Research Article

Improved Lower Bound of LFMD with Applications of Prism-Related Networks

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The different distance-based parameters are used to study the problems in various fields of computer science and chemistry such as pattern recognition, image processing, integer programming, navigation, drug discovery, and formation of different chemical compounds. In particular, distance among the nodes (vertices) of the networks plays a supreme role to study structural properties of networks such as connectivity, robustness, completeness, complexity, and clustering. Metric dimension is used to find the locations of machines with respect to minimum utilization of time, lesser number of the utilized nodes as places of the objects, and shortest distance among destinations. In this paper, lower bound of local fractional metric dimension for the connected networks is improved from unity and expressed in terms of ratio obtained by the cardinalities of the under-study network and the local resolving neighbourhood with maximum order for some edges of network. In the same context, the LFMDs of prism-related networks such as circular diagonal ladder, antiprism, triangular winged prism, and sun flower networks are computed with the help of obtained criteria. At the end, the bounded- and unboundedness of the obtained results is also shown numerically.

1. Introduction

For a connected network G , Salter introduced the concept of resolving (locating) set with the cardinality of minimum resolving set which is called the location number of G [1]. Harary and Melter introduced the concept of metric dimension for the connected networks [2]. The concept of metric independence number $mi(G)$ of a graph G is introduced by Currie and Oellermann [3]. The metric dimension has been applied to solve the problems involving percolation in hierarchical lattice [4], coin weighting, and robot navigation [5]. It is also applied in subject of chemistry to find the structures of chemical compounds having similar characteristics in functional groups. These functional groups play a vital role in chemical and pharmaceutical industries to predict the various chemical properties of the molecular

compounds that are used in the drug discovery [6]. Metric dimension of graph was formulated as integer programming problem by Charterand et al. [6]. Fehar et al. studied the metric dimension of Cayley digraphs [7]. For further studies of metric dimension of convex polytopes, Cayley and Toeplitz networks, see [8–11].

Currie and Oellermann defined the concept of fractional metric dimension (FMD) as an optimal solution of the linear relaxation of the integer programming problem (IPP) [3]. Later, Faher et al. presented the identical calculation of IPP with the help of FMD [7]. Arguman and Matthew introduced many different properties of FMD for connected networks with respect to their order [12]. FMDs of hierarchical product of graphs were computed by Feng and Wang [13]. Liu et al. [14] computed the FMD of generalized Jahangir graph. The concept of local fractional metric

dimension (LFMD) is introduced by Aisyah et al. [15]. They also computed it for the connected networks which are obtained by the operation of the corona product. The results for the LFMDs of some cycle-related networks and rotationally symmetric and planar networks can be found in [16, 17]. Javaid et al. (2020) computed the sharp bounds of LFMD of connected networks and illustrated the obtained results with the help of wheel-related networks. They also compared the bounded- and unboundedness of the obtained results [18].

In this paper, lower bound of LFMD for connected networks is improved from unity and expressed in terms of ratio obtained by the cardinalities of the under-study network and the local resolving neighbourhood with maximum order for some edges of network. In the outcome of the obtained result, the LFMDs of prism-related networks as exact values and sharp bounds are computed. The rest of the article is organised as follows: Section 2 consists the preliminaries, Section 3 consists of main results of LFMD of connected networks, Section 4 deals with the local resolving neighbourhoods of prism-related networks, Section 5 presents LFMD of prism-related networks and Section 6 consists of conclusion and comparison among the main results.

2. Preliminaries

Let $G = (V(G), E(G))$ be a network with $V(G)$ and $E(G)$ as set of vertices and edges, respectively. A walk is defined as a sequence of alternating vertices and edges. A walk in which the vertices are all distinct is a path between vertices a and b and a closed path is called a cycle. For any two vertices a and b of G , the distance $d(a, b)$ is the length of shortest path $a \sim b$ in G . A pair of vertices a and b in a network is a connected pair if there is a path between them and the network is a connected network. For a connected network G and $e = ab \in E(G)$, a vertex $x \in V(G)$ distinguishes two vertices a and b if $d(x, a) \neq d(x, b)$ is known as symmetric vertex. Moreover, x resolves the edge e in G if $d(x, a) \neq d(x, b)$. For $D = \{a_1, a_2, a_3, \dots, a_n\} \subseteq V(G)$ and $x \in V(G)$, the k -tuple metric form of D in terms of x is given by $r(x|D) = d(x, a_1), d(x, a_2), d(x, a_3), \dots, d(x, a_n)$. The set D becomes resolving set having n elements of graph G if each pair of vertices in G bears a distinct metric form with respect to D . The resolving set with least number of vertices is referred as metric basis for G and cardinality of such resolving set is called metric dimension of G defined by

$$\dim(G) = \min\{|D|: D \text{ is resolving set of } G\}. \quad (1)$$

For an edge $ab \in E(G)$, the local resolving neighbourhood (LRN) is defined as $LR(ab) = \{x \in V(G): d(x, a) \neq d(x, b)\}$, where $x \in V(G)$. A function is called an upper local resolving function (ULRF) if $f: V(G) \rightarrow [0, 1]$ and $f(LR(ab)) \geq 1$ for each $LR(ab)$ of G , where $f(LR(ab)) = \sum_{x \in LR(ab)} f(x)$. On the other hand, a function is called lower local resolving function (LLRF) if $\Psi: V(G) \rightarrow [0, 1]$ and $\Psi(LR(ab)) \leq 1$ for each $LR(ab)$ of G , where $\Psi(LR(ab)) = \sum_{x \in LR(ab)} \Psi(x)$. Then, LFMD is

defined as $\dim_{lf}(G) = \tau$, where τ is $\min\{|f|: f \text{ which is the upper local minimal resolving function of } G\}$ or $\max\{|g|: g \text{ is the lower local maximal resolving function of } G\}$.

For $1 \leq i \leq n$, now we present some prism-related networks. The circular diagonal ladder (CDL_n) is obtained from prism network D_n of order $2n$ and size $5n$ by adding some double crossing edges $a_i b_{(i+1)}$ and $a_{(i+1)} b_i$, as shown in Figure 1. The antiprism network (A_n) of order $2n$ and size $4n$ is obtained by prism network (D_n) by adding some crossing edges $a_i b_{(i+1)}$, see Figure 2. The sun flower network (SF_n) of order $2n$ and size $3n$, we mean a network, is isomorphic to the network obtained from A_n by deleting edges $b_i b_{(i+1)}$, see Figure 3 [19].

Theorem 1 (see [18]). *Let $G = (V(G), E(G))$ be a connected network. Let $LR(e)$ be a local resolving neighbourhood for the edge e of G . If $|LR(e) \cap Z| \geq \alpha$, $\forall e \in E(G)$, then*

$$1 \leq \dim_{lf}(G) \leq \frac{|Z|}{\alpha}, \quad (2)$$

where $Z = \cup \{LR(e): |LR(e)| = \alpha\}$, $\alpha = \min\{|LR(e)|: e \in E(G)\}$, and $2 \leq \alpha \leq |V(G)|$.

Proposition 1 (see [18]). *Let $G = (V(G), E(G))$ be a connected network. For each $e \in E(G)$, if $|LR(e) \cap Z| \geq 2$, then $\dim_{lf}(G) = (|Z|/2)$, where $Z = \cup \{LR(e): |LR(e)| = 2\}$, and $LR(e)$ is a LRN set of $e \in E(G)$.*

3. Main Results

Main results of LFMD are as follows.

Proposition 2. *Let $G = (V(G), E(G))$ be a connected network and $LR(e)$ be the local resolving neighbourhood set of the edge e of G . For $\beta = \max\{|LR(e)|: e \in E(G)\}$, if $Y = \cup \{LR(e): |LR(e)| = \beta\} \subseteq V(G)$, then $|LR(e) \cap Y| \leq \beta$ for each local resolving neighbourhood $LR(e)$ of G .*

Theorem 2. *Let $G = (V(G), E(G))$ be a connected network and $LR(e)$ be the local resolving neighbourhood set. Then,*

$$\frac{|V(G)|}{\beta} \leq \dim_{lf}(G), \quad (3)$$

where $\beta = \max\{|LR(e)|: e \in E(G)\}$ and $2 \leq \beta \leq |V(G)|$.

Proof. Define $\Phi: V(G) \rightarrow [0, 1]$ as $\phi(v) = (1/\beta)$ for $v \in V(G)$. By Proposition 2, for $e \in E(G)$, we have

$$\phi(LR(e)) = \sum_{x \in LR(e)} \phi(x) = \sum_{x \in LR(e) \cap V(G)} \frac{1}{\beta} = |LR(e) \cap V(G)| \frac{1}{\beta} \leq 1. \quad (4)$$

This shows that ϕ is a lower local resolving function (LLRF). To show that ϕ is maximal, suppose on contrary, there exists another LLRF Ψ such that $\Psi(x) \geq \Phi(x)$, where $\Psi(x) \neq \Phi(x)$, for at least one $x \in V(G)$. $\forall x \in LR(e)$ such that $|LR(e)| = \beta$, we have

FIGURE 1: Circular diagonal ladder CDL_n . FIGURE 2: Antiprism A_n . FIGURE 3: Sun flower network SF_n .

$$\Psi(LR(e)) = \sum_{x \in LR(e)} \Psi(x) > \sum_{x \in LR(e)} \phi(x) = 1. \quad (5)$$

Thus, $\Psi(LR(e)) > 1$. This shows that Ψ is not LLRF and consequently ϕ is maximal LLRF. Let ϕ' be another maximal LLRF of G . Then,

$$|\phi'| = \sum_{x \in V(G)} \phi'(x). \quad (6)$$

Now, we consider three cases (i) $\phi'(x) > (1/\beta)$ for each $x \in V(G)$, (ii) $\phi'(x) \leq (1/\beta)$ for each $x \in V(G)$, and (iii) $\phi'(x) > (1/\beta)$ for some $x \in V(G)$.

Case 1: if $\phi'(x) > (1/\beta)$, for each $x \in V(G)$. For $LR(e) \subseteq Y$ such that $|LR(e)| = \beta$, we have $\phi'(LR(e)) > 1$. This shows ϕ' is not LLRF. Thus, this case does not hold.

Case 2: let $\phi'(x) \leq (1/\beta) \forall x \in V(G)$. Then,

$$|\phi'| = \sum_{x \in V(G)} \phi'(x) \leq \frac{|V(G)|}{\beta} = |\phi|. \quad (7)$$

Consequently,

$$\dim_{lf}(G) = \frac{|V(G)|}{\beta}. \quad (8)$$

Case 3: assume that $\phi'(x) > (1/\beta)$ for some $x \in V(G)$. Suppose that $S = \{t \in V(G) : \phi'(t) > (1/\beta)\}$ and $Y = \bigcup \{LR(e) : |LR(e)| = \beta\}$. We note that $S \cap Y = \emptyset$; otherwise, for $|LR(e)| = \beta$, $\phi'(LR(x)) > 1$ which implies that ϕ' is not a LLRF. Consider

$$|\phi'| = \sum_{x \in V(G)} \phi'(x) = \sum_{x \in Y} \phi'(x) + \sum_{x \in V(G)-Y} \phi'(x). \quad (9)$$

Since $\sum_{x \in V(G)-Y} \phi'(x) \geq \sum_{x \in V(G)-Y} \phi(x)$, therefore,

$$\begin{aligned} |\phi'| &= \sum_{x \in V(G)} \phi'(x) = \sum_{x \in Y} \phi'(x) + \sum_{x \in V(G)-Y} \phi'(x) \geq \sum_{x \in Y} \phi(x) \\ &+ \sum_{x \in V(G)-Y} \phi(x) = \frac{|V(G)|}{\beta} = |\phi|. \end{aligned} \quad (10)$$

Consequently,

$$\dim_{lf}(G) = |\phi'| \geq |\phi| = \frac{|V(G)|}{\beta}. \quad (11)$$

Thus, from all the cases,

$$\frac{|V(G)|}{\beta} \leq \dim_{lf}(G), \text{ which completes the proof.} \quad (12)$$

Now, we present the following two corollaries as the direct consequences of the above result.

Corollary 1. Let $G = (V(G), E(G))$ be a connected network, $LR(e)$ be LRN of $e \in E(G)$, $\beta = \max\{|LR(e)| : e \in E(G)\}$,

$\alpha = \min\{|LR(e)|: e \in E(G)\}$, and $X = \cup \{LR(e): |LR(e)| = \alpha\}$. If $\alpha = \beta$ and $X = V(G)$, then

$$\dim_{lf}(G) = \frac{|V(G)|}{\beta}. \quad (13)$$

Proof. Since $\alpha = \beta$ and $X = V(G)$, therefore, by Theorem A, $\dim_{lf}(G) = (|V(G)|/\beta)$. Also, by Theorem 1, $\dim_{lf}(G) = (|V(G)|/\beta)$. Consequently,

$$\dim_{lf}(G) = \frac{|V(G)|}{\beta}. \quad (14)$$

Corollary 2. Let $G = (V(G), E(G))$ be a connected network, $LR(e)$ be the LRN of $e \in E(G)$, $\beta = \max\{|LR(e)|: e \in E(G)\}$, $\alpha = \min\{|LR(e)|: e \in E(G)\}$, and $X = \cup \{LR(e): |LR(e)| = \alpha\}$. If $\alpha = |X|$ and $\beta = |V(G)|$, then $\dim_{lf}(G) = 1$.

Proof. Since $\beta = |V(G)|$, then by Theorem 2, $1 \leq \dim_{lf}(G)$. Also, as $\alpha = |X|$, therefore, by Theorem 1, $\dim_{lf}(G) \leq 1$. Consequently, $\dim_{lf}(G) = 1$.

Remark 1. Corollary 2 strengthens the result proved in [18].

4. LRNs of the Prism-Related Networks

In this section, the local resolving neighbourhoods of prism-related networks are classified.

Lemma 1. Let CDL_n with $n \geq 4$ be a circular diagonal ladder network, where $n \equiv 0 \pmod{2}$ and $|V(CDL_n)| = 2n$, where $1 \leq i \leq n$. We have

- (a) $|LR(e_i)| = |LR(a_i b_i)| = 2$ and $|\cup_{i=1}^n LR(e_i)| = 2n$,
- (b) $|LR(e_i)| = |LR(b_i b_{(i+1)})|$ and $|LR(b_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (c) $|LR(e_i)| = |LR(a_i a_{(i+1)})|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (d) $|LR(e_i)| = |LR(a_i b_{(i+1)})|$ and $|LR(a_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (e) $|LR(e_i)| = |LR(b_i a_{(i+1)})|$ and $|LR(b_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (f) $|LR(e_i)| = |LR(a_i b_{(i-1)})|$ and $|LR(a_i b_{(i-1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$.

Proof. Assume that a_i and b_i are inner and outer vertices, respectively, for $1 \leq i \leq n$ and $n+1 \equiv 1 \pmod{n}$. We have the following:

- (a) Consider $LR(a_i b_i) = \{a_i, b_i\}$ with $|LR(e_i)| = |LR(a_i b_i)| = 2$. Moreover, $|\cup_{i=1}^n LR(e_i)| = |V(CDL_n)| = 2n$.
- (b) $LR(b_i b_{(i+1)}) = V(CDL_n) - \{a_i, a_{(i+1)}\}$ with $|LR(b_i b_{(i+1)})| = 2n - 2 > 2 = |LR(e_i)|$ and $|LR(b_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > 2 = |LR(e_i)|$.

- (c) $LR(a_i a_{(i+1)}) = V(CDL_n) - \{b_i, b_{(i+1)}\}$ with $|LR(a_i a_{(i+1)})| = 2n - 2 > 2 = |LR(e_i)|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > 2 = |LR(e_i)|$.
- (d) $LR(a_i b_{(i+1)}) = V(CDL_n) - \{a_{(i+1)}, b_i\}$ with $|LR(a_i b_{(i+1)})| = 2n - 2 > 2 = |LR(e_i)|$ and $|LR(a_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > 2 = |LR(e_i)|$.
- (e) $LR(b_i a_{(i+1)}) = V(CDL_n) - \{a_{(i+1)}, b_i\}$ with $|LR(b_i a_{(i+1)})| = 2n - 2 > 2 = |LR(e_i)|$ and $|LR(b_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > 2 = |LR(e_i)|$.
- (f) $LR(b_i a_{(i-1)}) = V(CDL_n) - \{a_{(i-1)}, b_{(i+1)}\}$ with $|LR(b_i a_{(i-1)})| = 2n - 2 > 2 = |LR(e_i)|$ and $|LR(b_i a_{(i-1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > 2 = |LR(e_i)|$.

Lemma 2. Let CDL_n with $n \geq 5$ be a circular diagonal ladder network, where $n \equiv 1 \pmod{2}$ and $|V(CDL_n)| = 2n$, where $1 \leq i \leq n$. We have

- (a) $|LR(e_i)| = |LR(a_i b_i)| = 2$ and $|\cup_{i=1}^n LR(e_i)| = 2n$,
- (b) $|LR(e_i)| < |LR(b_i b_{(i+1)})|$ and $|LR(b_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (c) $|LR(e_i)| < |LR(a_i a_{(i+1)})|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (d) $|LR(e_i)| < |LR(a_i b_{(i+1)})|$ and $|LR(a_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (e) $|LR(e_i)| < |LR(b_i a_{(i+1)})|$ and $|LR(b_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (f) $|LR(e_i)| < |LR(a_i b_{(i-1)})|$ and $|LR(a_i b_{(i-1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$.

Proof. Assume that a_i and b_i are inner and outer vertices, respectively, for $1 \leq i \leq n$ and $n+1 \equiv 1 \pmod{n}$, $(3n+1/2) \equiv 1 \pmod{n}$. We have the following:

- (a) Consider $LR(a_i b_i) = \{a_i, b_i\}$ with $|LR(e_i)| = |LR(a_i b_i)| = 2$. Moreover, $|\cup_{i=1}^n LR(e_i)| = |V(CDL_n)| = 2n$.
- (b) As $LR(b_i b_{(i+1)}) = V(CDL_n) - \{a_{(n+2i+1/2)}, a_{(n+2i+1/2)}, a_i, a_{(i+1)}\}$ with $|LR(b_i b_{(i+1)})| = 2n - 4 > 2 = |LR(e_i)|$ and $|LR(b_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 4 > 2 = |LR(e_i)|$.
- (c) As $LR(a_i a_{(i+1)}) = V(CDL_n) - \{a_{(n+2i+1/2)}, b_{(n+2i+1/2)}, b_i, b_{(i+1)}\}$ with $|LR(a_i a_{(i+1)})| = 2n - 4 > 2 = |LR(e_i)|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 4 > 2 = |LR(e_i)|$.
- (d) As $LR(a_i b_{(i+1)}) = V(CDL_n) - \{a_{(n+2i+1/2)}, b_{(n+2i+1/2)}, a_{(i+1)}, b_i\}$ with $|LR(a_i b_{(i+1)})| = 2n - 4 > 2 = |LR(e_i)|$ and $|LR(a_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 4 > 2 = |LR(e_i)|$.
- (e) As $LR(b_i a_{(i+1)}) = V(CDL_n) - \{a_{(i)}, b_{(i+1)}, a_{(n+2i+1/2)}, b_{(n+2i+1/2)}\}$ with $|LR(b_i a_{(i+1)})| = 2n - 4 > 2 = |LR(e_i)|$ and $|LR(b_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 4 > 2 = |LR(e_i)|$.
- (f) As $LR(a_i b_{(i-1)}) = V(CDL_n) - \{a_{i-1}, b_{i-1}, a_{(n+2i+1/2)}, b_{(n+2i+1/2)}\}$ with $|LR(a_i b_{(i-1)})| = 2n - 4 > 2 = |LR(e_i)|$.

and $|LR(b_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 4 > 2 = |LR(e_i)|$.

Lemma 3. Let A_n with $n \geq 4$ be an antiprism network, where $n \equiv 0 \pmod{2}$ and $|V(A_n)| = 2n$ and $1 \leq i \leq n$. We have

- (a) $|LR(e_i)| = |LR(a_i b_i)|$ and $|LR(a_i b_{(i+1)})| = n$ and $|\cup_{i=1}^n LR(e_i)| = 2n$,
- (b) $|LR(e_i)| = |LR(b_i b_{(i+1)})|$ and $|LR(b_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (c) $|LR(e_i)| = |LR(a_i a_{(i+1)})|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$.

Proof. Assume that a_i and b_i are inner and outer vertices, respectively, for $1 \leq i \leq n$, where $n+1 \equiv 1 \pmod{2}$, $(3n+2/2) \equiv 1 \pmod{n}$, and $(3n/2) \equiv 1 \pmod{n}$. We have

- (a) $LR(e_i) = LR(a_i b_i) = \{b_i, b_{(n+2i+2/2)}, b_{(n+2i+2/2)+1}, b_{(n+2i+2/2)+2}, \dots, b_{(n+i-1)}, a_i, a_{(i+1)}, a_{(i+2)}, \dots, a_{(n+2i-2/2)}\}$ and $LR(a_i b_{(i+1)}) = \{b_{(i+1)}, b_{(i+2)}, b_{(i+3)}, \dots, b_{(2i+n/2)}, a_i, a_{(n+2i+2/2)}, a_{(n+2i+2/2)+1}, a_{(n+2i+2/2)+2}, \dots, a_{(n+i-1)}\}$ with $|LR(e_i)| = n$. Moreover, $|\cup_{i=1}^n LR(e_i)| = |V(A_n)| = 2n$,
- (b) As $LR(b_i b_{(i+1)}) = V(A_n) - \{a_i, a_{(n+2i/2)}\}$ with $|LR(b_i b_{(i+1)})| = 2n - 2 > n = |LR(e_i)|$ and $|LR(b_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > n = |LR(e_i)|$,
- (c) As $LR(a_i a_{(i+1)}) = V(A_n) - \{b_{(i+2)}, b_{(n+2i+2/2)}\}$ with $|LR(a_i a_{(i+1)})| = 2n - 2 > n = |LR(e_i)|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > n = |LR(e_i)|$.

Lemma 4. Let A_n with $n \geq 3$ be an antiprism network, where $|V(A_n)| = 2n$ with $n \equiv 1 \pmod{2}$. For $1 \leq i \leq n$, we have

- (a) $|LR(e_i)| = |LR(a_i b_i)|$ and $|LR(a_i b_{(i+1)})| = n+1$ and $|\cup_{i=1}^n LR(e_i)| = 2n$,
- (b) $|LR(e_i)| = |LR(b_i b_{(i+1)})|$ and $|LR(b_i b_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$,
- (c) $|LR(e_i)| = |LR(a_i a_{(i+1)})|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$.

Proof. Assume that a_i and b_i are inner and outer vertices, respectively, for $1 \leq i \leq n$, where $n+1 \equiv 1 \pmod{2}$, $(3n+1/2) \equiv 1 \pmod{n}$, $(3n-1/2) \equiv 1 \pmod{n}$, and $2n-1 \equiv 1 \pmod{2}$. We have

- (a) $LR(a_i b_i) = \{a_i, a_{(i+1)}, a_{(i+2)}, \dots, a_{(n+2i-1/2)}, b_i, b_{(n+2i+1/2)}, b_{(n+2i+3/2)}, \dots, b_{(n+i-1)}\}$ and $LR(a_i b_{(i+1)}) = \{b_{(i+1)}, b_{(i+2)}, \dots, b_{(n+2i+1/2)}, a_i, a_{(n+2i+1/2)}, a_{(n+2i+1/2)+1}, \dots, a_{(n+i-1)}\}$ with $|LR(e_i)| = n+1$. Moreover, $|\cup_{i=1}^n LR(e_i)| = |V(A_n)| = 2n$.

- (b) $LR(a_i a_{(i+1)}) = V(A_n) - \{(a_{(n+2i+1/2)}, b_{(i+1)})\}$ and $|LR(a_i a_{(i+1)})| = 2n - 2 > n+1 = |LR(e_i)|$.
- (c) $LR(b_i b_{(i+1)}) = V(A_n) - \{a_i, b_{(n+2i+1/2)}\}$ and $|LR(b_i b_{(i+1)})| = 2n - 2 > n+1 = |LR(e_i)|$.

Lemma 5. Let SF_n with $n \geq 3$ be a sun flower network, where $|V(SF_n)| = 2n$ and $n \equiv 1 \pmod{2}$. For $1 \leq i \leq n$, we have

- (a) $|LR(e_i)| = |LR(a_i b_i)|$ and $|LR(b_i a_{(i+1)})| = n+1$ and $|\cup_{i=1}^n LR(e_i)| = 2n$,
- (b) $|LR(e_i)| = |LR(a_i a_{(i+1)})|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$.

Proof. Assume that a_i and b_i are inner and outer vertices, respectively, for $1 \leq i \leq n$, where $n+1 \equiv 1 \pmod{n}$, $n+2 \equiv 1 \pmod{n}$, $(3n+1/2) \equiv 1 \pmod{n}$, $(3n-1/2) \equiv 1 \pmod{n}$, and $2n-1 \equiv 1 \pmod{n}$. Now, we have

- (a) $LR(a_i b_i) = \{a_i, a_{(n+2i+1/2)}, a_{(n+2i+1/2)+1}, \dots, a_{(n+i-1)}, b_i, b_{(n+2i+1/2)}, b_{(n+2i+1/2)+1}, \dots, b_{(n+i-1)}\}$ and $LR(b_i a_{(i+1)}) = \{b_i, b_{(i+1)}, b_{(i+2)}, \dots, b_{((n+2i+1/2))}, a_{(i+1)}, a_{(i+2)}, a_{(i+3)}, \dots, a_{(n+i-1)}\}$ with $|LR(e_i)| = n+1$. Moreover, $|\cup_{i=1}^n LR(e_i)| = |V(SF_n)| = 2n$.
- (b) $LR(a_i a_{(i+1)}) = V(SF_n) - \{a_{(n+2i+1/2)}, b_i\}$ and $|LR(a_i a_{(i+1)})| = 2n - 2$ with $|LR(a_i a_{(i+1)})| = 2n - 2 > n+1 = |LR(e_i)|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > n+1 = |LR(e_i)|$.

Lemma 6. Let SF_n with $n \geq 4$ be a sun flower network, where $|V(SF_n)| = 2n$ and $n \equiv 0 \pmod{2}$. For $1 \leq i \leq n$, we have

- (a) $|LR(e_i)| = |LR(a_i b_i)|$ and $|LR(b_i a_{(i+1)})| = n+1$ and $|\cup_{i=1}^n LR(e_i)| = 2n$,
- (b) $|LR(e_i)| = |LR(a_i a_{(i+1)})|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| \geq |LR(e_i)|$.

Proof. Assume that a_i and b_i are inner and outer vertices, respectively, for $1 \leq i \leq n$, where $n+1 \equiv 1 \pmod{n}$, $n+2 \equiv 1 \pmod{n}$, $(3n+1/2) \equiv 1 \pmod{n}$, $(3n/2) \equiv 1 \pmod{n}$, and $2n-1 \equiv 1 \pmod{n}$. We have

- (a) $2n-1 \equiv 1 \pmod{n}$ and $LR(b_i a_{(i+1)}) = \{b_i, b_{(i+1)}, b_{(i+2)}, \dots, b_{(n+2i/2)}, a_{(i+1)}, a_{(i+2)}, a_{(i+3)}, \dots, a_{(n+2i/2)}\}$ with $|LR(e_i)| = n+1$. Moreover, $|\cup_{i=1}^n LR(e_i)| = |V(SF_n)| = 2n$.
- (b) $LR(a_i a_{(i+1)}) = V(SF_n) - \{b_{(n+2i/2)}, b_i\}$ and $|LR(a_i a_{(i+1)})| = 2n - 2$ with $|LR(a_i a_{(i+1)})| = 2n - 2 > n+1 = |LR(e_i)|$ and $|LR(a_i a_{(i+1)}) \cap \cup_{i=1}^n LR(e_i)| = 2n - 2 > n+1 = |LR(e_i)|$.

5. LFMD of Prism-Related Networks

The LFMD of prism-related networks is computed as follows.

Theorem 3. Let CDL_n with $n \geq 4$ be a circular diagonal ladder network and $|V(CDL_n)| = 2n$. Then, $\dim_{lf}(CDL_n) = n$.

Proof. Case 1: For $n = 4$, LRNs are as follows:

$$\begin{aligned} LR_1 &= LR(b_1b_2) = V(CDL_4) - \{a_1, a_2\}, \\ LR_2 &= LR(b_2b_3) = V(CDL_4) - \{a_2, a_3\}, \\ LR_3 &= LR(b_3b_4) = V(CDL_4) - \{a_3, a_4\}, \\ LR_4 &= LR(b_4b_1) = V(CDL_4) - \{a_4, a_1\}, \\ LR_5 &= LR(a_1a_2) = V(CDL_4) - \{b_1, b_2\}, \\ LR_6 &= LR(a_2a_3) = V(CDL_4) - \{b_2, b_3\}, \\ LR_7 &= LR(a_3a_4) = V(CDL_4) - \{b_3, b_4\}, \\ LR_8 &= LR(a_4a_1) = V(CDL_4) - \{b_4, b_1\}, \\ LR_9 &= LR(a_1b_2) = V(CDL_4) - \{a_2, b_1\}, \\ LR_{10} &= LR(a_2b_3) = V(CDL_4) - \{a_3, b_2\}, \\ LR_{11} &= LR(a_3b_4) = V(CDL_4) - \{a_4, b_3\}, \\ LR_{12} &= LR(a_4b_1) = V(CDL_4) - \{a_1, b_4\}, \\ LR_{13} &= LR(a_2b_1) = V(CDL_4) - \{a_1, b_2\}, \\ LR_{14} &= LR(a_3b_2) = V(CDL_4) - \{a_2, b_3\}, \\ LR_{15} &= LR(a_4b_3) = V(CDL_4) - \{a_3, b_4\}, \\ LR_{16} &= LR(a_1b_4) = V(CDL_4) - \{a_4, b_1\}, \\ LR_{17} &= LR(b_1a_2) = V(CDL_4) - \{b_2, a_1\}, \\ LR_{18} &= LR(b_2a_3) = V(CDL_4) - \{b_3, a_2\}, \\ LR_{19} &= LR(b_3a_4) = V(CDL_4) - \{b_4, a_3\}, \\ LR_{20} &= LR(b_4a_1) = V(CDL_4) - \{b_1, a_4\}, \\ LR_{21} &= LR(a_1b_1) = \{a_1, b_1\}, \\ LR_{22} &= LR(a_2b_2) = \{a_2, b_2\}, \\ LR_{23} &= LR(a_3b_3) = \{a_3, b_3\}, \\ LR_{24} &= LR(a_4b_4) = \{a_4, b_4\}. \end{aligned}$$

As, for $1 \leq i \leq 4$, $|LR(e_i)|$ is 2 such that $|LR(e_i)| < |LR(e_s)|$, where $1 \leq s \leq 20$. Moreover, $\bigcup_{i=1}^4 LR(e_i) = 8$ and $|LR_s \cap \bigcup_{i=1}^4 LRe_i| = |LRe_i| = 2$. Define $\phi: V(CDL_4) \rightarrow [0, 1]$ as $\phi(x) = (1/2)$ is a LRF with minimum cardinality for each $x \in V(CDL_4)$. Consequently, by Proposition 1, $\dim_{lf}(CDL_4) = (|V(CDL_4)|/2) = 4$.

Case 2: For $n = 5$, LRNs are as follows:

$$\begin{aligned} LR_1 &= LR(b_1b_2) = V(CDL_5) - \{b_4, a_4, a_1, a_2\}, \\ LR_2 &= LR(b_2b_3) = V(CDL_5) - \{b_5, a_5, a_2, a_3\}, \\ LR_3 &= LR(b_3b_4) = V(CDL_5) - \{b_6, a_6, a_3, a_4\}, \\ LR_4 &= LR(b_4b_5) = V(CDL_5) - \{b_2, a_2, a_4, a_5\}, \\ LR_5 &= LR(b_5b_1) = V(CDL_5) - \{b_3, a_1, a_3, a_5\}, \\ LR_6 &= LR(a_1a_2) = V(CDL_5) - \{b_4, a_4, b_1, b_2\}, \\ LR_7 &= LR(a_2a_3) = V(CDL_5) - \{b_5, a_5, b_2, b_3\}, \end{aligned}$$

$$\begin{aligned} LR_8 &= LR(a_3a_4) = V(CDL_5) - \{b_1, a_1, b_3, b_4\}, \\ LR_9 &= LR(a_4a_5) = V(CDL_5) - \{b_2, a_2, b_4, b_5\}, \\ LR_{10} &= LR(a_5a_1) = V(CDL_5) - \{b_1, a_3, b_3, b_5\}, \\ LR_{11} &= LR(a_1b_2) = V(CDL_5) - \{b_4, a_4, b_1, a_2\}, \\ LR_{12} &= LR(a_2b_3) = V(CDL_5) - \{b_5, a_5, b_2, a_3\}, \\ LR_{13} &= LR(a_3b_4) = V(CDL_5) - \{b_1, a_4, b_3, a_4\}, \\ LR_{14} &= LR(a_4b_5) = V(CDL_5) - \{b_4, a_4, b_1, a_2\}, \\ LR_{15} &= LR(a_5b_1) = V(CDL_5) - \{b_3, a_3, b_5, a_6\}, \\ LR_{16} &= LR(a_1b_5) = \{a_1, a_2, a_4, b_2, b_4, b_4\}, \\ LR_{17} &= LR(a_2b_1) = \{a_2, a_3, a_5, b_1, b_3, b_5\}, \\ LR_{18} &= LR(a_3b_2) = \{a_2, a_3, a_4, b_1, b_2, b_4\}, \\ LR_{19} &= LR(a_4b_3) = \{a_2, a_4, a_5, b_2, b_3, b_5\}, \\ LR_{20} &= LR(a_5b_4) = \{a_1, a_3, a_5, b_1, b_3, b_4\}, \\ LR_{21} &= LR(b_1a_2) = \{a_2, a_3, a_5, b_1, b_3, b_5\}, \\ LR_{22} &= LR(b_2a_3) = \{a_1, a_3, a_4, b_1, b_2, b_4\}, \\ LR_{23} &= LR(b_3a_4) = \{a_2, a_4, a_5, b_2, b_3, b_5\}, \\ LR_{24} &= LR(b_4a_5) = \{a_1, a_3, a_5, b_1, b_3, b_4\}, \\ LR_{25} &= LR(b_5a_1) = \{a_1, a_2, a_4, b_2, b_4, b_5\}, \\ LR_{26} &= LR(a_1b_1) = \{a_1, b_1\}, \\ LR_{27} &= LR(a_2b_2) = \{a_2, b_2\}, \\ LR_{28} &= LR(a_3b_3) = \{a_3, b_3\}, \\ LR_{29} &= LR(a_4b_4) = \{a_4, b_4\}, \\ LR_{30} &= LR(a_5b_5) = \{a_5, b_5\}. \end{aligned}$$

As, for $1 \leq i \leq 5$, $|LRN(e_i)| = 2$ such that $|LR(e_i)| < |LR(e_s)|$, where $1 \leq s \leq 25$. Moreover, $\bigcup_{i=1}^5 LR(e_i) = V(CDL_5)$; this implies $|\bigcup_{i=1}^5 LR(e_i)| = 10$ and $|LR_s \cap \bigcup_{i=1}^5 LRe_i| \geq |LRe_i| = 2$. Define $\phi: V(CDL_5) \rightarrow [0, 1]$ such that $\phi(x) = (1/2)$ is the LRF with minimum cardinality for each $x \in V(CDL_5)$. Consequently, by Proposition 1, $\dim_{lf}(CDL_5) = (|V(CDL_5)|/2) = 5$.

Case 3: For $n \geq 6$ and $1 \leq i \leq n$ by Lemma 1, $|LR(e_i)| = 2 = \alpha$ and $|R \cap \bigcup_{i=1}^n LR(e_i)| \geq 2$, where R are all other LRNs. Define $\phi: V(CDL_n) \rightarrow [0, 1]$ such that ϕ is the LRF with minimum cardinality and $|\phi| < |\phi'|$, as $\phi(v) = (1/2) \forall v \in V(CDL_n)$. $V(CDL_n) = \bigcup \{LR(e): |LR(e)| = 2\}$. Consequently, by Proposition 1, $\dim_{lf}(CDL_n) = (|V(CDL_n)|/2) = (2n/2) = n$.

Theorem 4. Let A_n with $n \geq 3$ be an antiprism network, where $n \equiv 1 \pmod{2}$ and $|V(A_n)| = 2n$. Then, $\dim_{lf}(A_n)$,

$$\frac{n}{n-1} \leq \dim_{lf}(A_n) \leq \frac{2n}{n+1}. \quad (15)$$

Proof. Case 1: For $n = 3$, LRNs are as follows:

$$\begin{aligned} LR_1 &= LR(a_1a_2) = \{a_1, a_2, b_1, b_3\}, \\ LR_2 &= LR(a_2a_3) = \{a_2, a_3, b_1, b_2\}, \\ LR_3 &= LR(a_3a_1) = \{a_3, a_1, b_2, b_3\}, \end{aligned}$$

$$\text{LR}_4 = \text{LR}(a_1b_1) = \{a_1, a_2, b_1, b_3\},$$

$$\text{LR}_5 = \text{LR}(a_2b_2) = \{a_2, a_3, b_1, b_2\},$$

$$\text{LR}_6 = \text{LR}(a_3b_3) = \{a_1, a_3, b_2, b_3\},$$

$$\text{LR}_7 = \text{LR}(a_1b_2) = \{a_1, a_3, b_1, b_2\},$$

$$\text{LR}_8 = \text{LR}(a_2b_3) = \{a_1, a_2, b_1, b_3\},$$

$$\text{LR}_9 = \text{LR}(a_3b_1) = \{a_2, a_3, b_1, b_2\}.$$

As, for $1 \leq i \leq 9$, $|\text{LR}(e_i)| = 4$. Moreover, $\cup_{i=1}^9 \text{LR}(e_i) = V(A_3) = 8$ such that $|\cup_{i=1}^9 \text{LR}(e_i)| = 6$ and $|\text{LRN}_s \cap \cup_{i=1}^9 \text{LR}(e_i)| > |\text{LR}(e_i)| = 4$. Define $\phi: V(A_3) \rightarrow [0, 1]$ as $\phi(x) = (1/4)$ is the LRF with minimum cardinality. Consequently, by Corollary 1, $\dim_{lf}(A_3) = (|V(A_3)|/4) = (3/2)$.

Case 2: For $n = 5$, LRNs are as follows:

$$\text{LR}_1 = \text{LR}(b_1b_2) = V(A_5) - \{a_1, b_4\},$$

$$\text{LR}_2 = \text{LR}(b_2b_3) = V(A_5) - \{a_3, b_5\},$$

$$\text{LR}_3 = \text{LR}(b_3b_4) = V(A_5) - \{a_3, b_1\},$$

$$\text{LR}_4 = \text{LR}(b_4b_5) = V(A_5) - \{a_4, b_2\},$$

$$\text{LR}_5 = \text{LR}(b_5b_1) = V(A_5) - \{a_5, b_3\},$$

$$\text{LR}_6 = \text{LR}(a_1a_2) = V(A_5) - \{a_4, b_2\},$$

$$\text{LR}_7 = \text{LR}(a_2a_3) = V(A_5) - \{a_5, b_3\},$$

$$\text{LR}_8 = \text{LR}(a_3a_4) = V(A_5) - \{a_1, b_4\},$$

$$\text{LR}_9 = \text{LR}(a_4a_5) = V(A_5) - \{a_2, b_5\},$$

$$\text{LR}_{10} = \text{LR}(a_5a_1) = V(A_5) - \{a_3, b_1\},$$

$$\text{LR}_{11} = \text{LR}(b_1a_1) = \{b_1, b_4, b_5, a_1, a_2, a_3\},$$

$$\text{LR}_{12} = \text{LR}(b_2a_2) = \{b_2, b_5, b_1, a_2, a_3, a_4\},$$

$$\text{LR}_{13} = \text{LR}(b_3a_3) = \{b_3, b_1, b_2, a_3, a_4, a_5\},$$

$$\text{LR}_{14} = \text{LR}(b_4a_4) = \{b_4, b_2, b_1, a_4, a_5, a_1\},$$

$$\text{LR}_{15} = \text{LR}(b_5a_5) = \{b_5, b_2, b_3, a_5, a_1, a_2\},$$

$$\text{LR}_{16} = \text{LR}(a_1b_2) = \{a_1, a_4, a_5, b_2, b_3, b_4\},$$

$$\text{LR}_{17} = \text{LR}(a_2b_3) = \{a_2, a_5, a_1, b_3, b_4, b_5\},$$

$$\text{LR}_{18} = \text{LR}(a_3b_4) = \{a_3, a_1, a_2, b_4, b_5, b_1\},$$

$$\text{LR}_{19} = \text{LR}(a_4b_5) = \{a_4, a_2, a_1, b_5, b_1, b_2\},$$

$$\text{LR}_{20} = \text{LR}(a_5b_1) = \{a_5, a_2, a_3, b_1, b_2, b_3\}.$$

As, for $1 \leq i \leq 10$, $|\text{LR}(e_i)| = 6$ such that $|\text{LR}(e_i)| < |\text{LR}(e_s)|$ and $1 \leq s \leq 10$. $\cup_{i=1}^{10} \text{LR}(e_i) = V(A_5)$; this implies $|\cup_{i=1}^{10} \text{LR}(e_i)| = 10$ and $|\text{LRN} \cap \cup_{i=1}^{10} \text{LR}(e_i)| > |\text{LR}(e_i)| = 6$. Define $(\phi: V(A_5) \rightarrow [0, 1])$ as $\phi(x) = (1/6)$ is the LRF with minimum cardinality for each $x \in V(A_5)$. Consequently, by Theorem 1, $\dim_{lf}(A_5) \leq (|V(A_5)|/6) \leq (5/3)$.

As, for $1 \leq i \leq 10$, $|\text{LR}(e_i)| = 8$, $|\text{LR}(e_i)| > |\text{LR}(e_s)|$, and $1 \leq s \leq 10$. Moreover, $\cup_{i=1}^{10} \text{LR}(e_i) = V(A_5)$. Define $(\phi': V(A_5) \rightarrow [0, 1])$ as $\phi'(x) = (1/8)$ is the LRF with minimum cardinality for each $x \in V(A_5)$. Therefore, by Theorem 2, $\dim_{lf}(A_5) \geq (|V(A_5)|/8) \geq (5/4)$. Consequently,

$$\frac{5}{4} \leq \dim_{lf}(A_5) \leq \frac{5}{3}. \quad (16)$$

Case 3: For $n \geq 6$ and $1 \leq i \leq n$ by Lemma 4, $|\text{LR}(e_i)| = n+1 = \alpha$ and $|R \cap \cup_{i=1}^n \text{LR}(e_i)| \geq n+1$, where R are all

other LRNs. There exists an ULRF $\phi: V(A_n) \rightarrow [0, 1]$, $|\phi| < |\phi'|$ which is defined as $\phi(v) = (1/\alpha) \forall v \in V(A_n)$, as $V(A_n) = \cup \{\text{LR}(e): |\text{LR}(e) = \alpha\}$. Consequently, by Theorem 1, $\dim_{lf}(A_n) \leq (|V(A_n)|/n+1) \leq (2n/n+1)$.

Case 4: For $|\text{LR}(b_i b_{i+1})| = 2n-2 \geq |\text{LR}(e_i)|$, moreover, $\cup_{i=1}^{2n} \text{LR}(e_i) = V(A_n)$. There exists a LLRF such that $\phi': V(A_n) \rightarrow [0, 1]$ is defined as $\phi'(x) = (1/2n-2)$ for each $x \in V(A_n)$. Therefore, by Theorem 2, we have $\dim_{lf}(A_n) \geq (n/n-1)$. Consequently,

$$\frac{n}{n-1} \leq \dim_{lf}(A_n) \leq \frac{2n}{n+1}. \quad (17)$$

Theorem 5. Let A_n with $n \geq 4$ be an antiprism network, where $n \equiv 0 \pmod{2}$ and $|V(A_n)| = 2n$. Then,

$$\frac{n}{n-1} \leq \dim_{lf}(A_4) \leq 2. \quad (18)$$

Proof. Case 1: For $n = 4$, LRNs are as follows:

$$\text{LR}_1 = \text{LR}(b_1b_2) = V(A_4) - \{a_1, a_3\},$$

$$\text{LR}_2 = \text{LR}(b_2b_3) = V(A_4) - \{a_2, a_4\},$$

$$\text{LR}_3 = \text{LR}(b_3b_4) = V(A_4) - \{a_1, a_3\},$$

$$\text{LR}_4 = \text{LR}(b_4b_1) = V(A_4) - \{a_2, a_4\},$$

$$\text{LR}_5 = \text{LR}(a_1a_2) = V(A_4) - \{b_2, b_4\},$$

$$\text{LR}_6 = \text{LR}(a_2a_3) = V(A_4) - \{b_3, b_1\},$$

$$\text{LR}_7 = \text{LR}(a_3a_4) = V(A_4) - \{b_2, b_4\},$$

$$\text{LR}_8 = \text{LR}(a_4a_1) = V(A_4) - \{b_1, b_3\},$$

$$\text{LR}_9 = \text{LR}(b_1a_1) = \{b_1, b_4, a_1, a_2\},$$

$$\text{LR}_{10} = \text{LR}(b_2a_2) = \{b_1, b_2, a_2, a_3\},$$

$$\text{LR}_{11} = \text{LR}(b_3a_3) = \{b_2, b_3, a_3, a_4\},$$

$$\text{LR}_{12} = \text{LR}(b_4a_4) = \{b_3, b_4, a_1, a_4\},$$

$$\text{LR}_{13} = \text{LR}(b_2a_1) = \{b_2, b_3, a_1, a_4\},$$

$$\text{LR}_{14} = \text{LR}(b_3a_2) = \{b_3, b_4, a_1, a_2\},$$

$$\text{LR}_{15} = \text{LR}(b_4a_3) = \{b_1, b_4, a_2, a_3\},$$

$$\text{LR}_{16} = \text{LR}(b_1a_4) = \{b_1, b_2, a_3, a_4\}.$$

As, for $1 \leq i \leq 8$, $|\text{LR}(e_i)| = 4$ such that $|\text{LR}(e_i)| < |\text{LR}(e_s)|$, where $1 \leq s \leq 8$. Moreover, $\cup_{i=1}^8 \text{LR}(e_i) = V(A_4)$ and $|\text{LR}_s \cap \cup_{i=1}^8 \text{LR}(e_i)| > |\text{LR}(e_i)| = 4$. There exists an ULRF $\phi: V(A_4) \rightarrow [0, 1]$ defined as $\phi(x) = (1/4)$ which is the LRF with minimum cardinality $\forall x \in V(A_4)$. Consequently, by Theorem 1, $\dim_{lf}(A_4) \leq (|V(A_4)|/4) \leq 2$.

As, for $1 \leq i \leq 8$, $|\text{LR}(e_i)| = 6$ such that $|\text{LR}(e_i)| > |\text{LR}(e_s)|$, where $1 \leq s \leq 8$. Moreover, $\cup_{i=1}^8 \text{LR}(e_i) = V(A_4)$. There exists a LLRF $\phi': V(A_4) \rightarrow [0, 1]$ which is defined as $\phi'(x) = (1/6)$ for each $x \in V(A_4)$. Therefore, by Theorem 1, we have $\dim_{lf}(A_4) \geq (|V(A_4)|/6) \geq (4/3)$. Consequently,

$$\frac{4}{3} \leq \dim_{lf}(A_4) \leq 2. \quad (19)$$

TABLE 1: LFMD of networks.

Networks	LFMDs	Comment
Antiprism network $[A_n \text{ for } n \equiv 0 \pmod{2}]$	$(n/n - 1) \leq \dim_{lf}(A_n) \leq 2.$	Bounded
Antiprism network A_n , where $n \equiv 1 \pmod{2}$	$(n/n - 1) \leq \dim_{lf}(A_n) \leq (2n/n + 1).$	Bounded
Sun flower network SF_n .	$(n/n - 1) \leq \dim_{lf}(SF_n) \leq (2n/n + 1).$	Bounded

Case 2: For $n \geq 5$, $1 \leq i \leq n$, by Lemma 3, $|\text{LR}(e_i)| = n = \alpha$ and $|R \cap \bigcup_{i=1}^n \text{LR}(e_i)| \geq n$, where R are all other LRNs.

There exists an ULRF $\phi: V(A_n) \rightarrow [0, 1]$ which is defined as $\phi(v) = (1/\alpha) \forall v \in V(A_n)$, as $V(G) = \bigcup \{|\text{LR}(e)|: |\text{LR}(e) = \alpha|\}$. Consequently, by Theorem 1, $\dim_{lf}(A_n) \leq (|V(A_n)|/n) \leq (2n/n) = 2$.

Case 3: As, $|\text{LR}(a_i a_{i+1})| = 2n - 2$, which is greater or equal to the cardinalities of all other LRNs, moreover, $\bigcup_{i=1}^{2n} \text{LR}(e_i) = V(A_n)$. There exists a LLRF $(\phi': V(A_n) \rightarrow [0, 1])$ which is defined as $\phi'(x) = (1/2n - 2) \forall v \in V(A_n)$. Therefore, by Theorem 2, $\dim_{lf}(A_n) \geq (n/n - 1)$. Consequently,

$$\frac{n}{n-1} \leq \dim_{lf}(A_n) \leq 2. \quad (20)$$

Theorem 6. Let SF_n with $n \geq 3$ be a sun flower network and $|V(SF_n)| = 2n$. Then,

$$\frac{n}{n-1} \leq \dim_{lf}(SF_n) \leq \frac{2n}{n+1}. \quad (21)$$

Proof. Case 1: For $n = 3$, LRNs are as follows:

$$\begin{aligned} \text{LR}_1 &= \text{LR}(a_1 a_2) = \{a_1, a_2, b_2, b_3\}, \\ \text{LR}_2 &= \text{LR}(a_2 a_3) = \{a_2, a_3, b_3, b_1\}, \\ \text{LR}_3 &= \text{LR}(a_3 a_1) = \{a_3, a_1, b_1, b_2\}, \\ \text{LR}_4 &= \text{LR}(a_1 b_1) = \{a_1, a_3, b_1, b_3\}, \\ \text{LR}_5 &= \text{LR}(a_2 b_2) = \{a_1, a_2, b_1, b_2\}, \\ \text{LR}_6 &= \text{LR}(a_3 b_3) = \{a_2, a_3, b_2, b_3\}, \\ \text{LR}_7 &= \text{LR}(a_2 b_1) = \{b_1, b_2, a_2, a_3\}, \\ \text{LR}_8 &= \text{LR}(a_3 b_2) = \{a_1, a_3, b_2, b_3\}, \\ \text{LR}_9 &= \text{LR}(a_1 b_3) = \{a_1, a_2, b_1, b_3\}. \end{aligned}$$

As, for $1 \leq i \leq 9$, $|\text{LR}(e_i)| = 4$ and $|\text{LRNs} \cap \bigcup_{i=1}^9 \text{LR}(e_i)| = |\text{LR}(e_i)| = 4$. There exists an ULRF $\phi: V(SF_3) \rightarrow [0, 1]$ which is defined by $\phi(x) = (1/4)$, $\forall v \in V(SF_3)$. Consequently, by Corollary 1, $\dim_{lf}(SF_3) = (3/2)$.

Case 2: For $n = 4$, LRNs are as follows:

$$\begin{aligned} \text{LR}_1 &= \text{LR}(a_1 a_2) = \{a_1, a_2, a_3, a_4, b_2, b_4\}, \\ \text{LR}_2 &= \text{LR}(a_2 a_3) = \{a_1, a_2, a_3, a_4, b_1, b_3\}, \\ \text{LR}_3 &= \text{LR}(a_3 a_4) = \{a_1, a_2, a_3, a_4, b_2, b_4\}, \\ \text{LR}_4 &= \text{LR}(a_4 a_1) = \{a_1, a_2, a_3, a_4, b_1, b_3\}, \\ \text{LR}_5 &= \text{LR}(a_1 b_1) = \{a_1, a_4, b_1, b_2, b_4\}, \\ \text{LR}_6 &= \text{LR}(a_2 b_2) = \{a_1, a_2, b_1, b_2, b_4\}, \\ \text{LR}_7 &= \text{LR}(a_3 b_3) = \{a_2, a_3, b_1, b_2, b_3\}, \end{aligned}$$

$$\begin{aligned} \text{LR}_8 &= \text{LR}(a_4 b_4) = \{a_3, a_4, b_2, b_3, b_4\}, \\ \text{LR}_9 &= \text{LR}(a_2 b_1) = \{a_2, a_3, b_1, b_2, b_3\}, \\ \text{LR}_{10} &= \text{LR}(a_3 b_2) = \{a_3, a_4, b_2, b_3, b_4\}, \\ \text{LR}_{11} &= \text{LR}(a_4 b_3) = \{a_1, a_4, b_2, b_3, b_4\}, \\ \text{LR}_{12} &= \text{LR}(a_1 b_4) = \{a_1, a_2, b_1, b_2, b_4\}. \end{aligned}$$

As, for $1 \leq i \leq 8$, $|\text{LR}(e_i)| = 5$, such that $|\text{LR}(e_i)| < |\text{LR}(e_s)|$, where $1 \leq s \leq 4$. Moreover, $\bigcup_{i=1}^8 \text{LR}(e_i) = V(SF_4)$. There exists $(\phi: V(SF_4) \rightarrow [0, 1])$ defined as $\phi(x) = (1/5)$ which is the LRF $\forall v \in V(SF_4)$, as $V(SF_4) = \bigcup \{|\text{LR}(e_i)|: |\text{LR}(e) = 5|\}$. Consequently, by Theorem 1, $\dim_{lf}(SF_4) \leq (8/5)$.

As, for $1 \leq i \leq 4$, $|\text{LR}(e_i)| = 6$ such that $|\text{LR}(e_i)| > |\text{LR}(e_s)|$, where $1 \leq s \leq 8$. There exists $\phi': V(SF_4) \rightarrow [0, 1]$ defined as $\phi(v) = (1/6) \forall v \in V(SF_4)$, which is the LLRF with maximum cardinality, where $|\phi'| < |\phi''|$, as $V(SF_4) = \bigcup \{|\text{LR}(e)|: |\text{LR}(e) = 6|\}$. By Theorem 1, $\dim_{lf}(SF_4) \geq (4/3)$. Consequently,

$$\frac{4}{3} \leq \dim_{lf}(SF_4) \leq \frac{8}{5}. \quad (22)$$

Case 3: As $n \geq 5$, $1 \leq i \leq n$, $|\text{LR}(e_i)| = n + 1 = \alpha$, and $|R \cap \bigcup_{i=1}^n \text{LR}(e_i)| \geq n + 1$, where R are all other LRNs. There exists an ULRF $\phi: V(SF_n) \rightarrow [0, 1]$ defined as $\phi(v) = (1/\alpha) \forall v \in V(SF_n)$, as $V(G) = \bigcup \{|\text{LR}(e)|: |\text{LR}(e) = \alpha|\}$. Consequently, by Theorem 1, $\dim_{lf}(SF_n) \leq (|V(SF_n)|/n + 1) \leq (2n/n + 1)$.

Case 4: As $|\text{LR}(a_i a_{i+1})| = 2n - 2$ which is greater than the cardinalities of all other LRNs of SF_n , there exists a LLRF $\phi': V(SF_n) \rightarrow [0, 1]$ defined by $\phi'(v) = (1/2n - 2)$, $\forall v \in V(SF_n)$, as $V(G) = \bigcup \{|\text{LR}(e)|: |\text{LR}(e) = 2n|\}$. By Theorem 2, $\dim_{lf}(SF_n) \geq (n/n - 1)$. Consequently,

$$\frac{n}{n-1} \leq \dim_{lf}(SF_n) \leq \frac{2n}{n+1}. \quad (23)$$

6. Conclusion

In this article, we studied the various aspects of the LFMDs for the different connected networks including the existence of the exact values, lower and upper bonds and bounded- and unbounded behaviours. Mainly, the lower bond of LFMD for the arbitrary connected networks is improved from unity. As the applications of the main result, LFMDs of the prism-related networks are as follows:

- (i) The exact value LFMD of CDL_n is n and of A_3 is $(3/2)$, where $n \geq 4$.

- (ii) The LFMDs of antiprism and sun flower networks are computed in bounds, see Table 1.

Moreover, we note that, for $n \rightarrow \infty$, the computed bounds are convergent which show their boundedness. Now, we close our discussion by proposing the following open problem: open problem. Characterize the connected networks whose LFMDs is $(|V(G)|/a)$, where $a < |V(G)|$ is some integral value.

Data Availability

The data used to support the findings of this study are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest

The authors have no conflicts of interest.

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