, and coefficient of skewness (SK) are calculated for different parameters' values and are given in Table 1.

It is observed from the entries in Table 1 and Figure 1 that TCMPD is positively skewed as we have SK > 0 for all the entries arranged in Table 1. We noticed that the variance of the TCMPD was a declining function of the mixture distribution's parameters.

4. Entropies

The entropy is a quantity of unspecified extent of evidence in a function. Here, we derive the expressions of the most commonly used entropy measures such as Shannon's entropy, β–entropy, and Re´nyi entropy, in this section. As said by Song [35], Shannon's entropy has a same behavior as a measure of kurtosis in equating the forms of different functions and computing substance of tails.

Shannon's entropy: Shannon's entropy for a random variable Y, which follows TCMPD, is

$$\begin{aligned} I_S(y) &= -\int_1^{\infty} f(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\log\{f(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}\mathrm{d}y, \\ I_S(y) &= -\int_1^{\infty}\left[\left\{w_1\lambda_1 e^{-(\lambda_1+1)\ln y} + w_2\lambda_2 e^{-(\lambda_2+1)\ln y} + (1-w_1-w_2)\lambda_3 e^{-(\lambda_3+1)\ln y}\right\}\right. \\ &\quad \left.\log\left\{w_1\lambda_1 e^{-(\lambda_1+1)\ln y} + w_2\lambda_2 e^{-(\lambda_2+1)\ln y} + (1-w_1-w_2)\lambda_3 e^{-(\lambda_3+1)\ln y}\right\}\right]\mathrm{d}y. \end{aligned} \tag{11}$$

Re´nyi entropy: the Rényi entropy (Rényi, [36]) is explained as

$$I_R(y) = \frac{1}{(1-\xi)}\log\left\{\int_1^{\infty} f^{\xi}(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\mathrm{d}y\right\}, \tag{12}$$

Table 1: Mean, median, variance, and coefficient of skewness of TCMPD.

$\lambda_1, \lambda_2, \lambda_3, w_1, w_2$	Mean	Median	Variance	SK
5, 6, 7, 0.1, 0.2	1.1817	1.1112	0.0503	4.1248
5, 6, 7, 0.2, 0.3	1.1933	1.1170	0.0593	4.3907
5, 6, 7, 0.3, 0.4	1.2050	1.1232	0.0680	4.4905
5, 6, 7, 0.4, 0.5	1.2167	1.1299	0.0764	4.5080
7, 6, 5, 0.2, 0.1	1.2283	1.1353	0.0879	4.7192
8, 7, 6, 0.3, 0.2	1.1762	1.1080	0.0466	3.8285
9, 8, 7, 0.4, 0.3	1.1429	1.0897	0.0282	3.3279
10, 9, 8, 0.5, 0.4	1.1198	1.0767	0.0186	2.9837
6, 5, 4, 0.5, 0.3	1.2417	1.1409	0.1083	6.5238
7, 6, 5, 0.5, 0.3	1.1933	1.1170	0.0593	4.3907
8, 7, 6, 0.5, 0.3	1.1614	1.1000	0.0377	3.6535
9, 8, 7, 0.5, 0.3	1.1387	1.0874	0.0262	3.2733

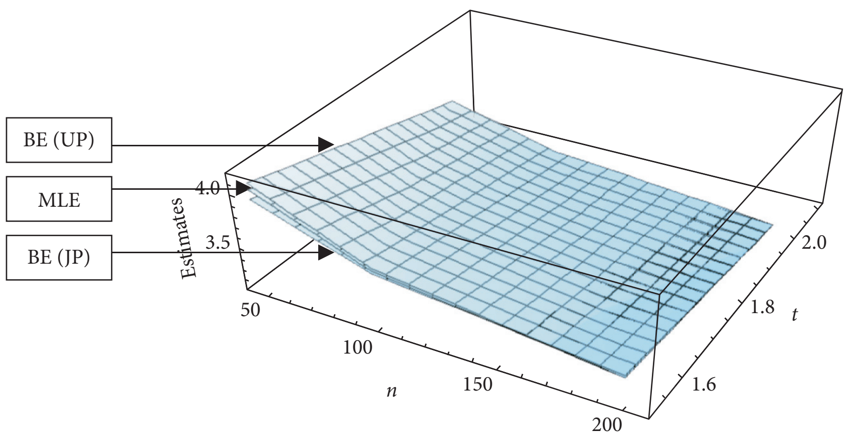

Figure 1: The graph of the BE and MLE of parameter $\lambda_1 = 3$.

Where $\xi > 0$, and $\xi \neq 1$. The Rényi entropy of a 3-CMPD is

$$I_R(y) = \frac{1}{(1-\xi)} \log\left[\int_1^\infty \left\{w_1\lambda_1 e^{-(\lambda_1+1)\ln y} + w_2\lambda_2 e^{-(\lambda_2+1)\ln y} + (1-w_1-w_2)\lambda_3 e^{-(\lambda_3+1)\ln y}\right\}^{\xi} \mathrm{d}y\right]. \tag{13}$$

β–Entropy: the β–entropy (Ullah, [37]) is written as

$$I_\beta(y) = \frac{1}{(\varphi-1)}\left\{1 - \int_1^\infty f^\varphi(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\mathrm{d}y\right\}, \tag{14}$$

Where $\varphi > 0$, and $\varphi \neq 1$. The β–entropy of a TCMPD is

$$I_\beta(y) = \frac{1}{(\varphi-1)}\left[1 - \int_1^\infty \left\{w_1\lambda_1 e^{-(\lambda_1+1)\ln y} + w_2\lambda_2 e^{-(\lambda_2+1)\ln y} + (1-w_1-w_2)\lambda_3 e^{-(\lambda_3+1)\ln y}\right\}^{\varphi} \mathrm{d}y\right], \tag{15}$$

Which can be evaluated by the numerical integration.

5. Inequality Measures

The most common income inequality indices are Bonferroni curve, Gini index, and Lorenz curve.

The Gini index: this index (Gini, [38]) for a TCMPD is

$$G=\frac{1}{E(Y)}\int_1^{\infty}F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\{1-F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}\mathrm{d}y,$$

$$G=\frac{\int_1^{\infty}\{w_1e^{-\lambda_1\ln y}+w_2e^{-\lambda_2\ln y}+(1-w_1-w_2)e^{-\lambda_3\ln y}\}\mathrm{d}y-\int_1^{\infty}\{w_1e^{-\lambda_1\ln y}+w_2e^{-\lambda_2\ln y}+(1-w_1-w_2)e^{-\lambda_3\ln y}\}^2\mathrm{d}y}{w_1\lambda_1(\lambda_1-1)^{-1}+w_2\lambda_2(\lambda_2-1)^{-1}+(1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}},$$

$$G=\frac{w_1(\lambda_1-1)^{-1}+w_2(\lambda_2-1)^{-1}+(1-w_1-w_2)(\lambda_3-1)^{-1}}{w_1\lambda_1(\lambda_1-1)^{-1}+w_2\lambda_2(\lambda_2-1)^{-1}+(1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}}-\frac{\left\{\begin{array}{c}w_1^2(2\lambda_1-1)^{-1}+w_2^2(2\lambda_2-1)^{-1}+(1-w_1-w_2)^2(2\lambda_3-1)^{-1}\\ +2w_1w_2((\lambda_1+\lambda_2)-1)^{-1}+2w_2(1-w_1-w_2)((\lambda_2+\lambda_3)-1)^{-1}\\ +2w_1(1-w_1-w_2)((\lambda_1+\lambda_3)-1)^{-1}\end{array}\right\}}{w_1\lambda_1(\lambda_1-1)^{-1}+w_2\lambda_2(\lambda_2-1)^{-1}+(1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}}. \tag{16}$$

The Lorenz curve: this curve (Lorenz, [39]) for a TCMPD is

$$L(p)=\frac{1}{E(Y)}\int_1^{y}y\,f(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\mathrm{d}y,$$
$$L(p)=1-\frac{w_1\lambda_1(\lambda_1-1)^{-1}e^{-(\lambda_1-1)\ln y}+w_2\lambda_2(\lambda_2-1)^{-1}e^{-(\lambda_2-1)\ln y}+(1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}e^{-(\lambda_3-1)\ln y}}{w_1\lambda_1(\lambda_1-1)^{-1}+w_2\lambda_2(\lambda_2-1)^{-1}+(1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}}. \tag{17}$$

Bonferroni curve: the Bonferroni [40] curve for a TCMPD is

$$BC(p)=\frac{L(p)}{F(y)}=\Big(1-w_1\lambda_1(\lambda_1-1)^{-1}e^{-(\lambda_1-1)\ln y}+w_2\lambda_2(\lambda_2-1)^{-1}e^{-(\lambda_2-1)\ln y}$$
$$+(1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}e^{-(\lambda_3-1)\ln y}/w_1\lambda_1(\lambda_1-1)^{-1}+w_2\lambda_2(\lambda_2-1)^{-1}+(1-w_1-w_2)\lambda_3(\lambda_3-1)^{-1}\Big)/$$
$$\cdot\left(\{1-w_1e^{-\lambda_1\ln y}-w_2e^{-\lambda_2\ln y}-(1-w_1-w_2)e^{-\lambda_3\ln y}\}\right). \tag{18}$$

6. Order Statistics

In this section, we derive $g(y_{k:n};\lambda_1,\lambda_2,\lambda_3,w_1,w_2)$ which is pdf of k^{th} order statistic $y_{k:n}$, assuming a sample of size n from the TCMPD. The r^{th} raw moment along with mean and variance of 1^{st} and n^{th} order statistics are obtained in this section.

Probability density function of k^{th} order statistic: the pdf of k^{th} order statistic is

$$g(y_{k:n};\lambda_1,\lambda_2,\lambda_3,w_1,w_2)=\frac{n!}{(k-1)!(n-k)!}f(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\{F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}^{k-1}$$
$$\{1-F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}^{n-k}, \tag{19}$$

where $f(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2) = w_1 f_1(y;\lambda_1) + w_2 f_2(y;\lambda_2) + (1-w_1-w_2) f_3(y;\lambda_3)$,

$$\begin{aligned}
&w_1, w_2 \geq 0, w_1 + w_2 \leq 1,\\
F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2) &= w_1 F_1(y;\lambda_1) + w_2 F_2(y;\lambda_2) + (1-w_1-w_2) F_3(y;\lambda_3),\\
\{F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}^{k-1} &= \left\{1 - w_1 e^{-\lambda_1 \ln y} - w_2 e^{-\lambda_2 \ln y} - (1-w_1-w_2) e^{-\lambda_3 \ln y}\right\}^{k-1},\\
\{F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}^{k-1} &= \sum_{a=0}^{k-1}\sum_{b=0}^{a}\sum_{c=0}^{b} (-1)^a \binom{k-1}{a}\binom{a}{b}\binom{b}{c} e^{-\lambda_1 (a-b)\ln y} e^{-\lambda_2 (b-c)\ln y}\\
&\quad e^{-\lambda_3 (c)\ln y} w_1^{a-b} w_2^{b-c} (1-w_1-w_2)^c,\\
\{1 - F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}^{n-k} &= \left\{w_1 e^{-\lambda_1 \ln y} + w_2 e^{-\lambda_2 \ln y} + (1-w_1-w_2) e^{-\lambda_3 \ln y}\right\}^{n-k},\\
\{1 - F(y;\lambda_1,\lambda_2,\lambda_3,w_1,w_2)\}^{n-k} &= \sum_{u=0}^{n-k}\sum_{v=0}^{u} \binom{n-k}{u}\binom{u}{v} e^{-\lambda_1 (n-k-u)\ln y} e^{-\lambda_2 (u-v)\ln y}\\
&\quad e^{-\lambda_3 (v)\ln y} w_1^{n-k-u} w_2^{u-v} (1-w_1-w_2)^v.
\end{aligned} \tag{20}$$

After little simplification, the pdf (19) of k^{th} order statistic can be written as

$$\begin{aligned}
g(y_{k:n};\lambda_1,\lambda_2,\lambda_3,w_1,w_2) &= \frac{n!}{(k-1)!(n-k)!} \sum_{a=0}^{k-1}\sum_{b=0}^{a}\sum_{c=0}^{b} (-1)^a \binom{k-1}{a}\binom{a}{b}\binom{b}{c} e^{-\lambda_1 (a-b)\ln y_k} e^{-\lambda_2 (b-c)\ln y_k}\\
&\quad \cdot e^{-\lambda_3 (c)\ln y_k} w_1^{a-b} w_2^{b-c} (1-w_1-w_2)^c \sum_{u=0}^{n-k}\sum_{v=0}^{u} \binom{n-k}{u}\binom{u}{v} e^{-\lambda_1 (n-k-u)\ln y_k}\\
&\quad \cdot e^{-\lambda_2 (u-v)\ln y_k} e^{-\lambda_3 (v)\ln y_k} w_1^{n-k-u} w_2^{u-v} (1-w_1-w_2)^v\\
&\quad \cdot \left\{w_1 \lambda_1 e^{-(\lambda_1+1)\ln y_k} + w_2 \lambda_2 e^{-(\lambda_2+1)\ln y_k} + (1-w_1-w_2)\lambda_3 e^{-(\lambda_3+1)\ln y_k}\right\},\\
g(y_{k:n};\lambda_1,\lambda_2,\lambda_3,w_1,w_2) &= \frac{n!}{(k-1)!(n-k)!} \sum_{a=0}^{k-1}\sum_{b=0}^{a}\sum_{c=0}^{b} (-1)^a \binom{k-1}{a}\binom{a}{b}\binom{b}{c} e^{-A_{01}\ln y_k} w_1^{B_{01}-1} w_2^{C_{01}-1}\\
&\quad \cdot (1-w_1-w_2)^{D_{01}-1} \sum_{u=0}^{n-k}\sum_{v=0}^{u} \binom{n-k}{u}\binom{u}{v} e^{-A_{02}\ln y_k} w_1^{B_{02}-1} w_2^{C_{02}-1} (1-w_1-w_2)^{D_{02}-1}\\
&\quad \cdot \left\{w_1 \lambda_1 e^{-(\lambda_1+1)\ln y_k} + w_2 \lambda_2 e^{-(\lambda_2+1)\ln y_k} + (1-w_1-w_2)\lambda_3 e^{-(\lambda_3+1)\ln y_k}\right\},
\end{aligned} \tag{21}$$

where $A_{01} = \lambda_1 (a-b) + \lambda_2 (b-c) + \lambda_3 (c)$, $B_{01} = a - b + 1$, $C_{01} = b - c + 1$, $D_{01} = c + 1$,

$$\begin{aligned}
A_{02} &= \lambda_1 (n-k-u) + \lambda_2 (u-v) + \lambda_3 (v),\\
B_{02} &= n - k - u + 1,\\
C_{02} &= u - v + 1,\\
D_{02} &= v + 1.
\end{aligned} \tag{22}$$

Probability density function of 1^{st} order statistic: substituting $k = 1$ in (21) and simplifying it, the pdf of 1^{st} order statistic is

$$g(y_{1:n};\lambda_1,\lambda_2,\lambda_3,w_1,w_2) = n \sum_{w=1}^{1} \lambda_w \sum_{u=0}^{n-1}\sum_{v=0}^{u} \binom{n-1}{u}\binom{u}{v} e^{-(A_{1w}+1)\ln y_1} w_1^{B_{1w}-1} w_2^{C_{1w}-1} (1-w_1-w_2)^{D_{1w}-1}, \tag{23}$$

where $A_{11} = \lambda_1 (n-u) + \lambda_2 (u-v) + \lambda_3 (v)$, $B_{11} = n-u+1$, $C_{11} = u-v+1$, $D_{11} = v+1$,

$$\begin{aligned} A_{12} &= \lambda_1 (n-1-u) + \lambda_2 (u-v+1) + \lambda_3 (v), \\ B_{12} &= n-u, \\ C_{12} &= u-v+2, \\ D_{12} &= v+1, \\ A_{13} &= \lambda_1 (n-1-u) + \lambda_2 (u-v) + \lambda_3 (v+1), \\ B_{13} &= n-u, \\ C_{13} &= u-v+1, \\ D_{13} &= v+2. \end{aligned} \tag{24}$$

Probability density function of n^{th} order statistic: substituting $k=n$ in (21) and after little algebraic simplification, the pdf of n^{th} order statistic is

$$g(y_{n:n}; \lambda_1, \lambda_2, \lambda_3, w_1, w_2) = n \sum_{w=1}^{1} \lambda_w \sum_{a=0}^{n-1} \sum_{b=0}^{a} \sum_{c=0}^{b} (-1)^a \binom{k-1}{a} \binom{a}{b} \binom{b}{c} e^{-(A_{2w}+1)\ln y_n} w_1^{B_{2w}-1} w_2^{C_{2w}-1} (1-w_1-w_2)^{D_{2w}-1}, \tag{25}$$

where $A_{21} = \lambda_1 (a-b+1) + \lambda_2 (b-c) + \lambda_3 (c)$, $B_{21} = a-b+2$, $C_{21} = b-c+1$, $D_{21} = c+1$,

$$\begin{aligned} A_{22} &= \lambda_1 (a-b) + \lambda_2 (b-c+1) + \lambda_3 (c), \\ B_{22} &= a-b+1, \\ C_{22} &= b-c+2, \\ D_{22} &= c+1, \\ A_{23} &= \lambda_1 (a-b) + \lambda_2 (b-c) + \lambda_3 (c+1), \\ B_{23} &= a-b+1, \\ C_{23} &= b-c+1, \\ D_{23} &= c+2. \end{aligned} \tag{26}$$

r^{th} moments, mean, and variance of 1^{st} order statistic: the r^{th} moment about the origin, mean, and variance of 1^{st} order statistic are

$$\begin{aligned} E(Y_1^r) &= n \sum_{w=1}^{3} \lambda_w \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} w_1^{B_{1w}-1} w_2^{C_{1w}-1} (1-w_1-w_2)^{D_{1w}-1} (A_{1w}-r)^{-1}, \\ E(Y_1) &= n \sum_{w=1}^{3} \lambda_w \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} w_1^{B_{1w}-1} w_2^{C_{1w}-1} (1-w_1-w_2)^{D_{1w}-1} (A_{1w}-1)^{-1}, \\ \operatorname{Var}(Y_1) &= n \sum_{w=1}^{3} \lambda_w \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} w_1^{B_{1w}-1} w_2^{C_{1w}-1} (1-w_1-w_2)^{D_{1w}-1} (A_{1w}-2)^{-1} \\ &\quad - \left\{ n \sum_{w=1}^{3} \lambda_w \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} w_1^{B_{1w}-1} w_2^{C_{1w}-1} (1-w_1-w_2)^{D_{1w}-1} (A_{1w}-1)^{-1} \right\}^2. \end{aligned} \tag{27}$$

r^{th} moments, mean, and variance of n^{th} order statistic: the r^{th} moment about the origin, mean, and variance of n^{th} order statistic are

$$
\begin{aligned}
E\left(Y_n^r\right) &= n\sum_{w=1}^{3}\lambda_w\sum_{a=0}^{n-1}\sum_{b=0}^{a}\sum_{c=0}^{b}(-1)^a\binom{k-1}{a}\binom{a}{b}\binom{b}{c}w_1^{B_{2w}-1}w_2^{C_{2w}-1}\left(1-w_1-w_2\right)^{D_{2w}-1}\left(A_{2w}-r\right)^{-1},\\
E\left(Y_n\right) &= n\sum_{w=1}^{3}\lambda_w\sum_{a=0}^{n-1}\sum_{b=0}^{a}\sum_{c=0}^{b}(-1)^a\binom{k-1}{a}\binom{a}{b}\binom{b}{c}w_1^{B_{2w}-1}w_2^{C_{2w}-1}\left(1-w_1-w_2\right)^{D_{2w}-1}\left(A_{2w}-1\right)^{-1},\\
\operatorname{Var}\left(Y_n\right) &= n\sum_{w=1}^{3}\lambda_w\sum_{a=0}^{n-1}\sum_{b=0}^{a}\sum_{c=0}^{b}(-1)^a\binom{k-1}{a}\binom{a}{b}\binom{b}{c}w_1^{B_{2w}-1}w_2^{C_{2w}-1}\left(1-w_1-w_2\right)^{D_{2w}-1}\left(A_{2w}-2\right)^{-1}\\
&\quad-\left\{n\sum_{w=1}^{3}\lambda_w\sum_{a=0}^{n-1}\sum_{b=0}^{a}\sum_{c=0}^{b}(-1)^a\binom{k-1}{a}\binom{a}{b}\binom{b}{c}w_1^{B_{2w}-1}w_2^{C_{2w}-1}\left(1-w_1-w_2\right)^{D_{2w}-1}\left(A_{2w}-1\right)^{-1}\right\}^2.
\end{aligned}
\tag{28}
$$

7. Parametric Estimation

Here, we discuss the parameter estimation methods. In particular, we use the ML and Bayesian methods of estimation to estimate unknown parameters under type I censored data. Before discussing the parameter estimation, we construct the likelihood function.

Assume n units, with fixed t as test termination time, from the TCMPD are used in a real-life experiment. Let $y_1, y_2, \ldots, y_\xi$ be the ordered values that can only be observed. The $n-\xi$ remaining largest values are censored from the study, i.e., their exact failure time cannot be recorded due to time constraint. So, $y_{11}, \ldots, y_{1\xi_1}$, $y_{21}, \ldots, y_{2\xi_2}$, and $y_{31}, \ldots, y_{3\xi_3}$ are failed observations relating to subpopulations I, II, and III, respectively. The observation y_ξ is assumed to be censored from each component, whereas the numbers ξ_1, ξ_2, and ξ_3 of failed values are obtained from the subpopulations. The $n_1-\xi_1$, $n_2-\xi_2$, and $n_3-\xi_3$ observations are assumed to be censored values from subpopulations, whereas $r_1+r_2+r_3=r$. So, the likelihood function using type I censored sample, $\mathbf{y}=\{(\mathbf{y}_1= y_{11}, \ldots, y_{1\xi_1})$, $(\mathbf{y}_2=y_{21}, \ldots, y_{2\xi_2})$, and $(\mathbf{y}_3=y_{31}, \ldots, y_{3\xi_3})$, for a TCMPD is $L(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y}) \propto \left\{\prod_{k=1}^{\xi_1} w_1 f_1\left(y_{1k}\right)\right\}\left\{\prod_{k=1}^{\xi_2} w_2 f_2\left(y_{2k}\right)\right\}\left\{\prod_{k=1}^{\xi_3}\left(1-w_1-w_2\right)f_3\left(y_{3k}\right)\right\}\{1-F(t)\}^{n-\xi}$.

After substitution and simplification, the likelihood function of TCMPD becomes

$$
\begin{aligned}
L(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y}) \propto{}& \sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}e^{-\lambda_1\left(\left(n-\xi-m_1\right)\ln t+\sum_{k=1}^{\xi_1}\ln y_{1k}\right)}e^{-\lambda_2\left(\left(m_1-m_2\right)\ln t+\sum_{k=1}^{\xi_2}\ln y_{2k}\right)}e^{-\lambda_3\left(\left(m_2\right)\ln t+\sum_{k=1}^{\xi_3}\ln y_{3k}\right)}\\
&\times\lambda_1^{\xi_1}\lambda_2^{\xi_2}\lambda_3^{\xi_3}w_1^{n-\xi-m_1+\xi_1}w_2^{m_1-m_2+\xi_2}\left(1-w_1-w_2\right)^{m_2+\xi_3}.
\end{aligned}
\tag{29}
$$

7.1. ML Estimators and Variances. The ML estimators of TCMPD for parameter $\Phi=(\lambda_1,\lambda_2,\lambda_3,w_1,w_2)$ are derived from the solution of nonlinear equations (30)–(34). The equations have been derived by partially differentiating the natural logarithm of the likelihood function as

$$
\frac{\partial\ln L(\boldsymbol{\Phi}|\mathbf{y})}{\partial\lambda_1}=\frac{\xi_1}{\lambda_1}-\sum_{k=1}^{r_1}\ln y_{1k}-\frac{(n-\xi)tw_1e^{-\lambda_1\ln t}}{w_1e^{-\lambda_1\ln t}+w_2e^{-\lambda_2\ln t}+\left(1-w_1-w_2\right)e^{-\lambda_3\ln t}}=0, \tag{30}
$$

$$
\frac{\partial\ln L(\boldsymbol{\Phi}|\mathbf{y})}{\partial\lambda_2}=\frac{\xi_2}{\lambda_2}-\sum_{k=1}^{r_2}\ln y_{2k}-\frac{(n-\xi)tw_2e^{-\lambda_2\ln t}}{w_1e^{-\lambda_1\ln t}+w_2e^{-\lambda_2\ln t}+\left(1-w_1-w_2\right)e^{-\lambda_3\ln t}}=0, \tag{31}
$$

$$
\frac{\partial\ln L(\boldsymbol{\Phi}|\mathbf{y})}{\partial\lambda_3}=\frac{\xi_3}{\lambda_3}-\sum_{k=1}^{r_3}\ln y_{3k}-\frac{(n-\xi)t\left(1-w_1-w_2\right)e^{-\lambda_3\ln t}}{w_1e^{-\lambda_1\ln t}+w_2e^{-\lambda_2\ln t}+\left(1-w_1-w_2\right)e^{-\lambda_3\ln t}}=0, \tag{32}
$$

$$
\frac{\partial\ln L(\boldsymbol{\Phi}|\mathbf{y})}{\partial w_1}=\frac{\xi_1}{w_1}-\frac{\xi_3}{\left(1-w_1-w_2\right)}+\frac{(n-\xi)\left\{e^{-\lambda_1\ln t}-e^{-\lambda_3\ln t}\right\}}{w_1e^{-\lambda_1\ln t}+w_2e^{-\lambda_2\ln t}+\left(1-w_1-w_2\right)e^{-\lambda_3\ln t}}=0, \tag{33}
$$

$$\frac{\partial \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_2} = \frac{\xi_2}{w_2} - \frac{\xi_3}{(1-w_1-w_2)} + \frac{(n-\xi)\left\{e^{-\lambda_2 \ln t} - e^{-\lambda_3 \ln t}\right\}}{w_1 e^{-\lambda_1 \ln t} + w_2 e^{-\lambda_2 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}} = 0. \quad (34)$$

It is very tough to find out an explicit solution from the nonlinear equations (30)–(34); therefore, to obtain ML estimators, any mathematical or statistical software such as Mathematica (Wolfram, [41]) can be used to solve them by an iterative procedure.

Let $\Phi = (\lambda_1, \lambda_2, \lambda_3, w_1, w_2)$, and by using multivariate central limit theorem, that is, $\widehat{\Phi} \sim \mathrm{N}(\Phi, I^{-1}(\Phi))$, one can get asymptotic variances, where the variance is given on diagonal of the inverted Fisher information matrix which is expectation of the negative Hessian as

$$I(\mathbf{\Phi}) = -E\begin{bmatrix} \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_1^2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_1 \partial \lambda_3} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_1 \partial w_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_1 \partial w_2} \\ \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_2^2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_2 \partial \lambda_3} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_2 \partial w_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_2 \partial w_2} \\ \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_3^2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_3 \partial w_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_3 \partial w_2} \\ \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_1 \partial \lambda_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_1 \partial \lambda_2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_1 \partial \lambda_3} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_1^2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_1 \partial w_2} \\ \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_2 \partial \lambda_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_2 \partial \lambda_2} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_2 \partial \lambda_3} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_2 p \lambda_1} & \frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_2^2} \end{bmatrix}, \quad (35)$$

where

$$\begin{aligned}
\frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_1^2} &= -\frac{\xi_1}{\lambda_1^2} + \frac{\left\{w_2 e^{-\lambda_2 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}\right\}(n-\xi)t^2 w_1 e^{-\lambda_1 \ln t}}{\left\{w_1 e^{-\lambda_1 \ln t} + w_2 e^{-\lambda_2 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}\right\}^2}, \\
\frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_2^2} &= -\frac{\xi_2}{\lambda_2^2} + \frac{\left\{w_1 e^{-\lambda_1 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}\right\}(n-\xi)t^2 w_2 e^{-\lambda_2 \ln t}}{\left\{w_1 e^{-\lambda_1 \ln t} + w_2 e^{-\lambda_2 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}\right\}^2}, \\
\frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial \lambda_3^2} &= -\frac{\xi_3}{\lambda_3^2} + \frac{\left\{w_1 e^{-\lambda_1 \ln t} + w_2 e^{-\lambda_2 \ln t}\right\}(n-\xi)t^2 (1-w_1-w_2)e^{-\lambda_3 \ln t}}{\left\{w_1 e^{-\lambda_1 \ln t} + w_2 e^{-\lambda_2 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}\right\}^2}, \\
\frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_1^2} &= -\frac{\xi_1}{w_1^2} - \frac{\xi_3}{(1-w_1-w_2)^2} - \frac{(n-\xi)\left\{e^{-\lambda_1 \ln t} - e^{-\lambda_3 \ln t}\right\}^2}{\left\{w_1 e^{-\lambda_1 \ln t} + w_2 e^{-\lambda_2 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}\right\}^2}, \\
\frac{\partial^2 \ln L(\mathbf{\Phi}|\mathbf{y})}{\partial w_2^2} &= -\frac{\xi_2}{w_2^2} - \frac{\xi_3}{(1-w_1-w_2)^2} - \frac{(n-\xi)\left\{e^{-\lambda_2 \ln t} - e^{-\lambda_3 \ln t}\right\}^2}{\left\{w_1 e^{-\lambda_1 \ln t} + w_2 e^{-\lambda_2 \ln t} + (1-w_1-w_2)e^{-\lambda_3 \ln t}\right\}^2}.
\end{aligned} \quad (36)$$

Next, we discuss the Bayesian estimation for the estimation of unknown parameters.

7.2. The Joint Prior and Posterior Distributions. Now, we discuss the Bayesian estimation of the unknown parameters. This method allows us to obtain an updated form of the knowledge which is calculated by combining the current and the prior knowledge. In particular, we use uniform and Jeffreys as noninformative priors, which are used when little or no formal prior knowledge on the parameters of concern is available. Box and Taio [42] stated a noninformative prior which gives slight information relative to the testing experiment. Bernardo and Smith [43] defined a noninformative prior has the least influence relative to the data. Jeffreys prior suggested by Jeffreys [44] is obtained by evaluating the Fisher information.

7.2.1. The Joint Prior and Posterior Distributions Assuming the UP. We take the improper UP for component parameters, i.e., $\lambda_d \sim U(0,\infty)$, $d = 1,2,3$. Also, we take the UP for the unknown proportion parameters, i.e., $w_s \sim U(0,1)$, $s = 1,2$. Under the assumption of independence of parameters $\lambda_1, \lambda_2, \lambda_3, w_1$, and w_2, the joint prior distribution is

$$\psi_1(\lambda_1,\lambda_2,\lambda_3,w_1,w_2) \propto 1. \tag{37}$$

By combining the likelihood function and the joint UP prior, i.e., $\psi_1(\lambda_1,\lambda_2,\lambda_3,w_1,w_2)$, we obtain the joint posterior distribution of parameters $\lambda_1,\lambda_2,\lambda_3,w_1$, and w_2 given data y as

$$\begin{aligned}
\xi_1(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y}) &= \frac{L(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})\psi_1(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})}{\int_{w_2}\int_{w_1}\int_{\lambda_3}\int_{\lambda_2}\int_{\lambda_1} L(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})\psi_1(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\lambda_3\mathrm{d}w_1\mathrm{d}w_2},\\
\xi_1(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y}) &= \frac{\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}e^{-B_{11}\lambda_1}e^{-B_{21}\lambda_2}e^{-B_{31}\lambda_3}w_1^{A_{01}-1}w_2^{B_{01}-1}(1-w_1-w_2)^{C_{01}-1}}{E_1\,\lambda_1^{1-A_{11}}\lambda_2^{1-A_{21}}\lambda_3^{1-A_{31}}},
\end{aligned} \tag{38}$$

where $A_{11} = \xi_1 + 1,\ A_{21} = \xi_2 + 1,\ A_{31} = \xi_3 + 1,$

$$\begin{aligned}
B_{11} &= (n-\xi-m_1)\ln t + \sum_{k=1}^{\xi_1}\ln y_{1k},\\
B_{21} &= (m_1-m_2)\ln t + \sum_{k=1}^{\xi_2}\ln y_{2k},\\
B_{31} &= (m_2)\ln t + \sum_{k=1}^{\xi_3}\ln y_{3k},\\
A_{01} &= n-\xi-m_1+\xi_1+1,\\
B_{01} &= m_1-m_2+\xi_2+1,\\
C_{01} &= m_2+\xi_3+1,\\
E_1 &= \sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})B_{11}^{-A_{11}}B_{21}^{-A_{21}}B_{31}^{-A_{31}}B(A_{01},B_{01},C_{01}).
\end{aligned} \tag{39}$$

7.2.2. The Joint Prior and Posterior Distributions Assuming the JP. The JP for component parameters λ_d, $d = 1,2,3$, is $p(\lambda_d) \propto \sqrt{|I(\lambda_d)|}$, where $I(\lambda_d)$ is the Fisher information and is $I(\lambda_d) = -E\{\partial^2 \ln L(\lambda_d;\mathbf{y}_d)/\partial\lambda_d^2\}$. The prior distributions of w_1 and w_2 are assumed as the UP, i.e., $w_s \sim U(0,1)$, $s = 1,2$. So, the joint prior distribution is

$$\psi_2(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y}) \propto \frac{1}{\lambda_1\lambda_2\lambda_3}. \tag{40}$$

Using the JP, the joint posterior distribution of λ_1, λ_2, λ_3, w_1, and w_2 that give data y is

$$\xi_2(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y}) = \frac{L(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})\psi_2(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})}{\int\limits_{w_2}\int\limits_{w_1}\int\limits_{\lambda_3}\int\limits_{\lambda_2}\int\limits_{\lambda_1} L(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})\psi_2(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y})d\lambda_1 d\lambda_2 d\lambda_3 dw_1 dw_2},$$
$$\xi_2(\lambda_1,\lambda_2,\lambda_3,w_1,w_2|\mathbf{y}) = \frac{\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}e^{-B_{12}\lambda_1}e^{-B_{22}\lambda_2}e^{-B_{32}\lambda_3}w_1^{A_{02}-1}w_2^{B_{02}-1}(1-w_1-w_2)^{C_{02}-1}}{E_2\lambda_1^{1-A_{12}}\lambda_2^{1-A_{22}}\lambda_3^{1-A_{32}}}, \tag{41}$$

where $A_{12}=\xi_1$, $A_{22}=\xi_2$, $A_{32}=\xi_3$,

$$\begin{aligned}
B_{12} &= (n-\xi-m_1)\ln t + \sum_{k=1}^{\xi_1}\ln y_{1k},\\
B_{22} &= (m_1-m_2)\ln t + \sum_{k=1}^{\xi_2}\ln y_{2k},\\
B_{32} &= (m_2)\ln t + \sum_{k=1}^{\xi_3}\ln y_{3k},\\
A_{02} &= n-\xi-m_1+\xi_1+1,\\
B_{02} &= m_1-m_2+\xi_2+1,\\
C_{02} &= m_2+\xi_3+1,\\
E_2 &= \sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})B_{12}^{-A_{12}}B_{22}^{-A_{22}}B_{32}^{-A_{32}}B(A_{02},B_{02},C_{02}).
\end{aligned} \tag{42}$$

7.3. Bayes Estimators and Posterior Variances. The Bayes estimators of the component and mixing proportion, i.e., λ_1, λ_2, λ_3, w_1, and w_2 using the UP are obtained as

$$\hat{\lambda}_\alpha|\mathbf{y} = \frac{1}{E_1}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{\alpha1}+1)\Gamma(A_{\beta1})\Gamma(A_{\gamma1})B(A_{01},C_{01})B(B_{01},A_{01}+C_{01})}{B_{\alpha1}^{(A_{\alpha1}+1)}B_{\beta1}^{A_{\beta1}}B_{\gamma1}^{A_{\gamma1}}}, \tag{43}$$

where α, β, and γ are defined as (i) $\alpha=1,\beta=2,\gamma=3$ (ii) $\alpha=2,\beta=1,\gamma=3$, and (iii) $\alpha=3,\beta=1,\gamma=2$.

$$\hat{w}_\delta|\mathbf{y} = \frac{1}{E_1}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})B(\Upsilon_{01},C_{01})B(\Lambda_{01}+1,\Upsilon_{01}+C_{01})}{B_{11}^{A_{11}}B_{21}^{A_{21}}B_{31}^{A_{31}}}, \tag{44}$$

where δ, Υ, and Λ are defined as (i) $\delta=1,\Upsilon=B,\Lambda=A$ and (ii) $\delta=2,\Upsilon=A,\Lambda=B$.

To measure the accuracy and efficiency of the Bayes estimators, we normally calculate the PVs of the parameters.

For the PVs of λ_1, λ_2, λ_3, w_1, and w_2, using the UP are derived as

$$\begin{aligned}\operatorname{Var}\left(\widehat{\lambda}_\alpha|\mathbf{y}\right) &= \frac{1}{E_1}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\binom{m_1}{m_2}\frac{\Gamma(A_{\alpha1}+2)\Gamma(A_{\beta1})\Gamma(A_{\gamma1})B(A_{01},C_{01})B(B_{01},A_{01}+C_{01})}{B_{\alpha1}^{(A_{\alpha1}+2)}B_{\beta1}^{A_{\beta1}}B_{\gamma1}^{A_{\gamma1}}}\\ &-\left\{\frac{1}{E_1}\sum_{m_1=0}^{n-r}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{\alpha1}+1)\Gamma(A_{\beta1})\Gamma(A_{\gamma1})B(A_{01},C_{01})B(B_{01},A_{01}+C_{01})}{B_{\alpha1}^{(A_{\alpha1}+1)}B_{\beta1}^{A_{\beta1}}B_{\gamma1}^{A_{\gamma1}}}\right\}^2,\\ \operatorname{Var}\left(\widehat{w}_\delta|\mathbf{y}\right) &= \frac{1}{E_1}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})B(\Upsilon_{01},C_{01})B(\Lambda_{01}+2,\Upsilon_{01}+C_{01})}{B_{11}^{A_{11}}B_{21}^{A_{21}}B_{31}^{A_{31}}}\\ &-\left\{\frac{1}{E_1}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})B(\Upsilon_{01},C_{01})B(\Lambda_{01}+1,\Upsilon_{01}+C_{01})}{B_{11}^{A_{11}}B_{21}^{A_{21}}B_{31}^{A_{31}}}\right\}^2.\end{aligned} \tag{45}$$

The Bayes estimators of λ_1, λ_2, λ_3, w_1, and w_2 using the JP are obtained as

$$\begin{aligned}\widehat{\lambda}_\alpha|\mathbf{y} &= \frac{1}{E_2}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{\alpha2}+1)\Gamma(A_{\beta2})\Gamma(A_{\gamma2})B(A_{02},C_{02})B(B_{02},A_{02}+C_{02})}{B_{\alpha2}^{(A_{\alpha2}+1)}B_{\beta2}^{A_{\beta2}}B_{\gamma2}^{A_{\gamma2}}},\\ \widehat{w}_\delta|\mathbf{y} &= \frac{1}{E_2}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})B(\Upsilon_{02},C_{02})B(\Lambda_{02}+1,\Upsilon_{02}+C_{02})}{B_{12}^{A_{12}}B_{22}^{A_{22}}B_{32}^{A_{32}}}.\end{aligned} \tag{46}$$

Also, the PVs of the parameters $\lambda_1, \lambda_2, \lambda_3, p_1$, and $p_2$using the JP are derived as

$$\begin{aligned}\operatorname{Var}\left(\widehat{\lambda}_\alpha|\mathbf{y}\right) &= \frac{1}{E_2}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{\alpha2}+2)\Gamma(A_{\beta2})\Gamma(A_{\gamma2})B(A_{02},C_{02})B(B_{02},A_{02}+C_{02})}{B_{\alpha2}^{(A_{\alpha2}+2)}B_{\beta2}^{A_{\beta2}}B_{\gamma2}^{A_{\gamma2}}}\\ &-\left\{\frac{1}{E_2}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{\alpha2}+1)\Gamma(A_{\beta2})\Gamma(A_{\gamma2})B(A_{02},C_{02})B(B_{02},A_{02}+C_{02})}{B_{\alpha2}^{(A_{\alpha2}+1)}B_{\beta2}^{A_{\beta2}}B_{\gamma2}^{A_{\gamma2}}}\right\}^2,\\ \operatorname{Var}\left(\widehat{w}_\delta|\mathbf{y}\right) &= \frac{1}{E_2}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})B(\Upsilon_{02},C_{02})B(\Lambda_{02}+2,\Upsilon_{02}+C_{02})}{B_{12}^{A_{12}}B_{22}^{A_{22}}B_{32}^{A_{32}}}\\ &-\left\{\frac{1}{E_2}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})B(\Upsilon_{02},C_{02})B(\Lambda_{02}+1,\Upsilon_{02}+C_{02})}{B_{12}^{A_{12}}B_{22}^{A_{22}}B_{32}^{A_{32}}}\right\}^2.\end{aligned} \tag{47}$$

7.4. The Bayesian Prediction. One of the main objectives of statistical modeling is the prediction of the future values. The Bayesian methodology allows us to obtain this in a natural way. In particular, the posterior predictive distribution (PPD) comprises the knowledge about future value $X = Y_{n+1}$ given data y. Al-Hussaini et al. [45], Bolstad [46], and Bansal [47] have discussed the usefulness of the prediction and predictive distribution comprehensively in the Bayesian framework.

7.4.1. The Posterior Predictive Distribution. For the future value $X = Y_{n+1}$, the PPD using the UP and the JP is

$$f(x|\mathbf{y}) = \int\limits_{w_2}\int\limits_{w_1}\int\limits_{\lambda_3}\int\limits_{\lambda_2}\int\limits_{\lambda_1} f(x \mid \lambda_1, \lambda_2, \lambda_3, w_1, w_2)\xi_v(\lambda_1, \lambda_2, \lambda_3, w_1, w_2|\mathbf{y})\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\lambda_3\mathrm{d}w_1\mathrm{d}w_2, \tag{48}$$

$$\begin{aligned} f(x|\mathbf{y}) = {} & \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})B(A_{0v}+1, C_{0v})B(B_{0v}, A_{0v}+C_{0v}+1)}{(B_{1v}+\ln x)^{(A_{1v}+1)}B_{2v}^{A_{2v}}B_{3v}^{A_{3v}}} \\ & + \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})B(A_{0v}, C_{0v})B(B_{0v}+1, A_{0v}+C_{0v}+1)}{B_{1v}^{A_{1v}}(B_{2v}+\ln x)^{(A_{2v}+1)}B_{3v}^{A_{3v}}} \\ & + \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)B(A_{0v}, C_{0v}+1)B(B_{0v}, A_{0v}+C_{0v}+1)}{B_{1v}^{A_{1v}}B_{2v}^{A_{2v}}(B_{3v}+\ln x)^{(A_{3v}+1)}}. \end{aligned} \tag{49}$$

In equation, we consider $v = 1$ for UP and $v = 2$ for JP, respectively.

7.4.2. Bayesian Predictive Intervals. To obtain the Bayesian predictive intervals (BPIs), let L and U are the two lower and upper endpoints of the BPI, which are obtained from (49). A $100(1-\alpha)\%$ BPI (L, U) using UP and JP can be obtained by simplifying the given expression:

$$\begin{aligned} & \int_1^L \left\{ \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})B(A_{0v}+1, C_{0v})B(B_{0v}, A_{0v}+C_{0v}+1)}{(B_{1v}+\ln x)^{(A_{1v}+1)}B_{2v}^{A_{2v}}B_{3v}^{A_{3v}}} \right. \\ & + \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})B(A_{0v}, C_{0v})B(B_{0v}+1, A_{0v}+C_{0v}+1)}{B_{1v}^{A_{1v}}(B_{2v}+\ln x)^{(A_{2v}+1)}B_{3v}^{A_{3v}}} \\ & \left. + \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)B(A_{0v}, C_{0v}+1)B(B_{0v}, A_{0v}+C_{0v}+1)}{B_{1v}^{A_{1v}}B_{2v}^{A_{2v}}(B_{3v}+\ln x)^{(A_{3v}+1)}} \right\}\mathrm{d}x = \frac{\alpha}{2}, \\ & \int_U^\infty \left\{ \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})B(A_{0v}+1, C_{0v})B(B_{0v}, A_{0v}+C_{0v}+1)}{(B_{1v}+\ln x)^{(A_{1v}+1)}B_{2v}^{A_{2v}}B_{3v}^{A_{3v}}} \right. \\ & + \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})B(A_{0v}, C_{0v})B(B_{0v}+1, A_{0v}+C_{0v}+1)}{B_{1v}^{A_{1v}}(B_{2v}+\ln x)^{(A_{2v}+1)}B_{3v}^{A_{3v}}} \\ & \left. + \frac{1}{xE_v}\sum_{m_1=0}^{n-\xi}\sum_{m_2=0}^{m_1}\binom{n-\xi}{m_1}\binom{m_1}{m_2}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)B(A_{0v}, C_{0v}+1)B(B_{0v}, A_{0v}+C_{0v}+1)}{B_{1v}^{A_{1v}}B_{2v}^{A_{2v}}(B_{3v}+\ln x)^{(A_{3v}+1)}} \right\}\mathrm{d}x = \frac{\alpha}{2}. \end{aligned} \tag{50}$$

8. Monte Carlo Simulation

Here, we tabulate a comprehensive simulation to check the performance of different estimation methods. As an analytical comparison of the Bayes and ML estimators is not possible, a Monte Carlo simulation is performed to measure the performance of the Bayes and ML estimators under different aspects. Through the following steps, we obtained the maximum likelihood estimates (MLEs), MLVs, Bayes estimates (BEs), and PVs as

(1) A sample of size n from mixtures may be taken as

  (i) Generate a sample of $w_1 n$ values randomly from $f_1(y;\lambda_1)$

  (ii) Generate a sample of $w_2 n$ values randomly from $f_2(y;\lambda_2)$

  (iii) Generate a sample of $(1-w_1-w_2)n$ values randomly from $f_3(y;\lambda_3)$

(2) Select the values that are larger than fixed t as censored values

(3) For a large number of times, say 1000, repeat steps 1 and 2 for the fixed t, n, and parametric values

(4) Evaluate the MLEs and MLVs of λ_1, λ_2, λ_3, p_1, and p_2 using nonlinear equations based on the samples obtained in step 3

(5) Calculate the BEs and PVs based on 1000 repetitions

The above procedure is adopted for 50, 100, and 200 as sample sizes n, parametric values $(\lambda_1,\lambda_2,\lambda_3,w_1,w_2) = \{(3,4,5,0.5,0.3),(5,6,7,0.5,0.3)\}$, and t = 1.5, 2. The choice of t = 1.5, 2 was done to have 8–20% censored rate in the resulting sample. Moreover, to have a perceptible indication about the behavior of the MLEs, BEs, MLVs, and PVs, we depicted some graphs in Figures 1–4 and Tables 2 and 3.

From Figures 1–4, it is observed that parameters λ_1, λ_2, λ_3, and w_2 are overestimated, but w_1 is underestimated at different values of t and n in both estimation methods, i.e., ML and Bayesian. Also, the degree of underestimation of λ_1, λ_2, λ_3, w_1, and w_2 is higher for a small n at various values of t, and an opposite behavior was observed for a large t at a given n. Furthermore, the parameters λ_1, λ_2, λ_3, w_1, and w_2 were observed overestimated to a larger extent when the true values of λ_1, λ_2, and λ_3 were smaller at different values of t for a fixed n. In addition, the similar pattern has been observed at different values of n for a fixed t. The difference of the MLEs and BEs of parameters λ_1, λ_2, λ_3, w_1, and w_2 from the nominal values becomes the minimum with the increase of t and n.

It can be seen from Tables 2 and 3 that at different values of t, the difference between the MLVs and the PVs (assuming the UP and the JP) diminishes by increasing sample size. The same remark is true for a large t at different values of n. Also, noticed that the MLVs and PVs of w_1 and w_2 are larger for smaller values of λ_1, λ_2, and λ_3 at different values of t and n. Also, it is pointed out that the performance of the Bayes estimators using JP is best than Bayes estimators using UP and ML estimators based on lesser associated PVs.

To the extent that the selection of an appropriate prior, Tables 2 and 3 revealed that the JP outperforms as compared to the UP because the variance of JP is smaller than the UP.

Table 4 and Figure 5 showcase the 90% BPI using the UP and the JP. It is pointed out that the width of 90% BPI increases with a decrease in n. The same conclusion was observed with a smaller t for varying values of n. The 90% BPIs, for larger values of λ_1, λ_2, and λ_3, were observed narrow at various values of t and n. Moreover, the BPIs using the JP were observed wider than the BPIs obtained by assuming the UP in the simulation study.

9. Real Data Application

To illustrate the proposed methodology, the mixture lifetime data, $\mathbf{z} = (z_{11}, z_{12}, \ldots, z_{1r_1}, z_{21}, z_{22}, \ldots, z_{2r_2}, z_{31}, z_{32}, \ldots, z_{3r_3})$ in thousand hours, was taken from Davis [48] on three factors, i.e., V805 Transmitter, Transmitter, and V600 Indicator Tube used in aircraft sets. For exponential distributed mixture data ($\mathbf{z}$), the suitable transformation $y = \exp(z)$ gives the Pareto distributed mixture data ($\mathbf{y}$). So, the mention transformation permits to utilize the given mixture data $\mathbf{z}$ for using the suggested ML and Bayesian estimation techniques. Thus, the proposed mixture of the Pareto distributions can be a fair choice to model the abovementioned data. Moreover, it is unidentified that which factor fails until a failure arises at or before 0.6 hours. To calculate the MLEs, MLVs, BEs, and PVs, the data summary is

$$\begin{aligned} n &= 1340, \\ \sum_{k=1}^{r_1} \ln(y_{1k}) &= \sum_{k=1}^{r_1} z_{1k} = 134.080, \\ r_1 &= 866, \\ \sum_{k=1}^{r_2} \ln(y_{2k}) &= \sum_{k=1}^{r_2} z_{2k} = 50.375, \\ r_2 &= 337, \\ \sum_{k=1}^{r_3} \ln(y_{3k}) &= \sum_{k=1}^{r_3} z_{3k} = 16.250, \\ r_3 &= 83, \\ r &= 1286, \\ n - r &= 54. \end{aligned} \tag{51}$$

The MLEs, MLVs, BEs, and PVs assuming the UP and the JP are presented in Table 5.

From Table 5, it is clear that the performance of the BEs using JP is the best as compared to the MLEs, as the variance of BEs is smaller than the counterpart. Moreover, the BEs using the UP have smaller variances for estimating the unknown parameters. Also, the JP was observed superior to the UP due to smaller associated PV for estimating the unknown parameters.

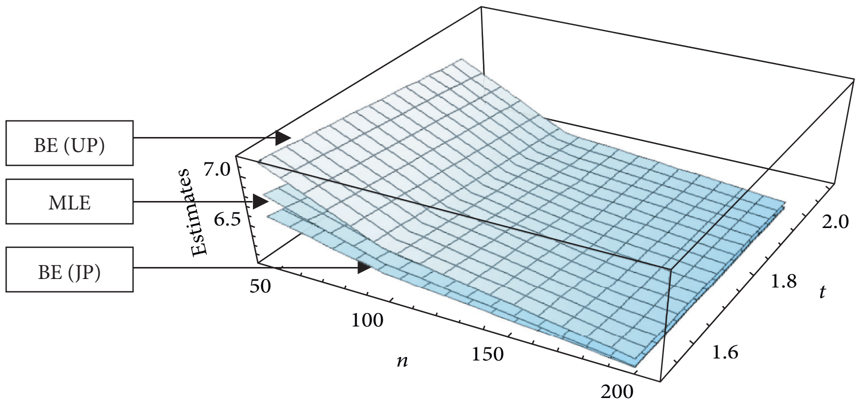

Figure 2: The graph of the BE and MLE of parameter $\lambda_2 = 6$.

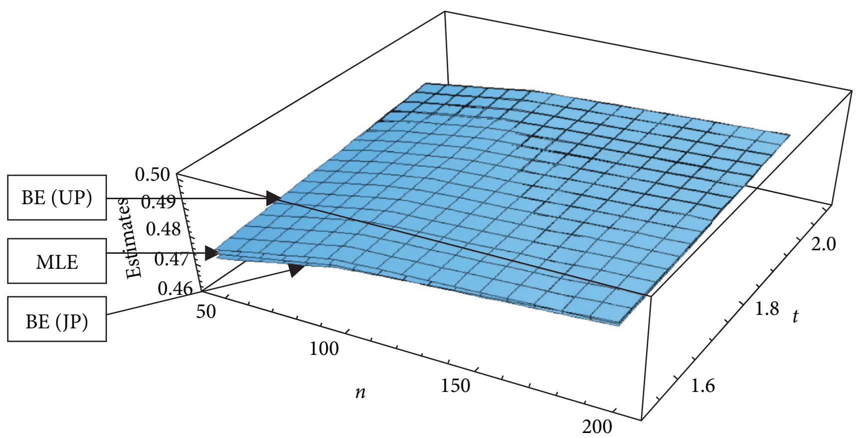

Figure 3: The graph of the BE and MLE of parameter $w_1 = 0.5$.

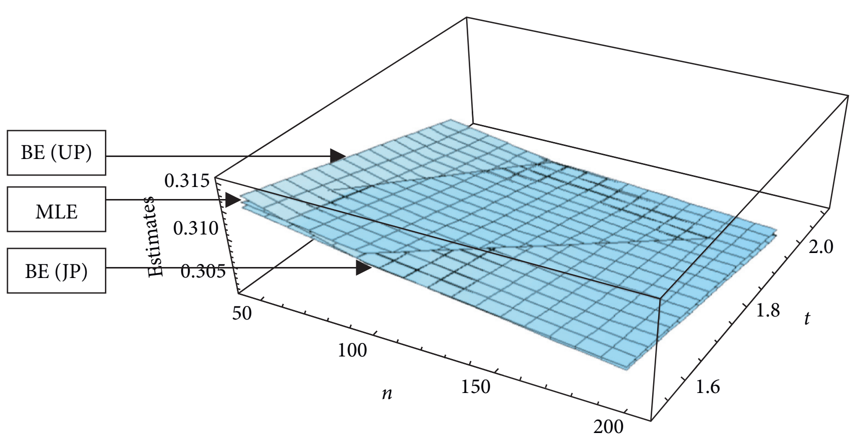

Figure 4: The graphs of the BE and MLE of parameter $w_2 = 0.3$.

Table 2: MLVs and PVs with $\lambda_1 = 3, \lambda_2 = 4, \lambda_3 = 5, w_1 = 0.5, w_2 = 0.3$, and $t = 1.5, 2$.

Variances	t	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{w}_1$	$\hat{w}_2$
MLV	1.5	50	2.320214	4.715420	8.914210	0.007794	0.006861
		100	1.070912	2.394020	4.312200	0.004701	0.003987
		200	0.504841	1.220180	2.175420	0.002699	0.002267
	2	50	0.877500	2.312550	4.832860	0.005245	0.004459
		100	0.402951	1.069200	2.242510	0.002806	0.002371
		200	0.182921	0.540710	1.085260	0.001448	0.001219
PV (UP)	1.5	50	2.516370	5.525710	10.38160	0.008028	0.006968
		100	1.110690	2.500540	4.549870	0.004774	0.004055
		200	0.504418	1.260720	2.263810	0.002761	0.002303
	2	50	0.930862	2.429000	5.280310	0.005255	0.004462
		100	0.418080	1.109120	2.276240	0.002809	0.002375
		200	0.183232	0.562381	1.114090	0.001450	0.001219
PV (JP)	1.5	50	2.248490	4.678600	8.667130	0.007740	0.006805
		100	1.052080	2.354530	4.287430	0.004681	0.003981
		200	0.503192	1.214890	2.149440	0.002698	0.002259
	2	50	0.859103	2.257300	4.765290	0.005241	0.004456
		100	0.400117	1.067430	2.225230	0.002805	0.002369
		200	0.182605	0.543210	1.074960	0.001448	0.001218

Table 3: MLVs and PVs with $\lambda_1 = 5$, $\lambda_2 = 6$, $\lambda_3 = 7, w_1 = 0.5$, $w_2 = 0.3$, and $t = 1.5$, 2.

Variances	t	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{w}_1$	$\hat{w}_2$
MLV	1.5	50	2.672150	5.994210	11.45520	0.005459	0.004672
		100	1.221460	2.898720	5.197530	0.002947	0.002483
		200	0.572900	1.419940	2.621400	0.001546	0.001293
	2	50	1.484210	3.532100	7.425120	0.004728	0.003996
		100	0.679930	1.635540	3.394550	0.002462	0.002075
		200	0.323310	0.806990	1.557430	0.001256	0.001057
PV (UP)	1.5	50	2.815860	6.274300	12.69820	0.005485	0.004675
		100	1.243740	2.991410	5.479060	0.002965	0.002492
		200	0.597448	1.450400	2.658400	0.001556	0.001306
	2	50	1.514350	3.867450	8.387800	0.004731	0.004000
		100	0.696689	1.700300	3.472690	0.002463	0.002076
		200	0.329232	0.816373	1.593910	0.001257	0.001057
PV (JP)	1.5	50	2.625140	5.725880	11.20090	0.005451	0.004670
		100	1.217280	2.814820	5.102490	0.002941	0.002480
		200	0.571080	1.410380	2.592280	0.001542	0.001289
	2	50	1.462580	3.461830	7.182980	0.004727	0.003995
		100	0.677330	1.607840	3.346260	0.002461	0.002075
		200	0.323280	0.806709	1.547360	0.001256	0.001056

Table 4: The BPI (L, U) with $\lambda_1 = 3, \lambda_2 = 4, \lambda_3 = 5,$, $w_1 = 0.5, w_2 = 0.3$, and $t = 1.5, 2$.

t	n	UP		JP	
		L	U	L	U
1.5	50	1.0124	2.7082	1.0134	3.0004
	100	1.0108	2.4246	1.0112	2.7094
	200	1.0129	2.3866	1.0133	2.5610
2	50	1.0130	2.5227	1.0139	2.6982
	100	1.0116	2.3975	1.0121	2.4609
	200	1.0132	2.3405	1.0134	2.3687

Table 5: MLEs, MLVs, BEs, and PVs assuming the UP and the JP with real-life data.

Estimates and variances	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{w}_1$	$\hat{w}_2$
MLE	5.6702	5.9699	3.5854	0.6689	0.2594
MLV	0.0514	0.1649	0.3810	0.00018146	0.00015131
BE (UP)	5.6673	5.9813	3.6470	0.6691	0.2594
PV (UP)	0.0519	0.1653	0.3939	0.00018126	0.00015134
BE (JP)	5.6721	5.9664	3.5539	0.6687	0.2594
PV (JP)	0.0512	0.1645	0.3799	0.00018151	0.00015129

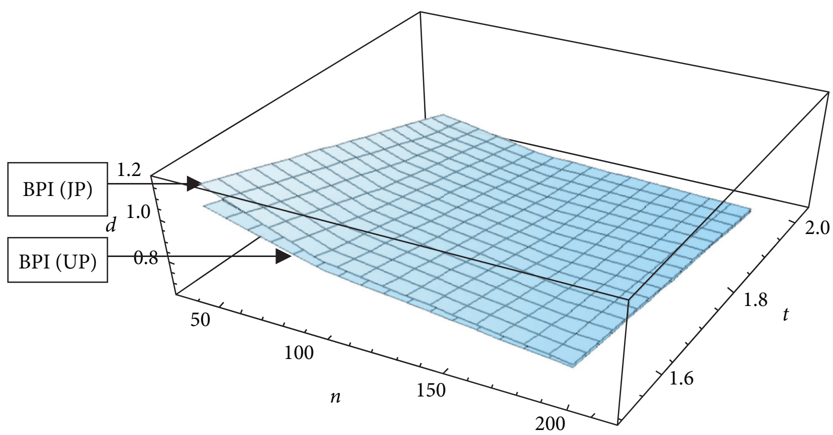

Figure 5: The graph of widths ($d = L - U$) of the BPIs with $\lambda_1 = 5$, $\lambda_2 = 6$, $\lambda_3 = 7$, $w_1 = 0.05, w_2$ and $t = 1.5$, 2.

10. Conclusion

In this study, we proposed TCMPD to model lifetime data. Parameter estimation assuming the type I censoring have been considered using the ML and Bayesian estimation methods. For Bayesian estimation, we assumed the non-informative priors and expressions of the Bayes estimators for the mixing proportion (w_1 and w_2) and component parameters (λ_1, λ_2, and λ_3) and PVs were derived. To examine the relative presentation of the Bayes and ML estimators under different scenario, a Monte Carlo simulation has been done. To illustrate a practical presentation of proposed mixture distribution, an example has also been analyzed.

From simulated results and depicted graphs, it has been noticed that an increase in t under a fixed n yields very efficient Bayes and ML estimators. It is also pointed that parameters $\lambda_1, \lambda_2, \lambda_3, w_1$, and w_2 are overestimated (underestimated) to a small (larger) extent with relatively larger (smaller) value of n (value of t). More specifically, the amount of overestimation (underestimation) of parameters is smaller for a relatively large parameter value. As the value of n (value of t) increases (decreases), the PVs decrease (increase) for a fixed t (fixed n). To address the problem of selection of a suitable prior, one can observe that the JP has smaller PVs than the UP. The results depend on real-life mixture data that also support the Monte Carlo simulation study. Finally, it is concluded that the Bayes estimators using the JP performed better as compared to the ML estimators because of smaller variances.

Data Availability

The dataset used to support the findings of this study is available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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