Research Article

Solving Complex Fuzzy Linear Matrix Equations

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In this paper, a kind of complex fuzzy linear matrix equation $A\tilde{X}B = \tilde{C}$, in which $\tilde{C}$ is a complex fuzzy matrix and A and B are crisp matrices, is investigated by using a matrix method. The complex fuzzy matrix equation is extended into a crisp system of matrix equations by means of arithmetic operations of fuzzy numbers. Two brand new and simplified procedures for solving the original fuzzy equation are proposed and the correspondingly sufficient condition for strong fuzzy solution are analysed. Some examples are calculated in detail to illustrate our proposed method.

1. Introduction

The uncertainty of the parameters is involved in the process of actual mathematical modeling, which is often represented and computed by the fuzzy numbers. The theory and computation of linear systems related with fuzzy numbers always play an important role in the fuzzy mathematics. In the past decades, there has a great enormous investigation in the study of fuzzy mathematics and its applications. The definition of fuzzy numbers and their arithmetic operations were first introduced by Zadeh [1], Dubois and Prade [2], and Nahmias [3]. A different approach to fuzzy numbers and the fuzzy number spaces was studied by Puri and Ralescu [4], Goetschell and Voxman [5], and Wu and Ma [6, 7].

In 1998, Behra and Chakraverty [8] investigated fuzzy linear systems $A\tilde{x} = \tilde{b}$ by an embedding approach of fuzzy set decomposition theorem. Later, Abbasbandy et al., Allahviranloo et al., and Zheng et al. studied some more complicated fuzzy linear systems [9–17]. In recent years, new approaches and theories for linear systems in which part or all parameters may be uncertain and can be represented and computed by the fuzzy numbers has been emerging one after another [8, 18–20].

It is well known that some matrix systems such as Lyapunov, Sylvester, and Stein matrix equations always have wide use in science and technology field. So, the investigation on fuzzy matrix systems has been paid attention by

some scholars in past decades. In 2009, Allahviranloo et al. [21] investigated the fuzzy matrix equation, $A\tilde{X}B = \tilde{C}$. In 2018, Amirfakhrian et al. [22] presented a new algorithm for calculating the fuzzy linear matrix equation with the form $A\tilde{X}B = \tilde{C}$ by another way. In 2011, Gong and Guo [23] discussed inconsistent fuzzy linear systems and studied its least squares fuzzy solution. In 2014, Gong et al. [24] studied the general dual fuzzy matrix systems $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$ based on the LR fuzzy numbers. In 2017, Guo et al. [25, 26] studied the fuzzy matrix system of the form $\tilde{X}A = \tilde{B}$ by a matrix method and made a further investigation to dual fuzzy matrix equation $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$. In 2018, Guo et al. introduced the complex fuzzy matrix equation $\tilde{Z}C = \tilde{W}$ and proposed a general model to deal with it. At the same year, they considered the approximate fuzzy inverse and its simple application in fully fuzzy linear systems [27]. In 2019, Guo and Shang [28] put up a new method for solving linear fuzzy matrix equations, $A\tilde{X}B = \tilde{C}$.

For complex fuzzy linear systems, few researchers have developed methods to investigate them in the past decades. The fuzzy complex numbers were introduced firstly by Buckley [29] in 1989. In 2000, Qiu et al. [30] restudied the sequence and series of fuzzy complex numbers and their convergence by considering the $n \times n$ fuzzy complex linear systems. In 2009, Rahgooy et al. [31] applied the fuzzy complex linear system of linear equations to described circuit analysis problem. In 2014, Behera and Chakraverty

discussed the fuzzy complex system of linear equations by the embedding method and modified arithmetic operations of the complex fuzzy numbers later [8, 18].

In this paper, we propose a matrix method to deal with complex fuzzy linear matrix equation, $A\tilde{X}B = \tilde{C}$. At first, we introduce the complex fuzzy matrix and its operation with the crisp number. Then, we convert the complex fuzzy matrix equation to a model which is a crisp linear system of function matrix equations. The fuzzy solution of the original fuzzy equation is gained by solving the model, that is, a crisp system of matrix equations. Then, conditions of the strong fuzzy solution are also discussed. Finally, two illustrating examples are given. Our results enrich fuzzy linear system theory.

2. Preliminaries

Definition 1 (see [1]). A fuzzy number is a fuzzy set such as $\tilde{u}: R \rightarrow I = [0, 1]$ which satisfies

- (1) $\tilde{u}$ is upper semicontinuous
- (2) $\tilde{u}$ is fuzzy convex, i.e., $\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\}$, for all $x, y \in R, \lambda \in [0, 1]$
- (3) $\tilde{u}$ is normal, i.e., there exists $x_0 \in R$ such that $\tilde{u}(x_0) = 1$
- (4) $\text{supp}\tilde{u} = \{x \in R | \tilde{u}(x) > 0\}$ is the support of the $\tilde{u}$, and its closure $\text{cl}(\text{supp}\tilde{u})$ is compact

Let E^1 be the set of all fuzzy numbers on R .

Definition 2 (see [22]). A fuzzy number $\tilde{u}$ in the parametric form is a pair $(\underline{u}, \bar{u})$ of functions $\underline{u}(r)$ and $\bar{u}(r)$, $0 \leq r \leq 1$, which satisfies the requirements:

- (1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function
- (2) $\bar{u}(r)$ is a bounded monotonic decreasing left continuous function
- (3) $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$

Definition 3 (see [8]). An arbitrary complex fuzzy number should be represented as $\tilde{x} = \tilde{p} + i\tilde{q}$, where $\tilde{p} = (\underline{p}(r), \bar{p}(r))$ and $\tilde{q} = (\underline{q}(r), \bar{q}(r))$, for all $0 \leq r \leq 1$. In this case, $\tilde{x}$ can be written as

$$\tilde{x} = (\underline{p}(r), \bar{p}(r)) + i(\underline{q}(r), \bar{q}(r)). \quad (1)$$

Definition 4 (see [8]). For any two arbitrary complex fuzzy numbers $\tilde{x} = \tilde{p} + i\tilde{q}$ and $\tilde{y} = \tilde{u} + i\tilde{v}$, where $\tilde{p}, \tilde{q}, \tilde{u}, \tilde{v}$ are fuzzy numbers, their arithmetic operations are as follows:

- (1) $\tilde{x} + \tilde{y} = (\tilde{p} + \tilde{u}) + i(\tilde{q} + \tilde{v})$
- (2) $\tilde{x} - \tilde{y} = (\tilde{p} - \tilde{u}) + i(\tilde{q} - \tilde{v})$
- (3) $k\tilde{x} = k\tilde{p} + ik\tilde{q}$, $k \in R$
- (4) $\tilde{x} \times \tilde{y} = (\tilde{p} \times \tilde{u} - \tilde{q} \times \tilde{v}) + i(\tilde{p} \times \tilde{v} + \tilde{q} \times \tilde{u})$

Definition 5. The matrix system:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{12} & \cdots & \tilde{x}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{12} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \\ = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1m} \\ \tilde{c}_{21} & \tilde{c}_{12} & \cdots & \tilde{c}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{c}_{n1} & \tilde{c}_{n2} & \cdots & \tilde{c}_{nm} \end{pmatrix}, \quad (2)$$

where a_{ij} and b_{ij} , $1 \leq i, j \leq n$ are crisp numbers and $\tilde{c}_{ij}$, $1 \leq i$ and $j \leq n$ are complex fuzzy numbers, which is called a complex fuzzy linear matrix equations (CFLMEs).

Using matrix notation, we have

$$A\tilde{X}B = \tilde{C}. \quad (3)$$

A complex fuzzy matrix,

$$\tilde{X} = (\tilde{x}_{ij} + i\tilde{y}_{ij}) = \left((\underline{x}_{ij}(r), \bar{x}_{ij}(r)) + i(\underline{y}_{ij}(r), \bar{y}_{ij}(r)) \right), \quad 1 \leq i, j \leq n, 0 \leq r \leq 1, \quad (4)$$

is called a solution of the fuzzy matrix equation (2) if and only if it satisfies $A\tilde{X}B = \tilde{C}$.

3. Solving Complex Fuzzy Linear Matrix Systems

Definition 6 (see [32]). A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a complex fuzzy matrix if each element $\tilde{a}_{ij}$ of $\tilde{A}$ is a complex fuzzy number. Let $\tilde{A} = (\tilde{a}_{ij}) = (\underline{m}_{ij}(r), \overline{m}_{ij}(r)) + i(\underline{n}_{ij}(r), \overline{n}_{ij}(r))$, $i, j = 1, 2, \dots, n$, and the complex fuzzy matrix $\tilde{A}$ can be represented by $\tilde{A} = (\underline{M}(r), \overline{M}(r)) + i(\underline{N}(r), \overline{N}(r))$.

Definition 7. For any two arbitrary complex fuzzy matrix $\tilde{X} = \tilde{P} + i\tilde{Q}$ and $\tilde{Y} = \tilde{U} + i\tilde{V}$, where $\tilde{P}, \tilde{Q}, \tilde{U}$, and $\tilde{V}$ are fuzzy numbers matrices; their arithmetic is as follows:

- (1) $\tilde{X} + \tilde{Y} = (\tilde{P} + \tilde{U}) + i(\tilde{Q} + \tilde{V})$
- (2) $\tilde{X} - \tilde{Y} = (\tilde{P} - \tilde{U}) + i(\tilde{Q} - \tilde{V})$
- (3) $k\tilde{X} = k\tilde{P} + ik\tilde{Q}, \quad k \in R$
- (4) $\tilde{X} \times \tilde{Y} = (\tilde{P} \times \tilde{U} - \tilde{Q} \times \tilde{V}) + i(\tilde{P} \times \tilde{V} + \tilde{Q} \times \tilde{U})$

Theorem 1. The fuzzy matrix equation (3) can be extended to a crisp function linear matrix system as follows:

$$\begin{cases} SX(r)B = D(r), \\ SY(r)B = E(r), \end{cases} \quad (5)$$

where

$$\begin{aligned} S &= \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}, \\ X(r) &= \begin{pmatrix} \underline{M}(r) \\ -\overline{M}(r) \end{pmatrix}, \\ D(r) &= \begin{pmatrix} \underline{U}(r) \\ -\overline{U}(r) \end{pmatrix}, \end{aligned} \quad (6)$$

and

$$\begin{aligned} Y(r) &= \begin{pmatrix} \underline{N}(r) \\ -\overline{N}(r) \end{pmatrix}, \\ E(r) &= \begin{pmatrix} \underline{V}(r) \\ -\overline{V}(r) \end{pmatrix}, \end{aligned} \quad (7)$$

in which the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by the following way: if $a_{ij} \geq 0$, $a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0$, $1 \leq i, j \leq n$; if $a_{ij} < 0$, $a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0$, $1 \leq i, j \leq n$.

Proof. Let $\tilde{C} = [\underline{U}(r), \overline{U}(r)] + i[\underline{V}(r), \overline{V}(r)] = ([\underline{u}_{ij}(r), \overline{u}_{ij}(r)] + i[\underline{v}_{ij}(r), \overline{v}_{ij}(r)])_{n \times n}$, $0 \leq r \leq 1$, and the unknown complex fuzzy matrix $\tilde{X} = [\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)] = ([\underline{m}_{ij}(r), \overline{m}_{ij}(r)] + i[\underline{n}_{ij}(r), \overline{n}_{ij}(r)])_{n \times n}$. We also let $A = A^+ + A^-$ in which the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by the following way: if $a_{ij} \geq 0$, $a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0$, $1 \leq i, j \leq n$; if $a_{ij} < 0$, $a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0$, $1 \leq i, j \leq n$ and also let $B = B^+ + B^-$ in the same way.

For complex fuzzy matrix equation, $A\tilde{X}B = \tilde{C}$, i.e.,

$$A([\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)])B = [\underline{U}(r), \overline{U}(r)] + i[\underline{V}(r), \overline{V}(r)]. \quad (8)$$

Supposing $A = A^+ + A^-$ and $B = B^+ + B^-$, we have

$$\begin{aligned} (A^+ + A^-)([\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)])(B^+ + B^-) \\ = [\underline{U}(r), \overline{U}(r)] + i[\underline{V}(r), \overline{V}(r)]. \end{aligned} \quad (9)$$

Since

$$\tilde{m}_{ij}k = \begin{cases} (k\underline{m}_{ij}(r), k\overline{m}_{ij}(r)), & k \geq 0, \\ (k\overline{m}_{ij}(r), k\underline{m}_{ij}(r)), & k < 0, \end{cases} \quad (10)$$

and

$$\tilde{M}B = \begin{cases} (\underline{M}(r)B, \overline{M}(r)B), & B \geq 0, \\ (\overline{M}(r)B, \underline{M}(r)B), & B < 0. \end{cases} \quad (11)$$

So, equation (9) can be rewritten as

$$\begin{aligned} A^+[\underline{M}(r), \overline{M}(r)]B^+ + A^+[\underline{M}(r), \overline{M}(r)]B^- + A^-[\underline{M}(r), \overline{M}(r)]B^+ + A^-[\underline{M}(r), \overline{M}(r)]B^- \\ + i(A^+[\underline{N}(r), \overline{N}(r)]B^+ + A^+[\underline{N}(r), \overline{N}(r)]B^- + A^-[\underline{N}(r), \overline{N}(r)]B^+ + A^-[\underline{N}(r), \overline{N}(r)]B^-) \\ = [\underline{U}(r), \overline{U}(r)] + i[\underline{V}(r), \overline{V}(r)]. \end{aligned} \quad (12)$$

In comparison with the coefficients of i , we obtain

$$\begin{aligned} [A^+ \underline{M}(r)B^+, A^+ \overline{M}(r)B^+] + [A^+ \overline{M}(r)B^-, A^+ \underline{M}(r)B^-] \\ + [A^- \overline{M}(r)B^+, A^- \underline{M}(r)B^+] + [A^- \underline{M}(r)B^-, A^- \overline{M}(r)B^-] \\ = [\underline{U}(r), \overline{U}(r)] \end{aligned} \quad (13)$$

and

$$\begin{aligned} [A^+ \underline{N}(r)B^+, A^+ \overline{N}(r)B^+] + [A^+ \overline{N}(r)B^-, A^+ \underline{N}(r)B^-] \\ + [A^- \overline{N}(r)B^+, A^- \underline{N}(r)B^+] + [A^- \underline{N}(r)B^-, A^- \overline{N}(r)B^-] \\ = [\underline{V}(r), \overline{V}(r)], \end{aligned} \quad (14)$$

i.e.,

$$\begin{cases} A^+ \underline{M}(r)B^+ + A^+ \overline{M}(r)B^- + A^- \overline{M}(r)B^+ + A^- \underline{M}(r)B^- = \underline{U}(r), \\ A^+ \overline{M}(r)B^+ + A^+ \underline{M}(r)B^- + A^- \underline{M}(r)B^+ + A^- \overline{M}(r)B^- = \overline{U}(r), \\ A^+ \underline{N}(r)B^+ + A^+ \overline{N}(r)B^- + A^- \overline{N}(r)B^+ + A^- \underline{N}(r)B^- = \underline{V}(r), \\ A^+ \overline{N}(r)B^+ + A^+ \underline{N}(r)B^- + A^- \underline{N}(r)B^+ + A^- \overline{N}(r)B^- = \overline{V}(r). \end{cases} \quad (15)$$

Denoting in the matrix form, the above matrix equations can be written as

$$\begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} \underline{M}(r) \\ -\overline{M}(r) \end{pmatrix} B = \begin{pmatrix} \underline{U}(r) \\ -\overline{U}(r) \end{pmatrix} \quad (16)$$

and

$$\begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} \underline{N}(r) \\ -\overline{N}(r) \end{pmatrix} B = \begin{pmatrix} \underline{V}(r) \\ -\overline{V}(r) \end{pmatrix}. \quad (17)$$

Thus, we obtain equations (5)–(7) as follows:

$$\begin{cases} SX(r)B = D(r), \\ SY(r)B = E(r), \end{cases} \quad (18)$$

where

$$\begin{aligned} S &= \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}, \\ X(r) &= \begin{pmatrix} \underline{M}(r) \\ -\overline{M}(r) \end{pmatrix}, \\ D(r) &= \begin{pmatrix} \underline{U}(r) \\ -\overline{U}(r) \end{pmatrix}, \end{aligned} \quad (19)$$

and

$$\begin{aligned} Y(r) &= \begin{pmatrix} \underline{N}(r) \\ -\overline{N}(r) \end{pmatrix}, \\ E(r) &= \begin{pmatrix} \underline{V}(r) \\ -\overline{V}(r) \end{pmatrix}. \end{aligned} \quad (20)$$

By the matrix operation, the above linear matrix equations are equivalent with

$$\begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} \underline{M}(r) & \underline{N}(r) \\ -\overline{M}(r) & -\overline{N}(r) \end{pmatrix} B = \begin{pmatrix} \underline{U}(r) & \underline{V}(r) \\ -\overline{U}(r) & -\overline{V}(r) \end{pmatrix}. \quad (21)$$

It is completed the proof.

In a similar way, we could obtain another model for solving equation (3).

Theorem 2. The fuzzy linear matrix equation (3) can be extended to a crisp function linear matrix system as follows:

$$\begin{cases} AX(r)T = F(r), \\ AY(r)T = G(r). \end{cases} \quad (22)$$

where

$$T = \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}, \quad (23)$$

$$\begin{aligned} X(r) &= [\underline{M}(r), -\overline{M}(r)], \\ F(r) &= [\underline{U}(r), -\overline{U}(r)], \end{aligned} \quad (24)$$

and

$$\begin{aligned} Y(r) &= [\underline{N}(r), -\overline{N}(r)], \\ G(r) &= [\underline{V}(r), -\overline{V}(r)], \end{aligned} \quad (25)$$

in which the elements b_{ij}^+ of matrix B^+ and b_{ij}^- of matrix B^- are determined by the following way: if $b_{ij} \geq 0$, $b_{ij}^+ = b_{ij}$ else $b_{ij}^+ = 0$, $1 \leq i, j \leq n$; if $b_{ij} < 0$, $b_{ij}^- = b_{ij}$ else $b_{ij}^- = 0$, $1 \leq i, j \leq n$.

By the matrix operation, the above linear matrix equations are equivalent with

$$A \begin{pmatrix} \underline{M}(r) & -\overline{M}(r) \\ \underline{N}(r) & -\overline{N}(r) \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix}. \quad (26)$$

Proof. The proof is similar to Theorem 1.

Theorem 3 (see [28]). Let S belong to $R^{m \times n}$, T belong to $R^{p \times q}$, and C belong to $R^{m \times q}$. Then, the minimal solution X^* of the matrix equation $SXT = C$ is expressed by

$$X^* = S^\dagger CT^\dagger. \quad (27)$$

In order to solve the fuzzy matrix equation (3), we need to consider the systems of linear equations (21) or (26). It seems that we have obtained the minimal solution of the function linear system (21) as

$$\begin{pmatrix} \underline{M}(r) & \underline{N}(r) \\ -\overline{M}(r) & -\overline{N}(r) \end{pmatrix} = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}^\dagger \begin{pmatrix} \underline{U}(r) & \underline{V}(r) \\ -\overline{U}(r) & -\overline{V}(r) \end{pmatrix} B^\dagger. \quad (28)$$

Meanwhile, we obtain the minimal solution of the function linear system (26) as

$$\begin{pmatrix} \underline{M}(r) & -\overline{M}(r) \\ \underline{N}(r) & -\overline{N}(r) \end{pmatrix} = A^\dagger \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger, \quad (29)$$

where $G^\dagger$ is the Moore–Penrose generalized inverse of matrix G .

However, the solution matrix

$$\tilde{X} = \tilde{M} + i\tilde{N} = [\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)], \quad (30)$$

from equations (28) or (29), may still not be an appropriate complex fuzzy matrix except that both $\tilde{M}$ and $\tilde{N}$ are appropriate fuzzy number matrices. Restricting the discussion to the complex triangular fuzzy numbers, i.e., $\underline{m}_{ij}(r)$, $\overline{m}_{ij}(r)$, $1 \leq i, j \leq n$, and consequently $\underline{n}_{ij}(r)$, $\overline{n}_{ij}(r)$, $1 \leq i, j \leq n$ are all linear functions of r , and having calculated $X(r)$ which solves (21) or (26), we define the

complex fuzzy minimal solution to the fuzzy linear matrix equation (3) as follows.

Definition 8. Let $X(r) = (\underline{m}_{ij}(r), \overline{m}_{ij}(r)) + i(\underline{n}_{ij}(r), \overline{n}_{ij}(r))$, $1 \leq i, j \leq n$ be the minimal solution of model (21) or (26). The complex fuzzy number matrix $\tilde{W} = \{(\underline{p}_{ij}(r), \overline{p}_{ij}(r)) + i(\underline{q}_{ij}(r), \overline{q}_{ij}(r)), 1 \leq i, j \leq n$ defined by

$$\begin{cases} \underline{p}_{ij}(r) = \min\{\underline{m}_{ij}(r), \overline{m}_{ij}(r), \underline{m}_{ij}(1), \overline{m}_{ij}(1)\}, \\ \overline{p}_{ij}(r) = \max\{\underline{m}_{ij}(r), \overline{m}_{ij}(r), \underline{m}_{ij}(1), \overline{m}_{ij}(1)\}, \\ 1 \leq i, j \leq n, 0 \leq r \leq 1, \end{cases} \quad (31)$$

and

$$\begin{cases} \underline{q}_{ij}(r) = \min\{\underline{n}_{ij}(r), \overline{n}_{ij}(r), \underline{n}_{ij}(1), \overline{n}_{ij}(1)\}, \\ \overline{q}_{ij}(r) = \max\{\underline{n}_{ij}(r), \overline{n}_{ij}(r), \underline{n}_{ij}(1), \overline{n}_{ij}(1)\}, \\ 1 \leq i, j \leq n, 0 \leq r \leq 1. \end{cases} \quad (32)$$

If $(\underline{m}_{ij}(r), \overline{m}_{ij}(r))$, $1 \leq i$ and $j \leq n$, and $(\underline{n}_{ij}(r), \overline{n}_{ij}(r))$, $1 \leq i$ and $j \leq n$, are all fuzzy numbers, then $\underline{p}_{ij}(r) = \underline{m}_{ij}(r)$, $\overline{p}_{ij}(r) = \overline{m}_{ij}(r)$, $\underline{q}_{ij}(r) = \underline{n}_{ij}(r)$, and $\overline{q}_{ij}(r) = \overline{n}_{ij}(r)$, $1 \leq i, j \leq n$, and $\tilde{W} = [\underline{P}(r), \overline{P}(r)] + i[\underline{Q}(r), \overline{Q}(r)]$ is called a strong complex fuzzy minimal solution of fuzzy linear matrix systems (3). Otherwise, $\tilde{W} = [\underline{P}(r), \overline{P}(r)] + i[\underline{Q}(r), \overline{Q}(r)]$ is called a weak complex fuzzy minimal solution of fuzzy linear matrix systems (3).

To illustrate expression (28) or (29) to be a fuzzy solution matrix, we now discuss the generalized inverses of the nonnegative symmetric matrix,

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}, \quad (33)$$

in a special structure [33].

Theorem 4. If

$$S^\dagger = \begin{pmatrix} E & F \\ F & E \end{pmatrix}, \quad (34)$$

where

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}, \quad (35)$$

then

$$\begin{cases} E = \frac{1}{2}((A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger), \\ F = \frac{1}{2}((A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger), \end{cases} \quad (36)$$

where $(A^+ + A^-)^\dagger$ and $(A^+ - A^-)^\dagger$ are Moore–Penrose inverses of matrices $A^+ + A^-$ and $A^+ - A^-$, respectively.

Proof. Let

$$SS^\dagger = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} E & F \\ F & E \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}. \quad (37)$$

We obtain

$$\begin{aligned} A^+E - A^-F &= I, \\ A^+F - A^-E &= O. \end{aligned} \quad (38)$$

By adding and then subtracting the two parts of the above equations, we obtain

$$\begin{aligned} E + F &= (A^+ - A^-)^\dagger, \\ E - F &= (A^+ + A^-)^\dagger, \end{aligned} \quad (39)$$

and consequently,

$$\begin{cases} E = \frac{1}{2}((A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger), \\ F = \frac{1}{2}((A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger). \end{cases} \quad (40)$$

The proof is completed.

The key points to make the solution matrix being a strong fuzzy solution is that

$$S^\dagger \begin{pmatrix} \underline{U}(r) & \underline{V}(r) \\ -\overline{U}(r) & -\overline{V}(r) \end{pmatrix} B^\dagger \quad (41)$$

or

$$A^\dagger \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix} T^\dagger \quad (42)$$

are appropriate fuzzy numbers matrices, i.e., each element in which is an appropriate triangular fuzzy number. By the following analysis, a sufficient condition is that $S^\dagger \geq O$, $B^\dagger \geq O$ or $A^\dagger \geq O$, $T^\dagger \geq O$. It is completed the proof.

For the model equation (26), we have the following result.

Theorem 5. If

$$\begin{aligned} A^\dagger &\geq O, \\ (B^+ + B^-)^\dagger &= B^\dagger \geq O, \end{aligned} \quad (43)$$

and

$$\begin{aligned} (B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger &\geq O, \\ (B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger &\geq O. \end{aligned} \quad (44)$$

The complex fuzzy matrix equation (3) has a strong complex fuzzy minimal solution as follows:

$$\tilde{X} = [\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)], \quad (45)$$

where

$$\left\{ \begin{array}{l} \underline{M}(r) = A^\dagger \underline{U}(r)E - A^\dagger \overline{U}(r)F, \\ \overline{M}(r) = -A^\dagger \underline{C}(r)F + A^\dagger \overline{U}(r)E, \\ \underline{N}(r) = A^\dagger \underline{V}(r)E - A^\dagger \overline{V}(r)F, \\ \overline{N}(r) = -A^\dagger \underline{V}(r)F + A^\dagger \overline{V}(r)E, \\ E = \frac{1}{2} \left((B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger \right), \\ F = \frac{1}{2} \left((B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger \right). \end{array} \right. \quad (46)$$

Proof. Let

$$T^\dagger = \begin{pmatrix} E & F \\ F & E \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (B^+ + B^-)^\dagger + (B^+ + B^-)^\dagger & (B^+ + B^-)^\dagger - (B^+ + B^-)^\dagger \\ (B^+ + B^-)^\dagger - (B^+ + B^-)^\dagger & (B^+ + B^-)^\dagger + (B^+ + B^-)^\dagger \end{pmatrix}. \quad (47)$$

We know the condition that $T^+ \geq 0$ is equivalent to conditions $E \geq 0$ and $F \geq 0$ by Theorem 4.

Since $\tilde{U} = [\underline{U}(r), \overline{U}(r)]$, $\underline{U}(r)$ is a bounded monotonic increasing left continuous function matrix and $\overline{U}(r)$ is a bounded monotonic decreasing left continuous function matrix with $\underline{U}(r) \leq \overline{U}(r)$, $0 \leq r \leq 1$, by Definition 6. For fuzzy matrix $\tilde{V} = [\underline{V}(r), \overline{V}(r)]$, the above property is the same.

According to equation (29), we have

$$\begin{aligned} \begin{pmatrix} \underline{M}(r) & -\overline{M}(r) \\ \underline{N}(r) & -\overline{N}(r) \end{pmatrix} &= A^\dagger \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger \\ &= A^\dagger \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix} \begin{pmatrix} E & F \\ F & E \end{pmatrix}. \end{aligned} \quad (48)$$

It admits a bounded monotonic increasing continuous function matrix.

On the contrary, we have

$$\begin{pmatrix} \underline{M}(r) \\ \underline{N}(r) \end{pmatrix} = A^\dagger \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix} \quad (49)$$

and

$$\begin{pmatrix} -\overline{M}(r) \\ -\overline{N}(r) \end{pmatrix} = A^\dagger \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix} \begin{pmatrix} F \\ E \end{pmatrix}, \quad (50)$$

i.e.,

$$\begin{aligned} \overline{M}(r) - \underline{M}(r) &= A^\dagger ((\underline{U}(r)F + \overline{U}(r)E) - (\underline{U}(r)E + \overline{U}(r)F)) = A^\dagger ((\overline{U}(r) - \underline{U}(r))(E - F)), \\ \overline{N}(r) - \underline{N}(r) &= A^\dagger ((\underline{V}(r)F + \overline{V}(r)E) - (\underline{V}(r)E + \overline{V}(r)F)) = A^\dagger ((\overline{V}(r) - \underline{V}(r))(E - F)). \end{aligned} \quad (51)$$

Now that $A^\dagger$, $E - F \geq 0$, and

$$\left(\begin{pmatrix} \overline{U}(r) \\ \overline{V}(r) \end{pmatrix} - \begin{pmatrix} \underline{U}(r) \\ \underline{V}(r) \end{pmatrix} \right) \geq 0. \quad (52)$$

We know that

$$\begin{pmatrix} \overline{M}(r) \\ \overline{N}(r) \end{pmatrix} - \begin{pmatrix} \underline{M}(r) \\ \underline{N}(r) \end{pmatrix} = A^\dagger \left(\begin{pmatrix} \overline{U}(r) \\ \overline{V}(r) \end{pmatrix} - \begin{pmatrix} \underline{U}(r) \\ \underline{V}(r) \end{pmatrix} \right) (E - F) \geq 0. \quad (53)$$

Thus, the above complex fuzzy matrix equation has a strong complex fuzzy minimal solution as

$$\tilde{X} = [\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)]. \quad (54)$$

It is completed the proof.

For the model equation (21), we have the following result by the similar analysis.

Theorem 6. *If*

$$\begin{aligned} B^\dagger &\geq O, \\ (A^+ - A^-)^\dagger &= A^\dagger \geq O, \end{aligned} \quad (55)$$

and

$$\begin{aligned} (A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger &\geq O, \\ (A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger &\geq O. \end{aligned} \quad (56)$$

The fuzzy matrix equation (3) has a strong fuzzy minimal solution as follows:

$$\tilde{X} = \begin{pmatrix} \underline{X}(r) \\ \overline{X}(r) \end{pmatrix}, \quad (57)$$

where

$$\begin{cases} \underline{X}(r) = E \underline{C}(r) B^\dagger + F \overline{C}(r) B^\dagger, \\ \overline{X}(r) = F \underline{C}(r) B^\dagger + E \overline{C}(r) B^\dagger, \\ E = \frac{1}{2} ((A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger), \\ F = \frac{1}{2} ((A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger). \end{cases} \quad (58)$$

$$\underline{M}(r) = A^\dagger \underline{U}(r) E - A^\dagger \overline{U}(r) F,$$

$$\overline{M}(r) = -A^\dagger \underline{C}(r) F + A^\dagger \overline{U}(r) E,$$

$$\underline{N}(r) = A^\dagger \underline{V}(r) E - A^\dagger \overline{V}(r) F,$$

$$\overline{N}(r) = -A^\dagger \underline{V}(r) F + A^\dagger \overline{V}(r) E.$$

Proof. The proof is similar to Theorem 5.

4. Numerical Examples

Example 1. Consider the following complex fuzzy linear matrix system:

$$\begin{aligned} &\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (1+r, 2-r) & (4+3r, 5-3r) \\ (1+2r, 3-2r) & (r, 2-r) \end{pmatrix} \\ &+ i \begin{pmatrix} (1+r, 3-r) & (2+r, 4-r) \\ (6+2r, 5-2r) & (r, 2-2r) \end{pmatrix}. \end{aligned} \quad (59)$$

By Theorem 2, the original fuzzy matrix equation is extended into the following system of linear matrix equation (17):

$$A \begin{pmatrix} \underline{M}(r) & -\overline{M}(r) \\ \underline{N}(r) & -\overline{N}(r) \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix}, \quad (60)$$

where

$$\begin{aligned} A^+ &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \\ A^- &= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (61)$$

and

$$\begin{aligned} \underline{U}(r) &= \begin{pmatrix} 1+r & 4+3r \\ 1+2r & r \end{pmatrix}, \\ \overline{U}(r) &= \begin{pmatrix} 2-r & 5-3r \\ 3-2r & 2-r \end{pmatrix}, \\ \underline{V}(r) &= \begin{pmatrix} 1+r & 2+r \\ 6+2r & r \end{pmatrix}, \\ \overline{V}(r) &= \begin{pmatrix} 3-r & 4-r \\ 5-2r & 2-r \end{pmatrix}. \end{aligned} \quad (62)$$

From (31), the solution of the computing model is

$$\begin{aligned}
\begin{pmatrix} \underline{M}(r) & \underline{N}(r) \\ -\overline{M}(r) & -\overline{N}(r) \end{pmatrix} &= \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}^\dagger \begin{pmatrix} \underline{U}(r) & \underline{V}(r) \\ -\overline{U}(r) & -\overline{V}(r) \end{pmatrix} B^\dagger \\
&= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^\dagger \begin{pmatrix} 1+r & 4+3r & 1+r & 2+r \\ 1+2r & r & 6+2r & r \\ -2+r & -5+3r & -3+r & -4+r \\ -3+2r & -2+r-5+2r & -2r & \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & 0 \end{pmatrix}^\dagger \\
&= \begin{pmatrix} 1.0 & 0.0 & 0.0 & -0.5 \\ 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & -0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{pmatrix} \begin{pmatrix} 1+r & 4+3r & 1+r & 2+r \\ 1+2r & r & 6+2r & r \\ -2+r & -5+3r & -3+r & -4+r \\ -3+2r & -2+r & -5+2r & -2r \end{pmatrix} \begin{pmatrix} 0.00 & 1.00 \\ 0.25 & 0.75 \end{pmatrix} \\
&= \begin{pmatrix} 1.250 + 0.625r & 6.250 + 1.875r & 0.750 + 0.000r & 5.750 + 0.000r \\ 0.000 + 0.125r & 0.500 + 1.375r & 0.000 + 0.125r & 3.000 + 1.375r \\ -1.250 + 0.625r & -6.250 + 1.875r & -1.00 + 0.125r & -9.00 + 0.375r \\ -0.250 + 0.125r & -2.250 + 1.375r & -0.25 + 0.250r & -3.25 + 1.750r \end{pmatrix}.
\end{aligned} \tag{63}$$

It means

$$\begin{aligned}
\underline{M}(r) &= \begin{pmatrix} 1.250 + 0.625r & 6.250 + 1.875r \\ 0.000 + 0.125r & 0.500 + 1.375r \end{pmatrix}, \\
\overline{M}(r) &= \begin{pmatrix} 1.250 - 0.625r & 6.250 - 1.875r \\ 0.250 - 0.125r & 2.250 - 1.375r \end{pmatrix},
\end{aligned} \tag{64}$$

$$\begin{aligned}
\underline{N}(r) &= \begin{pmatrix} 0.750 + 0.000r & 5.750 + 0.000r \\ 0.000 + 0.125r & 3.000 + 1.375r \end{pmatrix}, \\
\overline{N}(r) &= \begin{pmatrix} 1.00 - 0.125r & 9.00 - 0.375r \\ 0.25 - 0.250r & 3.25 - 1.750r \end{pmatrix}.
\end{aligned} \tag{65}$$

Since both,

and

$$\begin{aligned}
\tilde{M}(r) &= \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix} = \begin{pmatrix} (1.250 + 0.625r, 1.250 - 0.625r) & (6.250 + 1.875r, 6.250 - 1.875r) \\ (0.000 + 0.125r, 0.250 - 0.125r) & (0.500 + 1.375r, 2.250 - 1.375r) \end{pmatrix}, \\
\tilde{N}(r) &= \begin{pmatrix} \tilde{n}_{11} & \tilde{n}_{12} \\ \tilde{n}_{21} & \tilde{n}_{22} \end{pmatrix} = \begin{pmatrix} (0.750 + 0.000r, 1.00 - 0.125r) & (5.750 + 0.000r, 9.00 - 0.375r) \\ (0.000 + 0.125r, 0.25 - 0.250r) & (3.000 + 1.375r, 3.25 - 1.750r) \end{pmatrix},
\end{aligned} \tag{66}$$

are appropriate fuzzy numbers matrices, we can assert that the

$$\begin{aligned}
\tilde{X} &= [\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)] \\
&= \begin{pmatrix} (1.250 + 0.625r, 1.250 - 0.625r) & (6.250 + 1.875r, 6.250 - 1.875r) \\ (0.000 + 0.125r, 0.250 - 0.125r) & (0.500 + 1.375r, 2.250 - 1.375r) \end{pmatrix} \\
&\quad i \begin{pmatrix} (0.750 + 0.000r, 1.00 - 0.125r) & (5.750 + 0.000r, 9.00 - 0.375r) \\ (0.000 + 0.125r, 0.25 - 0.250r) & (3.000 + 1.375r, 3.25 - 1.750r) \end{pmatrix}
\end{aligned} \tag{67}$$

is strong complex fuzzy solution of the original fuzzy matrix system by Definition 8.

Example 2. Consider another complex fuzzy linear matrix system:

$$\begin{aligned}
& \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
& = \begin{pmatrix} (r, 2-r) & (-2+r, 1-r) \\ (0, 1-r) & (4+2r, 5-3r) \\ (1+r, 3-2r) & (r, 2-2r) \end{pmatrix} + i \begin{pmatrix} (1+r, 3-2r) & (2+r, 4-r) \\ (6+2r, 5-r) & (r, 2-r) \\ (0, 4-3r) & (3+2r, 2-2r) \end{pmatrix}.
\end{aligned} \tag{68}$$

Suppose

$$\begin{aligned}
\tilde{X} &= [\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)], \\
A &= A^+ + A^- = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \\
B &= B^+ + B^- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
\end{aligned} \tag{69}$$

and

$$\begin{aligned}
\tilde{U} &= (\underline{U}(r), \overline{U}(r)) = \left(\begin{pmatrix} r & -2+r \\ 0 & 4+2r \\ 1+r & r \end{pmatrix}, \begin{pmatrix} 2-r & 1-r \\ 1-r & 5-3r \\ 3-2r & 2-2r \end{pmatrix} \right), \\
\tilde{V} &= (\underline{V}(r), \overline{V}(r)) = \left(\begin{pmatrix} 1+r & 2+r \\ 6+2r & r \\ 0 & 3+2r \end{pmatrix}, \begin{pmatrix} 3-2r & 4-r \\ 5-r & 2-r \\ 4-3r & 2-2r \end{pmatrix} \right).
\end{aligned} \tag{70}$$

By Theorem 2, the original fuzzy matrix equation is extended into the following system of linear matrix equation (26):

$$A \begin{pmatrix} \underline{M}(r) & -\overline{M}(r) \\ \underline{N}(r) & -\overline{N}(r) \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix}. \tag{71}$$

From (32), the solution of the computing model is

$$\begin{aligned}
& \begin{pmatrix} \underline{M}(r) & -\overline{M}(r) \\ \underline{N}(r) & -\overline{N}(r) \end{pmatrix} = A^\dagger \begin{pmatrix} \underline{U}(r) & -\overline{U}(r) \\ \underline{V}(r) & -\overline{V}(r) \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger \\
& = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}^\dagger \begin{pmatrix} r & -2+r & -2+r & -1+r \\ 0 & 4+2r & -1+r & -5+3r \\ 1+r & r & -3+2r & -2+2r \\ 1+r & 2+r & -3+2r & -4+r \\ 6+2r & r & -5+r & -2+r \\ 0 & 3+2r & -4+3r & -2+2r \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^\dagger \\
& = \begin{pmatrix} -0.3393 + 0.3393r & -0.0893 + 0.3393r & -0.0893 - 0.3393r & -0.3393 - 0.3393r \\ 0.4464 + 0.0536r & -0.8036 + 0.0536r & -0.8036 + 0.0536r & 0.4464 + 0.0536r \\ 0.0536 + 0.3571r & -0.2857 + 0.3214r & -0.2857 + 0.3214r & 0.0536 + 0.3571r \\ -0.7679 - 0.0357r & 0.1786 + 0.3929r & 0.1786 + 0.3929r & -0.7679 - 0.0357r \end{pmatrix}.
\end{aligned} \tag{72}$$

It means

$$\begin{aligned}\underline{M}(r) &= \begin{pmatrix} -0.3393 + 0.3393r & -0.0893 + 0.3393r \\ 0.4464 + 0.0536r & -0.8036 + 0.0536r \end{pmatrix}, \\ \overline{M}(r) &= \begin{pmatrix} 0.0893 + 0.3393r & 0.3393 + 0.3393r \\ 0.8036 - 0.0536r & -0.4464 - 0.0536r \end{pmatrix},\end{aligned}\quad (73)$$

and

$$\begin{aligned}\underline{N}(r) &= \begin{pmatrix} 0.0536 + 0.3571r & -0.2857 + 0.3214r \\ -0.7679 - 0.0357r & 0.1786 + 0.3929r \end{pmatrix}, \\ \overline{N}(r) &= \begin{pmatrix} 0.2857 - 0.3214r & -0.0536 - 0.3571r \\ -0.1786 - 0.3929r & 0.7679 + 0.0357r \end{pmatrix}.\end{aligned}\quad (74)$$

Since both

$$\tilde{M} = \begin{pmatrix} (-0.3393 + 0.3393r, 0.2857 - 0.3214r) & (-0.0893 + 0.3393r, 0.3393 + 0.3393r) \\ (0.4464 + 0.0536r, 0.8036 - 0.0536r) & (-0.8036 + 0.0536r, -0.4464 - 0.0536r) \end{pmatrix}\quad (75)$$

and

$$\tilde{N} = \begin{pmatrix} (0.0536 + 0.3571r, 0.0893 + 0.3393r) & (-0.2857 + 0.3214r, -0.0536 - 0.3571r) \\ (-0.7679 - 0.0357r, -0.1786 - 0.3929r) & (0.1786 + 0.3929r, 0.7679 + 0.0357r) \end{pmatrix}\quad (76)$$

are not appropriate fuzzy numbers matrices, we can assert that the fuzzy solution of the original fuzzy system is

$$\begin{aligned}\tilde{X} &= [\underline{M}(r), \overline{M}(r)] + i[\underline{N}(r), \overline{N}(r)] \\ &= \begin{pmatrix} (-0.339300000000, 0.428600000000) & (-0.089300000000, 0.678600000000) \\ (0.4464 + 0.0536r, 0.8036 - 0.0536r) & (-0.8036 + 0.0536r, -0.4464 - 0.0536r) \end{pmatrix} \\ &\quad + i \begin{pmatrix} (0.05360000000000, 0.418600000000) & (-0.2857 + 0.3214r, -0.0536 - 0.3571r) \\ (-0.7679 - 0.0357r, -0.1786 - 0.3929r) & (0.178600000000, 0.803600000000) \end{pmatrix},\end{aligned}\quad (77)$$

which admits a weak fuzzy minimal solution by Definition 5.

5. Conclusion

In this paper, we put up a scheme for calculating complex fuzzy matrix equations $A\tilde{X}B = \tilde{C}$, where A and B are $m \times m$ and $n \times n$ crisp matrices, respectively, and $\tilde{C}$ is an $m \times n$ complex fuzzy matrix. The fuzzy approximate solution of the original fuzzy matrix equation was derived from solving the model which is a function linear matrix system. In addition, the conditions of strong fuzzy approximate solution were analysed. Numerical examples showed the effectiveness of the proposed method. We should point out that a fact, i.e., it is rather difficult to obtain the strong complex fuzzy solution of the system in general. Our method can be applied to solve all kinds of semicomplex fuzzy matrix equations. We will study the fully complex fuzzy matrix equation and its applications on traffic control and decision making on this basis in next step.

Data Availability

No data were used related with practical problems.

Conflicts of Interest

The authors declare that they have no conflicts of interest, financial, or otherwise.

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