Research Article

Efficient Estimation of the Generalized Quasi-Lindley Distribution Parameters under Ranked Set Sampling and Applications

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Ranked set sampling is a very useful method to collect data when the actual measurement of the units in a population is difficult or expensive. Recently, the generalized quasi-Lindley distribution is suggested as a new continuous lifetime distribution. In this article, the ranked set sampling method is considered to estimate the parameters of the generalized quasi-Lindley distribution. Several estimation methods are used, including the maximum likelihood, the maximum product of spacings, ordinary least squares, weighted least squares, Cramer–von Mises, and Anderson–Darling methods. The performance of the proposed ranked set sampling based estimators is achieved through a simulation study in terms of bias and mean squared errors compared to the simple random sample. Additional results are obtained based on real data for the survival times of 72 guinea pigs and 23 ball bearings. The simulation study results and the real data applications showed the superiority of the proposed ranked set sampling estimators compared to the simple random sample competitors based on the same number of measuring units.

1. Introduction

One of the significant interesting fields in statistics is the cost-effective sampling methods. The motivation of this field arises from its superiority in facilitating data collection, especially when collecting data of interest consumes a long time or is expensive. Over the past decades, researchers developed different sampling methods in order to achieve reliable results with low cost and more accuracy. McIntyre [1] proposed a new sampling method for estimating the mean of pasture and forage yields in Australia. This method is known as ranked set sampling (RSS) and has greater efficiency than the commonly used simple random sampling (SRS). Later, Halls and Dell [2]; Takahasi and Wakimoto [3]; and Dell and Clutter [4] published others studies on the RSS method. Due to its cost-effectiveness, it

is used in wide applications, including reliability, estimation of population parameters, statistical quality control, medicine, acceptance sampling plans, and so on (see Chen et al. [5–7], Haq et al. [8], and Al-Omari and Haq [9, 10]).

It is well documented that the RSS method is an attractive procedure and more acclimated to the nature of the underlying data. This advantage motivated researchers to propose and study new RSS schemes (see, for example, the double ranked set by Al-Saleh and Al-Kadiri [11]; median ranked set sampling by Muttlak [12]; neoteric ranked set sampling by Zamanzade and Al-Omari [13]; extreme ranked set sampling by Samawi et al. [14]; and L ranked set sampling by Al-Nasser [15]). For a good review of RSS and more motivations on this method, one can refer to Al-Omari and Bouza [16]; Al-Hadhrani and Al-Omari [17]; Haq et al. [18]; Santiago et al. [19]; and Haq et al. [20].

The problem of estimating distributions parameters, in general, is considered by many authors. For example, Nassar et al. [21] considered parameters estimation of the new extension of Weibull distribution. Nassar et al. [22] treated the estimation problem using the alpha power exponential distribution. Later, Afify and Mohamed [23] and Afify et al. [25] dealt with parameter estimation based on the new three-parameter exponential distribution and the Weibull Marshall-Olkin Lindley distribution, respectively. Recently, Alfaer et al. (2021) considered the extended log-logistic distribution and estimated its parameters. Although these works considered parameter estimations in related distributions, their works did not deal with sampling design techniques.

Undoubtedly, the parametric estimation method using the sampling design technique plays a vital role in statistical inference. Many studies considered the estimation of parameters based on RSS designs and their extensions using different estimation methods. Yousef and Al-Subh [26] estimated the Gumbel parameters using the maximum likelihood method, method of moment, and the method of regression. Hussian [27] used the Bayesian and maximum likelihood estimation methods to estimate the Kumaraswamy distribution parameters. Chen et al. [28] estimated the scale parameter for the scale distribution using moving extreme ranked set sampling, and Abu-Dayyeh et al. [29] considered the logistic method for parameter estimation based on both SRS and RSS. Pedroso et al. [30] considered the RSS to estimate the parameters of the two-parameter Birnbaum-Saunders distribution, and Akgul et al. [31] used the same RSS in system reliability estimation for generalized inverse Lindley distribution. (see also Taconeli and Bonat [32] for some estimation methods based on the RSS). For more details on parameter estimation, the readers are referred to Stokes [33]; Dey et al. [34]; Khamnei and Mayan [35]; and Al-Saleh and Al-Hadhrani [36].

This paper aims to study the performance of using RSS design in estimating the parameters of the generalized quasi-Lindley distribution (GQLD) introduced recently by Benchiha and Al-Omari [37]. A random variable X is said to follow a GQLD distribution with parameters θ and α if its pdf is given by

$$f(x; \alpha, \theta) = \frac{\theta^2((\theta^2 x^3/6) + \alpha \theta x^2 + \alpha^2 x)e^{-\theta x}}{(\alpha + 1)^2}; \quad (1)$$

$$x \geq 0, \alpha > -1, \theta \geq 0.$$

with cdf given by

$$F(x; \alpha, \theta) = 1 - \frac{(\theta^3 x^3 + 3(2\alpha + 1)\theta^2 x^2 + 6(\alpha + 1)^2(\theta x + 1))e^{-\theta x}}{6(\alpha + 1)^2}. \quad (2)$$

The first two moments of X , respectively, are

$$E(X) = \frac{6(\theta^2 + 1) + 6\theta + 6}{3(\theta + 1)^2 \alpha}, \quad (3)$$

$$E(X^2) = \frac{6(\theta + 1)^2 + 2(6\theta + 5) + 4}{(\theta + 1)^2 \alpha^2}.$$

The variance of the GQLD distribution is given by

$$V(X) = E(X^2) - (E(X))^2 = \frac{2(\theta^2 + 4\theta + 2)}{(\theta + 1)^2 \alpha^2}. \quad (4)$$

The corresponding reliability and hazard functions of the GQLD distribution, for $x > 0$, $\theta > -1$, $\alpha > 0$, are given, respectively, by

$$R_{\text{GQLD}}(x; \alpha, \theta) = \frac{(\alpha^3 x^3 + 3(2\theta + 1)\alpha^2 x^2 + 6(\theta + 1)^2 \alpha x + 6(\theta + 1)^2)e^{-\alpha x}}{6(\theta + 1)^2}, \quad (5)$$

$$H_{\text{GQLD}}(x; \alpha, \theta) = \frac{6\alpha^2((\alpha^2 x^3/6) + \theta \alpha x^2 + \theta^2 x)}{\alpha^3 x^3 + 3(2\theta + 1)\alpha^2 x^2 + 6(\theta + 1)^2 \alpha x + 6(\theta + 1)^2}.$$

To the best of our knowledge, there are no published papers which used the RSS in estimating the parameters of GQLD. The remainder of this paper is organized as follows. The RSS method is explained and the suggested various estimators for the GQLD are given in Section 2. In Section 3, a simulation study is provided to investigate the performance of the RSS estimators relative to the SRS counterparts based on the same number of measured units. Applications to real datasets fitted to the GQLD are given in Section 4. The paper is ended in Section 5 with concluding remarks and suggestions for future works.

2. Methods of Estimation

In this section, six methods of estimation are considered for estimating the unknown parameters θ and α of the GQLD

distribution using RSS design. These methods are the maximum likelihood method, method of maximum product of spacings, ordinary least squares method, weighted least squares method, Cramer-von Mises method, and Anderson-Darling method. The RSS strategy can be described as follows:

- (1) Select a simple random sample of size k^2 units from the desired population. Randomly partition them into k sets of each size k , where k is known as the set size.
- (2) Rank the units within each set of size k from smallest to largest with respect to the variable of interest.
- (3) Obtain the i th order statistic from the i th set, for $i = 1, 2, \dots, k$.

- (4) Repeat Steps (1)–(3), n times (cycles) if needed, to get a ranked set sample of size $N = nk$.

The resulting sample is denoted as $\{X_{[i]j}, i = 1, 2, \dots, k; j = 1, 2, \dots, n\}$, where $X_{[i]j}$ is the i th largest unit in a set of size k in the j th cycle. It is of interest to note that perfect ranking is assumed in this study.

2.1. Maximum Likelihood Estimation. Let $\{X_{[i]j}, i = 1, 2, \dots, k; j = 1, 2, \dots, r\}$ denote the i th order statistics

from the i th set of size k at the j th cycle, and take it as the RSS data for X of a sample size $n = kr$. Then, the maximum likelihood function based on the RSS sample is given by

$$L_{\text{RSS}}(\alpha, \theta) = \prod_{j=1}^r \prod_{i=1}^k f_{(i:k)}(x_{[i]j}, \alpha, \theta), \quad (6)$$

where

$$\begin{aligned} f_{(i:k)}(x_{[i]j}, \alpha, \theta) &= \frac{k!}{(i-1)!(k-i)!} [F(x_{[i]j})]^{i-1} [1 - F(x_{[i]j})]^{k-i} f(x_{[i]j}) \\ &= \frac{k!}{(i-1)!(k-i)!} \left[1 - \frac{(\theta^3 x^3 + 3(2\alpha + 1)\theta^2 x^2 + 6(\alpha + 1)^2(\theta x + 1))e^{-\theta x}}{6(\alpha + 1)^2} \right]^{i-1} \\ &\quad \times \left[\frac{(\theta^3 x^3 + 3(2\alpha + 1)\theta^2 x^2 + 6(\alpha + 1)^2(\theta x + 1))e^{-\theta x}}{6(\alpha + 1)^2} \right]^{k-i} \\ &\quad \times \left(\frac{\theta^2((\theta^2 x^3/6) + \alpha\theta x^2 + \alpha^2 x)e^{-\theta x}}{(\alpha + 1)^2} \right). \end{aligned} \quad (7)$$

The log-likelihood function, $\mathfrak{R}_{\text{RSS}} = \ln L_{\text{RSS}}(\alpha, \theta)$, is

$$\begin{aligned} \mathfrak{R}_{\text{RSS}} &= \sum_{j=1}^r \sum_{i=1}^k \log\{f_{i:k}(x_{[i]j}, \theta, \alpha)\} \\ &= \sum_{j=1}^r \sum_{i=1}^k \log\left(\frac{k!}{(i-1)!(k-i)!}\right) + \sum_{j=1}^r \sum_{i=1}^k (i-1)\log(F(x_{[i]j})) \\ &\quad + \sum_{j=1}^r \sum_{i=1}^k (k-i)\log(1 - F(x_{[i]j})) + \sum_{j=1}^r \sum_{i=1}^k \log f(x_{[i]j}). \end{aligned} \quad (8)$$

The estimators of α and θ of the GQLD using RSS can be obtained by solving the nonlinear equations

$$\begin{aligned} \frac{\partial \mathfrak{R}_{\text{RSS}}}{\partial \theta} &= 0, \\ \frac{\partial \mathfrak{R}_{\text{RSS}}}{\partial \alpha} &= 0. \end{aligned} \quad (9)$$

2.2. Method of Maximum Product of Spacings. Cheng and Amin [38, 39] introduced the method of maximum product of spacings (MPS). This method is based on maximization of the geometric mean of spacings in the data. The MPS is a

consistent and efficient in most general cases. Consider the ordered units $y_{(1:N)}, y_{(2:N)}, \dots, y_{(N:N)}$ that form a ranked set sample of size $N = nk$ where n is the number of cycle and k is the set size from the GQLD. Then, the uniform spacing is given by

$$D_i(\alpha, \theta) = F(y_{(i:N)}|\alpha, \theta) - F(y_{(i-1:N)}|\alpha, \theta), \quad i = 1, 2, \dots, N, \quad (10)$$

where $F(y_{(0:N)}|\alpha, \theta) = 0$ and $F(y_{(N+1:N)}|\alpha, \theta) = 1$. Clearly, $\sum_{i=1}^{N+1} D_i(\alpha, \theta) = 1$.

The MPS estimators, $\hat{\alpha}_{\text{MPS}}$ and $\hat{\theta}_{\text{MPS}}$, are the estimates of α and θ , which maximize the geometric mean of the spacing $H(\alpha, \theta|y)$, where

$$H(\alpha, \theta|y) = \left[\prod_{i=1}^{N+1} D_i(\alpha, \theta) \right]^{1/(N+1)}. \quad (11)$$

The natural logarithm of (11) is

$$M(\alpha, \theta|y) = \frac{1}{N+1} \sum_{i=1}^{N+1} \log D_i(\alpha, \theta). \quad (12)$$

The estimators $\hat{\alpha}_{\text{MPS}}$ and $\hat{\theta}_{\text{MPS}}$ of the parameters α and θ , respectively, can also be obtained by solving the nonlinear equations:

$$\begin{aligned}\frac{\partial}{\partial \alpha} M(\alpha, \theta) &= \frac{1}{N+1} \sum_{i=1}^{N+1} \frac{1}{D_i(\alpha, \theta)} [\Psi_1(y_{(i:N)}|\alpha, \theta) - \Psi_1(y_{(i-1:N)}|\alpha, \theta)] = 0, \\ \frac{\partial}{\partial \theta} M(\alpha, \theta) &= \frac{1}{N+1} \sum_{i=1}^{N+1} \frac{1}{D_i(\alpha, \theta)} [\Psi_2(y_{(i:N)}|\alpha, \theta) - \Psi_2(y_{(i-1:N)}|\alpha, \theta)] = 0,\end{aligned}\quad (13)$$

where

$$\begin{aligned}\Psi_1(y_{(i:N)}|\alpha, \theta) &= \frac{\partial}{\partial \alpha} F(y_{(i:N)}|\alpha, \theta) \\ &= \frac{\theta^2 y_{(i:N)}^2 e^{-\theta y_{(i:N)}} (3\alpha + \theta y_{(i:N)})}{3(\alpha + 1)^3}\end{aligned}\quad (14)$$

and

$$\begin{aligned}\Psi_2(y_{(i:N)}|\alpha, \theta) &= \frac{\partial}{\partial \theta} F(y_{(i:N)}|\alpha, \theta) \\ &= \frac{y_{(i:N)}^2 \theta (y_{(i:N)}^2 \theta^2 + 6\alpha y_{(i:N)} \theta + 6\alpha^2) e^{-y_{(i:N)} \theta}}{6(\alpha + 1)^2}\end{aligned}\quad (15)$$

can be then obtained numerically.

2.3. Methods of Least Squares. Swain et al. [40] was the first who used the method of least squares to estimate the parameters of beta distribution based on one of the famous

results in probability theory which indicates that $F(X_{(i:N)}) \sim \text{Beta}(i, N - i + 1)$ where F is a cumulative distribution function and $x_{(i:N)}$ is the i th order statistic of the random sample $(x_1, x_2, \dots, x_N)$. Therefore, in our case, we have

$$\mathbb{E}[F(x_{(i:N)})] = \frac{i}{N+1}, \quad (16)$$

$$\text{Var}[F(x_{(i:N)})] = \frac{i(N-i+1)}{(N+1)^2(N+2)}.$$

Using the above expectations and variances, we obtain two variants of the least squares methods.

2.3.1. Ordinary Least Squares. Let the ordered units $y_{(1:N)}, y_{(2:N)}, \dots, y_{(N:N)}$ constitute a ranked set sample of size $N = nk$. Then, the ordinary least squares (OLS) estimators, say $\hat{\alpha}_{\text{OLS}}$ and $\hat{\theta}_{\text{OLS}}$ of the parameters α and θ , respectively, can be obtained by minimizing the function:

$$\begin{aligned}\Omega(\alpha, \theta|y) &= \sum_{i=1}^N \left[F(y_{(i:N)}|\alpha, \theta) - \frac{i}{N+1} \right]^2 \\ &= \sum_{i=1}^N \left[1 - \frac{(y_{(i:N)}^3 \theta^3 + 3(2\alpha + 1)y_{(i:N)}^2 \theta^2 + 6(\alpha + 1)^2(y_{(i:N)} \theta + 1)) e^{-y_{(i:N)} \theta}}{6(\alpha + 1)^2} - \frac{i}{N+1} \right]^2,\end{aligned}\quad (17)$$

with respect to α and θ . Alternatively, these estimates can also be obtained by solving the following nonlinear equations:

$$\begin{aligned}\sum_{i=1}^N \left[F(y_{(i:N)}|\alpha, \theta) - \frac{i}{N+1} \right] \Psi_1(y_{(i:N)}|\alpha, \theta) &= 0, \\ \sum_{i=1}^N \left[F(y_{(i:N)}|\alpha, \theta) - \frac{i}{N+1} \right] \Psi_2(y_{(i:N)}|\alpha, \theta) &= 0,\end{aligned}\quad (18)$$

where $\Psi_1(y_{(i:N)}|\alpha, \theta)$ and $\Psi_2(y_{(i:N)}|\alpha, \theta)$ are defined as in (14) and (15), respectively.

2.3.2. Weighted Least Squares. Consider the RSS units $y_{(1:N)}, y_{(2:N)}, \dots, y_{(N:N)}$ that form ranked set sampling of size $N = nk$. Then, the weighted least squares (WLS) estimators of α and θ , say $\hat{\alpha}_{\text{WLS}}$ and $\hat{\theta}_{\text{WLS}}$, respectively, can be obtained by minimizing the following function:

$$\begin{aligned}
W(\alpha, \theta|y) &= \sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[F(y_{(i:N)}|\alpha, \theta) - \frac{i}{N+1} \right]^2 \\
&= \sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[1 - \frac{(y_{(i:N)}^3 \theta^3 + 3(2\alpha+1)y_{(i:N)}^2 \theta^2 + 6(\alpha+1)^2 y_{(i:N)} \theta + 6(\alpha+1)^2) e^{-y_{(i:N)} \theta}}{6(\alpha+1)^2} - \frac{i}{N+1} \right]^2,
\end{aligned} \tag{19}$$

with respect to α and θ . Equivalently, the estimates are the solution of the following nonlinear equations:

$$\begin{aligned}
&\sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[F(y_{(i:N)}|\alpha, \theta) - \frac{i}{N+1} \right], \\
&\Psi_1(y_{(i:N)}|\alpha, \theta) = 0, \\
&\sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[F(y_{(i:N)}|\alpha, \theta) - \frac{i}{N+1} \right],
\end{aligned} \tag{20}$$

$$\Psi_2(y_{(i:N)}|\alpha, \theta) = 0,$$

where $\Psi_1(y_{(i:N)}|\alpha, \theta)$ and $\Psi_2(y_{(i:N)}|\alpha, \theta)$ are specified as in (14) and (15), respectively.

2.4. Methods of Minimum Distances. Methods of estimation based on minimizing some famous goodness of fit statistics are useful in many cases and give good results. Here, two popular methods based on the minimization of test statistics between the theoretical and empirical cumulative distribution functions are considered. The methods are the Cramer-von Mises method and Anderson-Darling method (for more details, see D'Agostino and Stephens [41] and Luceño [42]).

2.4.1. Cramer-von Mises Method. Consider the ordered units $y_{(1:N)}, y_{(2:N)}, \dots, y_{(N:N)}$ that form a ranked set sample of size $N = nk$. Then, the Cramer-von Mises estimators (CV) $\hat{\alpha}$ and $\hat{\theta}$ of α and θ , respectively, are obtained by minimizing the function

$$\begin{aligned}
CV(\alpha, \theta) &= \frac{1}{12N} + \sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[F(y_{(i:N)}; \alpha, \theta) - \frac{2i-1}{2N} \right]^2 \\
&= \frac{1}{12N} + \sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[1 - \frac{(\theta^3 y_{(i:N)}^3 + 3(2\alpha+1)\theta^2 y_{(i:N)}^2 + 6(\alpha+1)^2 \theta y_{(i:N)} + 6(\alpha+1)^2) e^{-\theta y_{(i:N)}}}{6(\alpha+1)^2} - \frac{2i-1}{2N} \right]^2,
\end{aligned} \tag{21}$$

with respect to α and θ . Equivalently, the estimates are the solution of the following nonlinear equations:

$$\begin{aligned}
&\sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[F(y_{(i:N)}|\alpha, \theta) - \frac{2i-1}{2N} \right], \\
&\Psi_1(y_{(i:N)}|\alpha, \theta) = 0, \\
&\sum_{i=1}^N \frac{(N+1)^2(N+2)}{i(N-i+1)} \left[F(y_{(i:N)}|\alpha, \theta) - \frac{2i-1}{2N} \right], \\
&\Psi_2(y_{(i:N)}|\alpha, \theta) = 0,
\end{aligned} \tag{22}$$

where $\Psi_1(y_{(i:N)}|\alpha, \theta)$ and $\Psi_2(y_{(i:N)}|\alpha, \theta)$ are given in (14) and (15), respectively.

2.4.2. Anderson-Darling Method. Suppose that $y_{(1:N)}, y_{(2:N)}, \dots, y_{(N:N)}$ is a ranked set sample of size $N = nk$. Then, the estimates based on the Anderson-Darling (AD) method for the GQLD distribution parameters α and θ , denoted by $\hat{\alpha}_{AD}$ and $\hat{\theta}_{AD}$, can be obtained by minimizing the function

$$\begin{aligned}
A(\alpha, \theta) &= -N - \frac{1}{N} \sum_{i=1}^N (2i-1) \{ \log F(y_{(i:N)}|\alpha, \theta) \\
&\quad + \log \bar{F}(y_{(N-i+1:N)}|\alpha, \theta) \},
\end{aligned} \tag{23}$$

with respect to α and θ , or equivalently by solving the following two equations:

$$\begin{aligned}\frac{\partial A(\alpha, \theta)}{\partial \alpha} &= \sum_{i=1}^N (2i-1) \left\{ \frac{\Psi_1(y_{(i:N)}|\alpha, \theta)}{F(y_{(i:N)}|\alpha, \theta)} - \frac{\Psi_1(y_{(N-i+1:N)}|\alpha, \theta)}{\bar{F}(y_{(N-i+1:N)}|\alpha, \theta)} \right\} = 0, \\ \frac{\partial A(\alpha, \theta)}{\partial \alpha} &= \sum_{i=1}^N (2i-1) \left\{ \frac{\Psi_2(y_{(i:N)}|\alpha, \theta)}{F(y_{(i:N)}|\alpha, \theta)} - \frac{\Psi_2(y_{(N-i+1:N)}|\alpha, \theta)}{\bar{F}(y_{(N-i+1:N)}|\alpha, \theta)} \right\} = 0,\end{aligned}\quad (24)$$

where $\Psi_1(y_{(i:N)}|\alpha, \theta)$ and $\Psi_2(y_{(i:N)}|\alpha, \theta)$ are as specified in (14) and (15), respectively.

3. Simulation

In order to evaluate the performance of the estimation methods under RSS, a simulation study is conducted by using R software. 1000 samples are generated from the GQLD with different parameters values as (1, 1), (1, 3), (0.5, 1), and (0.8, 1.5) in different sizes for both RSS and SRS. For the RSS design, the number of cycles is selected to be $n = 3, 4$, and 5 while the set size k is taken as 5, 10, and 15. In each case, we combine n with all values of k to study the effect of set size and the number of cycles. For the SRS design, the size of SRS is $N = nk$ which is required for having the same size in each design. We have considered perfect ranking assumptions. The mean squared error (MSE) is calculated for each estimator in order to compare SRS and RSS. The MSE and the efficiency (Eff) are calculated by

$$\begin{aligned}\text{MSE}_{\hat{\alpha}} &= \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\alpha} - \alpha)^2, \\ \text{MSE}_{\hat{\theta}} &= \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta} - \theta)^2, \\ \text{Eff}_{\hat{\alpha}}(\hat{\alpha}_{\text{RSS}}, \hat{\alpha}_{\text{SRS}}) &= \frac{\text{MSE}_{\hat{\alpha}_{\text{SRS}}}}{\text{MSE}_{\hat{\alpha}_{\text{RSS}}}}, \\ \text{Eff}_{\hat{\theta}}(\hat{\theta}_{\text{RSS}}, \hat{\theta}_{\text{SRS}}) &= \frac{\text{MSE}_{\hat{\theta}_{\text{SRS}}}}{\text{MSE}_{\hat{\theta}_{\text{RSS}}}}.\end{aligned}\quad (25)$$

The results are reported in Tables 1–8 for the parameter estimates (Es), the MSE, and the Eff.

Based on simulation results presented in Tables 1–8, one can observe that

- (i) The values of the efficiency given in the tables are larger than 1. Hence, we can say that the suggested RSS estimators perform better than their SRS counterparts based on all methods considered in this study.
- (ii) Under the RSS design, where the number of cycles is increasing, the MSE values are decreasing. For example, when $k = 5$, the MSEs of the estimators of θ using the MLE method are 0.015, 0.011, and 0.009 for $n = 3, 4$, and 5, respectively.
- (iii) Under the RSS design, the MSE is decreasing when the set size k is increasing. For illustration, when

$n = 5$, the MSEs of estimators of α using AD method are 1.148, 0.189, and 0.082 for $k = 5, 10$, and 15, respectively.

- (iv) The MSE of the SRS estimators is decreasing when $N = nk$ is increasing.
- (v) In most cases, the efficiency is increasing when k is increasing. For instance, from Table 6, the values of efficiency of α using the CV method when $n = 3$ are 2.707, 5.114, and 8.812 for $k = 5, 10$, and 15, respectively.
- (vi) The MLE estimators perform better than the other methods under RSS and SRS designs for most cases presented in the tables.
- (vii) The values of the estimated parameters indicate that the bias values are negligible and go to zero as the sampling size increases.

4. Application to Real Datasets

In this section, we illustrate the performance of the suggested estimators based on RSS design for two well-known real datasets. The first dataset presents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [43]. The data were previously studied by Afify et al. [44]. The data observations are

Dataset 1. 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

The second dataset represents the number of million revolutions before failure for each of the 23 ball bearings in the life tests. It was presented by Lawless [45], and the data observations are

Dataset 2. 17.88, 28.92, 33.0, 41.52, 42.12, 45.6, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 105.84, 127.92, 128.04, 173.4.

First, we fitted the GQLD model to both datasets. Then, we considered the Kolmogorov–Smirnov (KS) test with its p value for quantifying the distance between the empirical distribution function of the real data and the cumulative distribution function using the estimators' parameters in each dataset. The results are summarized in Table 9. As

TABLE 1: Es, MSE, and Eff values using the MLE, MPS, and OLS methods for the GQLD parameters with $\alpha = 1$, $\theta = 1$ under SRS and RSS.

n	k	Method	MLE			MPS			OLS		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$ SRS	1.548	2.94	2.715	1.551	3.393	2.307	1.581	3.099	2.499
		$\hat{\alpha}$ RSS	1.261	1.083		1.242	1.471		1.296	1.240	
		$\hat{\theta}$ SRS	1.032	0.032	2.133	1.068	0.039	1.950	1.020	0.038	2.235
		$\hat{\theta}$ RSS	1.014	0.015		1.054	0.020		1.005	0.017	
	10	$\hat{\alpha}$ SRS	1.288	1.243	10.106	1.274	1.392	9.943	1.327	1.393	9.476
		$\hat{\alpha}$ RSS	1.057	0.123		1.020	0.140		1.072	0.147	
		$\hat{\theta}$ SRS	1.017	0.013	3.250	1.041	0.015	3.000	1.014	0.017	4.250
		$\hat{\theta}$ RSS	1.005	0.004		1.029	0.005		1.001	0.004	
	15	$\hat{\alpha}$ SRS	1.189	0.645	13.438	1.168	0.681	10.984	1.211	0.720	13.585
		$\hat{\alpha}$ RSS	1.024	0.048		1.000	0.062		1.033	0.053	
		$\hat{\theta}$ SRS	1.013	0.010	5.000	1.029	0.011	5.500	1.010	0.012	6.000
		$\hat{\theta}$ RSS	1.002	0.002		1.020	0.002		1.000	0.002	
4	5	$\hat{\alpha}$ SRS	1.420	2.113	2.919	1.400	2.327	2.577	1.439	2.224	2.801
		$\hat{\alpha}$ RSS	1.171	0.724		1.124	0.903		1.204	0.794	
		$\hat{\theta}$ SRS	1.022	0.021	1.909	1.051	0.025	1.786	1.014	0.026	2.000
		$\hat{\theta}$ RSS	1.008	0.011		1.041	0.014		1.003	0.013	
	10	$\hat{\alpha}$ SRS	1.195	0.687	7.387	1.175	0.757	6.759	1.216	0.739	7.465
		$\hat{\alpha}$ RSS	1.043	0.093		1.013	0.112		1.049	0.099	
		$\hat{\theta}$ SRS	1.017	0.010	3.333	1.035	0.011	2.750	1.011	0.012	4.000
		$\hat{\theta}$ RSS	1.003	0.003		1.023	0.004		1.000	0.003	
	15	$\hat{\alpha}$ SRS	1.098	0.328	7.810	1.079	0.347	7.229	1.103	0.339	7.370
		$\hat{\alpha}$ RSS	1.014	0.042		0.991	0.048		1.017	0.046	
		$\hat{\theta}$ SRS	1.004	0.007	7.000	1.018	0.007	3.500	1.001	0.008	4.000
		$\hat{\theta}$ RSS	1.000	0.001		1.015	0.002		0.998	0.002	
5	5	$\hat{\alpha}$ SRS	1.376	1.845	4.020	1.364	2.026	3.033	1.393	1.851	3.553
		$\hat{\alpha}$ RSS	1.128	0.459		1.111	0.668		1.155	0.521	
		$\hat{\theta}$ SRS	1.018	0.020	2.222	1.044	0.022	2.000	1.013	0.023	2.300
		$\hat{\theta}$ RSS	1.005	0.009		1.033	0.011		1.001	0.010	
	10	$\hat{\alpha}$ SRS	1.158	0.555	6.768	1.138	0.595	6.198	1.162	0.539	5.923
		$\hat{\alpha}$ RSS	1.036	0.082		1.012	0.096		1.044	0.091	
		$\hat{\theta}$ SRS	1.009	0.009	4.500	1.024	0.010	3.333	1.005	0.010	3.333
		$\hat{\theta}$ RSS	1.003	0.002		1.021	0.003		1.001	0.003	
	15	$\hat{\alpha}$ SRS	1.073	0.241	7.774	1.056	0.248	6.703	1.075	0.245	7.206
		$\hat{\alpha}$ RSS	1.017	0.031		0.998	0.037		1.021	0.034	
		$\hat{\theta}$ SRS	1.004	0.006	6.000	1.015	0.006	3.000	1.000	0.007	7.000
		$\hat{\theta}$ RSS	1.002	0.001		1.015	0.002		1.000	0.001	

shown in this table, the p value for the corresponding critical value in each dataset is greater than 5%, which indicates that the GQLD model fitted both datasets well.

To show the superiority of RSS over the SRS using the different estimation methods, we considered the Kolmogorov–Smirnov (KS) test but now for quantifying the distance between the empirical distribution function of the real data and the cumulative distribution function using the estimators' parameters in each design, based on the choice of n and k . Note that we used the KS here as an alternative to the mean squared error and relative bias, and it is defined in our case as

$$KS = \max \left\{ \max_{1 \leq i \leq n} \left[\frac{i}{n} - F(x_{(i)}, \hat{\theta}, \hat{\alpha}) \right], \max_{1 \leq i \leq n} \left[F(x_{(i)}, \hat{\theta}, \hat{\alpha}) - \frac{i-1}{n} \right] \right\}, \quad (26)$$

where n is the sample size. Of course, estimators with lower KS values and higher p value (greater than 5%) are better than the other competitions. Recall that the MLE estimators based on SRS for all datasets are considered the real population parameters.

For the first dataset, we considered the SRS design, and for each estimation method, we calculate the estimators using a sample of size $nk = 12$. Then, we used a sample of sizes $n = 2$ and $k = 6$ for calculating estimators using the RSS design, and based on the cycles shown in Table 10, we compare SRS and RSS designs in terms of the KS distance value and p value. The results are given in Table 11, and the corresponding fittings are displayed in Figure 1.

For the second dataset, the sample size is selected as $nk = 8$ in the SRS design, while we used $n = 2$ and $k = 4$ for calculating estimators based on the RSS design using the cycles in Table 12. The estimators' values, KS distance, and p values are computed and summarized in Table 13, and the corresponding fittings are displayed in Figure 2.

The results in Tables 11 and 13 indicate that for the estimates based on the RSS design, the KS distance values are less than their counterparts using SRS design, and the corresponding p values based on the RSS estimators are greater than those by SRS design. Figures 1 and 2 support this claim.

TABLE 2: Es, MSE, and Eff values using the WLS, CV, and AD methods for the GQLD model parameters with $\alpha = 1$, $\theta = 1$ under SRS and RSS.

n	k	Method	WLS			CV			AD		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$ SRS	1.564	2.945	2.694	1.556	2.982	2.696	1.555	2.863	2.706
		$\hat{\alpha}$ RSS	1.274	1.093		1.260	1.106		1.274	1.058	
		$\hat{\theta}$ SRS	1.019	0.036	2.250	1.027	0.037	2.312	1.019	0.033	2.200
		$\hat{\theta}$ RSS	1.003	0.016		1.012	0.016		1.005	0.015	
	10	$\hat{\alpha}$ SRS	1.309	1.296	9.818	1.302	1.310	10.000	1.305	1.252	10.097
		$\hat{\alpha}$ RSS	1.068	0.132		1.058	0.131		1.063	0.124	
		$\hat{\theta}$ SRS	1.013	0.016	4.000	1.018	0.016	4.000	1.013	0.015	3.750
		$\hat{\theta}$ RSS	1.000	0.004		1.005	0.004		1.001	0.004	
	15	$\hat{\alpha}$ SRS	1.197	0.635	12.700	1.191	0.638	12.760	1.197	0.637	13.271
		$\hat{\alpha}$ RSS	1.030	0.050		1.023	0.050		1.028	0.048	
		$\hat{\theta}$ SRS	1.010	0.011	5.500	1.013	0.011	5.500	1.009	0.011	5.500
		$\hat{\theta}$ RSS	0.999	0.002		1.003	0.002		1.000	0.002	
4	5	$\hat{\alpha}$ SRS	1.431	2.163	2.935	1.425	2.189	2.966	1.424	2.082	2.900
		$\hat{\alpha}$ RSS	1.189	0.737		1.175	0.738		1.183	0.718	
		$\hat{\theta}$ SRS	1.012	0.024	2.000	1.019	0.024	2.000	1.013	0.022	2.000
		$\hat{\theta}$ RSS	1.001	0.012		1.008	0.012		1.001	0.011	
	10	$\hat{\alpha}$ SRS	1.211	0.699	7.358	1.205	0.706	7.432	1.205	0.677	7.280
		$\hat{\alpha}$ RSS	1.050	0.095		1.042	0.095		1.048	0.093	
		$\hat{\theta}$ SRS	1.012	0.011	3.667	1.015	0.011	3.667	1.011	0.01	3.333
		$\hat{\theta}$ RSS	1.000	0.003		1.004	0.003		1.001	0.003	
	15	$\hat{\alpha}$ SRS	1.101	0.323	7.512	1.096	0.324	7.535	1.100	0.318	7.571
		$\hat{\alpha}$ RSS	1.017	0.043		1.011	0.043		1.016	0.042	
		$\hat{\theta}$ SRS	1.002	0.007	7.000	1.004	0.007	7.000	1.002	0.007	7.000
		$\hat{\theta}$ RSS	0.998	0.001		1.001	0.001		0.999	0.001	
5	5	$\hat{\alpha}$ SRS	1.386	1.810	3.635	1.380	1.840	3.651	1.381	1.792	3.913
		$\hat{\alpha}$ RSS	1.147	0.498		1.135	0.504		1.139	0.458	
		$\hat{\theta}$ SRS	1.012	0.021	2.333	1.018	0.022	2.444	1.011	0.020	2.222
		$\hat{\theta}$ RSS	1.000	0.009		1.006	0.009		1.000	0.009	
	10	$\hat{\alpha}$ SRS	1.160	0.523	6.226	1.155	0.529	6.298	1.157	0.513	6.256
		$\hat{\alpha}$ RSS	1.042	0.084		1.035	0.084		1.040	0.082	
		$\hat{\theta}$ SRS	1.005	0.010	3.333	1.008	0.010	3.333	1.005	0.010	3.333
		$\hat{\theta}$ RSS	1.001	0.003		1.004	0.003		1.001	0.003	
	15	$\hat{\alpha}$ SRS	1.073	0.236	7.613	1.069	0.237	7.645	1.074	0.237	7.645
		$\hat{\alpha}$ RSS	1.021	0.031		1.016	0.031		1.019	0.031	
		$\hat{\theta}$ SRS	1.000	0.006	6.000	1.002	0.006	6.000	1.000	0.006	6.000
		$\hat{\theta}$ RSS	1.000	0.001		1.003	0.001		1.001	0.001	

TABLE 3: Es, MSE, and Eff values using the MLE, MPS, and OLS methods for the GQLD model parameters with $\alpha = 1$, $\theta = 3$ under SRS and RSS.

n	k	Method	MLE			MPS			OLS		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$ SRS	1.523	2.913	2.715	1.533	3.385	2.287	1.591	3.074	2.465
		$\hat{\alpha}$ RSS	1.268	1.073		1.249	1.480		1.312	1.247	
		$\hat{\theta}$ SRS	3.082	0.279	2.098	3.192	0.337	1.872	3.062	0.339	2.159
		$\hat{\theta}$ RSS	3.048	0.133		3.164	0.180		3.025	0.157	
	10	$\hat{\alpha}$ SRS	1.235	1.059	8.209	1.210	1.142	7.513	1.243	1.069	7.476
		$\hat{\alpha}$ RSS	1.055	0.129		1.012	0.152		1.068	0.143	
		$\hat{\theta}$ SRS	3.041	0.123	3.514	3.106	0.141	3.000	3.015	0.145	3.816
		$\hat{\theta}$ RSS	3.012	0.035		3.082	0.047		3.003	0.038	
	15	$\hat{\alpha}$ SRS	1.191	0.63	13.125	1.169	0.672	11.200	1.212	0.709	13.130
		$\hat{\alpha}$ RSS	1.027	0.048		1.001	0.060		1.033	0.054	
		$\hat{\theta}$ SRS	3.044	0.084	6.000	3.094	0.094	4.273	3.034	0.102	6.375
		$\hat{\theta}$ RSS	3.007	0.014		3.064	0.022		3.000	0.016	

TABLE 3: Continued.

n	k	Method	MLE			MPS			OLS		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
4	5	$\hat{\alpha}$	SRS	1.402	2.025	1.397	2.339		1.431	2.132	
			RSS	1.094	0.485	4.175	1.042	0.656	1.126	0.548	3.891
		$\hat{\theta}$	SRS	3.064	0.192		3.153	0.228	3.039	0.243	
			RSS	2.998	0.084	2.286	3.092	0.107	2.981	0.099	2.455
	10	$\hat{\alpha}$	SRS	1.143	0.579	6.095	1.121	0.645	1.163	0.637	
			RSS	1.045	0.095		1.011	0.117	1.052	0.101	6.307
		$\hat{\theta}$	SRS	3.029	0.087		3.082	0.097	3.008	0.103	
			RSS	3.012	0.026	3.346	3.073	0.038	3.002	0.028	3.679
	15	$\hat{\alpha}$	SRS	1.106	0.333	9.000	1.087	0.353	1.117	0.366	
			RSS	1.009	0.037		0.985	0.043	1.013	0.041	8.927
		$\hat{\theta}$	SRS	3.022	0.058		3.063	0.064	3.011	0.070	
			RSS	3.000	0.011	5.273	3.042	0.016	2.994	0.013	5.385
5	5	$\hat{\alpha}$	SRS	1.330	1.321	3.294	1.314	1.480	1.362	1.417	
			RSS	1.118	0.401		1.095	0.519	1.141	0.440	3.220
		$\hat{\theta}$	SRS	3.066	0.159		3.145	0.186	3.051	0.182	
			RSS	3.025	0.072	2.208	3.116	0.099	3.010	0.082	2.220
	10	$\hat{\alpha}$	SRS	1.110	0.400	6.061	1.087	0.421	1.122	0.420	
			RSS	1.017	0.066		0.991	0.079	1.023	0.070	6.000
		$\hat{\theta}$	SRS	3.021	0.070		3.067	0.077	3.006	0.083	
			RSS	3.001	0.019	3.684	3.051	0.025	2.993	0.021	3.952
	15	$\hat{\alpha}$	SRS	1.098	0.230	7.419	1.081	0.238	1.115	0.272	
			RSS	1.019	0.031		0.996	0.036	1.023	0.034	8.000
		$\hat{\theta}$	SRS	3.019	0.047		3.054	0.051	3.020	0.059	
			RSS	3.007	0.009	5.222	3.042	0.013	3.002	0.010	5.900

TABLE 4: Es, MSE, and Eff values using the WLS, CV, and AD methods for the GQLD model parameters with $\alpha = 1$, $\theta = 3$ under SRS and RSS.

n	k	Method	WLS			CV			AD		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$	SRS	1.568	2.927	1.556	2.957		1.544	2.855	
			RSS	1.301	1.186	2.468	1.290	1.214	1.283	1.056	2.704
		$\hat{\theta}$	SRS	3.054	0.317		3.079	0.321	3.049	0.296	
			RSS	3.020	0.148	2.142	3.046	0.150	3.019	0.135	2.193
	10	$\hat{\alpha}$	SRS	1.239	1.031	7.931	1.234	1.049	1.238	1.026	
			RSS	1.064	0.130		1.054	0.130	1.061	0.130	7.892
		$\hat{\theta}$	SRS	3.016	0.136		3.031	0.137	3.017	0.132	
			RSS	3.000	0.035	3.886	3.015	0.036	3.001	0.035	3.771
	15	$\hat{\alpha}$	SRS	1.203	0.646	12.667	1.197	0.650	1.199	0.634	
			RSS	1.033	0.051		1.026	0.050	1.030	0.049	12.939
		$\hat{\theta}$	SRS	3.033	0.095		3.044	0.096	3.031	0.092	
			RSS	2.999	0.015	6.333	3.010	0.015	3.001	0.014	6.571
4	5	$\hat{\alpha}$	SRS	1.416	2.001	3.863	1.409	2.034	1.407	1.950	
			RSS	1.115	0.518		1.101	0.519	1.106	0.477	4.088
		$\hat{\theta}$	SRS	3.036	0.229	2.489	3.056	0.230	3.035	0.208	
			RSS	2.976	0.092		2.997	0.092	2.976	0.086	2.419
	10	$\hat{\alpha}$	SRS	1.154	0.578	5.959	1.147	0.584	1.154	0.587	
			RSS	1.053	0.097		1.045	0.097	1.050	0.096	6.115
		$\hat{\theta}$	SRS	3.009	0.095		3.021	0.096	3.009	0.093	
			RSS	3.003	0.027	3.519	3.015	0.027	3.004	0.026	3.577
	15	$\hat{\alpha}$	SRS	1.111	0.338	8.895	1.106	0.338	1.111	0.333	
			RSS	1.013	0.038		1.007	0.038	1.012	0.037	9.000
		$\hat{\theta}$	SRS	3.012	0.065		3.020	0.065	3.011	0.064	
			RSS	2.993	0.012	5.417	3.002	0.012	2.995	0.012	5.333

TABLE 4: Continued.

n	k	Method	WLS			CV			AD		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
5	5	$\hat{\alpha}$	SRS	1.347	1.291	1.338	1.299		1.338	1.276	
			RSS	1.134	0.413	3.126	1.123	0.414	1.128	0.399	3.198
		$\hat{\theta}$	SRS	3.048	0.172		3.065	0.174	3.047	0.164	
			RSS	3.008	0.076	2.263	3.025	0.077	3.007	0.073	2.247
	10	$\hat{\alpha}$	SRS	1.119	0.400	5.970	1.114	0.404	1.114	0.385	5.833
			RSS	1.023	0.067		1.016	0.067	1.022	0.066	
		$\hat{\theta}$	SRS	3.007	0.077		3.017	0.078	3.005	0.076	
			RSS	2.993	0.020	3.850	3.003	0.020	2.994	0.019	4.000
	15	$\hat{\alpha}$	SRS	1.107	0.247	7.719	1.103	0.246	1.106	0.244	7.871
			RSS	1.022	0.032		1.017	0.032	1.021	0.031	
		$\hat{\theta}$	SRS	3.018	0.054	5.400	3.025	0.054	3.016	0.053	5.889
			RSS	3.002	0.010		3.009	0.010	3.003	0.009	

TABLE 5: Es, MSE, and Eff values using the MLE, MPS, and OLS methods for the QLD model parameters with $\alpha = 0.5$, $\theta = 1$ under SRS and RSS.

n	k	Method	MLE			MPS			OLS		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$	SRS	0.702	0.727	0.658	0.806		0.759	0.857	
			RSS	0.610	0.268	2.713	0.534	0.282	0.638	0.313	2.738
		$\hat{\theta}$	SRS	1.021	0.024		1.053	0.028	1.013	0.030	
			RSS	1.015	0.013	1.846	1.049	0.017	1.006	0.015	2.000
	10	$\hat{\alpha}$	SRS	0.585	0.213	5.071	0.543	0.219	0.607	0.249	5.082
			RSS	0.520	0.042		0.470	0.050	0.531	0.049	
		$\hat{\theta}$	SRS	1.013	0.012	3.000	1.033	0.014	1.005	0.014	3.500
			RSS	1.003	0.004		1.025	0.005	0.999	0.004	
	15	$\hat{\alpha}$	SRS	0.574	0.132	8.800	0.539	0.131	0.590	0.152	8.941
			RSS	0.513	0.015		0.474	0.019	0.518	0.017	
		$\hat{\theta}$	SRS	1.014	0.008	8.000	1.029	0.009	1.011	0.010	5.000
			RSS	1.003	0.001		1.019	0.002	1.000	0.002	
4	5	$\hat{\alpha}$	SRS	0.649	0.507	4.487	0.607	0.558	0.677	0.503	4.123
			RSS	0.540	0.113		0.466	0.124	0.558	0.122	
		$\hat{\theta}$	SRS	1.015	0.018	2.250	1.042	0.021	1.007	0.022	2.444
			RSS	1.002	0.008		1.029	0.010	0.996	0.009	
	10	$\hat{\alpha}$	SRS	0.551	0.121	4.321	0.514	0.120	0.570	0.140	4.242
			RSS	0.520	0.028		0.477	0.033	0.530	0.033	
		$\hat{\theta}$	SRS	1.008	0.008	4.000	1.024	0.009	1.004	0.010	3.333
			RSS	1.004	0.002		1.022	0.003	1.002	0.003	
	15	$\hat{\alpha}$	SRS	0.535	0.079	6.583	0.508	0.079	0.551	0.093	7.154
			RSS	0.504	0.012		0.471	0.015	0.508	0.013	
		$\hat{\theta}$	SRS	1.006	0.006	6.000	1.018	0.006	1.003	0.007	7.000
			RSS	1.000	0.001		1.013	0.001	0.999	0.001	
5	5	$\hat{\alpha}$	SRS	0.650	0.378	4.447	0.606	0.392	0.683	0.424	4.283
			RSS	0.542	0.085		0.491	0.100	0.563	0.099	
		$\hat{\theta}$	SRS	1.022	0.016	2.286	1.045	0.019	1.018	0.018	2.250
			RSS	1.008	0.007		1.035	0.009	1.003	0.008	
	10	$\hat{\alpha}$	SRS	0.547	0.091	4.136	0.514	0.089	0.561	0.102	4.250
			RSS	0.504	0.022		0.466	0.026	0.510	0.024	
		$\hat{\theta}$	SRS	1.008	0.007	3.500	1.022	0.008	1.005	0.008	4.000
			RSS	1.000	0.002		1.014	0.002	0.997	0.002	
	15	$\hat{\alpha}$	SRS	0.534	0.063	6.300	0.509	0.063	0.547	0.079	7.182
			RSS	0.506	0.010		0.479	0.013	0.509	0.011	
		$\hat{\theta}$	SRS	1.004	0.004	4.000	1.014	0.005	1.004	0.006	6.000
			RSS	1.001	0.001		1.012	0.001	1.000	0.001	

TABLE 6: Es, MSE, and Eff values using the WLS, CV, and AD methods for the QLD model parameters with $\alpha = 0.5$, $\theta = 1$ under SRS and RSS.

n	k	Method	WLS			CV			AD		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$ SRS	0.747	0.788		0.730	0.777		0.732	0.739	
		RSS	0.634	0.294	2.680	0.617	0.287	2.707	0.629	0.273	2.707
		$\hat{\theta}$ SRS	1.011	0.028		1.018	0.028		1.010	0.026	
		RSS	1.005	0.014	2.000	1.013	0.014	2.000	1.006	0.013	2.000
	10	$\hat{\alpha}$ SRS	0.602	0.228	5.067	0.592	0.225	5.114	0.602	0.226	5.136
		RSS	0.530	0.045		0.519	0.044		0.527	0.044	
		$\hat{\theta}$ SRS	1.005	0.013		1.010	0.013		1.006	0.013	
		RSS	0.999	0.004	3.250	1.003	0.004	3.250	0.999	0.004	3.250
	15	$\hat{\alpha}$ SRS	0.586	0.142	8.875	0.578	0.141	8.812	0.586	0.140	9.333
		RSS	0.519	0.016		0.511	0.016		0.518	0.015	
		$\hat{\theta}$ SRS	1.010	0.009	9.000	1.013	0.009	9.000	1.010	0.009	9.000
		RSS	1.000	0.001		1.003	0.001		1.001	0.001	
4	5	$\hat{\alpha}$ SRS	0.670	0.467	4.026	0.657	0.468	4.105	0.665	0.452	3.930
		RSS	0.556	0.116		0.542	0.114		0.554	0.115	
		$\hat{\theta}$ SRS	1.006	0.020	2.500	1.012	0.021	2.625	1.007	0.019	2.375
		RSS	0.995	0.008		1.001	0.008		0.996	0.008	
	10	$\hat{\alpha}$ SRS	0.566	0.130	4.483	0.557	0.128	4.414	0.565	0.128	4.414
		RSS	0.528	0.029		0.519	0.029		0.527	0.029	
		$\hat{\theta}$ SRS	1.004	0.009	3.000	1.007	0.009	3.000	1.003	0.009	3.000
		RSS	1.002	0.003		1.005	0.003		1.002	0.003	
	15	$\hat{\alpha}$ SRS	0.546	0.086	7.167	0.540	0.084	7.000	0.547	0.085	7.083
		RSS	0.508	0.012		0.501	0.012		0.507	0.012	
		$\hat{\theta}$ SRS	1.003	0.006	6.000	1.005	0.006	6.000	1.003	0.006	6.000
		RSS	0.998	0.001		1.001	0.001		0.999	0.001	
5	5	$\hat{\alpha}$ SRS	0.675	0.380	4.086	0.662	0.375	4.121	0.670	0.383	4.303
		RSS	0.559	0.093		0.546	0.091		0.557	0.089	
		$\hat{\theta}$ SRS	1.017	0.017	2.429	1.022	0.018	2.571	1.016	0.017	2.429
		RSS	1.002	0.007		1.007	0.007		1.002	0.007	
	10	$\hat{\alpha}$ SRS	0.559	0.097	4.409	0.552	0.096	4.364	0.557	0.096	4.364
		RSS	0.511	0.022		0.503	0.022		0.509	0.022	
		$\hat{\theta}$ SRS	1.005	0.008	4.000	1.008	0.008	4.000	1.005	0.007	3.500
		RSS	0.998	0.002		1.001	0.002		0.998	0.002	
	15	$\hat{\alpha}$ SRS	0.542	0.070	7.000	0.536	0.069	6.900	0.542	0.070	7.000
		RSS	0.509	0.010		0.504	0.010		0.509	0.010	
		$\hat{\theta}$ SRS	1.004	0.005	5.000	1.006	0.005	5.000	1.003	0.005	5.000
		RSS	1.000	0.001		1.002	0.001		1.000	0.001	

TABLE 7: Es, MSE, and Eff values using the MLE, MPS, and OLS methods for the QLD model parameters with $\alpha = 0.8$, $\theta = 1.5$ under SRS and RSS.

n	k	Method	MLE			MPS			OLS		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$ SRS	2.358	5.294	2.322	2.416	6.034	2.155	2.404	5.459	2.319
		RSS	1.936	2.280		1.960	2.800		1.957	2.354	
		$\hat{\theta}$ SRS	0.829	0.022	2.444	0.861	0.028	2.154	0.823	0.025	2.500
		RSS	0.811	0.009		0.844	0.013		0.804	0.010	
	10	$\hat{\alpha}$ SRS	1.965	2.551	8.022	1.981	2.845	6.379	1.957	2.499	7.000
		RSS	1.599	0.318		1.592	0.446		1.615	0.357	
		$\hat{\theta}$ SRS	0.814	0.009	4.500	0.832	0.011	2.750	0.807	0.011	3.667
		RSS	0.803	0.002		0.823	0.004		0.800	0.003	
	15	$\hat{\alpha}$ SRS	1.719	1.132	9.593	1.723	1.265	8.490	1.737	1.210	9.237
		RSS	1.546	0.118		1.533	0.149		1.554	0.131	
		$\hat{\theta}$ SRS	0.805	0.005	5.000	0.819	0.006	3.000	0.801	0.007	7.000
		RSS	0.802	0.001		0.817	0.002		0.800	0.001	

TABLE 7: Continued.

n	k	Method	MLE			MPS			OLS		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
4	5	$\hat{\alpha}$	SRS	2.093	3.818	1.969	2.139	4.357	2.125	3.870	
			RSS	1.866	1.939		1.893	2.389	1.824	2.101	1.842
		$\hat{\theta}$	SRS	0.813	0.015	1.875	0.840	0.018		0.019	2.111
	10		RSS	0.809	0.008		0.836	0.010	1.800	0.804	0.009
		$\hat{\alpha}$	SRS	1.818	1.581	6.615	1.834	1.814		1.847	1.732
			RSS	1.575	0.239		1.571	0.309	5.871	1.588	0.269
		$\hat{\theta}$	SRS	0.810	0.006		0.825	0.007		0.805	0.008
			RSS	0.803	0.002	3.000	0.820	0.003	2.333	0.800	0.002
		$\hat{\alpha}$	SRS	1.732	0.879	8.969	1.737	0.981		1.752	0.978
	15		RSS	1.534	0.098		1.520	0.121	8.107	1.544	0.107
		$\hat{\theta}$	SRS	0.808	0.004	4.000	0.820	0.005		0.806	0.005
			RSS	0.801	0.001		0.812	0.001	5.000	0.800	0.001
5	5	$\hat{\alpha}$	SRS	1.984	2.788	2.339	2.013	3.207	1.991	2.844	
			RSS	1.761	1.192		1.812	1.679	1.910	1.788	2.254
		$\hat{\theta}$	SRS	0.812	0.011	1.833	0.833	0.013		0.805	0.013
	10		RSS	0.806	0.006		0.831	0.008	1.625	0.803	0.007
		$\hat{\alpha}$	SRS	1.700	1.103	5.805	1.699	1.197		1.706	1.098
			RSS	1.563	0.190		1.561	0.234	5.115	1.568	0.203
		$\hat{\theta}$	SRS	0.803	0.005	2.500	0.816	0.006		0.799	0.006
			RSS	0.802	0.002		0.817	0.002	3.000	0.800	0.002
		$\hat{\alpha}$	SRS	1.695	0.753	9.183	1.697	0.815		1.705	0.770
	15		RSS	1.528	0.082		1.523	0.099	8.232	1.533	0.088
		$\hat{\theta}$	SRS	0.805	0.004	4.000	0.814	0.004		0.804	0.005
			RSS	0.801	0.001		0.811	0.001	4.000	0.800	0.001

TABLE 8: Es, MSE, and Eff values using the WLS, CV, and AD methods for the GQLD model parameters with $\alpha = 0.8$, $\theta = 1.5$ under SRS and RSS.

n	k	Method	WLS			CV			AD		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
3	5	$\hat{\alpha}$	SRS	2.391	5.397	2.435	2.391	5.488	2.387	2.368	5.233
			RSS	1.942	2.216		1.945	2.299		1.928	2.148
		$\hat{\theta}$	SRS	0.821	0.024	2.400	0.828	0.025		0.820	0.023
	10		RSS	0.803	0.010		0.810	0.010	2.500	0.803	0.009
		$\hat{\alpha}$	SRS	1.953	2.445	7.570	1.955	2.494		1.943	2.363
			RSS	1.610	0.323		1.605	0.329	7.581	1.601	0.313
		$\hat{\theta}$	SRS	0.807	0.010	3.333	0.811	0.010		0.807	0.010
			RSS	0.800	0.003		0.804	0.003	3.333	0.800	0.002
		$\hat{\alpha}$	SRS	1.724	1.135	9.538	1.722	1.151		1.721	1.117
	15		RSS	1.550	0.119		1.545	0.119	9.672	1.548	0.118
		$\hat{\theta}$	SRS	0.800	0.006	6.000	0.803	0.006		0.800	0.006
			RSS	0.799	0.001		0.803	0.001	6.000	0.800	0.001
4	5	$\hat{\alpha}$	SRS	2.110	3.811	1.948	2.110	3.880	1.952	2.105	3.750
			RSS	1.878	1.956		1.875	1.988		1.867	1.892
		$\hat{\theta}$	SRS	0.807	0.017	1.889	0.813	0.018		0.807	0.016
	10		RSS	0.802	0.009		0.808	0.009	2.000	0.802	0.008
		$\hat{\alpha}$	SRS	1.832	1.623	6.734	1.832	1.652		1.826	1.579
			RSS	1.579	0.241		1.574	0.242	6.826	1.578	0.237
		$\hat{\theta}$	SRS	0.805	0.007	3.500	0.808	0.007		0.805	0.007
			RSS	0.800	0.002		0.803	0.002	3.500	0.800	0.002
		$\hat{\alpha}$	SRS	1.737	0.897	9.153	1.735	0.905		1.736	0.894
	15		RSS	1.538	0.098		1.534	0.098	9.235	1.536	0.098
		$\hat{\theta}$	SRS	0.806	0.005	5.000	0.808	0.005		0.806	0.005
			RSS	0.799	0.001		0.802	0.001	5.000	0.800	0.001

TABLE 8: Continued.

<i>n</i>	<i>k</i>	Method	WLS			CV			AD		
			Es	MSE	Eff	Es	MSE	Eff	Es	MSE	Eff
5	5	$\hat{\alpha}$ SRS	1.984	2.778	2.338	1.986	2.838	2.313	1.973	2.681	2.335
		$\hat{\alpha}$ RSS	1.776	1.188		1.774	1.227		1.763	1.148	
		$\hat{\theta}$ SRS	0.805	0.013	2.167	0.810	0.013	2.167	0.805	0.012	2.000
		$\hat{\theta}$ RSS	0.802	0.006		0.807	0.006		0.801	0.006	
	10	$\hat{\alpha}$ SRS	1.702	1.078	5.704	1.701	1.097	5.743	1.697	1.053	5.571
		$\hat{\alpha}$ RSS	1.564	0.189		1.560	0.191		1.565	0.189	
		$\hat{\theta}$ SRS	0.799	0.006	3.000	0.802	0.006	3.000	0.799	0.006	3.000
		$\hat{\theta}$ RSS	0.800	0.002		0.803	0.002		0.800	0.002	
	15	$\hat{\alpha}$ SRS	1.696	0.736	8.867	1.695	0.746	8.988	1.696	0.736	8.976
		$\hat{\alpha}$ RSS	1.531	0.083		1.528	0.083		1.529	0.082	
		$\hat{\theta}$ SRS	0.804	0.004	4.000	0.806	0.004	4.000	0.803	0.004	4.000
		$\hat{\theta}$ RSS	0.799	0.001		0.801	0.001		0.800	0.001	

TABLE 9: The goodness-of-fit tests for datasets.

Dataset	KS	<i>p</i> value
Dataset 1	0.0928	0.5646
Dataset 2	0.12244	0.8808

TABLE 10: RSS sample taken from dataset 1 for *n*=2 and *k*=6.

Cycle 1	Cycle 2
1.22	0.96
1.46	0.72
1.08	2.93
1.72	1.59
1.96	2.02
1.76	2.53

FIGURE 1: Continued.

FIGURE 1: Plots of estimated probability density functions under RSS and SRS based on dataset 1. (a) MLE. (b) MPS. (c) OLS. (d) WLS. (e) CV. (f) AD.

TABLE 11: The Es, KS, and p value in SRS and RSS designs for dataset 1.

Method	$\hat{\alpha}$		$\hat{\theta}$		KS		p value	
	SRS	RSS	SRS	RSS	SRS	RSS	SRS	RSS
MLE	0.3120	0.1701	1.8446	2.0829	0.1048	0.0787	0.3811	0.7341
MPS	2.4289	0.1692	1.2911	2.2342	0.1280	0.0834	0.1730	0.6666
OLS	2.5625	0.2182	1.2825	2.0813	0.1271	0.0677	0.1793	0.8736
WLS	2.3366	0.2132	1.2975	2.0895	0.1287	0.0680	0.1685	0.8705
CV	0.5572	0.1966	1.7033	2.1009	0.0993	0.0658	0.4479	0.8937
AD	0.6852	0.1917	1.6466	2.0988	0.1033	0.0674	0.3994	0.8776

TABLE 12: The RSS samples taken from dataset 2 for $n = 2$ and $k = 4$.

Cycle 1	Cycle 2
41.52	17.88
93.12	55.56
127.92	51.96
98.64	127.92

FIGURE 2: Continued.

FIGURE 2: Plots of the estimated probability density functions under RSS and SRS based on dataset 2. (a) MLE. (b) MPS. (c) OLS. (d) WLS. (e) CV. (f) AD.

TABLE 13: The Es, KS, and p value in SRS and RSS designs for dataset 2.

Method	$\hat{\alpha}$		$\hat{\theta}$		KS		p value	
	SRS	RSS	SRS	RSS	SRS	RSS	SRS	RSS
MLE	0.1100	0.0984	0.0407	0.0450	0.2562	0.1969	0.0809	0.2943
MPS	1.8074	0.1136	0.0275	0.0522	0.2463	0.1024	0.1028	0.9494
OLS	4.3593	0.0865	0.0232	0.0433	0.2615	0.2244	0.0708	0.1690
WLS	2.7256	0.1057	0.0253	0.0438	0.2518	0.2100	0.0901	0.2278
CV	0.5109	0.0851	0.0344	0.0447	0.2474	0.2051	0.1003	0.2516
AD	0.5394	0.1063	0.0341	0.0444	0.2473	0.2022	0.1003	0.2661

5. Conclusion

In this paper, RSS-based estimation is presented for the GQLD. Six estimation methods are considered, including the maximum likelihood, the maximum product of spacings, ordinary least squares, weighted least squares, Cramer-von Mises, and Anderson-Darling methods. The performances of the proposed estimators are compared with their SRS counterparts using a simulation study and two applications of real data. The numerical simulation results demonstrate that the proposed RSS estimators are better than their SRS counterparts in terms of the MSE for all results presented in the tables based on the same number of measuring units. The results of the real data also confirm the superiority of the RSS design over the SRS design.

For future works, the authors are interested in modifying the GQLD to the transmuted GQLD (see, for example, [46]) and estimating its parameters using the modified robust extreme ranked set sampling [47].

Data Availability

The real datasets related to guinea pigs and ball bearings used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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