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*Research Article*

# Premium Valuation of the Pension Benefit Guaranty Corporation with Regime Switching

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The Pension Benefit Guaranty Corporation (PBGC) provides insurance coverage for single-employer and multiemployer pension plans in private sector. It has played an important role in protecting the retirement security for over 1.5 million people since it was established about half a decade ago. PBGC collects insurance premiums from employers that sponsor insured pension plans for its coverage and receives funds from pension plans that it takes over. To address the issue of underfunded plans that the PBGC has, this work studies how to evaluate risk-based premiums for the PBGC. Inspired by a couple of existing work in which the premature termination of pension fund and distress termination of sponsor assets are analyzed separately, our work examines the two types of terminations under one framework and considers the occurrence of each termination dynamically. Given that market regime might have a big impact on the dynamics of both pension fund and sponsor's assets, we thus formulate our model using a continuous-time two-state Markov chain in which bull market and bear market are delineated. We thus formulate our model using a continuous-time two-state Markov Chain in which bull market and bear market are delineated. In other words, the pension fund and sponsor assets are market dependent in our work. Given that this additional uncertainty described by regime switching makes the market incomplete, we therefore utilize the Esscher transform to determine an equivalent martingale measure and apply the risk neutral pricing method to obtain the closed-form expressions for premium of PBGC. In addition, we carry out numerical analysis to demonstrate our results and observe that premium increases according to the retirement benefit irrespective of the type of terminations. In comparison to the case of early distress termination of sponsor assets, the premium goes up more quickly when premature termination of pension funds occurs first due to the fact that pension fund is the first venue of retirement security. Furthermore, we look at how the premium changes with respect to other key parameters as well and make some detailed observations in the section of numerical analysis.

## 1. Introduction

Different from defined contribution pension plans, where employees themselves bear the investment risk and where employees are not sure about the amount of benefit they would receive after retirement, sponsors of defined benefit plans offer their employees a definite amount of benefit by the time of retirement, regardless of the performance of the underlying investment pool. In contrast, a defined benefit plan gives employees a better sense of security, since people under this plan are always eligible to receive benefit as long as they are alive and pensioners know their benefit level ahead of time. However, as far as the plan' s sponsor is concerned, offering defined benefit plan is a huge financial commitment. One immediate problem is that if a private company offering defined benefit pension plan goes bankrupt, it would be difficult for its employees to get protection in this case. Without having insurance protection from elsewhere, employees in this troubled pension plan would suffer a lot. Because of this very reason, a federal agency called the Pension Benefit Guaranty Corporation (PBGC) was created by the Employee Retirement and Income Security Act of 1974 (ERISA) to protect defined benefit pension plans in private sector. Since its creation in 1974,

more than 1.4 million workers have relied on PBGC for their retirement income and most people receive the full benefit that they are expected to earn. For any plans covered by PBGC, sponsor of the plan pays premium to the PBGC. In return, the PBGC steps in and provides coverage to the pension plan if needed. However, the PBGC has most often operated at a deficit and U.S. Accountability Office thus designated the PBGC's single-employer program as high risk in July 2003 and added its multiemployer program as high risk in January 2009 due to its huge amount of deficit. Over the course of time, the problem becomes even more serious, the PBGC had a deficit of 35.6 billion in the year of 2013, and its deficit was 61.8 billion and 76.4 billion, respectively, at the end of fiscal year of 2014 and 2015. Annual report of the PBGC shows that there was a deficit of 79.4 billion in the year of 2016. As far as premium is concerned, the PBGC's premium rates are a key component of its funding and it has drawn attention of many scholars. [1–4] show that it is mispriceing PBGC to charge flat premium and the mispricing pension insurance harmfully motivate the company. Stevart [5] gives the economical rationale behind and discusses the resulting consequence when the premium is mispriced. Besides, the level of premiums has not kept pace with the risks that PBGC insures against. There have been proposals that legislative reform should authorize the PBGC board to adjust premiums and redesign a risk-based premium structure, such as consideration of a sponsor's financial health. These recommendations are also the motivations of our work. There have been a series of work on the premium calculation of the PBGC. The pioneering work can be traced back to [6]. In [6], it is assumed that PBGC is the first line of defense for the deficit of pension fund. Marcus [7] used contingent forward to model the PBGC's liability and PBGC is assumed to be allowed to gain surpluses from over funded terminated plans, which potentially means the liability can be negative. Levis and Pennacchi [8] stochastically modeled the PBGC's liability using contingent put option. The major problem of their work is that the maturity of the PBGC's insurance is assumed to be known and the fact that the pension fund will be terminated prematurely due to underfunding has not been taken into consideration. Kalra and Jain [9] firstly considered the premature termination of pension fund. Chen [10] extended their results under the assumption that the PBGC functions as a second line of defense, that is to say, the PBGC covers only the residual deficits of the pension fund that the sponsoring company fails to cover, and in the other related paper, Chen and Uzelac [11] examined the case that the distress termination is triggered due to sponsor's underfunding.

The differences between others' contributions and our work are evident. First, different from [9–11], we include the market regime for the first time into the premium calculations of PBGC's insurance program. We not only assume that the price dynamics of risky assets depend on the states of the economy but also assume that the proportion of pension fund invested in the risky asset also depends on the states of the economy. It is well known that the popularity of regime switching model has been supported by many empirical evidences for a while. The switching models reflect the changes of macroeconomic environment, such as the adjustment of economic structure, the change of market system, and the business cycle, which are exemplified by a continuous-time Markov chain. The market variable follows one risk model when one state of the economy is specified and transfers to other models when the market scheme transfers between different states. Some applications of regime switching models can be seen in [12, 13]. Second, different from [6, 7], Levis and Pennacchi [8], and others, we consider the risk-based premium that the PBGC should charge. In addition to examining the risk of premature termination for pension fund, we study the financial risk of sponso's assets as well. Finally, explicit solutions of the premiums are provided and numerical analysis is also carried out to demonstrate our results.

The rest of the paper is arranged as follows. Section 2 presents the problem formulation. Section 3 shows the Esscher transform under regime switching. Section 4 introduces the case when pension is underfunded and discusses the case that sponsor asset suffers from distress termination. Section 5 demonstrates how the premiums are calculated based on the two scenario cases. Section 6 provides some numerical examples to illustrate the effect of regime switching on the premium of PBGC. Section 7 concludes the paper with some further remarks.

## 2. Formulation

This section presents the formulation of the model of our interest. We begin with notation and assumptions. Given a probability space $(\Omega, \mathscr{F}, P)$, we use $R$ as a representative of beneficiary's retirement time. Let ${}_jP_R = P(T_R > j)$ be survival probability in which $T_R$ is the random variable representing the time until death for a representative beneficiary retires at age $R$ and $r$ be long-term interest rate applying to a typical retiree (for example, we can use 30 years' rate of interest without loss of generality based on the fact that most people retire at age of 60 or older and according to data compiled by the Social Security Administration, the average longevity of people is close to 87) and $B$ is the prescribed annual benefit, which depends on employee's years of -service, age, salary before retirement, and other possible factors. Because the focus of this work is on discussing how the PBGC charges the premiums under different scenario cases, we define the benefit that a typical beneficiary expects to receive similar to that in [10] and express it as below:

$$B_R = \sum_{j=R}^{\infty} {}_jP_R e^{-r(j-R)} B. \tag{1}$$

Note that we use a constant interest here to define the expected present value of the whole life annuity due starting at age $R$ for two reasons. (1) It is both mathematically elegant and practically important to obtain a closed-form solution. (2) It is convenient to choose a conservative interest rate in the first place, which, accordingly, contributes to making proactive provisions regarding all the uncertainties.

We further assume that there are three funding sources for the beneficiary's annual benefit and the pension fund is the primary funding resource for the pension benefit; sponsor company provides secondary support; then, the PBGC contributes the rest. These assumptions hold throughout the entire paper. We assume further that the portfolio of pension fund consists of two kinds of assets. One is risk free and the other is risky asset. The growth rate of the risk-free asset is risk-free interest rate and that of the risky asset is its expected rate of return. More often than not, at a given time, the expected rate of return of the risky asset is higher than that of the risk-free asset, which gives a positive risk premium for the risky asset. Different from the classical portfolio selection assumption, we want to take the market regime into consideration to reflect that the dynamics of a given product are different under different market regimes. In this paper, we consider such a case that there are two possible market regimes, which are bull and bear markets, respectively. Mathematically speaking, we assume that there is a continuous-time two-state Markov chain $\alpha(t)$ taking values in $M = \{0, 1\}$ with generator

$$Q_{2\times 2} = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}. \tag{2}$$

The Markov chain takes different values when the market is in different regimes. With this practical assumption, the risk-free interest rate and expected rate of return are functions of the market regime $\alpha(t)$. To be specific, the dynamics of the risk-free asset are

$$\mathrm{d}x_1(t) = rx_1(t)\mathrm{d}t, \tag{3}$$

where $x_1(0) = x_1$ and $r$ is the risk-free interest rate. Moreover, the dynamics of risky asset are given as follows:

$$\begin{aligned} &\mathrm{d}x_2(t) = \mu(\alpha(t))x_2(t)\mathrm{d}t + \sigma(\alpha(t))x_2(t)\mathrm{d}w_1(t), \\ &x_2(0) = x_2, \alpha(0) = i, \quad i = 0, 1, \end{aligned} \tag{4}$$

in which $w_1(t)$, a standard one-dimensional Brownian motion, is independent of $\alpha(t)$. Using $\pi(\alpha(t))$ to denote the proportion of pension fund invested in the risky asset and $1 - \pi(\alpha(t))$ to denote the proportion of pension fund invested in the risk-free asset, we can thus represent the dynamics of the total pension fund $x(t)$ as below:

$$\begin{aligned} \mathrm{d}x(t) &= \frac{x(t)(1-\pi(\alpha(t)))}{x_1(t)}\mathrm{d}x_1(t) + \frac{x(t)\pi(\alpha(t))}{x_2(t)}\mathrm{d}x_2(t), \\ &x(t)\left[(1-\pi(\alpha(t)))r + \pi(\alpha(t))\mu(\alpha(t))\mathrm{d}t + \pi(\alpha(t))\sigma(\alpha(t))\mathrm{d}w_1(t)\right], \end{aligned} \tag{5}$$

where $x(0) = x_0$.

The proportion of pension fund invested in the risky asset $\pi(\alpha(t))$ is dependent on the $\alpha(t)$ and it means there will be different proportion of pension fund distributed to risk free as well as risky asset at different times. Note that our interest in this work is about evaluation of premium for the PBGC, so we would assume that there is a given $\pi(\alpha(t))$ at time $t$ for the sake of calculation convenience. As far as the way to find out $\pi(\alpha(t))$, once the objective function is formulated, method of stochastic control can be utilized to find out the representation of $\pi(\alpha(t))$ and this could be one of our further works.

If the pension fund performs well enough, then all the benefit will be paid by the pension fund itself. If the pension fund is underfunded, then company's asset is a potential resource of paying benefit. The dynamics of employer's asset are given by

$$\frac{\mathrm{d}y(t)}{y(t)} = \mu_y(\alpha(t))\mathrm{d}t + \sigma_y(\alpha(t))\left(\rho\mathrm{d}w_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}w_2(t)\right), \tag{6}$$

where $y(0) = y_0$ and $w_2(t)$ is another standard Brownian motion and is independent of Brownian motion $w_1(t)$. We assume that the sponsoring corporation's assets are correlated to the pension fund's assets with a correlated coefficient $\rho \in [-1, 1]$ and how the correlation coefficient affects the premiums was analyzed in [10].

Note that typically corporation has corporate debt; we therefore assume that at any time $t$, the sponsor company always has to pay its debt $\theta y_0 e^{\nu t}$ in which $\theta$ is its equity-debt ratio and $\nu$ is the predetermined constant to illustrate the growth rate of the corporate debt. It is practically reasonable to assume that company has higher priority to pay back its corporate debt.

The third part of funding resources is the Pension Benefit Guaranty Corporate (PBGC). Let $S^p$ denote the possible contribution that PBGC makes. It is easy to see that $S^p$ depends on the performance of pension fund and sponsor's assets. We will derive the expression of $S^p$ and the premium that PBGC collects from sponsor company at the end of the following sections. To move forward, we will first find the risk neutral probability measure for the dynamic system with the help of Esscher transform in the following section.

## 3. Esscher Transform under Regime Switching

Since the additional uncertainty described by regime switching makes the market incomplete, there are infinitely many equivalent martingale measures. Here, we will adopt the Esscher transform to determine an equivalent martingale measure for pricing premium of the Pension Benefit

Guaranty Corporation. Esscher transform was first proposed by Gerber and Shiu [14], and it is widely used in the field of finance and insurance. For more details, refer to Bühlmann et al. [15, 16].

We denote $\{\mathcal{F}_t^{x_2}\}_{t\le R}$ as the filtration generated by $\{x_2(t)\}_{t\le R}$, $\{\mathcal{F}_t^{y}\}_{t\le R}$ the filtration generated by $\{y(t)\}_{t\le R}$, $\{\mathcal{F}_t^{\alpha}\}_{t\le R}$ the filtration generated by $\{\alpha(t)\}_{t\le R}$, and $\{\mathcal{G}_t\}_{t\le R}$ the filtration generated by $\mathcal{F}_t^{x_2}\vee\mathcal{F}_t^{y}\vee\mathcal{F}_t^{\alpha}$, respectively. The regime-switching Esscher transform $P^*\sim\mathcal{F}$ on $\mathcal{G}_t$ with respect to parameters $\varsigma_1(\alpha(\cdot))$ and $\varsigma_2(\alpha(\cdot))$ is given by

$$\frac{\mathrm{d}P^*}{\mathrm{d}\mathcal{P}}\Big|_{\mathcal{G}_t}=\frac{e^{\int_0^t\varsigma_1(\alpha(s))\mathrm{d}w_1(s)+\int_0^t\varsigma_2(\alpha(s))\mathrm{d}w_2(s)}}{E_{\mathcal{P}}\left[e^{\int_0^t\varsigma_1(\alpha(s))\mathrm{d}w_1(s)+\int_0^t\varsigma_2(\alpha(s))\mathrm{d}w_2(s)}\Big|\mathcal{F}_t^{\alpha}\right]}. \tag{7}$$

Thanks to the well-known result established in [17, 18], the absence of arbitrage opportunities is essentially equivalent to the existence of equivalent martingale, under which the discount price process is a martingale. We know $\{e^{-rt}x_2(t)\}$ and $\{e^{-rt}y(t)\}$ are martingales under the measure $P^*$. Thus, we have

$$\begin{aligned}\varsigma_1(\alpha(t))&=\frac{r-\mu(\alpha(t))}{\sigma(\alpha(t))},\\ \varsigma_2(\alpha(t))&=\frac{r-\mu_y(\alpha(t))}{\sigma_y(\alpha(t))\sqrt{1-\rho^2}}-\frac{\rho}{\sqrt{1-\rho^2}}\frac{r-\mu(\alpha(t))}{\sigma(\alpha(t))}.\end{aligned} \tag{8}$$

According to Girsanov's theorem, we know

$$\begin{aligned}w_1^*(t)&=w_1(t)-\int_0^t\frac{r-\mu(\alpha(s))}{\sigma(\alpha(s))}\mathrm{d}s,\\ w_2^*(t)&=w_2(t)-\int_0^t\left(\frac{r-\mu_y(\alpha(s))}{\sigma_y(\alpha(s))\sqrt{1-\rho^2}}-\frac{\rho}{\sqrt{1-\rho^2}}\frac{r-\mu(\alpha(s))}{\sigma(\alpha(s))}\right)\mathrm{d}s,\end{aligned} \tag{9}$$

are two independent standard Brownian motions under the measure $P^*$.

Let $z_x(t)=\ln(x(t)/x_0)$; then, we have

$$z_x(t)=\int_0^t\left(r-\frac{\pi(\alpha(s))^2\sigma^2(\alpha(s))}{2}\right)\mathrm{d}s+\int_0^t\pi(\alpha(s))\sigma(\alpha(s))\mathrm{d}w_1^*(s), \tag{10}$$

$$z_y(t)=\int_0^t\left(r-\frac{\sigma_y^2(\alpha(s))}{2}\right)\mathrm{d}s+\int_0^t\sigma_y(\alpha(s))\rho\mathrm{d}w_1^*(s)+\int_0^t\sigma_y(\alpha(s))\sqrt{1-\rho^2}\mathrm{d}w_2^*(s). \tag{11}$$

From (10) and (11), we find that, given $\mathcal{F}_t^{\alpha}$ under the measure $P^*$, $z_x(t)$ has normal distribution with mean $\mu_{z_x}(t)=\int_0^t(r-(\pi(\alpha(s))^2\sigma^2(\alpha(s))/2))\mathrm{d}s$ and variance $\sigma_{z_x}^2(t)=\int_0^t\pi(\alpha(s))^2\sigma^2(\alpha(s))\mathrm{d}s$. $z_y(t)$ has normal distribution with mean $\mu_{z_y}(t)=\int_0^t(r-(\sigma_y^2(\alpha(s))/2))\mathrm{d}s$ and variance $\sigma_{z_y}^2(t)=\int_0^t\sigma_y^2(\alpha(s))\mathrm{d}s$. Hence, we know that $(z_x(t),z_y(t))$ has bivariate normal distribution $N(\mu_{z_x}(t),\mu_{z_y}(t),\sigma_{z_x}^2(t),\sigma_{z_y}^2(t),\rho_{z_x,z_y}(t))$, with a correlation coefficient $\rho_{z_x,z_y}(t)=(\int_0^t\rho\pi(\alpha(s))\sigma(\alpha(s))\sigma_y(\alpha(s))\mathrm{d}s)/(\sigma_{z_x}(t)\sigma_{z_y}(t))$.

## 4. Scenario Case Analysis

In this section, we will focus on the premium calculations under different scenario cases. The first case is analyzing the pension fund has premature termination. We assume that there is a third-party external regulator (like pension actuary), who is in charge of monitoring the performance of pension fund. The pension fund can thus be forced to close prematurely if a certain threshold value is reached. The other case we will study is from the perspective of plan sponsor. When the performance of the sponsor asset is not good enough, the plan provider is not able to cover its debt and thus remains in business, and the distress termination would happen. We choose to examine a threshold value higher than its liability value to include the other possible expenses for the sponsor company to remain in business.

*4.1. Premature Termination of Pension Fund.* In this section, we consider the case of premature termination of pension fund. The threshold value at time $t$ is assumed to be $\eta B_R e^{-(R-t)r}$, where $\eta$ is a positive constant less than 1. Therefore, we can define the first hitting time as

$$\tau=\inf\left\{t\mid x(t)\le\eta B_R e^{-(R-t)r}\right\}. \tag{12}$$

Starting from here, we will consider two cases as below.

The first case is premature termination of pension fund happens prior to retirement time $R$, i.e., $\tau<R$. When $\tau<R$, pension fund is underfunded, and the possible outcomes for the sponsor assets are

Sponsor company is defaulted, which means $y(\tau)<\theta y_0e^{\nu\tau}$. In this case, PBGC takes the whole obligation to pay the part that pension fund fails to cover.

Sponsor company is partially solvent:

$$\theta y_0e^{\nu\tau}\le y(\tau)<\theta y_0e^{\nu\tau}+B_Re^{-(R-\tau)r}-x(\tau). \tag{13}$$

In this case, PBGC provides the rest of the part that sponsor company is unable to pay.

Sponsor company is performing very well:

$$y(\tau) \geq \theta y_0 e^{\nu\tau} + B_R e^{-(R-\tau)r} - x(\tau). \tag{14}$$

All the benefit to beneficiary can be paid without the help of PBGC. Note that in our work, we are mainly interested in finding how much premium that PBGC should collect from sponsor company to provide the corresponding protection. In this case, PBGC does not provide anything, and therefore, we will derive the premium just based on the first two scenarios.

Based on the above analysis, we can model the support from sponsor's company $S^c(\tau)I_{\{\tau<R\}}$ as below:

$$S^c(\tau)I_{\{\tau<R\}} = \begin{cases} 0, & y(\tau) < \theta y_0 e^{\nu\tau}; \\ y(\tau) - \theta y_0 e^{\nu\tau}, & \theta y_0 e^{\nu\tau} \leq y(\tau) < \theta y_0 e^{\nu\tau} + B_R e^{-(R-\tau)r} - x(\tau); \\ B_R e^{-(R-\tau)r} - x(\tau), & y(\tau) \geq \theta y_0 e^{\nu\tau} + B_R e^{-(R-\tau)r} - x(\tau). \end{cases} \tag{15}$$

The second case is that pension fund falls below the threshold after time $R$, i.e., $\tau \geq R$; this implies $x(R) \geq \eta B_R$. When $\tau > R$, the pension fund is naturally closed at the maturity date $R$. Note that our assumption is PBGC collects premium up to time $R$ when beneficiary gets retired. Therefore, we just need to consider the case of $\tau = R$. Also, note that if $x(R) \geq B_R$, both sponsor company and PBGC do not need to provide anything, and therefore, we just focus on the case that $\eta B_R \leq x(R) < B_R$ to proceed and we can represent $S^c(\tau)I_{\{\tau\geq R\}}$ as below:

$$S^c(R)I_{\{\tau\geq R\}} = \begin{cases} 0, & y(R) < \theta y_0 e^{\nu R}, & \eta B_R \leq x(R) < B_R; \\ y(R) - \theta y_0 e^{\nu R}, & \theta y_0 e^{\nu R} \leq y(R) < \theta y_0 e^{\nu R} + B_R - x(R), & \eta B_R \leq x(R) < B_R; \\ B_R - x(R), & y(R) \geq \theta y_0 e^{\nu R} + B_R - x(R), & \eta B_R \leq x(R) < B_R. \end{cases} \tag{16}$$

Therefore, the entire support provided by sponsor company is

$$S^c = S^c(\tau)I_{\{\tau<R\}} + S^c(R)I_{\{\tau\geq R\}}. \tag{17}$$

Accordingly, the entire support provided by PBGC is the sum of the following two parts:

$$\begin{aligned} S^p(\tau)I_{\{\tau<R\}} &= B_R e^{-(R-\tau)r} - x(\tau) - S^c(\tau)I_{\{\tau<R\}}, \\ S^p(R)I_{\{\tau\geq R\}} &= B_R - x(R) - S^c(R)I_{\{\tau\geq R\}}. \end{aligned} \tag{18}$$

Thus, the total contribution supported by PBGC is

$$S^p = S^p(\tau)I_{\{\tau<R\}} + S^p(R)I_{\{\tau\geq R\}}. \tag{19}$$

*4.2. Distress Termination of Sponsor Asset.* In this section, we consider the case that sponsor asset can be underfunded and it is called "distress termination" in [11]. Therefore, in this case, we use a stopping time $\tau^p$ to describe the first time that the sponsor asset falls below or across the threshold $\vartheta y_0 e^{\nu t}$, given that the sponsor company has corporate debt $\theta y_0 e^{\nu t}$ at time $t$. We assume $\vartheta \geq \theta$, similar to the assumption used in [11]. Under this framework, the definition of $\tau^p$ is given as

$$\tau^p = \inf\left\{t \mid y(t) \leq \vartheta y_0 e^{\nu t}\right\}. \tag{20}$$

Similar to what we have discussed previously, we consider the case that $\tau^p < R$ and $\tau^p \geq R$ in the following paragraphs. Similarly, our analysis breaks into two cases as below.

Sponsor asset falls below the threshold before time $R$, $\tau^p < R$. When $\tau^p < R$, the possible outcomes for the pension fund are

The pension fund performs good enough, that is, the value of pension fund is worth more than the discounted promised pension benefit payment. Both sponsor company and PBGC need to do nothing in the case.

The pension fund is not sufficient to cover all the discounted pension benefit, i.e., $x(\tau^p) < B_R \exp(-(R-\tau^p)r)$.

Hence, we can represent $S^c(\tau^p)I_{\{\tau^p<R\}}$ as below:

$$S^c(\tau^p)I_{\{\tau^p<R\}} = \begin{cases} B_R e^{-(R-\tau^p)r} - x(\tau^p), & 0 < B_R e^{-(R-\tau^p)r} - x(\tau^p) < (\vartheta-\theta) y_0 e^{\nu\tau^p}; \\ (\vartheta-\theta) y_0 e^{\nu\tau^p}, & B_R e^{-(R-\tau^p)r} - x(\tau^p) \geq (\vartheta-\theta) y_0 e^{\nu\tau^p}. \end{cases} \tag{21}$$

Sponsor asset falls below the threshold not earlier than $R$, $\tau^p \geq R$; this implies $y(R) > \vartheta y_0 e^{\nu R}$. When $\tau^p \geq R$, the pension fund is naturally closed at the maturity date $R$. Note that PBGC collects premium up to time $R$ when beneficiary is retired. Therefore, we essentially consider the case of $\tau^p = R$. Also, note that if $x(R) \geq B_R$, both sponsor company and the PBGC do not need to provide anything, and therefore, we just focus on the case that $x(R) < B_R$ to proceed.

Thus, $S^c(R)I_{\{\tau^p \geq R\}}$ is described as follows:

$$S^c(R)I_{\{\tau^p \geq R\}} = \begin{cases} B_R - x(R), & y(R) \geq \vartheta y_0 e^{\nu R} + B_R - x(R), \quad B_R > x(R); \\ y(R) - \theta y_0 e^{\nu R}, & y(R) < \vartheta y_0 e^{\nu R} + B_R - x(R), \quad B_R > x(R). \end{cases} \tag{22}$$

Combining what we have above together, the entire support provided by the sponsor company is

$$S^c = S^c(\tau^p)I_{\{\tau^p < R\}} + S^c(R)I_{\{\tau^p \geq R\}}. \tag{23}$$

Accordingly, the entire support provided by PBGC is

$$\begin{aligned} &S^p(\tau^p)I_{\{\tau^p < R\}} \\ &= \left(B_R e^{-(R-\tau^p)r} - x(\tau^p) - S^c(\tau^p)I_{\{\tau^p < R\}}\right) \\ &= \left(B_R e^{-(R-\tau^p)r} - x(\tau^p) - (\vartheta - \theta)y_0 e^{\nu\tau^p}\right)I_{\left\{B_R e^{-(R-\tau^p)r} - x(\tau^p) \geq (\vartheta-\theta)y_0 e^{\nu\tau^p}\right\}}, \\ &S^p(R)I_{\{\tau^p \geq R\}} \\ &= \left(B_R - x(R) - S^c(R)_{\{\tau^p \geq R\}}\right) \\ &= \left(B_R - x(R) - y(R) + \theta y_0 e^{\nu R}\right)I_{\{y(R) < \theta y_0 e^{\nu R} + B_R - x(R), B_R > x(R)\}}. \end{aligned} \tag{24}$$

Thus, the entire support provided by PBGC is summarized as

$$S^p = S^p(\tau^p)I_{\{\tau^p < R\}} + S^p(R)I_{\{\tau^p \geq R\}}. \tag{25}$$

## 5. Premium Calculations

Now we can proceed to find the premium for PBGC. Using the no arbitrage idea, the premium paid by plan sponsor to PBGC is the expected discounted insurance payoff under the risk neutral probability measure. Before we discuss the calculation of premium, let us introduce some notations first. Let $J_t = \int_0^t I_{\{\alpha(s)=0\}} ds$ denote the occupation time of Markov chain at state 0 during $[0, t]$. In the calculation below, we will employ the method mentioned in [19] to analyze the case of stopping time being reached before and after retirement time $R$, $\{\tau^p < R\}$ and $\{\tau^p \geq R\}$ separately due to calculation convenience.

For the sake of simplicity, we first give some symbols. We denote

$$\begin{aligned} m_x(t,R) &= \left(r - \frac{1}{2}\pi(0)^2\sigma^2(0)\right)t + \left(r - \frac{1}{2}\pi(1)^2\sigma^2(1)\right)(R-t) \\ &= \frac{1}{2}\left(\pi(1)^2\sigma^2(1) - \pi(0)^2\sigma^2(0)\right)t + \left(r - \frac{1}{2}\pi(1)^2\sigma^2(1)\right)R, \\ v_x(t,R) &= \sqrt{\left(\pi(0)^2\sigma^2(0)\right)t + \pi(1)^2\sigma^2(1)(R-t)} \\ &= \sqrt{\left(\pi(0)^2\sigma^2(0) - \pi(1)^2\sigma^2(1)\right)t + \pi(1)^2\sigma^2(1)R}, \\ m_y(t,R) &= \left(r - \frac{1}{2}\sigma_y^2(0)\right)t + \left(r - \frac{1}{2}\sigma_y^2(1)\right)(R-t) \\ &= \frac{1}{2}\left(\sigma_y^2(1) - \sigma_y^2(0)\right)t + \left(r - \frac{1}{2}\sigma_y^2(1)\right)R, \\ v_y(t,R) &= \sqrt{\sigma_y^2(0)t + \sigma_y^2(1)(R-t)} = \sqrt{\left(\sigma_y^2(0) - \sigma_y^2(1)\right)t + \sigma_y^2(1)R}, \\ v_{xy}(t,R) &= \pi(0)\sigma(0)\sigma_y(0)\rho t + \pi(1)\sigma(1)\sigma_y(1)\rho(R-t). \end{aligned} \tag{26}$$

Then, $(z_x(R), z_y(R) \mid J_R = t)$ has bivariate normal distribution, with probability density function

$$\phi(z_1, z_2, t) = \frac{1}{2\pi v_x(t,R) v_y(t,R)\sqrt{1-\overline{\rho}^2}} \exp\left(-\frac{z}{2\left(1-\overline{\rho}^2\right)}\right), \tag{27}$$

where

$$z = \frac{\left(z_1 - m_x(t,R)\right)^2}{v_x^2(t,R)} - 2\overline{\rho}\frac{\left(z_1 - m_x(t,R)\right)\left(z_2 - m_y(t,R)\right)}{v_x(t,R)v_y(t,R)} + \frac{\left(z_2 - m_y(t,R)\right)^2}{v_y^2(t,R)},$$
$$\overline{\rho} = \frac{v_{xy}(t,R)}{v_x(t,R)v_y(t,R)}. \tag{28}$$

From Yoon et al. [20], we can obtain the following lemma.

**Lemma 1.** *Let $f_i(t,u)$ be the probability density of $J_t$ at initial state i, and we have*

$$f_0(t,u) = e^{-\lambda_1(t-u)-\lambda_0 u}\left(\left[\frac{u\lambda_1\lambda_0}{t-u}\right]^{1/2} I_1\left[2\left(\lambda_0\lambda_1 u(t-u)\right)^{1/2}\right] + \lambda_0 I_0\left[2\left(\lambda_0\lambda_1 u(t-u)\right)^{1/2}\right]\right),$$
$$f_1(t,u) = e^{-\lambda_1(t-u)-\lambda_0 u}\left(\left[\frac{(t-u)\lambda_1\lambda_0}{u}\right]^{1/2} I_1\left[2\left(\lambda_0\lambda_1 u(t-u)\right)^{1/2}\right] + \lambda_1 I_0\left[2\left(\lambda_0\lambda_1 u(t-u)\right)^{1/2}\right]\right), \tag{29}$$

*where $f_0(t,0) = 0$, $f_0(t,t) = e^{-\lambda_0 t}$, $f_1(t,0) = e^{-\lambda_1 t}$, $f_1(t,t) = 0$, and $I_\alpha(z)$ is the modified Bessel function ($\alpha = 0, 1$) of the first type such that*

$$I_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(\alpha+n+1)}. \tag{30}$$

Let $F_\tau(t; J_t) = P^*(\tau \le t \mid \mathscr{F}_R^\alpha)$ and $F_{\tau^p}(t; J_t) = P^*(\tau^p \le t \mid \mathscr{F}_R^\alpha)$ denote the probability distribution function of $\tau$ and $\tau^p$ given on $\mathscr{F}_R^\alpha$. Then, we have the following lemma.

**Lemma 2.** *Let $g(t; J_t)$ and $h(t; J_t)$ denote the condition density functions of $\tau$ and $\tau^p$; $g(t; J_t)$ and $h(t; J_t)$ are then given by (32) and (34).*

*Proof.* It follows from [21, 22] that

$$F_\tau(t; J_t) = N\left(\frac{\ln\left(\eta B_R e^{-rR}/x_0\right) + \left(v_x(J_t;t)^2/2\right)}{v_x(J_t;t)}\right) + \frac{x_0}{\eta B_R e^{-rR}} N\left(\frac{\ln\left(\eta B_R e^{-rR}/x_0\right) - \left(v_x(J_t;t)^2/2\right)}{v_x(J_t;t)}\right). \tag{31}$$

By taking derivative with respect to variable $t$ of $F_\tau(t; J_t)$, we have

$$g(t; J_t) = \frac{\pi(1)^2\sigma(1)^2\left(0.5 v_x(t;J_t)^2 - \ln\left(\eta B_R e^{-rR}/x_0\right)\right)}{2\sqrt{2\pi} v_x(t;J_t)^3} \exp\left\{-0.5\left(\frac{\ln\left(\eta B_R e^{-rR}/x_0\right) + \left(v_x(t;J_t)^2/2\right)}{v_x(t;J_t)}\right)^2\right\}$$
$$+ \frac{\pi(1)^2\sigma(1)^2 x_0\left(0.5 v_x(t;J_t)^2 + \ln\left(\eta B_R e^{-rR}/x_0\right)\right)}{2\sqrt{2\pi}\eta B_R e^{-rR} v_x(t;J_t)^3} \exp\left\{-0.5\left(\frac{\ln\left(\eta B_R e^{-rR}/x_0\right) - \left(v_x(t;J_t)^2/2\right)}{v_x(t;J_t)}\right)^2\right\}. \tag{32}$$

By adopting the same method, we have

$$F_{\tau^P}(t;J_t) = N\left(\frac{\ln\vartheta-(r-\nu)t+\left(v_y(t;J_t)^2/2\right)}{v_y(t;J_t)}\right)+\vartheta^{\left(2(r-\nu)t-v_y(J_t;t)^2\right)/v_y(J_t;t)^2}N\left(\frac{\ln\vartheta+(r-\nu)t-\left(v_y(J_t;t)^2/2\right)}{v_y(J_t;t)}\right). \tag{33}$$

Then,

$$\begin{aligned}
h(t;J_t) = {} & \frac{\left(\sigma(1)^2-\sigma(0)^2\right)\left[0.5v_y(J_t;t)^2-\ln\vartheta+(r-\nu)t\right]-2(r-\nu)v_y(J_t;t)^2}{2\sqrt{2\pi}v_y(J_t;t)^3} \\
& \times \exp\left\{-0.5\left(\frac{\ln\vartheta-(r-\nu)t+\left(v_y(J_t;t)^2/2\right)}{v_y(J_t;t)}\right)^2\right\} \\
& + \vartheta^{\left(2(r-\nu)t-v_y(J_t;t)^2\right)/v_y(J_t;t)^2}\frac{\left(\sigma(1)^2-\sigma(0)^2\right)\left[0.5v_y(J_t;t)^2+\ln\vartheta+(r-\nu)t\right]+2(r-\nu)v_y(J_t;t)^2}{2\sqrt{2\pi}v_y(J_t;t)^3} \\
& \times \exp\left\{-0.5\left(\frac{\ln\vartheta+(r-\nu)t-\left(v_y(J_t;t)^2/2\right)}{v_y(J_t;t)}\right)^2\right\}+\vartheta^{\left(2(r-\nu)t-v_y(J_t;t)^2\right)/v_y(J_t;t)^2} \\
& \times N\left(\frac{\ln\vartheta+(r-\nu)t-\left(v_y(J_t;t)^2/2\right)}{v_y(J_t;t)}\right)\ln\vartheta\left\{\frac{2(r-\nu)\left(v_y(J_t;t)^2+t\left(\sigma(1)^2-\sigma(0)^2\right)\right)}{v_y(J_t;t)^4}\right\}.
\end{aligned} \tag{34}$$

□

**Theorem 1.** *If the Markov chain initial state $\alpha(0)=i$, then the premium received by PBGC with premature closure of pension fund under the risk neutral probability measure $P^*$ is given by (A.14).*

*Proof.* The closed-form solution is given in Appendix A. For details, see Appendix A. □

**Corollary 1.** *If the Markov chain initial state $\alpha(0)=i$, then the risk-based premium of PBGC with early termination of sponsor assets under the risk neutral probability measure $P^*$ is given by (B.6).*

*Proof.* The closed-form solution is given in Appendix B. For details, see Appendix B. □

*Remark 1.*

In reality, sometimes both sponsor company and the PBGC only provide a capped retirement income when the employee gets retired early or when the pension fund is highly underfunded. In this case, we can model the support from PBGC by a constant $C$ with $C<B_R$; by using the similar method as [11], the premium can be calculated.

The premium that sponsor company collects from plan participants can also be calculated by the similar method used before.

## 6. Numerical Analysis

In this section, we make numerical analysis of the explicit formula derived in the previous section. For numerical demonstration, we take the following parameters:

$$\begin{aligned}
& r(1)=0.05, \sigma(1)=0.16, x(0)=600, \eta=0.8, \theta=0.6, \\
& \pi(0)=0.6, \pi(1)=0.3, \rho=0.5, y(0)=800, \\
& R=15, B_R=762.12, r=0.035, \nu=0.03, \\
& r(2)=0.02, \sigma(2)=0.4, \sigma_y(1)=0.18, \sigma_y(2)=0.48, \vartheta=0.63, \\
& \mu(1)=0.08, \mu(2)=0.04, \mu_y(1)=0.07, \mu_y(2)=0.05.
\end{aligned} \tag{35}$$

Figures 1 and 2 demonstrate the effects of $B_R$ on premiums under different situations. To be more specific, Figure 1 illustrates how the premium changes against $B_R$ when premature termination of pension fund occurs and Figure 2 shows the impact of $B_R$ on premium for the case of distress termination of sponsor assets. We make a couple of observations regarding the two graphs. First, the premium is an increasing function of $B_R$ in both graphs. Given that $B_R$ is the retirement benefit that one expects to receive, it makes sense to observe that a higher $B_R$ implies a bigger financial responsibility for the employer to hold. Therefore, it is expected for the employer to pay more premium to the PBGC to transfer the pension risk. Second, we notice that the

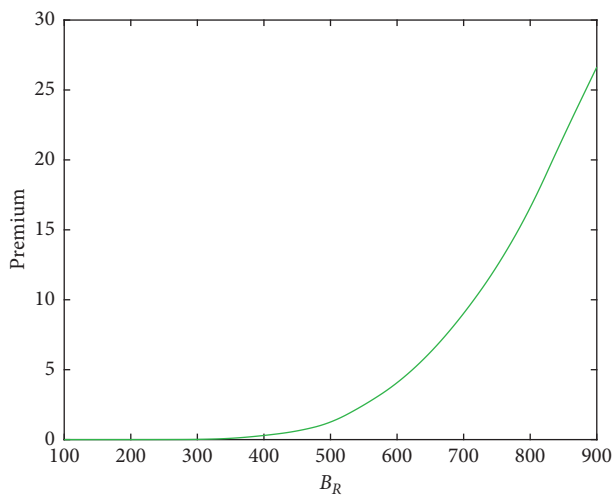


Figure 1: The impact of $B_R$ on the premium with premature termination of pension fund.

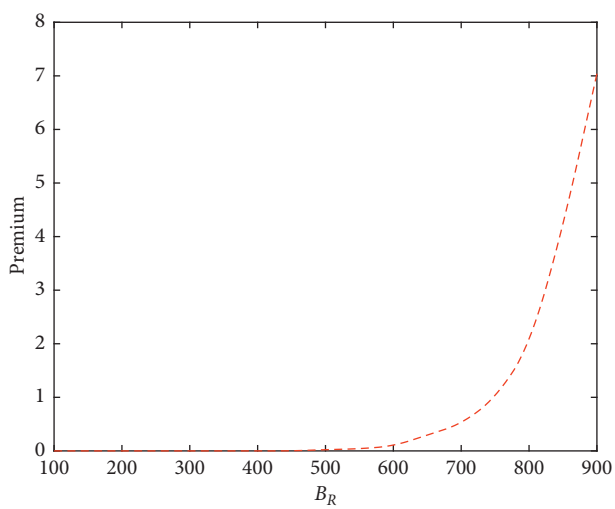


Figure 2: The impact of $B_R$ on the premium with distress termination of sponsor asset.

premium increases much more significantly in Figure 1, compared with its trend in Figure 2. Note that pension fund is the first and foremost pool of fund for postretirement payment regardless of financial soundness of the sponsor company, and thus it is reasonable to witness that premium increases more quickly when there is a risk of premature termination of pension fund.

In Figure 3, we demonstrate how the premium changes with respect to $\eta$ while premature termination of pension is under consideration. Recall that $\eta$ is the trigger ratio of the pension fund and is also referred to as the regulatory parameter. We know that, on the one hand, termination would never happen if $\eta < 0$ given that the pension fund is nonnegative. On the other hand, $\eta > 1$ does not make sense since it implies that more than enough reserve should be set aside in the pension fund pool. Thus, we assume that $\eta \in (0, 1)$. We can see that the dynamics of premium according to $\eta$ are not monotonic with our choice of parameters. The increase

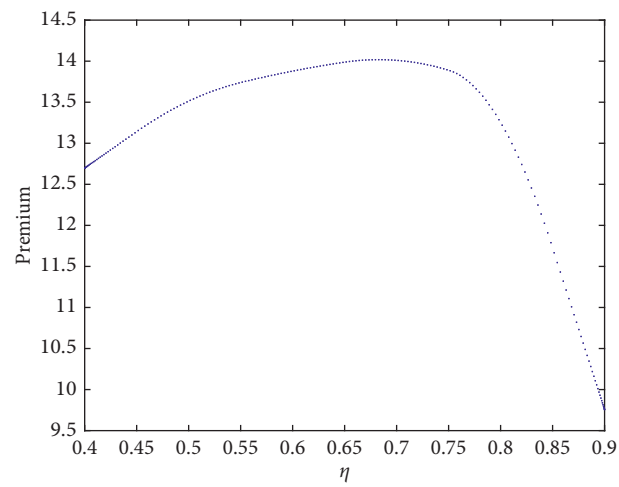


Figure 3: The impact of $\eta$ on the premium with premature termination of pension fund.

of regulatory parameter first leads to a very mild increase of premium and there is a dip of the premium afterwards. The possible reasons behind this interesting pattern are explored as follows. When the regulatory parameter $\eta$ is small, the pension fund value is small while the threshold value is hit. It is more likely that the PBGC needs to step in when pension fund has premature termination. Our conjecture for the mild increase portion of premium is that it is a reflection of the expected compensation that the PBGC predicts for its higher probability of providing the coverage. With the increase of regulatory parameter, the premium starts to go down to adjust for stronger regulation requirement after an "optimal" value of it being reached first.

Figure 4 is about the relation between premium and threshold value $\vartheta$ for the case of distress termination of the sponsor assets. We have the requirements about the $\vartheta$ such that it not only satisfies $\vartheta < 1$ but also meets the condition that $\vartheta < \nu$. We need $\vartheta < 1$ to reflect the assumption that the sponsoring company is not in default yet in the beginning. $\vartheta > \nu$ is necessary to take account of the fact that the pension sponsor has the moral obligation to cover some deficits of the claimed pension benefit. Our result shows a decrease of premium in regard to the increase of $\vartheta$ in the figure. Note that the three sequential lines of protections are assumed to be the pension fund, the sponsor assets, and the PBGC. Higher threshold value of the sponsor company at distress termination shows that the employer is capable to provide more financial support irrespective of performance of the pension fund, and the PBGC thus charges less premium accordingly with less financial burden.

## 7. Further Remarks

In this work, we focus on finding the closed-form formula for the risk-based premium of the PBGC with regime switching. The explicit solutions of the premiums are derived.

Further efforts can be directed to the portfolio selection for the pension fund in which stochastic control and Markov

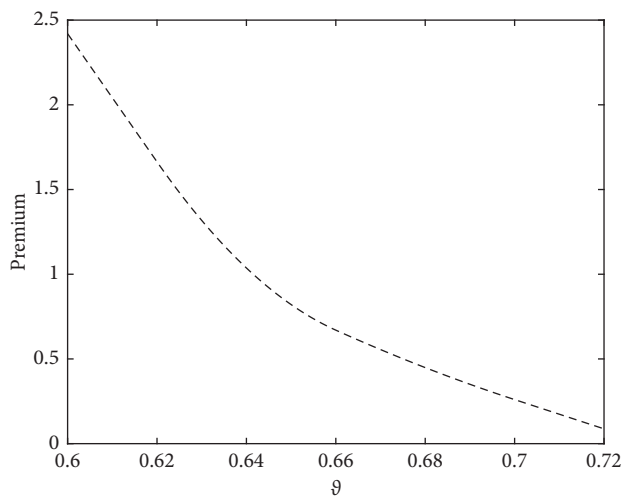


Figure 4: The impact of $\vartheta$ on the premium with distress termination of sponsor asset.

chain approximation seem to be reasonable methods to use. Further effort in this direction deserves more thoughts and considerations.

## Appendix

## A. Proof of Theorem 1

By the risk neutral pricing theory, the premium for PBGC with premature closure of pension fund is the expected discounted insurance payoff under the risk neutral probability measure $P^*$ and is given by

$$S_0^p = E^*\left[e^{-r\tau}S^p(\tau)I_{\{\tau<R\}}\right] + E^*\left[e^{-rR}S^p(R)I_{\{\tau\geq R\}}\right]. \quad (A.1)$$

Hence, the premium for the insurance of the PBGC can be decomposed into two parts:

(i) $E^*[e^{-r\tau}S^p(\tau)I_{\{\tau<R\}}]$.

(ii) $E^*[e^{-rR}S^p(R)I_{\{\tau\geq R\}}]$.

(1) Regarding (i), we calculate it by iterated expectation formula as follows:

$$\begin{aligned}
&E^*\left[e^{-r\tau}S^p(\tau)I_{\{\tau<R\}}\right] \\
&= E^*\Big[e^{-r\tau}\Big(\left(B_R e^{-r(R-\tau)} - x(\tau)\right)I_{\{y(\tau)<\theta y_0 e^{\nu\tau}\}} \\
&\quad + \left(B_R e^{-r(R-\tau)} - x(\tau) - y(\tau) + \theta y_0 e^{\nu\tau}\right)I_{\{\theta y_0 e^{\nu\tau}\leq y(\tau)<\theta y_0 e^{\nu\tau}+(1-\eta)B_R e^{r(R-\tau)}\}}I_{\{\tau<R\}} \\
&= E^*\left[e^{-r\tau}\left(B_R e^{-r(R-\tau)}\right)(1-\eta)I_{\{y(\tau)\leq\theta y_0 e^{\nu\tau}\}}I_{\{\tau<R\}}\right] \\
&\quad + E^*\Big[\Big(e^{-r\tau}\left((1-\eta)B_R e^{-r(R-\tau)} - \left(y(\tau) - \theta y_0 e^{\nu\tau}\right)\right) \times I_{\{\theta y_0 e^{\nu\tau}\leq y(\tau)<\theta y_0 e^{\nu\tau}+(1-\eta)B_R e^{-r(R-\tau)}\}}I_{\{\tau<R\}}.
\end{aligned} \quad (A.2)$$

Note that here we used the fact that $x(\tau) = \eta B_R e^{-r(R-\tau)}$ in the above. On the one hand,

$$\begin{aligned}
&E^*\left[e^{-r\tau}\left(B_R e^{-r(R-\tau)}\right)(1-\eta)I_{\{y(\tau)\leq\theta y_0 e^{\nu\tau}\}}I_{\{\tau<R\}}\right] \\
&= \int_0^R\int_0^u\int_{-\infty}^{\ln\theta+\nu u} e^{-rR}B_R(1-\eta)g(u;t)f_i(u,t)\phi_2(z_2;u,t)\mathrm{d}z_2\mathrm{d}t\mathrm{d}u \\
&\quad + \Gamma_1(0,u;\theta,R,\nu,B_R,\lambda) + \Gamma_1(1,0;\theta,R,\nu,B_R,\lambda),
\end{aligned} \quad (A.3)$$

in which

$$\Gamma_1(j,t;\theta,R,\nu,B_R,\lambda)=\int_0^R\int_{-\infty}^{\ln\theta+\nu u}\delta_j(i)e^{-\lambda_j u}e^{-rR}B_R(1-\eta)g(u;t)\phi_2(z_2;u,t)\mathrm{d}z_2\mathrm{d}u,$$
$$\phi_2(z_2;u,t)=\frac{1}{\sqrt{2\pi v_y(t,u)}}\exp\left\{\frac{\left(z_2-m_y(t,u)\right)^2}{2v_y(t,u)^2}\right\},\tag{A.4}$$
$$\delta_j(i)=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

On the other hand,

$$\begin{aligned}&E^*\left[\left[e^{-r\tau}\left((1-\eta)B_Re^{-r(R-\tau)}-\left(y(\tau)-\theta y_0e^{\nu\tau}\right)\right)\right]\right]\\&\quad\times I_{\left\{\theta y_0e^{\nu\tau}\le y(\tau)<\theta y_0e^{\nu\tau}+(1-\eta)B_Re^{-r(R-\tau)}\right\}}I_{\{\tau<R\}}\\&=\int_0^R\int_0^u\int_{\ln\theta+\nu u}^{\ln\left(\left(\theta y_0e^{\nu u}+(1-\eta)B_Re^{-r(R-u)}\right)/y_0\right)}\left(e^{-rR}B_R(1-\eta)-e^{-ru}\left(y_0e^{z_2}-\theta y_0e^{\nu u}\right)\right)g(u;t)f_i(u,t)\times\phi_2(z_2;u,t)\mathrm{d}z_2\mathrm{d}t\mathrm{d}u\\&\quad+\Gamma_2(0,u;\theta,R,\nu,B_R,\lambda,y_0)+\Gamma_2(1,0;\theta,R,\nu,B_R,\lambda,y_0),\end{aligned}\tag{A.5}$$

in which

$$\Gamma_2(j,t;\theta,R,\nu,B_R,\lambda,y_0)=\int_0^R\int_{\ln\theta+\nu u}^{\ln\left(\left(\theta y_0e^{\nu u}+(1-\eta)B_Re^{-r(R-u)}\right)/y_0\right)}\delta_j(i)e^{-\lambda_j u}\left(e^{-rR}B_R(1-\eta)-e^{-ru}\left(y_0e^{z_2}-\theta y_0e^{\nu u}\right)\right)g(u;t)\phi_2(z_2;u,t)\mathrm{d}z_2\mathrm{d}u.\tag{A.6}$$

According to relationships (A.2), (A.3), and (A.5), we can obtain

$$\begin{aligned}&E^*\left[e^{-r\tau}S^p(\tau)I_{\{\tau<R\}}\right]\\&=\int_0^R\int_0^u\int_{-\infty}^{\ln\theta+\nu u}e^{-rR}B_R(1-\eta)g(u;t)f_i(u,t)\phi_2(z_2;u,t)\mathrm{d}z_2\mathrm{d}t\mathrm{d}u\\&\quad+\Gamma_1(0,u;\theta,R,\nu,B_R,\lambda)+\Gamma_1(1,0;\theta,R,\nu,B_R,\lambda)\\&\quad+\int_0^R\int_0^u\int_{\ln\theta+\nu u}^{\ln\left(\left(\theta y_0e^{\nu u}+(1-\eta)B_Re^{-r(R-u)}\right)/y_0\right)}\left(e^{-rR}B_R(1-\eta)-e^{-ru}\left(y_0e^{z_2}-\theta y_0e^{\nu u}\right)\right)g(u;t)f_i(u,t)\times\phi_2(z_2;u,t)\mathrm{d}z_2\mathrm{d}t\mathrm{d}u\\&\quad+\Gamma_2(0,u;\theta,R,\nu,B_R,\lambda,y_0)+\Gamma_2(1,0;\theta,R,\nu,B_R,\lambda,y_0).\end{aligned}\tag{A.7}$$

(2) As to (ii), we have

$$\begin{aligned}&E^*\left[e^{-rR}S^p(R)I_{\{\tau\ge R\}}\right]\\&\quad=E^*\left[E^*\left[e^{-rR}S^p(R)I_{\{\tau\ge R\}}|\tau\ge R,\mathscr{F}_R^\alpha\right]\right]\\&\quad=E^*\Bigg[I_{\{\tau\ge R\}}E^*\Bigg[e^{-rR}\Bigg(\left(B_R-x(R)\right)I_{\left\{y(R)<\theta y_0e^{\nu R}\right\}}I_{\left\{\eta B_R<x(R)<B_R\right\}}\\&\qquad+\left(B_R-x(R)-\left(y(R)-\theta y_0e^{\nu R}\right)\right)I_{\left\{\theta y_0e^{\nu R}\le y(R)<\theta y_0e^{\nu R}+B_R-x(R)\right\}}\\&\qquad\times I_{\left\{\eta B_R<x(R)<B_R\right\}}\Bigg)|\tau\ge R,\mathscr{F}_R^\alpha\Bigg]\Bigg].\end{aligned}\tag{A.8}$$

Therefore, according to Lemma 1, for $0<t<R$ and $i=1,2$, we get

$$\begin{aligned}
&E^*\left[I_{\{\tau\geq R\}}E^*\left[e^{-rR}\left(B_R-x(R)\right)I_{\{y(R)\leq\theta y_0e^{\nu R}\}}I_{\{\eta B_R<x(R)<B_R\}}|\tau\geq R,\mathscr{F}_R^\alpha\right]\right]\\
&=\int_0^R\int_{\ln\left(\eta B_R/x_0\right)}^{\ln\left(B_R/x_0\right)}\int_{-\infty}^{\nu R+\ln\theta}\left(1-F_\tau(R;t)\right)e^{-rR}\left(B_R-x_0e^{z_1}\right)\phi\left(z_1,z_2,t\right)f_i(R,t)\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}t\\
&\quad+\Gamma_3\left(0,R;\theta,R,\nu,B_R,\lambda,x_0\right)+\Gamma_3\left(1,0;\theta,R,\nu,B_R,\lambda,x_0\right),
\end{aligned}\tag{A.9}$$

in which

$$\begin{aligned}
\Gamma_3\left(j,t;\theta,R,\nu,B_R,\lambda,x_0\right)&=\delta_j(i)e^{-\lambda_jR-rR}\\
&\quad\times\int_{\ln\left(\eta B_R/x_0\right)}^{\ln\left(B_R/x_0\right)}\int_{-\infty}^{\nu R+\ln\theta}\left(1-F_\tau(R;t)\right)\left(B_R-x_0e^{z_1}\right)\phi\left(z_1,z_2,t\right)\cdot\mathrm{d}z_2\mathrm{d}z_1.
\end{aligned}\tag{A.10}$$

On the other hand,

$$\begin{aligned}
&E^*\left[I_{\{\tau\geq R\}}E^*\left(e^{-rR}\left(B_R-x(R)-t\left(y(R)-\theta y_0e^{\nu R}\right)\right)\right)\times I_{\{\theta y_0e^{\nu R}\leq y(R)<\theta y_0e^{\nu R}+B_R-x(R)\}}I_{\{\eta B_R<x(R)<B_R\}}|\tau\geq R,\mathscr{F}_R^\alpha\right]\\
&=\int_0^R\int_{\ln\left(\eta B_R/x_0\right)}^{\ln\left(B_R/x_0\right)}\int_{\nu R+\ln\theta}^{\ln\left(\left(\theta y_0e^{\nu R}+B_R-x_0e^{z_1}\right)/y_0\right)}\left(1-F_\tau(R;t)\right)e^{-rR}\times\left(B_R-x_0e^{z_1}-y_0e^{z_2}+\theta y_0e^{\nu R}\right)\phi\left(z_1,z_2,t\right)f_i(R,t)\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}t\\
&\quad+\Gamma_4\left(0,R;\theta,R,\nu,B_R,\lambda,x_0,y_0\right)+\Gamma_4\left(1,0;\theta,R,\nu,B_R,\lambda,x_0,y_0\right),
\end{aligned}\tag{A.11}$$

in which

$$\begin{aligned}
\Gamma_4\left(j,t;\theta,R,\nu,B_R,\lambda,x_0,y_0\right)&=\delta_j(i)e^{-\lambda_jR-rR}\int_{\ln\left(\eta B_R/x_0\right)}^{\ln\left(B_R/x_0\right)}\int_{\nu R+\ln\theta}^{\ln\left(\left(\theta y_0e^{\nu R}+B_R-x_0e^{z_1}\right)/y_0\right)}\left(1-F_\tau(R;t)\right)\\
&\quad\times\left(B_R-x_0e^{z_1}-y_0e^{z_2}+\theta y_0e^{\nu R}\right)\phi\left(z_1,z_2,t\right)\mathrm{d}z_2\mathrm{d}z_1.
\end{aligned}\tag{A.12}$$

Combining (A.9) with (A.11), we have

$$\begin{aligned}
E^*\left[e^{-rR}S^p(R)I_{\{\tau\geq R\}}\right]&=\int_0^R\int_{\ln\left(\eta B_R/x_0\right)}^{\ln\left(B_R/x_0\right)}\int_{-\infty}^{\nu R+\ln\theta}\left(1-F_\tau(R;t)\right)e^{-rR}\left(B_R-x_0e^{z_1}\right)\phi\left(z_1,z_2,t\right)f_i(R,t)\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}t\\
&\quad+\Gamma_3\left(0,R;\theta,R,\nu,B_R,\lambda,x_0\right)+\Gamma_3\left(1,0;\theta,R,\nu,B_R,\lambda,x_0\right)\\
&\quad+\int_0^R\int_{\ln\left(\eta B_R/x_0\right)}^{\ln\left(B_R/x_0\right)}\int_{\nu R+\ln\theta}^{\ln\left(\left(\theta y_0e^{\nu R}+B_R-x_0e^{z_1}\right)/y_0\right)}\left(1-F_\tau(R;t)\right)e^{-rR}\left(B_R-x_0e^{z_1}-y_0e^{z_2}+\theta y_0e^{\nu R}\right)\times\phi\left(z_1,z_2,t\right)f_i(R,t)\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}t\\
&\quad+\Gamma_4\left(0,R;\theta,R,\nu,B_R,\lambda,x_0,y_0\right)+\Gamma_4\left(1,0;\theta,R,\nu,B_R,\lambda,x_0,y_0\right).
\end{aligned}\tag{A.13}$$

From (A.7) and (A.13), the pricing formula of the premium of the PBGC with premature closure is given as follows:

$$\begin{aligned}
S_0^p = &\int_0^R \int_0^u \int_{-\infty}^{\ln\theta+\nu u} e^{-rR} B_R (1-\eta) g(u;t) f_i(u,t) \phi_2(z_2;u,t) \mathrm{d}z_2 \mathrm{d}t \mathrm{d}u + \Gamma_1(0,u;\theta,R,\nu,B_R,\lambda) \\
&+ \Gamma_1(1,0;\theta,R,\nu,B_R,\lambda) + \int_0^R \int_0^u \int_{\ln\theta+\nu u}^{\ln\left(\left(\theta y_0 e^{\nu u}+(1-\eta)B_R e^{-r(R-u)}\right)/y_0\right)} \left(e^{-rR} B_R (1-\eta)\right. \\
&\left. - e^{-ru}\left(y_0 e^{z_2} - \theta y_0 e^{\nu u}\right)\right) g(u;t) f_i(u,t) \times \phi_2(z_2;u,t) \mathrm{d}z_2 \mathrm{d}t \mathrm{d}u \\
&+ \Gamma_2(0,u;\theta,R,\nu,B_R,\lambda,y_0) + \Gamma_2(1,0;\theta,R,\nu,B_R,\lambda,y_0) \\
&+ \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{-\infty}^{\nu R+\ln\theta} (1-F_\tau(R;t)) e^{-rR} (B_R - x_0 e^{z_1}) \phi(z_1,z_2,t) f_i(R,t) \mathrm{d}z_2 \mathrm{d}z_1 \mathrm{d}t \\
&+ \Gamma_3(0,R;\theta,R,\nu,B_R,\lambda,x_0) + \Gamma_3(1,0;\theta,R,\nu,B_R,\lambda,x_0) \\
&+ \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{\nu R+\ln\theta}^{\ln\left(\theta y_0 e^{\nu R}+B_R-x_0 e^{z_1}/y_0\right)} (1-F_\tau(R;t)) e^{-rR} \left(B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}\right) \times \phi(z_1,z_2,t) f_i(R,t) \mathrm{d}z_2 \mathrm{d}z_1 \mathrm{d}t \\
&+ \Gamma_4(0,R;\theta,R,\nu,B_R,\lambda,x_0,y_0) + \Gamma_4(1,0;\theta,R,\nu,B_R,\lambda,x_0,y_0).
\end{aligned} \tag{A.14}$$

## B. Proof of Corollary 1

Similarly, we get $S_0^p$ by the risk neutral pricing theory as follows:

$$\begin{aligned}
S_0^p &= E^*\left[e^{-r\tau^p} S^p(\tau^p) I_{\{\tau^p<R\}}\right] + E^*\left[e^{-rR} S^p(R) I_{\{\tau^p\geq R\}}\right] \\
&= E^*\left[e^{-r\tau^p}\left(B_R e^{-(R-\tau^p)r} - x(\tau^p) - (\vartheta-\theta) y_0 e^{\nu\tau^p}\right) I_{\left\{B_R e^{-(R-\tau^p)r} - x(\tau^p) \geq (\vartheta-\theta) y_0 e^{\nu\tau^p}\right\}} I_{\{\tau^p<R\}}\right] \\
&\quad + E^*\left[e^{-rR}\left(B_R - x(R) - y(R) + \theta y_0 e^{\nu R}\right) I_{\{y(R)<\theta y_0 e^{\nu R}+B_R-x(R),\quad B_R>x(R)\}} I_{\{\tau^p\geq R\}}\right].
\end{aligned} \tag{B.1}$$

We will deal with the two parts similarly as in the previous section. The details are presented with the detailed calculation omitted.

$$\begin{aligned}
&E^*\left[e^{-r\tau^p} S^p(\tau^p) I_{\{\tau^p<R\}}\right] \\
&= \int_0^R \int_0^u \int_{-\infty}^{\ln\left(B_R e^{-rR-u} - \vartheta - \theta y_0 e^{\nu u}\right)/x_0} e^{-rR} B_R - e^{-ru} x_0 e^{z_1} - e^{-ru}\vartheta - \theta y_0 e^{\nu u} hu; t \times f_i(u,t)\, \phi_1(z_1,u,t)\, \mathrm{d}z_1 \mathrm{d}t \mathrm{d}u \\
&\quad + \Gamma_5(0,u;\theta,\vartheta,R,\nu,B_R,\lambda,x_0) + \Gamma_5(1,0;\theta,\vartheta,R,\nu,B_R,\lambda,x_0),
\end{aligned} \tag{B.2}$$

in which

$$\Gamma_5(j,t;\theta,\vartheta,R,\nu,B_R,\lambda,x_0) = \int_0^R \int_{-\infty}^{\ln\left(\left(B_R e^{-r(R-u)}-(\vartheta-\theta)y_0 e^{\nu u}\right)/x_0\right)} \delta_j(i)e^{-\lambda_j u} \times\left(e^{-rR}B_R - e^{-ru}x_0 e^{z_1} - e^{-ru}\left((\vartheta-\theta)y_0 e^{\nu u}\right)\right)h(u;t)\phi_1(z_1,u,t)\mathrm{d}z_1\mathrm{d}u, \tag{B.3}$$

$$\phi_1(z_2;u,t) = \frac{1}{\sqrt{2\pi v_x(t,u)}}\exp\left\{\frac{(z_1-m_x(t,u))^2}{2v_x(t,u)^2}\right\},$$

$$\begin{aligned} &E^*\left[e^{-rR}S^p(R)I_{\{\tau^p\geq R\}}\right] \\ &= \int_0^R \int_{-\infty}^{\ln(B_R/x_0)} \int_{-\infty}^{\ln\left(\left(\theta y_0 e^{\nu R}+B_R-x_0 e^{z_1}\right)/y_0\right)} (1-F_{\tau^p}(R;t))e^{-rR}\left(B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}\right)\times\phi(z_1,z_2,t)f_i(R,t)\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}t \\ &\quad + \Gamma_6(0,R;\theta,\vartheta,R,\nu,B_R,\lambda,x_0,y_0) + \Gamma_6(1,0;\theta,\vartheta,R,\nu,B_R,\lambda,x_0,y_0), \end{aligned} \tag{B.4}$$

in which

$$\begin{aligned} \Gamma_6(j,t;\theta,\vartheta,R,\nu,B_R,\lambda,x_0,y_0) = \delta_j(i)e^{-\lambda_j R - rR} &\int_{-\infty}^{\ln(B_R/x_0)} \int_{-\infty}^{\ln\left(\left(\theta y_0 e^{\nu R}+B_R-x_0 e^{z_1}\right)/y_0\right)} (1-F_{\tau^p}(R;t)) \\ &\times\left(B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}\right)\phi(z_1,z_2,t)\mathrm{d}z_2\mathrm{d}z_1. \end{aligned} \tag{B.5}$$

By combining the results of (B.1), (B.2), and (B.4), we thus get

$$\begin{aligned} S_0^p &= \int_0^R \int_0^u \int_{-\infty}^{\ln\left(\left(B_R e^{-r(R-u)}-(\vartheta-\theta)y_0 e^{\nu u}\right)/x_0\right)} \left(e^{-rR}B_R - e^{-ru}x_0 e^{z_1} - e^{-ru}\left((\vartheta-\theta)y_0 e^{\nu u}\right)\right)h(u;t)\times f_i(u,t)\phi_1(z_1,u,t)\mathrm{d}z_1\mathrm{d}t\mathrm{d}u \\ &\quad + \Gamma_5(0,u;\theta,\vartheta,R,\nu,B_R,\lambda,x_0) + \Gamma_5(1,0;\theta,\vartheta,R,\nu,B_R,\lambda,x_0) \\ &\quad + \int_0^R \int_{-\infty}^{\ln(B_R/x_0)} \int_{-\infty}^{\ln\left(\left(\theta y_0 e^{\nu R}+B_R-x_0 e^{z_1}\right)/y_0\right)} (1-F_{\tau^p}(R;t))e^{-rR}\left(B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}\right)\times\phi(z_1,z_2,t)f_i(R,t)\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}t \\ &\quad + \Gamma_6(0,R;\theta,\vartheta,R,\nu,B_R,\lambda,x_0,y_0) + \Gamma_6(1,0;\theta,\vartheta,R,\nu,B_R,\lambda,x_0,y_0). \end{aligned} \tag{B.6}$$

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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