Research Article

Lie Symmetry Analysis and Dynamics of Exact Solutions of the (2+1)-Dimensional Nonlinear Sharma–Tasso–Olver Equation

Sachin Kumar ¹, Ilyas Khan ^{2,3}, Setu Rani,¹ and Behzad Ghanbari⁴

¹Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi-110007, India

²Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia

³Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 72915, Vietnam

⁴Department of Engineering Science, Kermanshah University of Technology, Kermanshah, Iran

Correspondence should be addressed to Ilyas Khan; ilyaskhan@tdtu.edu.vn

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In soliton theory, the dynamics of solitary wave solutions may play a crucial role in the fields of mathematical physics, plasma physics, biology, fluid dynamics, nonlinear optics, condensed matter physics, and many others. The main concern of this present article is to obtain symmetry reductions and some new explicit exact solutions of the (2 + 1)-dimensional Sharma–Tasso–Olver (STO) equation by using the Lie symmetry analysis method. The infinitesimals for the STO equation were achieved under the invariance criteria of Lie groups. Then, the two stages of symmetry reductions of the governing equation are obtained with the help of an optimal system. Meanwhile, this Lie symmetry method will reduce the STO equation into new partial differential equations (PDEs) which contain a lesser number of independent variables. Based on numerical simulation, the dynamical characteristics of the solitary wave solutions illustrate multiple-front wave profiles, solitary wave solutions, kink wave solitons, oscillating periodic solitons, and annihilation of parabolic wave structures via 3D plots.

1. Introduction

Most of the physical problems are inherently nonlinear because of their nature. These types of nonlinear problems are of much interest to engineers, mathematicians, physicists, and many other researchers because of their physical applications in scientific areas such as engineering, biology, theoretical physics, plasma physics, and condensed matter physics (see [1]). To predict the possible behaviors of physical phenomena, exact solutions of these nonlinear PDEs are very important. The various types of rational solutions were investigated by analytical and numerical methods such as the Lie symmetry method [2, 3], homogeneous balance method [4], tanh function method [5], Hirota bilinear method [6], Backlund transformation method [7], Darboux transformation method [8], the truncated Painlevé expansion [9], and F-expansion method [10]. In solitary wave theory, a large variety of traveling waves are of much concern. Some important traveling waves are the kink waves and the solitons. The kink waves fall or

rise from one asymptotic state to others, and the latter is the localized traveling waves which are asymptotically zero at large distances. Some potential applications of these wave solutions may be found in coastal areas and open oceans.

The (1 + 1)-dimensional STO equation is of the form

$$u_t + \alpha u_x^3 + \frac{3}{2} \alpha u_{xx}^2 + \alpha u_{xxx} = 0, \quad (1)$$

where α is some constant and the dissipative term u_{xxx} provides damping of a small scale. A lot of researchers worked on the STO equation (1) due to its huge appearance in scientific applications. The STO equation is similar to the KdV equation and has been used to describe a wide range of physical phenomena of the evolution and interaction with the nonlinear waves, such as solitons and turbulence, fluid dynamics, continuum mechanics, and aerodynamics. The STO equation can also be used to describe how do nonlinear dispersive waves propagate in inhomogeneous media. It possesses the bi-Hamiltonian formulation and an infinite number of symmetries. Different nonlinear waves are

described by the exact solutions for (1) with different forms. The generalized Kaup–Newell-type hierarchy of nonlinear evolution equations is related to the Sharma–Tasso–Olver equation from [11]. A large number of integrability properties of (1) are achieved in [12, 13]. In [14], equation (1) was managed by Yan using the Cole–Hopf transformation method. By considering the improved tanh function method, fission and fusion for the Sharma–Tasso–Olver equation have some exact solutions in [15]. In ref. [16–18], some powerful techniques such as the Hirota direct method, extended hyperbolic function method, and Backlund transformation method are used for finding the explicit solutions of (1), respectively. Also, this equation was managed by using the tanh method and the extended tanh method (see [16]).

This research aims to find the exact-soliton solutions of $(2+1)$ -dimensional STO equation. It is well known that the $(2+1)$ -dimensional KP equation is an extended version of $(1+1)$ -dimensional Korteweg–de Vries (KdV) equation. Analogously, $(2+1)$ -dimensional STO equation is an extended version of $(1+1)$ -dimensional STO equation. The extended form of $(2+1)$ -dimensional STO equation is introduced in [19], which is accomplished by adding a new term u_{yy} in $(1+1)$ -dimensional STO equation (1)

$$\left(u_t + \alpha u_x^3 + \frac{3}{2} \alpha u_{xx}^2 + \alpha u_{xxx}\right)_x + \beta u_{yy} = 0, \quad (2)$$

where α and β are two arbitrary constants and the unknown function u depends on the spatial variables x , y and temporal variable t . This equation encloses both nonlinear and linear terms. Using the tanh function method, the traveling wave solution of STO equation (2) is found as

$$u(x, y, t) = c_5 \mp \sqrt{6} \tanh\left(\mp \frac{x}{\sqrt{6}} \pm y c_6 \pm \frac{1}{3} \sqrt{\frac{2}{3}} t (\alpha + 9c_6^2 \beta)\right), \quad (3)$$

where c_5 and c_6 are arbitrary constants. Here, we will solve a new $(2+1)$ -dimensional equation using the similarity transformation method. Felix Klein (1849–1925) and M. Sophus Lie (1842–1899) introduced this method to construct integration theory for ODEs. For further details, one can refer to textbooks [20, 21] or some articles [3, 22–27] as well as references therein. During this process, some variable transformations take place, which is used to obtain new solutions from the existing ones using differential operators, called the generators of the symmetry group. With the help of similarity forms, STM reduces the number of independent variables of the original PDE by one. As a result, the equation was reduced into a nonlinear PDE having one less independent variable. In the same way, one can reduce this nonlinear PDE into nonlinear ODE by reducing one independent variable, which can be solved mathematically.

Being motivated by the aforementioned references/works, we would like to study to construct the explicit exact-soliton solutions by using the Lie symmetry approach. Besides, we derive various Lie symmetry reductions and localized solitary wave solutions for the considered equation.

Recently, Ben and Ma [19] investigated $(2+1)$ -dimensional STO equation and found some rational solutions using the resulting trilinear form. Our analytic results in this paper are more general in terms of independent functions and entirely different from their works/findings. Furthermore, the physical behaviors of the soliton solutions are demonstrated through their evolution profiles.

The remaining manuscript is structured as follows. In Section 1, we discuss various algorithms for solving the STO equation and a new form of the equation by adding one term u_{yy} . The Lie symmetries, vector fields, infinitesimals, and an optimal system of one-dimensional subalgebra for the STO equation are depicted in Section 2. In Section 3, Lie symmetry reductions and analytical wave solutions are furnished. The dynamical analysis of attained soliton solutions is analyzed physically in Section 4. Lastly, a conclusion is drawn in Section 5.

2. Lie Symmetry Analysis

The motive of this present section is to describe the main steps used in obtaining the infinitesimal, Lie symmetries, and optimal system of one-dimensional subalgebra for STO equation (2). To find the invariant solutions of (2), the STM is applied. Since equation (2) is highly nonlinear, therefore to solve equation (2), we need to introduce one parameter (ε) Lie group of infinitesimals transformation as

$$\begin{aligned} \tilde{x}_i^* &= x_i + \varepsilon \xi_i(x, u) + O(\varepsilon^2), \\ \tilde{u}^* &= u + \varepsilon \eta(x, u) + O(\varepsilon^2), \end{aligned} \quad (4)$$

where the infinitesimals $\xi_1 = \xi^{(1)}$, $\xi_2 = \xi^{(2)}$, $\xi_3 = \tau$, and η are the infinitesimals transformations for x , y , t , and u , respectively. The linked vector field V with the above group of transformation is defined as

$$V = \xi^{(1)} \frac{\partial}{\partial x} + \xi^{(2)} \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}, \quad (5)$$

where the obtained functions $\xi^{(1)}$, $\xi^{(2)}$, τ , and η are found under the constraint:

$$\begin{aligned} \xi^{(1)} &= \frac{d\tilde{x}}{d\varepsilon}\bigg|_{\varepsilon=0}, \\ \xi^{(2)} &= \frac{d\tilde{y}}{d\varepsilon}\bigg|_{\varepsilon=0}, \\ \tau &= \frac{d\tilde{t}}{d\varepsilon}\bigg|_{\varepsilon=0}, \\ \eta &= \frac{d\tilde{u}}{d\varepsilon}\bigg|_{\varepsilon=0}. \end{aligned} \quad (6)$$

If equation (2) is symmetrical about the vector field (5), then V must fulfill the condition

$$Pr^{(4)}V(\Delta)|_{\Delta=0} = 0, \quad (7)$$

where $Pr^{(4)}V$ represents the fourth-order prolongation and is given by [21]

$$\begin{aligned}
V^{(4)} = & V + \eta^x \frac{\partial}{\partial u_x} + \eta^y \frac{\partial}{\partial u_y} + \eta^t \frac{\partial}{\partial u_t} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xt} \frac{\partial}{\partial u_{xt}} + \cdots + \eta^{yy} \frac{\partial}{\partial u_{yy}} + \cdots + \eta^{tt} \frac{\partial}{\partial u_{tt}} \\
& + \eta^{xxx} \frac{\partial}{\partial u_{xxx}} + \eta^{xxy} \frac{\partial}{\partial u_{xxy}} + \cdots + \eta^{yyy} \frac{\partial}{\partial u_{yyy}} + \cdots + \eta^{ttt} \frac{\partial}{\partial u_{ttt}} + \eta^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \eta^{xxxxt} \frac{\partial}{\partial u_{xxxxt}} + \cdots + \eta^{yyyyy} \frac{\partial}{\partial u_{yyyyy}} + \cdots + \eta^{ttttt} \frac{\partial}{\partial u_{ttttt}}.
\end{aligned} \quad (8)$$

Applying the above prolongation formula to (2) under invariance condition, the symmetry condition is

$$\begin{aligned}
& \eta^{xt} + 3\alpha u_x^2 \eta^{xx} + 6\alpha u_x u_{xx} \eta^x + 3\alpha u_{xxx} \eta^{xx} + 3\alpha u_{xx} \eta^{xxx} \\
& + \alpha \eta^{xxxx} + \beta \eta^{yy} = 0,
\end{aligned} \quad (9)$$

$$\begin{aligned}
\eta^x &= D_x(\eta) - u_x(D_x \xi^{(1)}) - u_y(D_x \xi^{(2)}) - u_t(D_x \tau), \\
\eta^y &= D_y(\eta) - u_x(D_y \xi^{(1)}) - u_y(D_y \xi^{(2)}) - u_t(D_y \tau), \\
\eta^{xx} &= D_x(\eta^x) - u_{xx}(D_x \xi^{(1)}) - u_{xy}(D_x \xi^{(2)}) - u_{xt}(D_x \tau), \\
\eta^{xt} &= D_t(\eta^x) - u_{xx}(D_t \xi^{(1)}) - u_{xy}(D_t \xi^{(2)}) - u_{xt}(D_t \tau), \\
\eta^{yy} &= D_y(\eta^y) - u_{yx}(D_y \xi^{(1)}) - u_{yy}(D_y \xi^{(2)}) - u_{yt}(D_y \tau), \\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxx}(D_x \xi^{(1)}) - u_{xxy}(D_x \xi^{(2)}) - u_{xxt}(D_x \tau), \\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxx}(D_x \xi^{(1)}) - u_{xxxxy}(D_x \xi^{(2)}) - u_{xxxxt}(D_x \tau)
\end{aligned} \quad (10)$$

are the coefficients of $\text{Pr}^{(4)}V(\Delta) = 0$ and D_x , D_y , and D_t are total derivative operators that can be expressed as in [28].

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad \text{for } i = 1, 2, 3, 4. \quad (11)$$

Using (10) and (9), then the determining equations

$$\begin{aligned}
(\eta_u)_u &= (\eta_u)_x = (\eta_u)_{yy} = 0, \\
(\xi_t)_t &= (\xi_t)_u = (\xi_t)_x = (\xi_t)_y = 0, \\
(\xi_x)_t &= (\xi_x)_u = (\xi_x)_x = 0, \\
(\xi_x)_y &= \frac{-(\xi_y)_t}{2\beta}, \\
(\xi_y)_u &= (\xi_y)_x = (\xi_y)_y = (\xi_y)_{tt} = 0.
\end{aligned} \quad (12)$$

On solving equation (12), we obtain infinitesimal generators as

$$\xi^{(1)} = \frac{c_1 y}{2\beta} + c_4, \quad \xi^{(2)} = c_1 t + c_2, \quad (13)$$

$$\tau = c_3, \quad \eta = f_1(t)y + f_2(t),$$

where $c_i; 1 \leq i \leq 4$ are arbitrary constants and $f_1(t)$ and $f_2(t)$ are functions obtained in the processing of

where

infinitesimals. The choice of f'_i s can achieve some physical structures of solutions of equation (2). The symmetry algebra of equation (2) can be obtained by the given vectors

$$\begin{aligned}
V_1 &= -\frac{y}{2\beta} \frac{\partial}{\partial x} + t \frac{\partial}{\partial y}, \\
V_2 &= \frac{\partial}{\partial y}, \\
V_3 &= \frac{\partial}{\partial t}, \\
V_4 &= \frac{\partial}{\partial x}, \\
V_5 &= y f_1(t) \frac{\partial}{\partial u}, \\
V_6 &= f_2(t) \frac{\partial}{\partial u},
\end{aligned} \quad (14)$$

such that $V = c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 + V_5(f_1) + V_6(f_2)$.

It has been noted from the commutative relations of vector fields that the Lie group of transformations formed an infinite-dimensional Lie algebra with functions $f_1(t)$ and $f_2(t)$ and its commutation table (Table 1) is skew-symmetric with each diagonal entry zero.

TABLE 1: Commutative table.

$[V_i V_j]$	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	$(V_4/2\beta)$	$-V_2$	0	$t(f_1(t)/f_2(t))V_6$	0
V_2	$-(V_4/2\beta)$	0	0	0	$(f_1(t)/f_2(t))V_6$	0
V_3	V_2	0	0	0	$(f_1'(t)/f_1(t))V_5$	$(f_2'(t)/f_2(t))V_6$
V_4	0	0	0	0	0	0
V_5	$-t(f_1(t)/f_2(t))V_6$	$-(f_1(t)/f_2(t))V_6$	$-(f_1'(t)/f_1(t))V_5$	0	0	0
V_6	0	0	$-(f_2'(t)/f_2(t))V_6$	0	0	0

The Lie series to calculate the adjoint relation is furnished as

$$\text{Ad}(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_i, V_j]] + \dots \quad (15)$$

Furthermore, an optimal system of one-dimensional subalgebra is achieved by means of Olver's technique [21]. The desired optimal system for STO equation (2) is obtained by using adjoint table (Table 2). Following is the given one-dimensional subalgebras for STO equation (2):

$$\begin{aligned} \mathcal{T}_1 &= V_2 + V_4, \\ \mathcal{T}_2 &= V_2 + V_3 + V_4, \\ \mathcal{T}_3 &= V_4 + V_5, \\ \mathcal{T}_4 &= V_4 + V_6, \\ \mathcal{T}_5 &= c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 + V_5 + V_6, \\ \mathcal{T}_6 &= c_2 V_2 + c_3 V_3 + c_4 V_4 + V_5 + V_6. \end{aligned} \quad (16)$$

3. Symmetry Reductions and Explicit Solutions

In this section, we will construct some exact-soliton solutions with the aid of 1-dimensional optimal system of symmetry subalgebras. To achieve exact-invariant solutions for the considered equation (2), the corresponding Lagrange's system is chosen as

$$\frac{dx}{\xi^{(1)}} = \frac{dy}{\xi^{(2)}} = \frac{dt}{\tau} = \frac{du}{\eta}. \quad (17)$$

3.1. Subalgebra $\mathcal{T}_1 = V_2 + V_4 = (\partial/\partial y) + (\partial/\partial x)$. Using vector field $V_2 + V_4$, equation (2) is turned into

$$\alpha U_{xxxx} + 3\alpha U_{xx}U_{xxx} + 3\alpha U_{xx}U_x^2 + \beta U_{xx} + U_{XT} = 0, \quad (18)$$

where $u(x, y, t) = U(X, T)$ and $X = x - y$, $T = t$.

Since equation (18) is also a nonlinear PDE and contains one dependent and two independent variables, we will again

apply STM on this equation to construct new infinitesimal generators of the form

$$\begin{aligned} \xi_X &= k_2, \\ \xi_T &= k_1, \\ \eta_U &= g_1(T), \end{aligned} \quad (19)$$

where k_1 and k_2 are arbitrary constants and $g_1(T)$ is an arbitrary function. Then, the required characteristic equation is

$$\frac{dX}{k_2} = \frac{dT}{k_1} = \frac{dU}{g_1(T)}. \quad (20)$$

On solving equation (20), we get the following similarity form

$$U(X, T) = \frac{1}{k_1} \int g_1(T) dT + U_1(X_1), \quad (21)$$

where $X_1 = X - (k_2/k_1)T$. By using (21) into (18), then reduced form of ODE is

$$\alpha k_1 U_1'' + 3\alpha k_1 U_1'' U_1' + 3\alpha k_1 U_1'^2 U_1'' + (k_1\beta - k_2)U_1'' = 0, \quad (22)$$

which on integration gives

$$\alpha k_1 U_1'' + \frac{3}{2} \alpha k_1 U_1''^2 + \alpha k_1 U_1'^3 + (k_1\beta - k_2)U_1' = C, \quad (23)$$

where C is constant of integration and $'$ represents the derivative of U_1 w.r.t X_1 . One of the particular solutions of equation (23) is

$$U_1(X_1) = a_0 - X_1 \left(\frac{T}{3\sqrt{[3]} 2\alpha k_1} + \frac{\sqrt{[3]} 2(k_2 - \beta k_1)}{T} \right), \quad (24)$$

where

$$T = \sqrt{[3]} \sqrt{729\alpha^4 C^2 k_1^6 - 108\alpha^3 k_1^3 (k_2 - \beta k_1)^3 - 27\alpha^2 C k_1^3}.$$

Hence, the exact explicit solution of equation (2) is given as

$$u(x, y, t) = a_0 + \left(\frac{T}{3\sqrt{[3]} 2\alpha k_1} + \frac{\sqrt{[3]} 2(k_2 - \beta k_1)}{T} \right) \left(\frac{k_2 t}{k_1} - x + y \right) + \frac{\int g_1(t) dt}{k_1}. \quad (25)$$

TABLE 2: Adjoint table.

[Ad]	V_1	V_2	V_3	V_4	V_5	V_6
V_1	V_1	$V_2 - \varepsilon(V_4/2\beta)$	$V_3 + \varepsilon V_2$	V_4	$V_5 - \varepsilon(t f_1(t)/f_2(t))V_6$	V_6
V_2	$V_1 + \varepsilon(V_4/2\beta)$	V_2	V_3	V_4	$V_5 - \varepsilon(f_1(t)/f_2(t))V_6$	V_6
V_3	$V_1 - \varepsilon V_2$	V_2	V_3	V_4	$V_5 - \varepsilon(f_1'(t)/f_1(t))V_5$	$V_6 - \varepsilon(f_2'(t)/f_2(t))V_6$
V_4	V_1	V_2	V_3	V_4	V_5	V_6
V_5	$V_1 + \varepsilon(t f_1(t)/f_2(t))V_6$	$V_2 + \varepsilon(f_1(t)/f_2(t))V_6$	$V_3 + \varepsilon(f_1'(t)/f_1(t))V_5$	V_4	V_5	V_6
V_6	V_1	V_2	$V_3 + \varepsilon(f_2'(t)/f_2(t))V_6$	V_4	V_5	V_6

3.2. *Subalgebra* $\mathcal{T}_2 = V_2 + V_3 + V_4 = (\partial/\partial y) + (\partial/\partial t) + (\partial/\partial x)$. For $V_2 + V_3 + V_4$, equation (2) is converted into

$$\beta U_{YY} + U_{XX}(-1 + 3\alpha(U_X^2 + U_{XXX})) + \alpha U_{XXX} - U_{XY} = 0, \quad (26)$$

where $u(x, y, t) = U(X, Y)$ with $X = x - t$, $Y = y - t$. Again applying STM on equation (26), obtained form of infinitesimals is

$$\xi_X = k_4, \quad \xi_Y = k_3, \quad \eta_U = k_5 Y + k_6. \quad (27)$$

where k_3, k_4, k_5 , and k_6 are constants. Then, the associated Lagrange's system is found as

$$\frac{dX}{k_4} = \frac{dY}{k_3} = \frac{dU}{k_5 Y + k_6}. \quad (28)$$

On solving equation (28), we get a new similarity form of U as

$$U(X, Y) = \frac{((k_5 Y^2)/2) + k_6 Y}{K_3} + U_2(X_2), \quad (29)$$

with $X_2 = X - (k_4/k_3)Y$ as a similarity variable. By using equation (29) into (26), we get an ODE of the form

$$\alpha k_3^2 U_2'' + U_2''(\beta k_4^2 + k_4 k_3 - k_3^2 + 3\alpha k_3^2(U_2'' + U_2'^2)) + \beta k_3 k_5 = 0. \quad (30)$$

Since equation (30) is tedious nonlinear ODE, therefore by taking $k_5 = 0$ in equation (30), we attain one particular solution of the type

$$U_2(X_2) = b_1 X_2 + b_0, \quad (31)$$

where b_0 and b_1 are arbitrary constants. Thus, the explicit exact solution for STO equation (2) is

$$u(x, y, t) = b_1 \left(-\frac{k_4(y-t)}{k_3} - t + x \right) + b_0 + \frac{k_6(y-t)}{k_3}. \quad (32)$$

3.3. *Subalgebra* $\mathcal{T}_3 = V_4 + V_5 = (\partial/\partial x) + y f_1(t)(\partial/\partial u)$. Using vector field $V_4 + V_5$, equation (2) is transformed into

$$\beta U_{YY} + Y f_1'(T) = 0, \quad (33)$$

with $u(x, y, t) = x Y f_1(T) + U(Y, T)$ and $Y = y$, $T = t$. The invariant solution of PDE (33) is

$$U(Y, T) = -\frac{Y^3 f_1'(t)}{6\beta} + Y h_1(T) + h_2(T). \quad (34)$$

Therefore, the explicit exact solution of STO equation (2) is

$$u(x, y, t) = -\frac{y^3 f_1'(t)}{6\beta} + x y f_1(t) + y h_1(t) + h_2(t). \quad (35)$$

3.4. *Subalgebra* $\mathcal{T}_4 = V_4 + V_6 = (\partial/\partial x) + f_2(t)(\partial/\partial u)$. Using $V_4 + V_6$, then, symmetry reduction is easily obtained as

$$\beta U_{YY} + f_2'(T) = 0, \quad (36)$$

where $u(x, y, t) = x f_2(T) + U(Y, T)$ with $Y = y$, $T = t$. Thus, reduced PDE has the solution

$$U(Y, T) = -\frac{Y^2 f_2'(t)}{2\beta} + Y h_1(T) + h_2(T). \quad (37)$$

Thus, leading analytic solution for the STO equation is

$$u(x, y, t) = -\frac{y^2 f_2'(t)}{2\beta} + x f_2(t) + y h_1(t) + h_2(t). \quad (38)$$

3.5. *Subalgebra* $\mathcal{T}_5 = c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 + V_5 + V_6 = -(c_1 y/2\beta)(\partial/\partial x) + c_1 t(\partial/\partial y) + c_2(\partial/\partial y) + c_3(\partial/\partial t) + c_4(\partial/\partial x) + y f_1(t)(\partial/\partial u) + f_2(t)(\partial/\partial u)$. Using (13) and (17), we will get the following Lagrange system:

$$\frac{dx}{-(c_1 y/2\beta) + c_4} = \frac{dy}{c_1 t + c_2} = \frac{dt}{c_3} = \frac{du}{f_1(t)y + f_2(t)}. \quad (39)$$

It follows the similarity form as

$$u(x, y, t) = \int \frac{(y f_1(t) + f_2(t))}{c_3} dt + U(X, Y), \quad (40)$$

where

$$X = x + \frac{3c_1 c_2 t^2 + c_1^2 t^3 + 6c_1 c_3 t Y - 12c_3 c_4 t \beta}{12c_3^2 \beta}, \quad (41)$$

$$Y = y - \frac{c_2 t}{c_3} + \frac{c_1 t^2}{2c_3}$$

are similarity variables with $U(X, Y)$ as a similarity function. Insert the above value of $u(x, y, t)$ from (40) into STO equation (2). Then, the reduced equation is given as

$$2c_3\alpha\beta U_{XXXX} + 2c_3\beta^2 U_{YY} - 2c_2\beta U_{XY} + U_{XX}(c_1Y - 2c_4\beta + 6c_3\alpha\beta(U_X^2 + U_{XX})) = 0. \quad (42)$$

Again applying STM, equation (42) admits infinitesimals as

$$\begin{aligned} \hat{\xi}^1 &= a_1, \\ \hat{\xi}^2 &= 0, \\ \hat{\eta} &= a_2Y + a_3, \end{aligned} \quad (43)$$

where a_i ; $1 \leq i \leq 3$ are arbitrary constants.

3.5.1. If $a_2 \neq 0$ in Equation (43). Then, the characteristic equation is as follows:

$$\frac{dX}{A_1} = \frac{dY}{0} = \frac{dU}{Y + A_3}, \quad (44)$$

where $A_1 = (a_1/a_2)$ and $A_3 = (a_3/a_2)$. Integration of equation (44) yields the following form of U

$$\begin{aligned} u(x, y, t) = & \frac{1}{12A_1c_3\beta} \left[(2c_2t + c_1t^2 - 2c_3y)(3c_2^2t + 3c_1c_2t^2 + c_1^2t^3 - 3c_2c_3y - 3c_1c_3ty + 6c_3(c_4t - c_3x)\beta) \right. \\ & + A_3c_3(c_1t(-3c_2t - 2c_1t^2 + 6c_3y) + 12c_3(-c_4t + c_3x)\beta) + 6A_1c_3^2\beta \left(-t(2c_2 + c_1t)B + 2c_3(A + yB) + 2 \int y f_1(t) \right. \\ & \left. \left. + f_2(t)dt \right) \right]. \end{aligned} \quad (48)$$

Taking the appropriate choice of $A = 1$ and $B = 2\sqrt{(c_2/(2A_1c_3\beta))}$ in equation (47), we get

$$U_3(X_3) = \left(\sqrt{\frac{c_2}{2A_1c_3\beta}} X_3 + 1 \right)^2. \quad (49)$$

Thus, from equations (40), (45), and (49), the leading solution is of the form

$$\begin{aligned} u(x, y, t) = & \frac{1}{12A_1c_3\beta} \left[(2c_2t + c_1t^2 - 2c_3y)(3c_2^2t + 3c_1c_2t^2 + c_1^2t^3 - 3c_2c_3y - 3c_1c_3ty + 6c_3(c_4t - c_3x)\beta) \right. \\ & + A_3c_3(c_1t(-3c_2t - 2c_1t^2 + 6c_3y) + 12c_3(-c_4t + c_3x)\beta) \\ & \left. + 6A_1c_3^2\beta \left(-\sqrt{2}t(2c_2 + c_1t)\sqrt{\frac{c_2}{c_3\beta A_1}} + 2 \left(c_3 + \sqrt{2}c_3y\sqrt{\frac{c_2}{c_3\beta A_1}} \right) + 2 \int y f_1(t) + f_2(t)dt \right) \right]. \end{aligned} \quad (50)$$

3.5.2. If $a_2 = 0$ in Equation (43). Then, from (43), the Lagrange equation takes the form

$$\frac{dX}{a_1} = \frac{dY}{0} = \frac{dU}{a_3}. \quad (51)$$

Therefore, the similarity form is

$$U(X, Y) = \frac{a_3X}{a_1} + U_4(X_4), \quad (52)$$

where $X_4 = Y$ and $U_4(X_4)$ are the similarity variable and similarity function, respectively. Substituting equation (52) in (42), one gets the following ODE:

$$U(X, Y) = \frac{XY}{A_1} + \frac{XA_3}{A_1} + U_3(X_3), \quad (45)$$

where $X_3 = Y$. Using (45) into (42), we get an ODE as

$$U_3''(X_3) = \frac{c_2}{A_1c_3\beta}. \quad (46)$$

The solution of ODE (46) is given as

$$U_3(X_3) = \frac{c_2X_3^2}{2A_1c_3\beta} + BX_3 + A, \quad (47)$$

where A and B are integrating constants. On comparing equations (40), (45), and (47), the explicit solution for equation (2) is obtained as

$$U_4''(X_4) = 0. \quad (53)$$

On solving equation (53),

$$U_4(X_4) = CX_4 + D, \quad (54)$$

where C and D are called integrating constants. From (40), (52), and (54), we will obtain the analytic solution of the STO equation

$$u(x, y, t) = \frac{1}{12a_1c_3\beta} \left(a_3c_1t(-3c_2t - 2c_1t^2 + 6c_3y) + 6c_3(-2a_3c_4t - a_1tC(2c_2 + c_1t) + 2a_3c_3x + 2a_1c_3(Cy + D))\beta \right) + \int \frac{yf_1(t) + f_2(t)}{c_3} dt. \quad (55)$$

3.6. *Subalgebra* $\mathcal{T}_6 = c_2V_2 + c_3V_3 + c_4V_4 + V_5 + V_6 = c_2(\partial/\partial y) + c_3(\partial/\partial t) + c_4(\partial/\partial x) + yf_1(t)(\partial/\partial u) + f_2(t)(\partial/\partial u)$. The required characteristic equation is

$$\frac{dx}{c_4} = \frac{dy}{c_2} = \frac{dt}{c_3} = \frac{du}{yf_1(t) + f_2(t)}. \quad (56)$$

Solution of (56) predicts the similarity function

$$u(x, y, t) = U(X, Y) + \int \frac{yf_1(t) + f_2(t)}{c_3} dt, \quad (57)$$

with similarity variables $X = x - (c_4t/c_3)$ and $Y = y - (c_2t/c_3)$. Incorporating equation (57) into equation (2) yields a PDE of the form

$$c_3\beta U_{YY} + 3c_3\alpha U_{XX}U_X^2 + 3c_3\alpha U_{XX}U_{XXX} + c_3\alpha U_{XXX} - c_2U_{XY} - c_4U_{XX} = 0. \quad (58)$$

By applying the STM on (58), we get a new set of infinitesimals as

$$\begin{aligned} \hat{\xi}^1 &= a_2, \\ \hat{\xi}^2 &= a_1, \\ \hat{\eta} &= a_3Y + a_4, \end{aligned} \quad (59)$$

where a_i ; $1 \leq i \leq 4$ are arbitrary constants. The Lagrange form for (59) reads

$$\frac{dX}{a_2} = \frac{dY}{a_1} = \frac{dU}{a_3Y + a_4}. \quad (60)$$

3.6.1. If $a_3 \neq 0$ in Equation (59). Then, we have

$$\frac{dX}{B_2} = \frac{dY}{B_1} = \frac{dU}{Y + B_3}, \quad (61)$$

where $B_1 = (a_1/a_3)$, $B_2 = (a_2/a_3)$, and $B_3 = (a_4/a_3)$. To get the solution of (2), the new similarity form of (61) is

$$U(X, Y) = \frac{Y^2}{2B_1} + \frac{B_3Y}{B_1} + U_5(X_5), \quad (62)$$

with $X_5 = X - (B_2Y/B_1)$ as the similarity variable. Using the value of (62) into (58), we get an ODE of the form

$$c_3\alpha B_1^2U_5'' + 3c_3\alpha B_1^2U_5''U_5' + 3c_3\alpha B_1^2U_5'^2U_5'' - c_4B_1^2U_5'' + c_2B_1B_2U_5'' + c_3\beta B_2^2U_5'' + c_3\beta B_1 = 0. \quad (63)$$

Integration of (63) provides

$$c_3\alpha B_1^2U_5'' + \frac{3}{2}c_3\alpha B_1^2U_5''^2 + c_3\alpha B_1^2U_5'^3 - c_4B_1^2U_5' + c_2B_1B_2U_5' + c_3\beta B_2^2U_5' + c_3\beta B_1X_1 = D, \quad (64)$$

where D is constant of integration. It is clearly seen that equation (64) is difficult to solve.

3.6.2. If $a_3 = 0$ in Equation (59). Then, given characteristic equation is

$$\frac{dX}{a_2} = \frac{dY}{a_1} = \frac{dU}{a_4}. \quad (65)$$

It follows the similarity form as

$$U(X, Y) = \frac{a_4Y}{a_1} + U_6(X_6), \quad (66)$$

where $X_6 = X - (a_2Y/a_1)$. From equations (66) and (58), we obtain the ODE as

$$a_1c_3\alpha U_6'' + U_6'' \left(3a_1c_3\alpha \left(U_6'^2 + U_6'' \right) - a_1c_4 + a_2 \left(c_2 + \frac{a_2c_3\beta}{a_1} \right) \right) = 0. \quad (67)$$

By setting the particular values of arbitrary constants as $a_1 = 1$, $a_2 = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $\alpha = 1$, and $\beta = 1$, the reduced form of ODE is

$$U_6'' + 3U_6''U_6'' + 3U_6'^2U_6'' + U_6'' = 0, \quad (68)$$

which on integration gives

$$U_6'' + \frac{3}{2}U_6''^2 + U_6'^3 + U_6' = C, \quad (69)$$

FIGURE 1: Three-dimensional shapes of annihilation of kink wave profiles for equation (3) at different values of t . (a) 3D profile of u with $t = 1$. (b) 3D profile of u with $t = 11$.

where C is constant of integration. One of the particular solutions of equation (69) is found as

$$U_6(X_6) = A_0 + \frac{\left(\sqrt{[3]} 2(\sqrt{81C^2 + 12} + 9C)^{(2/3)} - 2\sqrt{[3]} 3\right)X_6}{6^{(2/3)}\sqrt{[3]}\sqrt{81C^2 + 12 + 9C}}, \quad (70)$$

where A_0 is an arbitrary constant. Thus, the explicit exact solution of (2) is given as

$$u(x, y, t) = \frac{\left(\sqrt{[3]} 2(\sqrt{81C^2 + 12} + 9C)^{(2/3)} - 2\sqrt{[3]} 3\right)(a_1(c_3x - c_4t) + a_2(c_2t - c_3y))}{6^{(2/3)}a_1c_3\sqrt{[3]}\sqrt{81C^2 + 12 + 9C}} + \frac{a_4(y - (c_2t/c_3))}{a_1} + A_0 + \frac{\int (yf_1(t) + f_2(t))dt}{c_3}. \quad (71)$$

3.6.3. If $a_3 = 0$ and $a_1 = 0$. Then, Lagrange equation is

$$\frac{dX}{a_2} = \frac{dY}{0} = \frac{dU}{a_4}, \quad (72)$$

which on solving proposes the following form of U :

$$U(X, Y) = \frac{a_4X}{a_2} + U_7(X_7), \quad (73)$$

where $X_7 = Y$. Substituting the value of (73) into (58), we get ODE as

$$U_7'' = 0. \quad (74)$$

The solution of this ODE is

$$U_7(X_7) = \alpha_0X_7 + \alpha_1, \quad (75)$$

where α_0 and α_1 are arbitrary constants. Thus, from equations (75), (73), and (57), the analytic exact solution of (2) can be furnished as

FIGURE 2: Three-dimensional shapes of periodic solitary waves for (35) with different values of y . (a) 3D profile of u with $y=1$. (b) 3D profile of u with $y=6$.

FIGURE 3: Three-dimensional shapes of oscillating solitary waves (multiple-front profiles) for (38) with different values of y . (a) 3D profile of u with $y=2$. (b) 3D profile of u with $y=9$.

FIGURE 4: Three-dimensional shapes of annihilation of parabolic profile for equation (48) which gets annihilated at $t = 20$. (a) 3D profile of u with $t = 2$. (b) 3D profile of u with $t = 20$.

FIGURE 5: Three-dimensional shapes of asymptotic wave structures of equation (50) with different values of y . (a) 3D profile of u with $y = 1$. (b) 3D profile of u with $y = 20$.

$$u(x, y, t) = \alpha_1 + \frac{-(\alpha_0 a_2 c_2 + a_4 c_4)t + a_4 c_3 x}{a_2 c_3} + \alpha_0 y + \frac{\int (y f_1(t) + f_2(t)) dt}{c_3}, \quad (76)$$

4. Analysis and Discussion

In the present section, a discussion about graphical structures of solutions is carried out. Since mathematical expressions do not produce the wave propagation of flow, we need physical as well as dynamical analysis of these solutions. Here, explicit analytic solutions of the STO equation (2) has been given by equations (3), (25), (32), (35), (38), (48), (50), (55), (71), and (76). All the obtained solutions contain arbitrary functions as well as constants. Therefore, by taking the adequate adjustment of these functions and constants, then we have analyzed the dynamical characteristics of obtained results in the following manner.

Figure 1 represents the physical nature of solution (3) with time. Kink nature of the solution is obtained by taking the values of constants as $c_5 = 1$, $c_6 = 1$, $\alpha = 1$, and $\beta = 1$. This profile is traced at $t = 1$, for $-20 \leq x \leq 20$, $-20 \leq y \leq 20$. The kink solution is a localized solution, i.e., u tends to a finite number when $t \rightarrow \infty$. It was observed by numerical simulation that the behavior of exiting wave depends on time variation. It turns into single solitons at $t = 11$ and is then annihilated after $t = 16$.

Figure 2 demonstrates periodic solitary wave profiles of u given by equation (35). The choice of arbitrary functions is taken as $f_1(t) = \tan(t)$, $h_1(t) = \cot(b_0 t + b_1)$, $h_2(t) = \sin(t)$, and arbitrary constants as $b_0 = 5$, $b_1 = 10$, and $\beta = 4$ for space range $-25 \leq x \leq 25$, $-25 \leq t \leq 25$. In Figure 2(b), the corresponding 3D plot is shown for $y = 6$. It was observed that periodicity decreases as the value of y increases. This nonlinear behavior of soliton solution generally reveals the intersection of Riemann wave and long wave.

Figure 3 interprets the multiple-front solitary wave solution profile of u given by equation (38) at $y = 2$, for range $-25 \leq x \leq 25$, $-25 \leq t \leq 25$. The choice of arbitrary functions is taken as $h_1(t) = \cos(t)$, $h_2(t) = (\tan(t^2 + t + 1)/(2t + 1))$, $f_2(t) = \sin(t)$, and parameter $\beta = -0.5$. Initially, wave profile reveals curve-front wave nature of solution which gets converted into plane-front wave as y increases.

Figure 4 shows the parabolic profile for the solution given by equation (48). Solution profile is observed at $t = 2$, for $-15 \leq x \leq 15$, $-40 \leq y \leq 40$, and the values of arbitrary constants are taken as $A_1 = 0.110$, $A_3 = 0.115$, $c_1 = 2$, $c_2 = 0.001$, $c_3 = 1$, $c_4 = -0.12$, $A = 3$, $B = 1$, $\beta = 1$, $b_0 = 10$, and $b_1 = 0.91$, and the choice of arbitrary functions is taken as $f_1(t) = b_0 t + b_1$ and $f_2(t) = ((2b_0 t + b_1)/(\tan(b_0 t^2 + b_1 t + 5)))$. The nonlinearity of solution (48) is annihilated at $t = 20$. This parabolic profile of solution can be utilized in making the design of ballistic missiles.

Figure 5 is plotted between spatial and temporal axes $x - t$ which represent the asymptotic wave profile for solution (50) and is illustrated at $y = 1$, for $-15 \leq x \leq 15$, $-40 \leq t \leq 40$. The values of constants are taken as $A_1 = 0.110$, $A_3 = 1.019$, $c_1 = 2$, $c_2 = 0.001$, $c_3 = 1$, $c_4 = 0.12$, $\beta = 1$,

$b_0 = 10$, and $b_1 = 0.91$, and those of functions are taken as $f_1(t) = ((2b_0 t + b_1)/(b_0 t^2 + b_1 t + 5))$ and $f_2(t) = \tanh(t)$.

5. Conclusion

In this research work, we have obtained some closed-form explicit solutions for the (2 + 1)-dimensional Sharma-Tasso-Olver (STO) equation by means of Lie approach, which are given by equations (3), (25), (32), (35), (38), (48), (50), (55), (71), and (76). These achieved solutions are entirely distinct from the earlier findings [19]. Also, these reported results are revealed to be more general because it contains arbitrary constants and independent functions $f_1(t)$, $f_2(t)$, $h_1(t)$, and $h_2(t)$. All the generated results are analyzed physically through Figures 1, 2, 3, 4, and 5 which shows parabolic, periodic solitary waves, oscillating traveling waves, kink wave profiles, etc. The dynamics of solitary wave profiles are vividly exhibited via Mathematica. This research shows that the Lie transformation method can be extended for obtaining explicit exact-analytic solutions of many other NLEEs that are emerging in optical physics, plasma physics, chemical physics, acoustics, etc. In summary, the proficient Lie symmetry technique that we implemented is robust, competent, and efficient to solve analytically.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

- [1] A.-M. Wazwaz, *Partial Differential Equations and Solitary Wave Theory*, Springer, New York, NY, USA, 2009.
- [2] P. Devi and K. Singh, "Classical Lie symmetries and similarity reductions of the (2+1)-dimensional dispersive long wave system," *AEJM-journal*, vol. 14, 2020.
- [3] S. Kumar and S. Rani, "Lie symmetry reductions and dynamics of soliton solutions of (2 + 1)-dimensional Pavlov equation," *Pramana-Journal of Physics*, vol. 94, p. 116, 2020.
- [4] E. Fan and H. Zhang, "A note on the homogeneous balance method," *Physics Letters A*, vol. 246, no. 5, pp. 403–406, 1998.
- [5] S.-J. Yu et al., *J. Phys. A*, 31, 3337 (1998).
- [6] W. X. Ma, Z. Qin, and X. Lü, "Lump solutions to dimensionally reduced $\vec{p}$ -gKP and $\vec{p}$ -gBKP equations," *Nonlinear Dynamics*, vol. 84, no. 2, pp. 923–931, 2016.
- [7] J.-G. Liu, Y.-Z. Li, and G. M. Wei, "Extended generalized hyperbolic-function method and new exact solutions of the

- generalized hamiltonian and NNV equations by the symbolic computation,” *Chinese Physics Letters*, vol. 23, p. 1670, 2006.
- [8] M. Wadati, H. Sanuki, and K. Konno, “Relationships among inverse method, Backlund transformation and an infinite number of conservation laws,” *Progress of Theoretical Physics*, vol. 53, no. 2, pp. 419–436, 1975.
- [9] W. Hong and Y.-D. Jung, “Auto-Bäcklund transformation and analytic solutions for general variable-coefficient KdV equation,” *Physics Letters A*, vol. 257, no. 3-4, pp. 149–152, 1999.
- [10] Z. Sheng, “The periodic wave solutions for the (2+1)-dimensional Konopelchenko-Dubrovsky equations,” *Chaos, Solitons & Fractals*, vol. 30, no. 5, pp. 1213–1220, 2006.
- [11] Y. C. Hon and E. Fan, “Uniformly constructing finite-band solutions for a family of derivative nonlinear Schrödinger equations,” *Chaos, Solitons & Fractals*, vol. 24, no. 4, pp. 1087–1096, 2005.
- [12] W.-X. Ma and M. Chen, “Do symmetry constraints yield exact solutions?” *Chaos, Solitons & Fractals*, vol. 32, no. 4, pp. 1513–1517, 2007.
- [13] Z. J. Lian and S. Y. Lou, “Symmetries and exact solutions of the Sharma-Tasso-Olver equation,” *Nonlinear Analysis*, vol. 63, p. 1167, 2005.
- [14] Z. Yan, *MM Research*. 22, 302 (2003)..
- [15] Y. Ugurlu and D. Kaya, *Phys. Lett. A*. 370, 251 (2007).
- [16] A.-M. Wazwaz, “New solitons and kinks solutions to the Sharma-Tasso-Olver equation,” *Applied Mathematics and Computation*, vol. 188, no. 2, pp. 1205–1213, 2007.
- [17] L. Jianming, D. Jie, and Y. Wenjun, “Solitary wave solutions for the boussinesq and fisher equations by the modified simple equation method,” *Mathematical Letters*, vol. 2, p. 8, Article ID 935710, 2011.
- [18] S. Wang, X.-y. Tang, and S.-Y. Lou, “Soliton fission and fusion: burgers equation and Sharma-Tasso-Olver equation,” *Chaos, Solitons & Fractals*, vol. 21, no. 1, pp. 231–239, 2004.
- [19] B. Ren and W.-X. Ma, “Rational solutions of a (2+1)-dimensional Sharma-Tasso-Olver equation,” *Chinese Journal of Physics*, vol. 60, pp. 153–157, 2019.
- [20] G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equation*, Springer-Verlag, New York, NY, USA, 1974.
- [21] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, NY, USA, 1993.
- [22] D. Kumar and S. Kumar, “Some new periodic solitary wave solutions of (3+1)-dimensional generalized shallow water wave equation by Lie symmetry approach,” *Computers & Mathematics with Applications*, vol. 78, no. 3, pp. 857–877, 2019.
- [23] M. Kumar and A. K. Tiwari, “Soliton solutions of BLMP equation by Lie symmetry approach,” *Computers & Mathematics with Applications*, vol. 75, no. 4, pp. 1434–1442, 2018.
- [24] S. Kumar, M. Niwas, and A. M. Wazwaz, “Lie symmetry analysis, exact analytical solutions and dynamics of solitons for (2 + 1)-dimensional NNV equations,” *Physica Scripta*, vol. 95, no. 9, Article ID 095204, 2020.
- [25] S. Kumar and D. Kumar, “Lie symmetry analysis and dynamical structures of soliton solutions for the (2 + 1)-dimensional modified CBS equation,” *International Journal of Modern Physics B*, vol. 34, no. 25, Article ID 2050221, 2020.
- [26] S. Kumar and Y. K. Gupta, “Generalized invariant solutions for spherical symmetric non-conformally flat fluid distributions of embedding class one,” *International Journal of Theoretical Physics*, vol. 53, no. 6, pp. 2041–2050, 2014.
- [27] S. Kumar and A. Kumar, “Dynamical structures of solitons and some new types of exact solutions for the (2+1)-dimensional DJKM equation using Lie symmetry analysis,” *Modern Physics Letter B*, vol. 34, 2020.
- [28] P. Hydon, *Symmetry Methods for Differential Equations*, Cambridge University Press, Cambridge, UK, 2000.