Appendix A. Proof of Lemma 1

Lemma 1. *Let assumptions **A1** and **A2** be true. If the vector* $(p, q_0, q_1, \ldots, q_n)$ *is the equilibrium exterior for the given influence coefficients* $(\nu_0, \nu_1, \ldots, \nu_n)$ *then the relation* $p > p_0$ *is equivalent to the fact that all production volumes are positive, i.e.,* $q_i > 0 \ \forall i \in \{0, 1, \ldots, n\}$.

Proof: Let $(p, q_0, q_1, \ldots, q_n)$ be an exterior equilibrium.

Assume that $q_i > 0 \ \forall i \in \{0, 1, \ldots, n\}$, then, from the optimality condition (10) and the assumption **A2** we have

$$p = \nu_i q_i + f_i'(q_i) \geq f_i'(q_i) > f_i'(0), \quad \forall i \in \{1, \ldots, n\}, \tag{A.1}$$

from which we get $p > \max\limits_{i \in \{1,\ldots,n\}} \{f_i'(0)\} = p_0$.

On the other hand, if $p > p_0$, we get from assumption **A2** that

$$f_i'(0) \leq \max_{i \in \{1,\ldots,n\}} \{f_i'(0)\} = p_0, \quad \forall i \in \{1, \ldots, n\}, \tag{A.2}$$

and

$$-\beta \left(\sum_{i=1}^{n} q_i \right) \nu_0 + f_0'(0) \leq f_0'(0) \leq \max_{i \in \{1,\ldots,n\}} \{f_i'(0)\} = p_0, \tag{A.3}$$

which implies that inequalities

$$p \leq f_i'(0), \quad i \in \{1, \ldots, n\}, \tag{A.4}$$

and

$$p \leq -\beta \left(\sum_{i=1}^{n} q_i \right) \nu_0 + f_0'(0), \tag{A.5}$$

from the optimality condition (10) and (12), are impossible, thus, it must holds that $q_i > 0 \ \forall i \in \{1, \ldots, n\}$, and $q_0 > 0$■

Appendix B. Proof of Theorem 1

Theorem 1. *Under assumptions **A1**, **A2** and **A3**, for any* $\beta \in (0, 1]$, $D \geq 0$, $\nu_i \geq 0$, $i \in \{1, \ldots, n\}$, *and* $\nu_0 \in [0, \overline{\nu}_0)$, *there exists uniquely the exterior* *equilibrium* $(p, q_0, q_1, \ldots, q_n)$*, which is continuously differentiable with respect to the parameters* β*,* D *and* ν_i*,* $i \in \{0, 1, \ldots, n\}$*. Moreover,* $p > p_0$ *and*

$$\frac{\partial p}{\partial D} = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + f_0''(q_0)} + \dfrac{\nu_0 + f_0''(q_0)}{(1-\beta)\nu_0 + f_0''(q_0)} \sum_{i=1}^{n} \dfrac{1}{\nu_i + f_i''(q_i)} - G'(p)}. \quad (17)$$

Proof: Since we are looking for an exterior equilibrium with $q_i > 0$, $\forall i \in \{1, \ldots, n\}$, from the optimality conditions (10), we can consider the equality

$$\varphi_i(p, q_i, \nu_i) = p - [\nu_i q_i + f_i'(q_i)] = 0, \quad \forall i \in \{1, \ldots, n\}, \quad \text{(B.1)}$$

where, from the assumption **A2**, φ_i is continuously differentiable with respect to (p, q_i, ν_i).

Therefore, for all $\nu_i \geq 0$, $i \in \{1, \ldots, n\}$, we have that the function

$$g_i(q_i) = \nu_i q_i + f_i'(q_i), \quad \text{(B.2)}$$

is continuously differentiable and strictly increasing for all $q_i \geq 0$ (by the assumption **A2**). In addition,

$$g_i(0) = f_i'(0) \leq p_0 = f_i'(q_i^0) \leq g_i(q_i^0), \quad \text{(B.3)}$$

and there exists the limit (which can be either, finite or infinite)

$$p_i^1(\nu_i) := \lim_{q_i \to +\infty} g_i(q_i) = \lim_{q_i \to +\infty} [\nu_i q_i + f_i'(q_i)] > p_0. \quad \text{(B.4)}$$

Therefore, for each $p \in \left[p_0, p_i^1(\nu_i)\right)$, there exists a unique $q_i \geq 0$ such that $\varphi_i(p, q_i, \nu_i) = 0$.

Moreover, since

$$\frac{\partial \varphi_i}{\partial q_i} = -\nu_i - f_i''(q_i) < 0, \quad \forall q_i, \nu_i \geq 0, p \in \left[p_0, p_i^1(\nu_i)\right), \quad \text{(B.5)}$$

by the implicit function theorem, the mapping

$$q_i(p, \nu_i) := \{q_i \geq 0 \mid \varphi_i(p, q_i, \nu_i) = 0\}, \quad \text{(B.6)}$$

is a continuously differentiable function with respect to $\nu_i \geq 0$ and $p \in \left[p_0, p_i^1(\nu_i)\right)$, with the relationships

$$\frac{\partial q_i}{\partial p} = -\frac{\dfrac{\partial \varphi_i}{\partial p}}{\dfrac{\partial \varphi_i}{\partial q_i}} = \frac{1}{\nu_i + f_i''(q_i)} > 0 \quad \text{(B.7)}$$

and

$$\frac{\partial q_i}{\partial \nu_i} = -\frac{\dfrac{\partial \varphi_i}{\partial \nu_i}}{\dfrac{\partial \varphi_i}{\partial q_i}} = -\frac{q_i}{\nu_i + f_i''(q_i)} \leq 0, \tag{B.8}$$

which means that $q_i(p,\nu_i)$ is strictly increasing with respect to $p \in \big[p_0, p_i^1(\nu_i)\big)$ (which implies that $q_i(p,\nu_i) > 0$ if $p > p_0$) and decreasing with respect to $\nu_i \geq 0$. Moreover, from (B.1) and (B.4) we can see that

$$q_i(p_0,\nu_i) \leq q_i(p_0,0) = q_i^0 \tag{B.9}$$

and

$$\lim_{p \uparrow p_i^1(\nu_i)} q_i(p,\nu_i) = +\infty. \tag{B.10}$$

Next, from the optimality conditions (12), we can consider the equation

$$\begin{aligned}\varphi_0(p, q_0, \beta, \nu_0, \nu_1, \ldots, \nu_n) = &\left[p + \beta\nu_0 \sum_{i=1}^{n} q_i(p,\nu_i)\right] \\ &- [(1-\beta)\nu_0 q_0 + f_0'(q_0)] = 0,\end{aligned} \tag{B.11}$$

where, from the above arguments and the assumption **A2**, $\varphi_0(p, q_0, \beta, \nu_0, \nu_1, \ldots, \nu_n)$ is continuously differentiable with respect to $(p, q_0, \beta, \nu_0, \nu_1, \ldots, \nu_n)$.

Now, consider an arbitrary $\nu_i \geq 0,\ \forall i \in \{1,\ldots,n\}$. If we consider $\nu_0 > 0$, then, by the assumption **A2**, the following inequalities hold

$$\left[p_0 + \beta\nu_0 \sum_{i=1}^{n} q_i(p_0,\nu_i)\right] \geq p_0 \geq f_0'(0) = [(1-\beta)\nu_0(0) + f_0'(0)], \tag{B.12}$$

and we can define

$$p^1(\nu_1,\ldots,\nu_n) := \min_{i \in \{1,\ldots,n\}} \{p_i^1(\nu_i)\}. \tag{B.13}$$

Now we are going to show that there exists a value $p_0^1(\beta,\nu_0,\nu_1,\ldots,\nu_n) > p_0$, finite or infinite, but not exceeding $p^1(\nu_1,\ldots,\nu_n)$, such that

$$\lim_{p \uparrow p_0^1(\beta,\nu_0,\nu_1,\ldots,\nu_n)} \left[p + \beta\nu_0 \sum_{i=1}^{n} q_i(p,\nu_i)\right] = \lim_{q_0 \to +\infty} [(1-\beta)\nu_0 q_0 + f_0'(q_0)]. \tag{B.14}$$

The proof is divided into two cases: $\beta = 1$ and $\beta \in (0,1)$.

First, we can observe that the function $p+\beta\nu_0 \sum_{i=1}^{n} q_i(p,\nu_i)$ is continuous and strictly increasing (as shown above) with respect to $p \in \big[p_0, p^1(\nu_1,\ldots,\nu_n)\big)$, and the function $(1-\beta)\nu_0 q_0 + f_0'(q_0)$ is continuous and strictly increasing (by the assumption **A2**) with respect to $q_0 \geq 0$.

Case 1: Let $\beta = 1$. Under these conditions, the equality (B.14) is rewritten as

$$\lim_{p\uparrow p_0^1(\beta,\nu_0,\nu_1,\ldots,\nu_n)} \left[p + \nu_0 \sum_{i=1}^{n} q_i(p,\nu_i)\right] = \lim_{q_0\to+\infty} f_0'(q_0). \tag{B.15}$$

If $\sum_{i=1}^{n} q_i^0 > 0$, from (16) we have that

$$\nu_0 < \frac{f_0'(G(p_0) - \sum_{i=1}^{n} q_i^0) - p_0}{\sum_{i=1}^{n} q_i^0}, \tag{B.16}$$

which implies the chain of inequalities

$$\begin{aligned} & p_0 + \nu_0 \sum_{i=1}^{n} q_i(p_0,\nu_i) \leq p_0 + \nu_0 \sum_{i=1}^{n} q_i^0 \\ & < p_0 + \frac{f_0'(G(p_0) - \sum_{i=1}^{n} q_i^0) - p_0}{\sum_{i=1}^{n} q_i^0} \sum_{i=1}^{n} q_i^0 = f_0'(G(p_0) - \sum_{i=1}^{n} q_i^0) \\ & < \lim_{q_0\to+\infty} f_0'(q_0) \leq +\infty = \lim_{p\uparrow p^1(\nu_1,\ldots,\nu_n)} \left[p + \nu_0 \sum_{i=1}^{n} q_i(p,\nu_i)\right], \end{aligned} \tag{B.17}$$

which guarantees the existence of the desired value of $p_0^1(\beta,\nu_0,\nu_1,\ldots,\nu_n)$ previously mentioned.

Otherwise, if $\sum_{i=1}^{n} q_i^0 = 0$, then, evidently

$$\begin{aligned} & p_0 + \nu_0 \sum_{i=1}^{n} q_i(p_0,\nu_i) \leq p_0 + \nu_0 \sum_{i=1}^{n} q_i^0 = p_0 < \lim_{q_0\to+\infty} f_0'(q_0) \\ & \leq +\infty = \lim_{p\uparrow p^1(\nu_1,\ldots,\nu_n)} \left[p + \nu_0 \sum_{i=1}^{n} q_i(p,\nu_i)\right], \end{aligned} \tag{B.18}$$

which again guarantees the existence of the desired value $p_0^1(\beta,\nu_0,\nu_1,\ldots,\nu_n)$ mentioned above.

Case 2: Now let $\beta \in (0,1)$. In this case

$$\lim_{p \uparrow p^1(\nu_1,\dots,\nu_n)} \left[p + \beta\nu_0 \sum_{i=1}^{n} q_i(p,\nu_i) \right] = \lim_{q_0 \to +\infty} [(1-\beta)\nu_0 q_0 + f_0'(q_0)] = +\infty, \tag{B.19}$$

which implies that $p_0^1(\beta,\nu_0,\nu_1,\dots,\nu_n) = p^1(\nu_1,\dots,\nu_n)$.

Thus, the proof of the existence of the value $p_0^1(\beta,\nu_0,\nu_1,\dots,\nu_n)$ mentioned above is completed.

Therefore, from relationships (B.12) and (B.14), we can conclude that for every $p \in \left[p_0, p_0^1(\beta,\nu_0,\nu_1,\dots,\nu_n)\right)$ there exists a unique $q_0 \geq 0$ such that $\varphi_0(p,q_0,\beta,\nu_0,\nu_1,\dots,\nu_n) = 0$.

On the other hand if $\nu_0 = 0$, then, the equation $\varphi_0(p,q_0,\beta,\nu_0,\nu_1,\dots,\nu_n) = 0$ takes the form

$$p - f_0'(q_0) = 0. \tag{B.20}$$

So we can define the value

$$p_0^1(\beta,\nu_0,\nu_1,\dots,\nu_n) = \lim_{q_0 \to +\infty} f_0'(q_0), \tag{B.21}$$

and by the assumption **A2**, we can guarantee that for all $p \in \left[p_0, p_0^1(\beta,\nu_0,\nu_1,\dots,\nu_n)\right)$ there exists a unique $q_0 \geq 0$ such that $\varphi_0(p,q_0,\beta,\nu_0,\nu_1,\dots,\nu_n) = 0$.

Moreover, we have that

$$\frac{\partial \varphi_0}{\partial q_0} = -(1-\beta)\nu_0 - f_0''(q_0) < 0, \tag{B.22}$$

for all $q_0,\nu_0,\nu_1,\dots,\nu_n \geq 0$, $p \in \left[p_0, p_0^1(\beta,\nu_0,\nu_1,\dots,\nu_n)\right)$.

Then, by the implicit function theorem, the mapping

$$q_0(p,\beta,\nu_0,\nu_1,\dots,\nu_n) := \{q_0 \geq 0 \mid \varphi_0(p,q_0,\beta,\nu_0,\nu_1,\dots,\nu_n) = 0\} \tag{B.23}$$

is a continuously differentiable function with respect to $\beta \in (0,1]$, $\nu_0,\nu_1,\dots,\nu_n \geq 0$, and $p \in \left[p_0, p_0^1(\beta,\nu_0,\nu_1,\dots,\nu_n)\right)$, with the following properties

$$\frac{\partial q_0}{\partial p} = -\frac{\dfrac{\partial \varphi_0}{\partial p}}{\dfrac{\partial \varphi_0}{\partial q_0}} = \frac{1 + \beta\nu_0 \sum\limits_{i=1}^{n} \dfrac{\partial q_i}{\partial p}}{(1-\beta)\nu_0 + f_0''(q_0)} = \frac{1 + \beta\nu_0 \sum\limits_{i=1}^{n} \dfrac{1}{\nu_i + f_i''(q_i)}}{(1-\beta)\nu_0 + f_0''(q_0)} > 0, \tag{B.24}$$

$$\frac{\partial q_0}{\partial \nu_i} = -\frac{\dfrac{\partial \varphi_0}{\partial \nu_i}}{\dfrac{\partial \varphi_0}{\partial q_0}} = \frac{\beta \nu_0 \dfrac{\partial q_i}{\partial \nu_i}}{(1-\beta)\nu_0 + f_0''(q_0)} = -\frac{\beta \nu_0 \dfrac{q_i}{\nu_i + f_i''(q_i)}}{(1-\beta)\nu_0 + f_0''(q_0)} \leq 0, \tag{B.25}$$

for all $i \in \{1, \dots, n\}$, and

$$\frac{\partial q_0}{\partial \nu_0} = -\frac{\dfrac{\partial \varphi_0}{\partial \nu_0}}{\dfrac{\partial \varphi_0}{\partial q_0}} = \frac{\beta \sum\limits_{i=1}^{n} q_i(p, \nu_i) - (1-\beta) q_0}{(1-\beta)\nu_0 + f_0''(q_0)}, \tag{B.26}$$

which allow us to conclude that $q_0(p, \beta, \nu_0, \nu_1, \dots, \nu_n)$ is strictly increasing with respect to $p \in \big[p_0, p_0^1(\beta, \nu_0, \nu_1, \dots, \nu_n)\big)$ (which implies that $q_0(p, \beta, \nu_0, \nu_1, \dots, \nu_n) > 0$ if $p > p_0$) and decreasing with respect to $\nu_i \geq 0$, $\forall i \in \{1, \dots, n\}$. In addition, from the relationships (B.11), (B.14) and (B.21), it is easy to see that

$$q_0(p_0, \beta, 0, \nu_1, \dots, \nu_n) = q_0^0 \tag{B.27}$$

and

$$\lim_{p \uparrow p_0^1(\beta, \nu_0, \nu_1, \dots, \nu_n)} q_0(p, \beta, \nu_0, \nu_1, \dots, \nu_n) = +\infty. \tag{B.28}$$

Now let $\beta \in (0, 1]$, $\nu_1, \dots, \nu_n, D \geq 0$, and consider the equation

$$\begin{aligned} \Gamma(p, \beta, \nu_0, \nu_1, \dots, \nu_n, D) = &\left[q_0(p, \beta, \nu_0, \nu_1, \dots, \nu_n) + \sum_{i=1}^{n} q_i(p, \nu_i) \right] \\ &- [G(p) + D] = 0, \end{aligned} \tag{B.29}$$

which, by (B.1)-(B.28) and the assumption **A1**, is continuously differentiable with respect to all $\beta \in (0, 1]$, $\nu_i \geq 0$, $\forall i \in \{1, \dots, n\}$, $D \geq 0$ and $p \in \big[p_0, p_0^1(\beta, \nu_0, \nu_1, \dots, \nu_n)\big)$.

Moreover, by (B.1)-(B.28), the function $q_0(p, \beta, \nu_0, \nu_1, \dots, \nu_n) + \sum\limits_{i=1}^{n} q_i(p, \nu_i)$ is continuously differentiable and strictly increasing with respect to all $p \in \big[p_0, p_0^1(\beta, \nu_0, \nu_1, \dots, \nu_n)\big)$, and the following relationship holds

$$\lim_{p \uparrow p_0^1(\beta, \nu_0, \nu_1, \dots, \nu_n)} \left[q_0(p, \beta, \nu_0, \nu_1, \dots, \nu_n) + \sum_{i=1}^{n} q_i(p, \nu_i) \right] = +\infty. \tag{B.30}$$

In addition, by **A1**, the function $G(p)$ is continuously differentiable and strictly decreasing for all $p > 0$.

Then, by (B.1)-(B.28) and the assumption **A3**, we have that

$$\begin{aligned} 0 &\leq q_0(p_0, \beta, \nu_0, \nu_1, \ldots, \nu_n) + \sum_{i=1}^{n} q_i(p_0, \nu_i) \\ &\leq q_0(p_0, \beta, \nu_0, 0, \ldots, 0) + \sum_{i=1}^{n} q_i(p_0, 0) \\ &= q_0(p_0, \beta, \nu_0, 0, \ldots, 0) + \sum_{i=1}^{n} q_i^0. \end{aligned} \tag{B.31}$$

Now, we are going to analyze the sign of the derivative

$$\frac{\partial q_0}{\partial \nu_0}(p_0, \beta, \nu_0, 0, \ldots, 0) = \frac{\beta \sum\limits_{i=1}^{n} q_i^0 - (1-\beta) q_0(p_0, \beta, \nu_0, 0, \ldots, 0)}{(1-\beta)\nu_0 + f_0''(q_0)}. \tag{B.32}$$

This analysis is done separately for the Cases A, B, C, D and E.

Case A: Let $\beta = 1$ and $\sum\limits_{i=1}^{n} q_i^0 = 0$. In this case, it is clear that

$$\frac{\partial q_0}{\partial \nu_0}(p_0, \beta, \nu_0, 0, \ldots, 0) = 0, \quad \forall \nu_0 \geq 0, \tag{B.33}$$

thus, $q_0(p_0, \beta, \nu_0, 0, \ldots, 0)$ is constant with respect to ν_0 and

$$\begin{aligned} q_0(p_0, \beta, \nu_0, 0, \ldots, 0) + \sum_{i=1}^{n} q_i^0 &= q_0(p_0, \beta, 0, 0, \ldots, 0) + \sum_{i=1}^{n} q_i^0 \\ &= \sum_{i=0}^{n} q_i^0 < G(p_0) \leq G(p_0) + D. \end{aligned} \tag{B.34}$$

Case B: Let $\beta = 1$ and $\sum\limits_{i=1}^{n} q_i^0 > 0$. In this case, we have that

$$\frac{\partial q_0}{\partial \nu_0}(p_0, \beta, \nu_0, 0, \ldots, 0) = \frac{\sum\limits_{i=1}^{n} q_i^0}{f_0''(q_0)} > 0, \quad \forall \nu_0 \geq 0, \tag{B.35}$$

hence, $q_0(p_0, \beta, \nu_0, 0, \ldots, 0)$ is strictly increasing with respect to ν_0.

In addition, from (16) we have that

$$\nu_0 < \frac{f_0'(G(p_0) - \sum\limits_{i=1}^{n} q_i^0) - p_0}{\sum\limits_{i=1}^{n} q_i^0}, \tag{B.36}$$

from which one has the inequality

$$q_0(p_0,\beta,\nu_0,0,\ldots,0) < q_0\left(p_0,\beta,\frac{f_0'(G(p_0)-\sum\limits_{i=1}^{n} q_i^0)-p_0}{\sum\limits_{i=1}^{n} q_i^0},0,\ldots,0\right), \tag{B.37}$$

and from the relationship (B.11) one can see that

$$q_0\left(p_0,\beta,\frac{f_0'(G(p_0)-\sum\limits_{i=1}^{n} q_i^0)-p_0}{\sum\limits_{i=1}^{n} q_i^0},0,\ldots,0\right) = G(p_0)-\sum_{i=1}^{n} q_i^0. \tag{B.38}$$

Therefore,

$$\begin{aligned} & q_0(p_0,\beta,\nu_0,0,\ldots,0)+\sum_{i=1}^{n} q_i^0 \\ & \quad < q_0\left(p_0,\beta,\frac{f_0'(G(p_0)-\sum\limits_{i=1}^{n} q_i^0)-p_0}{\sum\limits_{i=1}^{n} q_i^0},0,\ldots,0\right)+\sum_{i=1}^{n} q_i^0 \\ & \quad = G(p_0)-\sum_{i=1}^{n} q_i^0+\sum_{i=1}^{n} q_i^0 = G(p_0) \le G(p_0)+D. \end{aligned} \tag{B.39}$$

Case C: Let $\beta \in (0,1)$ and $\sum\limits_{i=1}^{n} q_i^0 > \max\left\{(1-\beta)G(p_0), \frac{1-\beta}{\beta}q_0^0\right\}$. For this case, we have the relationship:

$$\sum_{i=1}^{n} q_i^0 > \frac{1-\beta}{\beta}q_0^0, \tag{B.40}$$

which is equivalent to

$$\frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0 > q_0^0. \tag{B.41}$$

Now, suppose that there exists the value ν_0 such that

$$\beta\sum_{i=1}^{n} q_i^0-(1-\beta)q_0(p_0,\beta,\nu_0,0,\ldots,0) \le 0, \tag{B.42}$$

the latter inequality implies that

$$q_0(p_0,\beta,\nu_0,0,\ldots,0) \ge \frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0. \tag{B.43}$$

From the equation (B.11) we have that

$$p_0 = (1-\beta)\nu_0 \left[q_0(p_0, \beta, \nu_0, 0, \ldots, 0) - \frac{\beta}{1-\beta} \sum_{i=1}^{n} q_i^0 \right] + f_0'(q_0(p_0, \beta, \nu_0, 0, \ldots, 0)), \tag{B.44}$$

and applying the inequalities (B.41) and (B.43) to (B.44) yields the following relationship

$$\begin{aligned} p_0 &\geq (1-\beta)\nu_0 \left[\frac{\beta}{1-\beta} \sum_{i=1}^{n} q_i^0 - \frac{\beta}{1-\beta} \sum_{i=1}^{n} q_i^0 \right] + f_0' \left(\frac{\beta}{1-\beta} \sum_{i=1}^{n} q_i^0 \right) \\ &= f_0' \left(\frac{\beta}{1-\beta} \sum_{i=1}^{n} q_i^0 \right) > f_0'(q_0^0) = p_0, \end{aligned} \tag{B.45}$$

which cannot be valid.

Thus, the assumption (B.42) cannot hold, which implies that

$$\beta \sum_{i=1}^{n} q_i^0 - (1-\beta) q_0(p_0, \beta, \nu_0, 0, \ldots, 0) > 0, \quad \forall \nu_0 \in [0, \overline{\nu}_0), \tag{B.46}$$

therefore,

$$\frac{\partial q_0}{\partial \nu_0}(p_0, \beta, \nu_0, 0, \ldots, 0) = \frac{\beta \sum_{i=1}^{n} q_i^0 - (1-\beta) q_0(p_0, \beta, \nu_0, 0, \ldots, 0)}{(1-\beta)\nu_0 + f_0''(q_0)} > 0, \tag{B.47}$$

for all $\nu_0 \in [0, \overline{\nu}_0)$, so the function $q_0(p_0, \beta, \nu_0, 0, \ldots, 0)$ is strictly increasing with respect to $\nu_0 \in [0, \overline{\nu}_0)$.

In addition, for this case we have

$$\overline{\nu}_0 = \frac{f_0'(G(p_0) - \sum_{i=1}^{n} q_i^0) - p_0}{\sum_{i=1}^{n} q_i^0 - (1-\beta) G(p_0)} \tag{B.48}$$

where $G(p_0) - \sum_{i=1}^{n} q_i^0 > 0$ by **A3**, and $\sum_{i=1}^{n} q_i^0 - (1-\beta)G(p_0) > 0$ is one of the assumptions for this **Case C**. Moreover, from equation (B.11) we have that

$$q_0(p_0, \beta, \overline{\nu}_0, 0, \ldots, 0) = G(p_0) - \sum_{i=1}^{n} q_i^0. \tag{B.49}$$

Hence,

$$\begin{aligned} q_0(p_0,\beta,\nu_0,0,\ldots,0)+\sum_{i=1}^{n}q_i^0 &< q_0(p_0,\beta,\overline{\nu}_0,0,\ldots,0)+\sum_{i=1}^{n}q_i^0 \\ &= G(p_0)-\sum_{i=1}^{n}q_i^0+\sum_{i=1}^{n}q_i^0 = G(p_0)\le G(p_0)+D. \end{aligned} \tag{B.50}$$

Case D: Let $\beta \in (0,1)$ and $\sum\limits_{i=1}^{n} q_i^0 \le \dfrac{1-\beta}{\beta} q_0^0$. Then,

$$\frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0 \le q_0^0. \tag{B.51}$$

Now, suppose that there exists the value ν_0 such that

$$\beta\sum_{i=1}^{n} q_i^0-(1-\beta)q_0(p_0,\beta,\nu_0,0,\ldots,0)>0, \tag{B.52}$$

the latter inequality implies that

$$q_0(p_0,\beta,\nu_0,0,\ldots,0)<\frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0. \tag{B.53}$$

Applying the inequalities (B.51) and (B.53) to (B.44) yields the following relationship

$$\begin{aligned} p_0 &< (1-\beta)\nu_0\left[\frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0-\frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0\right]+f_0'\left(\frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0\right) \\ &= f_0'\left(\frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0\right)\le f_0'(q_0^0)=p_0, \end{aligned} \tag{B.54}$$

which cannot be valid.

Thus, the assumption (B.52) cannot hold, which implies that

$$\beta\sum_{i=1}^{n} q_i^0-(1-\beta)q_0(p_0,\beta,\nu_0,0,\ldots,0)\le 0, \quad \forall \nu_0\in[0,\overline{\nu}_0), \tag{B.55}$$

therefore,

$$\frac{\partial q_0}{\partial \nu_0}(p_0,\beta,\nu_0,0,\ldots,0)=\frac{\beta\sum\limits_{i=1}^{n} q_i^0-(1-\beta)q_0(p_0,\beta,\nu_0,0,\ldots,0)}{(1-\beta)\nu_0+f_0''(q_0)}\le 0, \tag{B.56}$$

for all $\nu_0\in[0,\overline{\nu}_0)$, so the function $q_0(p_0,\beta,\nu_0,0,\ldots,0)$ is decreasing with respect to $\nu_0\in[0,\overline{\nu}_0)$.

Hence,

$$\begin{aligned} q_0(p_0,\beta,\nu_0,0,\ldots,0)+\sum_{i=1}^{n} q_i^0 &\le q_0(p_0,\beta,0,0,\ldots,0)+\sum_{i=1}^{n} q_i^0 \\ &= q_0^0+\sum_{i=1}^{n} q_i^0=\sum_{i=0}^{n} q_i^0 < G(p_0)\le G(p_0)+D. \end{aligned} \tag{B.57}$$

Case E: Let $\beta \in (0,1)$ and $\frac{1-\beta}{\beta}q_0^0 < \sum_{i=1}^{n} q_i^0 \le (1-\beta)G(p_0)$. Analogous to **Case C**, using the inequality $\sum_{i=1}^{n} q_i^0 > \frac{1-\beta}{\beta}q_0^0$, we can show that that $q_0(p_0,\beta,\nu_0,0,\ldots,0)$ is strictly increasing with respect to $\nu_0 \ge 0$.

Now, from equation (B.11) we have that

$$\begin{aligned} &[p_0 - f_0'(q_0(p_0,\beta,\nu_0,0,\ldots,0))] \\ &\quad + \beta\nu_0\left[\sum_{i=1}^{n} q_i^0 - \frac{1-\beta}{\beta}q_0(p_0,\beta,\nu_0,0,\ldots,0)\right]=0. \end{aligned} \tag{B.58}$$

In addition, if $\nu_0 > 0$, then,

$$\begin{aligned} &p_0 - f_0'(q_0(p_0,\beta,\nu_0,0,\ldots,0)) < p_0 - f_0'(q_0(p_0,\beta,0,0,\ldots,0)) \\ &\quad = p_0 - f_0'(q_0^0) = p_0 - p_0 = 0, \end{aligned} \tag{B.59}$$

thus,

$$p_0 - f_0'(q_0(p_0,\beta,\nu_0,0,\ldots,0)) < 0, \quad \forall \nu_0 > 0. \tag{B.60}$$

Moreover, since $\sum_{i=1}^{n} q_i^0 > \frac{1-\beta}{\beta}q_0^0$ and $q_0(p_0,\beta,\nu_0,0,\ldots,0)$ is strictly increasing with respect to ν_0, for any $\nu_0 > 0$ the inequality

$$\sum_{i=1}^{n} q_i^0 - \frac{1-\beta}{\beta}q_0(p_0,\beta,\nu_0,0,\ldots,0) > 0 \tag{B.61}$$

implies that

$$q_0^0 < q_0(p_0,\beta,\nu_0,0,\ldots,0) < \frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0. \tag{B.62}$$

Hence, from (B.58), (B.60), (B.62) and the strictly increasing behavior of $q_0(p_0,\beta,\nu_0,0,\ldots,0)$ with respect to ν_0, we can see that

$$\lim_{\nu_0\to+\infty} q_0(p_0,\beta,\nu_0,0,\ldots,0) = \frac{\beta}{1-\beta}\sum_{i=1}^{n} q_i^0. \tag{B.63}$$

Therefore, we have that

$$q_0(p_0, \beta, \nu_0, 0, \ldots, 0) + \sum_{i=1}^{n} q_i^0 < \frac{\beta}{1-\beta} \sum_{i=1}^{n} q_i^0 + \sum_{i=1}^{n} q_i^0 = \frac{1}{1-\beta} \sum_{i=1}^{n} q_i^0. \quad \text{(B.64)}$$

Applying the condition $\sum_{i=1}^{n} q_i^0 \le (1-\beta)G(p_0)$ of this **Case E** to (B.64), we get

$$q_0(p_0, \beta, \nu_0, 0, \ldots, 0) + \sum_{i=1}^{n} q_i^0 < \frac{1}{1-\beta} \sum_{i=1}^{n} q_i^0 \le G(p_0) \le G(p_0) + D. \quad \text{(B.65)}$$

Thus, as it follows from the above Cases A, B, C, D and E, we have shown that

$$q_0(p_0, \beta, \nu_0, 0, \ldots, 0) + \sum_{i=1}^{n} q_i^0 < G(p_0) + D, \quad \forall \beta \in (0, 1], \nu_0 \in [0, \overline{\nu}_0). \quad \text{(B.66)}$$

Finally, the latter inequality (B.66), together with (B.31), yield the following inequality

$$q_0(p_0, \beta, \nu_0, \nu_1, \ldots, \nu_n) + \sum_{i=1}^{n} q_i(p_0, \nu_i) < G(p_0) + D \quad \text{(B.67)}$$

for all $\beta \in (0, 1]$, $\nu_0 \in [0, \overline{\nu}_0)$ and $\nu_1, \ldots, \nu_n \ge 0$.

Therefore, from the relationships (B.30), (B.67), the strictly increasing behavior of the continuous functions q_i, $\forall i \in \{0, 1, \ldots, n\}$, with respect to p, and the strictly decreasing behavior of the continuous function $G(p)$ with respect to p, we can guarantee the existence of a unique price value $p \in \big(p_0, p_0^1(\beta, \nu_0, \nu_1, \ldots, \nu_n)\big)$ that solves (B.29), i.e., such that

$$\Gamma(p, \beta, \nu_0, \nu_1, \ldots, \nu_n, D) = 0. \quad \text{(B.68)}$$

Moreover, from (B.29) we have that

$$\frac{\partial \Gamma}{\partial p} = \frac{\partial q_0}{\partial p} + \sum_{i=1}^{n} \frac{\partial q_i}{\partial p} - G'(p) > 0, \quad \text{(B.69)}$$

for all $\beta \in (0, 1]$, $\nu_0 \in [0, \overline{\nu}_0)$, $\nu_1, \ldots, \nu_n, D \ge 0$ and $p \in \big[p_0, p_0^1(\beta, \nu_0, \nu_1, \ldots, \nu_n)\big)$.

Hence, by the implicit function theorem, the mapping

$$\begin{aligned} &p(\beta, \nu_0, \nu_1, \ldots, \nu_n, D) \\ &\quad := \big\{p \in \big[p_0, p_0^1(\beta, \nu_0, \nu_1, \ldots, \nu_n)\big) \;\big|\; \Gamma(p, \beta, \nu_0, \nu_1, \ldots, \nu_n, D) = 0\big\} \end{aligned} \quad \text{(B.70)}$$

is a continuously differentiable function with respect to $\beta \in (0, 1]$, $\nu_0 \in [0, \overline{\nu}_0)$, $\nu_i \geq 0$, $\forall i \in \{1, \dots, n\}$, and $D \geq 0$, i.e., for all $\beta \in (0, 1]$, $\nu_0 \in [0, \overline{\nu}_0)$, $\nu_i \geq 0$, $\forall i \in \{1, \dots, n\}$, and $D \geq 0$, there exist unique values $p = p(\beta, \nu_0, \nu_1, \dots, \nu_n, D) \in \big(p_0, p_0^1(\beta, \nu_0, \nu_1, \dots, \nu_n)\big)$, $q_0 = q_0(p(\beta, \nu_0, \nu_1, \dots, \nu_n, D), \nu_0, \nu_1, \dots, \nu_n) > 0$ and $q_i = q_i(p(\beta, \nu_0, \nu_1, \dots, \nu_n, D), \nu_i) > 0$, $\forall i \in \{1, \dots, n\}$, such that the vector $(p, q_0, q_1, \dots, q_n)$ is the exterior equilibrium, with $q_i > 0$ $\forall i \in \{0, 1, \dots, n\}$, for the influence coefficients $(\nu_0, \nu_1, \dots, \nu_n)$. Moreover, if $p \leq p_0$ there is no exterior equilibrium with $q_i > 0$ $\forall i \in \{0, 1, \dots, n\}$, and if $p \geq p_0^1(\beta, \nu_0, \nu_1, \dots, \nu_n)$ then some of the optimality conditions (10) and/or (12) cannot hold, which implies that the vector $(p, q_0, q_1, \dots, q_n)$ is the unique exterior equilibrium for the influence coefficients $(\nu_0, \nu_1, \dots, \nu_n)$.

Finally, for this value $p = p(\beta, \nu_0, \nu_1, \dots, \nu_n, D)$, taking into account the formulas (B.7), (B.24), (B.29), and (B.69), we can calculate

$$\frac{\partial p}{\partial D} = -\frac{\frac{\partial \Gamma}{\partial D}}{\frac{\partial \Gamma}{\partial p}} = \frac{1}{\frac{1}{(1-\beta)\nu_0 + f_0''(q_0)} + \frac{\nu_0 + f_0''(q_0)}{(1-\beta)\nu_0 + f_0''(q_0)} \sum_{i=1}^{n} \frac{1}{\nu_i + f_i''(q_i)} - G'(p)}. \quad \text{(B.71)}$$

The proof of Theorem 1 is complete■

Appendix C. Proof of Theorem 2

Theorem 2. *Under assumptions **A1**, **A2** and **A3**, there exists the interior equilibrium.*

Proof: Let us fix the values of $\beta \in (0, 1]$ and $D \geq 0$. We are going to prove that there exists the interior equilibrium for any $n \geq 1$.

By Theorem 1, for any $\nu_0 \in [0, \overline{\nu}_0)$ and $\nu_i \geq 0$, $\forall i \in \{1, \dots, n\}$, there exists a unique exterior equilibrium $(p, q_0, q_1, \dots, q_n)$, with $p > p_0$ and $q_i > 0$, $\forall i \in \{0, 1, \dots, n\}$, and in addition, the functions $p = p(\nu_0, \nu_1, \dots, \nu_n)$, $q_0 = q_0(\nu_0, \nu_1, \dots, \nu_n)$, and $q_i = q_i(\nu_0, \nu_1, \dots, \nu_n)$, $\forall i \in \{1, \dots, n\}$, are continuously differentiable with respect to $\nu_0 \in [0, \overline{\nu}_0)$ and $\nu_i \geq 0$, $\forall i \in \{1, \dots, n\}$.

Since $p > p_0$ and $G(p)$ is strictly decreasing, we have that

$$\sum_{i=0}^{n} q_i = G(p) + D < G(p_0) + D, \quad \text{(C.1)}$$

then, $q_i \in (0, G(p_0)+D)$, $\forall i \in \{0,1,\ldots,n\}$. Moreover, $f_i''(q_i) \in (0,\alpha)$, $\forall i \in \{0,1,\ldots,n\}$, where $\alpha = \max_{i\in\{0,1,\ldots,n\}} \{f_i''(G(p_0)+D)\} > 0$.

Now, consider the functions

$$F_0(\nu_0,\nu_1,\ldots,\nu_n) = \frac{1}{\sum_{i=1}^{n} \frac{1}{\nu_i + f_i''(q_i)} - G'(p)} > 0, \tag{C.2}$$

$$F_i(\nu_0,\nu_1,\ldots,\nu_n) = \frac{1}{\frac{1}{(1-\beta)\nu_0 + f_0''(q_0)} + \frac{\nu_0 + f_0''(q_0)}{(1-\beta)\nu_0 + f_0''(q_0)} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\nu_j + f_j''(q_i)} - G'(p)} > 0, \tag{C.3}$$

for all $i \in \{1,\ldots,n\}$.

By assumptions **A1** and **A2**, these functions are continuous with respect to $(\nu_0,\nu_1,\ldots,\nu_n)$.

Now let us define the constants

$$M_0 = \frac{2\alpha}{\beta+n-1} > 0, \tag{C.4}$$

$$M = \frac{[(1-\beta)+n]\alpha}{\beta+n-1} > 0. \tag{C.5}$$

If we select $\nu_0 \in [0, M_0]$ and $\nu_i \in [0, M]$, $\forall i \in \{1,\ldots,n\}$, then,

$$\begin{aligned} F_0(\nu_0,\nu_1,\ldots,\nu_n) &= \frac{1}{\sum_{i=1}^{n} \frac{1}{\nu_i + f_i''(q_i)} - G'(p)} < \frac{1}{\sum_{i=1}^{n} \frac{1}{M+\alpha}} = \frac{1}{\frac{n}{M+\alpha}} \\ &= \frac{M+\alpha}{n} = \frac{\frac{[(1-\beta)+n]\alpha}{\beta+n-1} + \alpha}{n} = \frac{2\alpha}{\beta+n-1} = M_0, \end{aligned} \tag{C.6}$$

$$\begin{aligned} F_i(\nu_0,\nu_1,\ldots,\nu_n) &= \frac{1}{\frac{1}{(1-\beta)\nu_0 + f_0''(q_0)} + \frac{\nu_0 + f_0''(q_0)}{(1-\beta)\nu_0 + f_0''(q_0)} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\nu_j + f_j''(q_i)} - G'(p)} \\ &< \frac{1}{\frac{1}{(1-\beta)M_0 + \alpha}} = (1-\beta)M_0 + \alpha \\ &= (1-\beta)\frac{2\alpha}{\beta+n-1} + \alpha = \frac{[(1-\beta)+n]\alpha}{\beta+n-1} = M, \end{aligned} \tag{C.7}$$

for all $i \in \{1,\ldots,n\}$.

In other words $F_0(\nu_0,\nu_1,\ldots,\nu_n) \in [0, M_0]$ and $F_i(\nu_0,\nu_1,\ldots,\nu_n) \in [0, M]$, $\forall i \in \{1,\ldots,n\}$, whenever $\nu_0 \in [0, M_0]$ and $\nu_i \in [0, M]$, $\forall i \in \{1,\ldots,n\}$, which means that the continuous function $H = (F_0, F_1, \dots, F_n)$ maps the convex compact set $[0, M_0] \times [0, M]^n$ into itself, therefore, by Brouwer's fixed-point theorem, the function $H = (F_0, F_1, \dots, F_n)$ has a fixed point $(\nu_0^*, \nu_1^*, \dots, \nu_n^*)$, i.e., there exists $(\nu_0^*, \nu_1^*, \dots, \nu_n^*) > 0$ such that

$$F_0(\nu_0^*, \nu_1^*, \dots, \nu_n^*) = \nu_0^*, \tag{C.8}$$

$$F_i(\nu_0^*, \nu_1^*, \dots, \nu_n^*) = \nu_i^*, \quad \forall i \in \{1, \dots, n\}. \tag{C.9}$$

Now, for these influence coefficients $(\nu_0^*, \nu_1^*, \dots, \nu_n^*)$ we calculate their exterior equilibrium $(p^*, q_0^*, q_1^*, \dots, q_n^*)$, implying that

$$\nu_0^* = \frac{1}{\sum\limits_{i=1}^{n} \frac{1}{\nu_i^* + f_i''(q_i^*)} - G'(p^*)}, \tag{C.10}$$

$$\nu_i^* = \frac{1}{\frac{1}{(1-\beta)\nu_0^* + f_0''(q_0^*)} + \frac{\nu_0^* + f_0''(q_0^*)}{(1-\beta)\nu_0^* + f_0''(q_0^*)} \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{\nu_j^* + f_j''(q_i^*)} - G'(p^*)}, \tag{C.11}$$

for all $i \in \{1, \dots, n\}$.

Finally, the equations (C.10) and (C.11) imply that $(p^*, q_0^*, q_1^*, \dots, q_n^*, \nu_0^*, \nu_1^*, \dots, \nu_n^*)$ is the interior equilibrium, which proves Theorem 2■

Appendix D. Proof of Theorem 3

Theorem 3. *Under assumptions **A4**, **A5** and **A6**, for every $\beta \in (0, 1]$ there exists uniquely the interior equilibrium $(p^*, q_0^*, q^*, \nu_0^*, \nu^*)$.*

Proof: For our particular case, the consistency criterion takes the following form:

$$\nu_0 = \frac{1}{\frac{n}{\nu + a} + K}, \tag{D.1}$$

$$\nu = \frac{1}{\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \frac{n-1}{\nu + a} + K}, \tag{D.2}$$

then, the functions (C.2) and (C.3), within the proof of Theorem 2, are rewritten as follows

$$F_0(\nu_0, \nu) = \frac{1}{\frac{n}{\nu + a} + K} > 0, \tag{D.3}$$

$$F(\nu_0, \nu) = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\dfrac{n-1}{\nu + a} + K} > 0, \tag{D.4}$$

We can see that the functions F_0 and F are continuously differentiable with respect to $\beta \in (0, 1]$ and $\nu_0, \nu \geq 0$, and by Theorem 2 we know that the function $H = (F_0, F)$ has a fixed point, i.e., the system of equations

$$F_0(\nu_0, \nu) = \nu_0, \tag{D.5}$$

$$F(\nu_0, \nu) = \nu, \tag{D.6}$$

has a solution.

In order to prove that the solution of (D.5) and (D.6) is unique, we are going to show that the function $H = (F_0, F)$ is a nonexpansive mapping.

Given that H is a continuously differentiable function, in order to prove that H is a nonexpansive mapping, it is enough to show that $\|\nabla H\|_\infty < 1$ (i.e., the infinite norm of the Jacobian matrix of H is less than 1).

From the function (D.3) and (D.4) we have that

$$\frac{\partial F_0}{\partial \nu_0} = 0, \tag{D.7}$$

$$\frac{\partial F_0}{\partial \nu} = \frac{n}{(\nu + a)^2 \left(\dfrac{n}{\nu + a} + K\right)^2} = \frac{n}{[n + (\nu + a)K]^2} > 0, \tag{D.8}$$

$$\begin{aligned}
\frac{\partial F}{\partial \nu_0} &= \frac{\dfrac{1-\beta}{[(1-\beta)\nu_0 + a_0]^2} - \dfrac{(1-\beta)\nu_0 + a_0 - (1-\beta)(\nu_0 + a_0)}{[(1-\beta)\nu_0 + a_0]^2}\dfrac{n-1}{\nu + a}}{\left[\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\dfrac{n-1}{\nu + a} + K\right]^2}, \\
&= \frac{(1-\beta) - \beta a_0 \dfrac{n-1}{\nu + a}}{[(1-\beta)\nu_0 + a_0]^2 \left[\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\dfrac{n-1}{\nu + a} + K\right]^2} \qquad \text{(D.9)} \\
&= \frac{(1-\beta) - \beta a_0 \dfrac{n-1}{\nu + a}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu + a} + [(1-\beta)\nu_0 + a_0]K\right\}^2},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial F}{\partial \nu} &= \frac{\dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\dfrac{n-1}{(\nu+a)^2}}{\left[\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\dfrac{n-1}{\nu+a} + K\right]^2}, \\
&= \frac{[(1-\beta)\nu_0 + a_0](\nu_0 + a_0)\dfrac{n-1}{(\nu+a)^2}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \geq 0.
\end{aligned}
\tag{D.10}
$$

Therefore,

$$
\begin{aligned}
&\left|\frac{\partial F_0}{\partial \nu_0}\right| + \left|\frac{\partial F_0}{\partial \nu}\right| = |0| + \left|\frac{n}{[n + (\nu+a)K]^2}\right| = \frac{n}{[n + (\nu+a)K]^2} \\
&< \frac{n}{n^2} = \frac{1}{n} \leq 1,
\end{aligned}
\tag{D.11}
$$

$$
\begin{aligned}
\left|\frac{\partial F}{\partial \nu_0}\right| + \left|\frac{\partial F}{\partial \nu}\right| &= \left|\frac{(1-\beta) - \beta a_0 \dfrac{n-1}{\nu+a}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2}\right| \\
&+ \left|\frac{[(1-\beta)\nu_0 + a_0](\nu_0 + a_0)\dfrac{n-1}{(\nu+a)^2}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2}\right| \\
&= \frac{\left|(1-\beta) - \beta a_0 \dfrac{n-1}{\nu+a}\right| + [(1-\beta)\nu_0 + a_0](\nu_0 + a_0)\dfrac{n-1}{(\nu+a)^2}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&\leq \frac{(1-\beta) + \beta a_0 \dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0](\nu_0 + a_0)\dfrac{n-1}{(\nu+a)^2}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&< \frac{1 + 2(\nu_0 + a_0)\dfrac{n-1}{\nu+a} + \left[(\nu_0 + a_0)\dfrac{n-1}{\nu+a}\right]^2}{\left[1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a}\right]^2} = 1.
\end{aligned}
\tag{D.12}
$$

Hence,

$$
\|\nabla H\|_\infty = \max\left\{\left|\frac{\partial F_0}{\partial \nu_0}\right| + \left|\frac{\partial F_0}{\partial \nu}\right|, \left|\frac{\partial F}{\partial \nu_0}\right| + \left|\frac{\partial F}{\partial \nu}\right|\right\} < 1. \tag{D.13}
$$

Therefore, the function $H = (F_0, F)$ is a nonexpansive mapping and the fixed point (ν_0^*, ν^*) is unique. In addition, since the corresponding exterior equilibrium (p^*, q_0^*, q^*) is unique, too, the interior equilibrium $(p^*, q_0^*, q^*, \nu_0^*, \nu^*)$ is also unique■

Appendix E. Proof of Theorem 4

Theorem 4. *The interior equilibrium $(p^*(\beta), q_0^*(\beta), q^*(\beta), \nu_0^*(\beta), \nu^*(\beta))$ and the function $\pi^*(\beta)$ are continuously differentiable with respect to $\beta \in (0, 1]$. Moreover, the functions $p^*(\beta)$, $\nu_0^*(\beta)$ and $\nu^*(\beta)$ are strictly decreasing for all $\beta \in (0, 1]$.*

Proof: First, we will show that the influence coefficients $\nu_0^* = \nu_0^*(\beta)$ and $\nu^* = \nu^*(\beta)$ are continuously differentiable functions with respect to $\beta \in (0, 1]$.

To do that, we consider again the functions F_0 and F, but this time taking into account their dependence with respect to β, i.e.,

$$F_0(\beta, \nu_0, \nu) = \frac{1}{\dfrac{n}{\nu + a} + K}, \tag{E.1}$$

$$F(\beta, \nu_0, \nu) = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\dfrac{n-1}{\nu + a} + K}. \tag{E.2}$$

Now, we define the function $\Phi = (\phi_0, \phi)$, where

$$\phi_0(\beta, \nu_0, \nu) = \nu_0 - F_0(\beta, \nu_0, \nu), \tag{E.3}$$

$$\phi(\beta, \nu_0, \nu) = \nu - F(\beta, \nu_0, \nu). \tag{E.4}$$

By Theorem 3, the equation

$$\Phi(\beta, \nu_0, \nu) = 0 \tag{E.5}$$

has a unique solution $\nu_0^*(\beta), \nu^*(\beta) > 0$, for every $\beta \in (0, 1]$.

In addition, it is easy to see that Φ is continuously differentiable with respect to $\beta \in (0, 1]$ and $\nu_0, \nu \geq 0$, with

$$\nabla_{(\nu_0,\nu)}\Phi = \frac{\partial \Phi}{\partial(\nu_0, \nu)} = \begin{pmatrix} \dfrac{\partial \phi_0}{\partial \nu_0} & \dfrac{\partial \phi_0}{\partial \nu} \\ \dfrac{\partial \phi}{\partial \nu_0} & \dfrac{\partial \phi}{\partial \nu} \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\partial F_0}{\partial \nu_0} & -\dfrac{\partial F_0}{\partial \nu} \\ -\dfrac{\partial F}{\partial \nu_0} & 1 - \dfrac{\partial F}{\partial \nu} \end{pmatrix}, \tag{E.6}$$

and by (D.13), together with (D.7) and (D.10), we have that

$$\left|\frac{\partial\phi_0}{\partial\nu_0}\right| = \left|1-\frac{\partial F_0}{\partial\nu_0}\right| = |1| = 1 > \left|\frac{\partial F_0}{\partial\nu}\right| = \left|\frac{\partial\phi_0}{\partial\nu}\right|, \tag{E.7}$$

$$\left|\frac{\partial\phi}{\partial\nu}\right| = \left|1-\frac{\partial F}{\partial\nu}\right| = 1-\frac{\partial F}{\partial\nu} > \left|\frac{\partial F}{\partial\nu_0}\right| = \left|\frac{\partial\phi}{\partial\nu_0}\right|. \tag{E.8}$$

Then, we can conclude that the matrix $\nabla_{(\nu_0,\nu)}\Phi$ is diagonally dominant, and, therefore, invertible for every $\beta \in (0,1]$.

Hence, by the implicit function theorem, the functions $\nu_0^*(\beta)$ and $\nu^*(\beta)$ are continuously differentiable with respect to $\beta \in (0,1]$, and, in addition,

$$\frac{d(\nu_0^*,\nu^*)}{d\beta} = -\left[\frac{\partial\Phi}{\partial(\nu_0,\nu)}\right]^{-1}\frac{\partial\Phi}{\partial\beta} = \begin{pmatrix} 1-\dfrac{\partial F_0}{\partial\nu_0} & -\dfrac{\partial F_0}{\partial\nu} \\ -\dfrac{\partial F}{\partial\nu_0} & 1-\dfrac{\partial F}{\partial\nu}\end{pmatrix}^{-1}\begin{pmatrix}\dfrac{\partial F_0}{\partial\beta} \\ \dfrac{\partial F}{\partial\beta}\end{pmatrix}, \tag{E.9}$$

where

$$\frac{\partial F_0}{\partial\beta} = 0, \tag{E.10}$$

$$\begin{aligned}\frac{\partial F}{\partial\beta} &= \frac{\dfrac{-\nu_0}{[(1-\beta)\nu_0+a_0]^2}\left[1+(\nu_0+a_0)\dfrac{n-1}{\nu+a}\right]}{\left[\dfrac{1}{(1-\beta)\nu_0+a_0}+\dfrac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\dfrac{n-1}{\nu+a}+K\right]^2} \\ &= -\frac{\nu_0\left[1+(\nu_0+a_0)\dfrac{n-1}{\nu+a}\right]}{\left\{1+(\nu_0+a_0)\dfrac{n-1}{\nu+a}+[(1-\beta)\nu_0+a_0]K\right\}^2} < 0.\end{aligned} \tag{E.11}$$

Therefore, we have that

$$\frac{d(\nu_0^*,\nu^*)}{d\beta} = \begin{pmatrix}\dfrac{d\nu_0^*}{d\beta} \\ \dfrac{d\nu^*}{d\beta}\end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{\partial F_0}{\partial\nu} \\ -\dfrac{\partial F}{\partial\nu_0} & 1-\dfrac{\partial F}{\partial\nu}\end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \dfrac{\partial F}{\partial\beta}\end{pmatrix}. \tag{E.12}$$

Now, since $\nabla_{(\nu_0,\nu)}\Phi$ is invertible for all $\beta \in (0,1]$, then,

$$\det\left(\nabla_{(\nu_0,\nu)}\Phi\right) = 1-\frac{\partial F}{\partial\nu}-\frac{\partial F_0}{\partial\nu}\frac{\partial F}{\partial\nu_0} \neq 0, \quad \forall\beta\in(0,1], \tag{E.13}$$

and

$$\begin{pmatrix} 1 & -\dfrac{\partial F_0}{\partial\nu} \\ -\dfrac{\partial F}{\partial\nu_0} & 1-\dfrac{\partial F}{\partial\nu}\end{pmatrix}^{-1} = \frac{1}{1-\dfrac{\partial F}{\partial\nu}-\dfrac{\partial F_0}{\partial\nu}\dfrac{\partial F}{\partial\nu_0}}\begin{pmatrix} 1-\dfrac{\partial F}{\partial\nu} & \dfrac{\partial F_0}{\partial\nu} \\ \dfrac{\partial F}{\partial\nu_0} & 1\end{pmatrix}, \tag{E.14}$$

thus,

$$\begin{pmatrix} \dfrac{d\nu_0^*}{d\beta} \\ \dfrac{d\nu^*}{d\beta} \end{pmatrix} = \frac{1}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}} \begin{pmatrix} 1 - \dfrac{\partial F}{\partial \nu} & \dfrac{\partial F_0}{\partial \nu} \\ \dfrac{\partial F}{\partial \nu_0} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{\partial F}{\partial \beta} \end{pmatrix}. \tag{E.15}$$

Next, from (E.8) we can see that that

$$1 - \frac{\partial F}{\partial \nu} > 0. \tag{E.16}$$

In addition for $\beta = 1$ we have that

$$\left.\frac{\partial F}{\partial \nu_0}\right|_{\beta=1} = \frac{-a_0 \dfrac{n-1}{\nu + a}}{\left[1 + (\nu_0 + a_0)\dfrac{n-1}{\nu + a} + a_0 K\right]^2} \leq 0. \tag{E.17}$$

Then, from (D.8), (E.16) and (E.17),

$$1 - \frac{\partial F}{\partial \nu} - \frac{\partial F_0}{\partial \nu}\frac{\partial F}{\partial \nu_0} > 0, \quad \text{for } \beta = 1. \tag{E.18}$$

Moreover, we know that F_0 and F are continuously differentiable with respect to $\beta \in (0, 1]$, therefore, $1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}$ is continuous respect to $\beta \in (0, 1]$, thus, from the inequalities (E.13) and (E.18), we have that

$$1 - \frac{\partial F}{\partial \nu} - \frac{\partial F_0}{\partial \nu}\frac{\partial F}{\partial \nu_0} > 0, \quad \forall \beta \in (0, 1]. \tag{E.19}$$

Therefore, by the relationships (D.8), (E.11) and (E.19),

$$\frac{d\nu_0^*}{d\beta} = \frac{\dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \beta}}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}} < 0, \tag{E.20}$$

$$\frac{d\nu^*}{d\beta} = \frac{\dfrac{\partial F}{\partial \beta}}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}} < 0, \tag{E.21}$$

so the functions $\nu_0^*(\beta)$ and $\nu^*(\beta)$ are strictly decreasing with respect to $\beta \in (0, 1]$.

Because the influence coefficients $\nu_0^*(\beta)$ and $\nu^*(\beta)$ are continuously differentiable with respect to $\beta \in (0, 1]$, and the exterior equilibrium $(p(\beta, \nu_0, \nu), q_0(\beta, \nu_0, \nu), q(\beta, \nu_0, \nu))$

is continuously differentiable with respect to $\beta \in (0,1]$ and $\nu_0, \nu \geq 0$ (by Theorem 1), then, the equilibrium

$$(p^*(\beta), q_0^*(\beta), q^*(\beta)) = (p(\beta, \nu_0^*, \nu^*), q_0(\beta, \nu_0^*, \nu^*), q(\beta, \nu_0^*, \nu^*)) \tag{E.22}$$

is continuously differentiable with respect to $\beta \in (0,1]$. Moreover, since $\pi(p,q)$ is continuously differentiable with respect to any $p, q \geq 0$ (as we can see from equation (26)), then, $\pi^*(\beta) = \pi(p^*(\beta), q^*(\beta))$ is also continuously differentiable with respect to $\beta \in (0,1]$.

Next, since $p^*(\beta) = p(\beta, \nu_0^*, \nu^*)$, we can apply the chain rule to obtain

$$\frac{dp^*}{d\beta} = \frac{\partial p}{\partial \beta} + \frac{\partial p}{\partial \nu_0}\frac{d\nu_0^*}{d\beta} + \frac{\partial p}{\partial \nu}\frac{d\nu^*}{d\beta}. \tag{E.23}$$

Now, consider the function

$$\Gamma(p, \beta, \nu_0, \nu) = [q_0(p, \beta, \nu_0, \nu) + nq(p, \nu)] - G(p) = 0, \tag{E.24}$$

where, for our particular case,

$$q(p, \nu) = \frac{p - b}{\nu + a}, \tag{E.25}$$

$$q_0(p, \beta, \nu_0, \nu) = \frac{(p - b_0) + \beta n \nu_0 q(p, \nu)}{(1 - \beta)\nu_0 + a_0}, \tag{E.26}$$

$$\begin{aligned} \frac{\partial \Gamma}{\partial p} &= \frac{\partial q_0}{\partial p} + n\frac{\partial q}{\partial p} - G'(p) = \frac{1 + \beta\nu_0 \dfrac{n}{\nu + a}}{(1-\beta)\nu_0 + a_0} + \frac{n}{\nu + a} + K \\ &= \frac{1 + (\nu_0 + a_0)\dfrac{n}{\nu + a}}{(1-\beta)\nu_0 + a_0} + K > 0. \end{aligned} \tag{E.27}$$

In addition,

$$\begin{aligned} \frac{\partial \Gamma}{\partial \beta} &= \frac{\partial q_0}{\partial \beta} \\ &= \frac{n\nu_0 q(p,\nu)[(1-\beta)\nu_0 + a_0] + \nu_0[(p - b_0) + \beta n \nu_0 q(p,\nu)]}{[(1-\beta)\nu_0 + a_0]^2} \\ &= \frac{\nu_0[(p - b_0) + n(\nu_0 + a_0)q(p,\nu)]}{[(1-\beta)\nu_0 + a_0]^2} > 0, \end{aligned} \tag{E.28}$$

$$\begin{aligned} \frac{\partial \Gamma}{\partial \nu_0} &= \frac{\partial q_0}{\partial \nu_0} \\ &= \frac{\beta n q(p,\nu)[(1-\beta)\nu_0 + a_0] - (1-\beta)[(p - b_0) + \beta n \nu_0 q(p,\nu)]}{[(1-\beta)\nu_0 + a_0]^2} \\ &= \frac{\beta n a_0 q(p,\nu) - (1-\beta)(p - b_0)}{[(1-\beta)\nu_0 + a_0]^2}, \end{aligned} \tag{E.29}$$

$$\frac{\partial\Gamma}{\partial\nu}=\frac{\partial q_0}{\partial\nu}+n\frac{\partial q}{\partial\nu}=\frac{\beta n\nu_0}{(1-\beta)\nu_0+a_0}\frac{\partial q}{\partial\nu}+n\frac{\partial q}{\partial\nu}=\frac{n(\nu_0+a_0)}{(1-\beta)\nu_0+a_0}\frac{\partial q}{\partial\nu}$$
$$=-\frac{n(\nu_0+a_0)}{(1-\beta)\nu_0+a_0}\frac{p-b}{(\nu+a)^2}<0. \tag{E.30}$$

Therefore, by the implicit function theorem,

$$\frac{\partial p}{\partial\beta}=-\frac{\frac{\partial\Gamma}{\partial\beta}}{\frac{\partial\Gamma}{\partial p}},\quad \frac{\partial p}{\partial\nu_0}=-\frac{\frac{\partial\Gamma}{\partial\nu_0}}{\frac{\partial\Gamma}{\partial p}},\quad \frac{\partial p}{\partial\nu}=-\frac{\frac{\partial\Gamma}{\partial\nu}}{\frac{\partial\Gamma}{\partial p}}, \tag{E.31}$$

then, from equalities (E.23) and (E.31),

$$p^{*\prime}(\beta)=\frac{dp^*}{d\beta}=-\frac{\frac{\partial\Gamma}{\partial\beta}+\frac{\partial\Gamma}{\partial\nu_0}\frac{d\nu_0^*}{d\beta}+\frac{\partial\Gamma}{\partial\nu}\frac{d\nu^*}{d\beta}}{\frac{\partial\Gamma}{\partial p}}, \tag{E.32}$$

and from the equalities (E.28) and (E.29), we can see that

$$\begin{aligned}\frac{\partial\Gamma}{\partial\beta}+\frac{\partial\Gamma}{\partial\nu_0}\frac{d\nu_0^*}{d\beta}&=\frac{\nu_0^*[(p^*-b_0)+n(\nu_0^*+a_0)q(p^*,\nu^*)]}{[(1-\beta)\nu_0^*+a_0]^2}\\&\quad+\frac{\beta n a_0 q(p^*,\nu^*)-(1-\beta)(p^*-b_0)}{[(1-\beta)\nu_0^*+a_0]^2}\nu_0^{*\prime}\\&=\frac{\nu_0^*(p^*-b_0)+nq(p^*,\nu^*)[\nu_0^{*2}+a_0(\nu_0^*+\beta\nu_0^{*\prime})]}{[(1-\beta)\nu_0^*+a_0]^2}\\&\quad+\frac{(1-\beta)(p^*-b_0)}{[(1-\beta)\nu_0^*+a_0]^2}(-\nu_0^{*\prime}).\end{aligned} \tag{E.33}$$

Now, we are going to analyze the sign of $\nu_0^*+\beta\nu_0^{*\prime}$.

First, from the equality (D.8), we can see that

$$\frac{\partial F_0}{\partial\nu}=\frac{n}{[n+(\nu+a)K]^2}<\frac{n}{n^2}=\frac{1}{n}\leq 1. \tag{E.34}$$

Second, from the identities (D.9) and (E.11), we have that

$$\begin{aligned}
\frac{\partial F}{\partial \nu_0} - \frac{\beta}{\nu_0}\frac{\partial F}{\partial \beta} &= \frac{(1-\beta) - \beta a_0 \dfrac{n-1}{\nu+a}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&\quad + \frac{\beta}{\nu_0}\frac{\nu_0\left[1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a}\right]}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&= \frac{(1-\beta) - \beta a_0 \dfrac{n-1}{\nu+a} + \beta\left[1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a}\right]}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&= \frac{1 + \beta\nu_0\dfrac{n-1}{\nu+a}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} > 0.
\end{aligned} \tag{E.35}$$

And third, from the equalities (D.10) and (E.35), we get

$$\begin{aligned}
\frac{\partial F}{\partial \nu} + \frac{\partial F}{\partial \nu_0} - \frac{\beta}{\nu_0}\frac{\partial F}{\partial \beta} &= \frac{[(1-\beta)\nu_0 + a_0](\nu_0 + a_0)\dfrac{n-1}{(\nu+a)^2}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&\quad + \frac{1 + \beta\nu_0\dfrac{n-1}{\nu+a}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&= \frac{1 + \beta\nu_0\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0](\nu_0 + a_0)\dfrac{n-1}{(\nu+a)^2}}{\left\{1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a} + [(1-\beta)\nu_0 + a_0]K\right\}^2} \\
&< \frac{1 + 2(\nu_0 + a_0)\dfrac{n-1}{\nu+a} + (\nu_0 + a_0)^2\dfrac{(n-1)^2}{(\nu+a)^2}}{\left[1 + (\nu_0 + a_0)\dfrac{n-1}{\nu+a}\right]^2} = 1.
\end{aligned} \tag{E.36}$$

Then, using the relationships (E.19), (E.20), (E.34), (E.35) (E.36), we obtain

$$
\begin{aligned}
\nu_0^* + \beta\nu_0^{*\prime} &= \nu_0^* + \beta\frac{\dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \beta}}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}} \\
&= \frac{\nu_0^*\left(1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}\right) + \beta\dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \beta}}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}} \\
&= \frac{\nu_0^*}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}}\left[1 - \frac{\partial F}{\partial \nu} - \frac{\partial F_0}{\partial \nu}\left(\frac{\partial F}{\partial \nu_0} - \frac{\beta}{\nu_0^*}\frac{\partial F}{\partial \beta}\right)\right] \\
&> \frac{\nu_0^*}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}}\left[1 - \frac{\partial F}{\partial \nu} - \left(\frac{\partial F}{\partial \nu_0} - \frac{\beta}{\nu_0^*}\frac{\partial F}{\partial \beta}\right)\right] \\
&= \frac{\nu_0^*}{1 - \dfrac{\partial F}{\partial \nu} - \dfrac{\partial F_0}{\partial \nu}\dfrac{\partial F}{\partial \nu_0}}\left[1 - \left(\frac{\partial F}{\partial \nu} + \frac{\partial F}{\partial \nu_0} - \frac{\beta}{\nu_0^*}\frac{\partial F}{\partial \beta}\right)\right] > 0.
\end{aligned}
\tag{E.37}
$$

Finally, applying the inequalities (E.20), (E.37), and the assumption **A5**, to (E.33), we can see that

$$
\begin{aligned}
\frac{\partial \Gamma}{\partial \beta} + \frac{\partial \Gamma}{\partial \nu_0}\frac{d\nu_0^*}{d\beta} = &\frac{\nu_0^*(p^* - b_0) + nq(p^*, \nu^*)[\nu_0^{*2} + a_0(\nu_0^* + \beta\nu_0^{*\prime})]}{[(1-\beta)\nu_0^* + a_0]^2} \\
&+ \frac{(1-\beta)(p^* - b_0)}{[(1-\beta)\nu_0^* + a_0]^2}(-\nu_0^{*\prime}) > 0.
\end{aligned}
\tag{E.38}
$$

Therefore, applying the inequalities (E.21), (E.27), (E.30) and (E.38), to (E.32), we obtain

$$
p^{*\prime}(\beta) = -\frac{\dfrac{\partial \Gamma}{\partial \beta} + \dfrac{\partial \Gamma}{\partial \nu_0}\dfrac{d\nu_0^*}{d\beta} + \dfrac{\partial \Gamma}{\partial \nu}\dfrac{d\nu^*}{d\beta}}{\dfrac{\partial \Gamma}{\partial p}} < 0, \tag{E.39}
$$

which proves that $p^*(\beta)$ is strictly decreasing with respect to $\beta \in (0, 1]$■

Appendix F. Proof of Theorem 5

Theorem 5. *Under assumptions **A4**, **A5** and **A6**, the exterior equilibrium $(p^c(\beta), q_0^c(\beta), q^c(\beta))$ and the function $\pi^c(\beta)$ are continuously differentiable with respect to $\beta \in (0, 1]$. Moreover, the function $p^c(\beta)$ is strictly decreasing for all $\beta \in (0, 1]$.*

Proof: From Theorem 1 we have that the exterior equilibrium $(p^c(\beta), q_0^c(\beta), q^c(\beta))$, corresponding to the influence coefficients $(\nu_0^c, \nu^c) = \left(\frac{1}{K}, \frac{1}{K}\right)$, is continuously differentiable with respect to $\beta \in (0, 1]$. Moreover, since $\pi(p, q)$ is continuously differentiable with respect to any $p, q \geq 0$ (as we can see from equation (26)), then, $\pi^c(\beta) = \pi(p^c(\beta), q^c(\beta))$ is also continuously differentiable with respect to $\beta \in (0, 1]$.

Next, in the same way as in the proof of Theorem 4, we consider the functions $q_0(p, \beta, \nu_0, \nu)$, $q(p, \nu)$ and $\Gamma(p, \beta, \nu_0, \nu)$, given by (E.24), (E.25) and (E.26), and define the implicit function

$$\begin{aligned} \Gamma^c(p, \beta) &= \Gamma\left(p, \beta, \frac{1}{K}, \frac{1}{K}\right) \\ &= \left[q_0\left(p, \beta, \frac{1}{K}, \frac{1}{K}\right) + nq\left(p, \frac{1}{K}\right)\right] - G(p) = 0, \end{aligned} \tag{F.1}$$

where (similarly to the proof of Theorem 4)

$$\frac{\partial \Gamma^c}{\partial p} = \frac{1 + \left(\frac{1}{K} + a_0\right)\dfrac{n}{\frac{1}{K} + a}}{(1 - \beta)\frac{1}{K} + a_0} + K > 0, \tag{F.2}$$

and

$$\frac{\partial \Gamma^c}{\partial \beta} = \frac{\frac{1}{K}\left[(p - b_0) + n\left(\frac{1}{K} + a_0\right) q\left(p, \frac{1}{K}\right)\right]}{\left[(1 - \beta)\frac{1}{K} + a_0\right]^2} > 0. \tag{F.3}$$

Hence, by the implicit function theorem,

$$p^{c\prime}(\beta) = \frac{dp^c}{d\beta} = -\frac{\dfrac{\partial \Gamma^c}{\partial \beta}}{\dfrac{\partial \Gamma^c}{\partial p}} < 0, \tag{F.4}$$

which proves that $p^c(\beta)$ is strictly decreasing with respect to $\beta \in (0, 1]$ ■

Appendix G. Proof of Theorem 6

Theorem 6. *Under assumptions **A4**, **A5** and **A6**, the exterior equilibrium* $(p^t(\beta), q_0^t(\beta), q^t(\beta))$ *and the function* $\pi^t(\beta)$ *are constant with respect to* $\beta \in (0, 1]$.

Proof: From Theorem 1 we have that the exterior equilibrium $(p^t(\beta), q_0^t(\beta), q^t(\beta))$, corresponding to the influence coefficients $(\nu_0^t, \nu^t) = (0,0)$, is continuously differentiable with respect to $\beta \in (0,1]$.

Again, in the same way as in the proof of Theorem 4, we consider the functions $q_0(p,\beta,\nu_0,\nu)$, $q(p,\nu)$ and $\Gamma(p,\beta,\nu_0,\nu)$, given by (E.24), (E.25) and (E.26), and define the implicit function

$$\begin{aligned}\Gamma^t(p,\beta) &= \Gamma\left(p,\beta,0,0\right) \\ &= \left[q_0\left(p,\beta,0,0\right) + nq\left(p,0\right)\right] - G(p) = 0,\end{aligned} \tag{G.1}$$

where (similarly to the proof of Theorem 4)

$$\frac{\partial \Gamma^t}{\partial p} = \frac{1 + a_0 \dfrac{n}{a}}{a_0} + K > 0, \tag{G.2}$$

and

$$\frac{\partial \Gamma^t}{\partial \beta} = 0. \tag{G.3}$$

Hence, by the implicit function theorem,

$$p^{t\prime}(\beta) = \frac{dp^t}{d\beta} = -\frac{\dfrac{\partial \Gamma^t}{\partial \beta}}{\dfrac{\partial \Gamma^t}{\partial p}} = 0, \tag{G.4}$$

which proves that $p^t = p^t(\beta)$ is constant with respect to $\beta \in (0,1]$.

Substituting the values p^t and $\nu_0^t = \nu^t = 0$, in the expressions of $q_0(p,\beta,\nu_0,\nu)$ and $q(p,\nu)$ given by (E.25) and (E.26), we obtain

$$q_0^t(\beta) = q_0(p^t,\beta,0,0) = \frac{p^t - b_0}{a_0}, \tag{G.5}$$

$$q^t(\beta) = q(p^t,0) = \frac{p^t - b}{a}, \tag{G.6}$$

then, $q_0^t = q_0^t(\beta)$ and $q^t = q^t(\beta)$ are also constant with respect to $\beta \in (0,1]$, which implies that $\pi^t = \pi^t(\beta) = \pi(p^t, q^t)$ is constant with respect to $\beta \in (0,1]$, too■

Appendix H. Proof of Theorem 7

Theorem 7. *Under the assumptions* ***A4****,* ***A5*** *and* ***A6****, the following inequalities hold:*

$$\lim_{\beta\downarrow 0} p^c(\beta) > \lim_{\beta\downarrow 0} p^*(\beta) > p^t. \tag{40}$$

Proof: From the equations (E.24), (E.25), (E.26) and the assumption **A4**, we have that

$$\begin{aligned}
\Gamma(p,\beta,\nu_0,\nu) &= \frac{(p-b_0)+\beta n\nu_0\dfrac{p-b}{\nu+a}}{(1-\beta)\nu_0+a_0} + n\frac{p-b}{\nu+a} + Kp - T \\
&= \frac{(p-b_0)+n(\nu_0+a_0)\dfrac{p-b}{\nu+a}}{(1-\beta)\nu_0+a_0} + Kp - T \\
&= p\left[\frac{1+(\nu_0+a_0)\dfrac{n}{\nu+a}}{(1-\beta)\nu_0+a_0}+K\right] \\
&\quad - \left[\frac{b_0+(\nu_0+a_0)\dfrac{nb}{\nu+a}}{(1-\beta)\nu_0+a_0}+T\right] = 0,
\end{aligned} \tag{H.1}$$

from which we can isolate p to obtain

$$p(\beta,\nu_0,\nu) = \frac{\dfrac{b_0+(\nu_0+a_0)\dfrac{nb}{\nu+a}}{(1-\beta)\nu_0+a_0}+T}{\dfrac{1+(\nu_0+a_0)\dfrac{n}{\nu+a}}{(1-\beta)\nu_0+a_0}+K}. \tag{H.2}$$

Now, we are going to prove that

$$\lim_{\beta\downarrow 0} p^c(\beta) > \lim_{\beta\downarrow 0} p^*(\beta) > p^t, \tag{40}$$

where

$$p^*(\beta) = p(\beta,\nu_0^*(\beta),\nu^*(\beta)) = \frac{\dfrac{b_0+(\nu_0^*+a_0)\dfrac{nb}{\nu^*+a}}{(1-\beta)\nu_0^*+a_0}+T}{\dfrac{1+(\nu_0^*+a_0)\dfrac{n}{\nu^*+a}}{(1-\beta)\nu_0^*+a_0}+K}, \tag{H.3}$$

$$p^c(\beta) = p\left(\beta, \frac{1}{K}, \frac{1}{K}\right) = \frac{\dfrac{b_0 + \left(\frac{1}{K} + a_0\right)\dfrac{nb}{\frac{1}{K} + a}}{(1-\beta)\frac{1}{K} + a_0} + T}{\dfrac{1 + \left(\frac{1}{K} + a_0\right)\dfrac{n}{\frac{1}{K} + a}}{(1-\beta)\frac{1}{K} + a_0} + K}, \tag{H.4}$$

$$p^t \equiv p(\beta, 0, 0) = \frac{\dfrac{b_0 + a_0\dfrac{nb}{a}}{a_0} + T}{\dfrac{1 + a_0\dfrac{n}{a}}{a_0} + K}. \tag{H.5}$$

First, we are going to show that

$$\lim_{\beta\downarrow 0} p^c(\beta) > \lim_{\beta\downarrow 0} p^*(\beta). \tag{H.6}$$

In order to do that, we are going to introduce the notation

$$\tilde{\nu}_0^* = \lim_{\beta\downarrow 0} \nu_0^*(\beta), \tag{H.7}$$

$$\tilde{\nu}^* = \lim_{\beta\downarrow 0} \nu^*(\beta). \tag{H.8}$$

From the equations (30) and (31) of the consistency criterion for our particular case, we can see that $\nu_0^*(\beta), \nu^*(\beta) < \dfrac{1}{K}$ for all $\beta \in (0, 1]$, which, together with the strictly decreasing behavior of $\nu_0^*(\beta)$ and $\nu^*(\beta)$ with respect to $\beta \in (0, 1]$, imply that the values $\tilde{\nu}_0^*$ and $\tilde{\nu}^*$ exist.

Then, we can calculate the difference

$$\begin{aligned}
&\lim_{\beta\downarrow 0} p^c(\beta) - \lim_{\beta\downarrow 0} p^*(\beta) \\
&= \frac{\dfrac{b_0 + \left(\frac{1}{K} + a_0\right)\dfrac{nb}{\frac{1}{K} + a}}{\frac{1}{K} + a_0} + T}{\dfrac{1 + \left(\frac{1}{K} + a_0\right)\dfrac{n}{\frac{1}{K} + a}}{\frac{1}{K} + a_0} + K} - \frac{\dfrac{b_0 + (\tilde{\nu}_0^* + a_0)\dfrac{nb}{\tilde{\nu}^* + a}}{\tilde{\nu}_0^* + a_0} + T}{\dfrac{1 + (\tilde{\nu}_0^* + a_0)\dfrac{n}{\tilde{\nu}^* + a}}{\tilde{\nu}_0^* + a_0} + K} \\
&= \frac{\dfrac{b_0}{\frac{1}{K} + a_0} + \dfrac{nb}{\frac{1}{K} + a} + T}{\dfrac{1}{\frac{1}{K} + a_0} + \dfrac{n}{\frac{1}{K} + a} + K} - \frac{\dfrac{b_0}{\tilde{\nu}_0^* + a_0} + \dfrac{nb}{\tilde{\nu}^* + a} + T}{\dfrac{1}{\tilde{\nu}_0^* + a_0} + \dfrac{n}{\tilde{\nu}^* + a} + K} = \frac{S_1}{S_2},
\end{aligned} \tag{H.9}$$

where

$$S_1 = \left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{b_0}{\frac{1}{K} + a_0} + \frac{nb}{\frac{1}{K} + a} + T\right) - \left(\frac{1}{\frac{1}{K} + a_0} + \frac{n}{\frac{1}{K} + a} + K\right)\left(\frac{b_0}{\tilde{\nu}_0^* + a_0} + \frac{nb}{\tilde{\nu}^* + a} + T\right), \tag{H.10}$$

$$S_2 = \left(\frac{1}{\frac{1}{K} + a_0} + \frac{n}{\frac{1}{K} + a} + K\right)\left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n}{\tilde{\nu}^* + a} + K\right) > 0. \tag{H.11}$$

Now, we analyze the sign of S_1:

$$\begin{aligned}
S_1 &= \left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n}{\tilde{\nu}^* + a}\right)\left(\frac{b_0}{\frac{1}{K} + a_0} + \frac{nb}{\frac{1}{K} + a}\right) \\
&\quad + K\left(\frac{b_0}{\frac{1}{K} + a_0} + \frac{nb}{\frac{1}{K} + a}\right) + \left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n}{\tilde{\nu}^* + a}\right)T \\
&\quad + KT - \left(\frac{1}{\frac{1}{K} + a_0} + \frac{n}{\frac{1}{K} + a}\right)\left(\frac{b_0}{\tilde{\nu}_0^* + a_0} + \frac{nb}{\tilde{\nu}^* + a}\right) \\
&\quad - K\left(\frac{b_0}{\tilde{\nu}_0^* + a_0} + \frac{nb}{\tilde{\nu}^* + a}\right) - \left(\frac{1}{\frac{1}{K} + a_0} + \frac{n}{\frac{1}{K} + a}\right)T \\
&\quad - KT \\
&= -nb\left(\frac{1}{\tilde{\nu}^* + a}\frac{1}{\frac{1}{K} + a_0} - \frac{1}{\tilde{\nu}_0^* + a_0}\frac{1}{\frac{1}{K} + a}\right) \\
&\quad + nb_0\left(\frac{1}{\tilde{\nu}^* + a}\frac{1}{\frac{1}{K} + a_0} - \frac{1}{\tilde{\nu}_0^* + a_0}\frac{1}{\frac{1}{K} + a}\right) \\
&\quad - K\left[nb\left(\frac{1}{\tilde{\nu}^* + a} - \frac{1}{\frac{1}{K} + a}\right) + b_0\left(\frac{1}{\tilde{\nu}_0^* + a_0} - \frac{1}{\frac{1}{K} + a_0}\right)\right] \\
&\quad + T\left[n\left(\frac{1}{\tilde{\nu}^* + a} - \frac{1}{\frac{1}{K} + a}\right) + \left(\frac{1}{\tilde{\nu}_0^* + a_0} - \frac{1}{\frac{1}{K} + a_0}\right)\right] \\
&= nG(b)\left(\frac{1}{\tilde{\nu}^* + a} - \frac{1}{\frac{1}{K} + a}\right) \\
&\quad - n(b - b_0)\left(\frac{1}{\tilde{\nu}^* + a}\frac{1}{\frac{1}{K} + a_0} - \frac{1}{\tilde{\nu}_0^* + a_0}\frac{1}{\frac{1}{K} + a}\right) \\
&\quad + G(b_0)\left(\frac{1}{\tilde{\nu}_0^* + a_0} - \frac{1}{\frac{1}{K} + a_0}\right)
\end{aligned} \tag{H.12}$$

Since the limits $\tilde{\nu}_0^* = \lim_{\beta \downarrow 0} \nu_0^*(\beta)$ and $\tilde{\nu}^* = \lim_{\beta \downarrow 0} \nu^*(\beta)$ exist, it is easy to see that the equations (30) and (31) must hold for the values $\nu_0 = \tilde{\nu}_0^*$ and $\nu = \tilde{\nu}^*$, implying that

$$\tilde{\nu}_0^*, \tilde{\nu}^* < \frac{1}{K}, \tag{H.13}$$

therefore,

$$\frac{1}{\tilde{\nu}_0^* + a_0} > \frac{1}{\frac{1}{K} + a_0}, \tag{H.14}$$

$$\frac{1}{\tilde{\nu}^* + a} > \frac{1}{\frac{1}{K} + a}. \tag{H.15}$$

In addition, by the assumption **A5** and **A6**, we have that

$$G(b) > \frac{b - b_0}{a_0} \geq 0 \tag{H.16}$$

which, together with the fact that $G(p)$ is strictly decreasing with respect to p, imply that

$$G(b_0) \geq G(b) > 0, \tag{H.17}$$

and

$$G(b) - \frac{b - b_0}{\tilde{\nu}_0^* + a_0} > 0. \tag{H.18}$$

Then, by the inequalities (H.14), (H.15), (H.17), (H.18), and the assumption **A5**, we obtain

$$\begin{aligned} S_1 \geq\ & nG(b) \left(\frac{1}{\tilde{\nu}^* + a} - \frac{1}{\frac{1}{K} + a} \right) \\ & - n(b - b_0) \left(\frac{1}{\tilde{\nu}^* + a} \frac{1}{\tilde{\nu}_0^* + a_0} - \frac{1}{\tilde{\nu}_0^* + a_0} \frac{1}{\frac{1}{K} + a} \right) \\ & + G(b_0) \left(\frac{1}{\tilde{\nu}_0^* + a_0} - \frac{1}{\frac{1}{K} + a_0} \right) \\ =\ & n \left(G(b) - \frac{b - b_0}{\tilde{\nu}_0^* + a_0} \right) \left(\frac{1}{\tilde{\nu}^* + a} - \frac{1}{\frac{1}{K} + a} \right) \\ & + G(b_0) \left(\frac{1}{\tilde{\nu}_0^* + a_0} - \frac{1}{\frac{1}{K} + a_0} \right) > 0. \end{aligned} \tag{H.19}$$

Therefore, applying the relationships (H.11) and (H.19) to (H.9), we get

$$\lim_{\beta \downarrow 0} p^c(\beta) - \lim_{\beta \downarrow 0} p^*(\beta) = \frac{S_1}{S_2} > 0. \tag{H.20}$$

In other words, $\lim_{\beta \downarrow 0} p^c(\beta) > \lim_{\beta \downarrow 0} p^*(\beta)$.

Now, we show that

$$\lim_{\beta \downarrow 0} p^*(\beta) > p^t. \tag{H.21}$$

In order to do that, we calculate the difference

$$
\begin{aligned}
&\lim_{\beta \downarrow 0} p^*(\beta) - p^t \\
&= \frac{\dfrac{b_0 + (\tilde{\nu}_0^* + a_0)\dfrac{nb}{\tilde{\nu}^* + a}}{\tilde{\nu}_0^* + a_0} + T}{\dfrac{1 + (\tilde{\nu}_0^* + a_0)\dfrac{n}{\tilde{\nu}^* + a}}{\tilde{\nu}_0^* + a_0} + K} - \frac{\dfrac{b_0 + a_0\dfrac{nb}{a}}{a_0} + T}{\dfrac{1 + a_0\dfrac{n}{a}}{a_0} + K} \\
&= \frac{\dfrac{b_0}{\tilde{\nu}_0^* + a_0} + \dfrac{nb}{\tilde{\nu}^* + a} + T}{\dfrac{1}{\tilde{\nu}_0^* + a_0} + \dfrac{n}{\tilde{\nu}^* + a} + K} - \frac{\dfrac{b_0}{a_0} + \dfrac{nb}{a} + T}{\dfrac{1}{a_0} + \dfrac{n}{a} + K} = \frac{R_1}{R_2},
\end{aligned}
\tag{H.22}
$$

where

$$
\begin{aligned}
R_1 = {} & \left(\frac{1}{a_0} + \frac{n}{a} + K\right)\left(\frac{b_0}{\tilde{\nu}_0^* + a_0} + \frac{nb}{\tilde{\nu}^* + a} + T\right) \\
& - \left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{b_0}{a_0} + \frac{nb}{a} + T\right),
\end{aligned}
\tag{H.23}
$$

$$
R_2 = \left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{1}{a_0} + \frac{n}{a} + K\right) > 0. \tag{H.24}
$$

Now, we analyze the sign of R_1:

$$\begin{aligned}
R_1 = & \left(\frac{1}{a_0}+\frac{n}{a}\right)\left(\frac{b_0}{\tilde{\nu}_0^*+a_0}+\frac{nb}{\tilde{\nu}^*+a}\right)+K\left(\frac{b_0}{\tilde{\nu}_0^*+a_0}+\frac{nb}{\tilde{\nu}^*+a}\right) \\
& +\left(\frac{1}{a_0}+\frac{n}{a}\right)T+KT \\
& -\left(\frac{1}{\tilde{\nu}_0^*+a_0}+\frac{n}{\tilde{\nu}^*+a}\right)\left(\frac{b_0}{a_0}+\frac{nb}{a}\right)-K\left(\frac{b_0}{a_0}+\frac{nb}{a}\right) \\
& -\left(\frac{1}{\tilde{\nu}_0^*+a_0}+\frac{n}{\tilde{\nu}^*+a}\right)T-KT \\
= & -nb\left(\frac{1}{a}\frac{1}{\tilde{\nu}_0^*+a_0}-\frac{1}{a_0}\frac{1}{\tilde{\nu}^*+a}\right)+nb_0\left(\frac{1}{a}\frac{1}{\tilde{\nu}_0^*+a_0}-\frac{1}{a_0}\frac{1}{\tilde{\nu}^*+a}\right) \\
& -K\left[nb\left(\frac{1}{a}-\frac{1}{\tilde{\nu}^*+a}\right)+b_0\left(\frac{1}{a_0}-\frac{1}{\tilde{\nu}_0^*+a_0}\right)\right] \\
& +T\left[n\left(\frac{1}{a}-\frac{1}{\tilde{\nu}^*+a}\right)+\left(\frac{1}{a_0}-\frac{1}{\tilde{\nu}_0^*+a_0}\right)\right] \\
= & \, nG(b)\left(\frac{1}{a}-\frac{1}{\tilde{\nu}^*+a}\right) \\
& -n(b-b_0)\left(\frac{1}{a}\frac{1}{\tilde{\nu}_0^*+a_0}-\frac{1}{a_0}\frac{1}{\tilde{\nu}^*+a}\right) \\
& +G(b_0)\left(\frac{1}{a_0}-\frac{1}{\tilde{\nu}_0^*+a_0}\right)
\end{aligned} \tag{H.25}$$

Again, from the equations (30) and (31), it is easy to see that

$$\tilde{\nu}_0^*, \tilde{\nu}^* > 0, \tag{H.26}$$

therefore,

$$\frac{1}{a_0} > \frac{1}{\tilde{\nu}_0^*+a_0}, \tag{H.27}$$

$$\frac{1}{a} > \frac{1}{\tilde{\nu}^*+a}, \tag{H.28}$$

and, from inequality (H.16), we have that

$$G(b) - \frac{b-b_0}{a_0} > 0. \tag{H.29}$$

Then, by the inequalities (H.17), (H.27), (H.28), (H.29), and the assumption

A5, we obtain

$$\begin{aligned}R_1 &\geq nG(b)\left(\frac{1}{a}-\frac{1}{\tilde{\nu}^*+a}\right)-n(b-b_0)\left(\frac{1}{a}\frac{1}{a_0}-\frac{1}{a_0}\frac{1}{\tilde{\nu}^*+a}\right)\\&\quad+G(b_0)\left(\frac{1}{a_0}-\frac{1}{\tilde{\nu}_0^*+a_0}\right)\\&=n\left(G(b)-\frac{b-b_0}{a_0}\right)\left(\frac{1}{a}-\frac{1}{\tilde{\nu}^*+a}\right)\\&\quad+G(b_0)\left(\frac{1}{a_0}-\frac{1}{\tilde{\nu}_0^*+a_0}\right)>0.\end{aligned}\tag{H.30}$$

Therefore, applying the relationships (H.24) and (H.30) to (H.22), we get

$$\lim_{\beta\downarrow 0}p^*(\beta)-p^t=\frac{R_1}{R_2}>0.\tag{H.31}$$

In other words, $\lim_{\beta\downarrow 0}p^*(\beta)>p^t$ ■

Appendix I. Proof of Theorem 8

Theorem 8. *Under the assumptions **A4**, **A5** and **A6**, for any $\beta\in(0,1]$, if $\pi^c(\beta)\geq\pi^*(\beta)$, then, it is satisfied that $p^*(\beta)<p^c(\beta)$.*

Proof: Let $\beta\in(0,1]$ such that $\pi^c(\beta)\geq\pi^*(\beta)$.

First, we substitute the value of $q^*=q^*(\beta)=q(p^*(\beta),\nu^*(\beta))$, given by (E.25), in the expression of $\pi^*(\beta)=\pi(p^*(\beta),q^*(\beta))$, given by (26):

$$\begin{aligned}\pi^*(\beta)&=p^*q^*-\frac{1}{2}aq^{*2}-bq^*=\left(p^*-b-\frac{1}{2}aq^*\right)q^*\\&=\left[(\nu^*+a)\frac{p^*-b}{\nu^*+a}-\frac{1}{2}a\frac{p^*-b}{\nu^*+a}\right]\frac{p^*-b}{\nu^*+a}\\&=\left(\nu^*+\frac{1}{2}a\right)\left(\frac{p^*-b}{\nu^*+a}\right)^2.\end{aligned}\tag{I.1}$$

Similarly, we can get

$$\pi^c(\beta)=p^cq^c-\frac{1}{2}aq^{c2}-bq^c=\left(\frac{1}{K}+\frac{1}{2}a\right)\left(\frac{p^c-b}{\frac{1}{K}+a}\right)^2.\tag{I.2}$$

Now, suppose that $p^*(\beta) \geq p^c(\beta)$, then,

$$\pi^*(\beta) - \pi^c(\beta) = \left(\nu^* + \frac{1}{2}a\right)\left(\frac{p^* - b}{\nu^* + a}\right)^2 - \left(\frac{1}{K} + \frac{1}{2}a\right)\left(\frac{p^c - b}{\frac{1}{K} + a}\right)^2$$

$$\geq \left(\nu^* + \frac{1}{2}a\right)\left(\frac{p^c - b}{\nu^* + a}\right)^2 - \left(\frac{1}{K} + \frac{1}{2}a\right)\left(\frac{p^c - b}{\frac{1}{K} + a}\right)^2$$

$$= \left[\frac{\nu^* + \frac{1}{2}a}{(\nu^* + a)^2} - \frac{\frac{1}{K} + \frac{1}{2}a}{\left(\frac{1}{K} + a\right)^2}\right](p^c - b)^2 \tag{I.3}$$

$$= \frac{\left[\left(\frac{1}{K} + a\right)^2\left(\nu^* + \frac{1}{2}a\right) - (\nu^* + a)^2\left(\frac{1}{K} + \frac{1}{2}a\right)\right]}{(\nu^* + a)^2\left(\frac{1}{K} + a\right)^2}(p^c - b)^2$$

$$= \frac{\left(\frac{1}{K} - \nu^*\right)\left(\frac{1}{K}\nu^* + \frac{1}{2}\nu^* a + \frac{1}{2}\frac{1}{K}a\right)}{(\nu^* + a)^2\left(\frac{1}{K} + a\right)^2}(p^c - b)^2 > 0,$$

which implies that $\pi^c(\beta) < \pi^*(\beta)$, contradicting the hypothesis $\pi^c(\beta) \geq \pi^*(\beta)$.

Therefore, the assumption $p^*(\beta) \geq p^c(\beta)$ is false, i.e., $p^*(\beta) < p^c(\beta)$, proving the theorem■

Appendix J. Proof of Theorem 9

Theorem 9. *Suppose that assumptions **A4**, **A5** and **A6** are true. If the relationship*

$$\frac{(n-1)a}{n + aK} \geq a_0 \tag{41}$$

is valid, then, there exists the value $\hat{\beta} \in (0, 1)$ *such that* $\pi^c(\hat{\beta}) = \pi^*(\hat{\beta})$ *and* $p^*(\hat{\beta}) < p^c(\hat{\beta})$.

Proof: We see that the inequality (41) implies that

$$a > a_0, \tag{J.1}$$

since we consider $n \geq 2$.

Let $\tilde{\nu}_0^* = \lim_{\beta \downarrow 0} \nu_0^*(\beta)$ and $\tilde{\nu}^* = \lim_{\beta \downarrow 0} \nu^*(\beta)$ be the same limits introduced in (H.7) and (H.8) from the proof of Theorem 7.

From the equalities (30) and (31), we have that

$$\begin{aligned}
\tilde{\nu}_0^* - \tilde{\nu}^* &= \frac{1}{\frac{n}{\tilde{\nu}^* + a} + K} - \frac{1}{\frac{1}{(1-\beta)\tilde{\nu}_0^* + a_0} + \frac{\tilde{\nu}_0^* + a_0}{(1-\beta)\tilde{\nu}_0^* + a_0}\frac{n-1}{\tilde{\nu}^* + a} + K} \\
&> \frac{1}{\frac{n}{\tilde{\nu}^* + a} + K} - \frac{1}{\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n-1}{\tilde{\nu}^* + a} + K} \\
&= \frac{\left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n-1}{\tilde{\nu}^* + a} + K\right) - \left(\frac{n}{\tilde{\nu}^* + a} + K\right)}{\left(\frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n-1}{\tilde{\nu}^* + a} + K\right)} \\
&= \frac{\frac{1}{\tilde{\nu}_0^* + a_0} - \frac{1}{\tilde{\nu}^* + a}}{\left(\frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n-1}{\tilde{\nu}^* + a} + K\right)} \\
&= \frac{-(\tilde{\nu}_0^* - \tilde{\nu}^*) + (a - a_0)}{(\tilde{\nu}_0^* + a_0)(\tilde{\nu}^* + a)\left(\frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n-1}{\tilde{\nu}^* + a} + K\right)} \\
&> \frac{-(\tilde{\nu}_0^* - \tilde{\nu}^*)}{(\tilde{\nu}_0^* + a_0)(\tilde{\nu}^* + a)\left(\frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n-1}{\tilde{\nu}^* + a} + K\right)}.
\end{aligned} \tag{J.2}$$

Then,

$$\tilde{\nu}_0^* - \tilde{\nu}^* > \frac{-(\tilde{\nu}_0^* - \tilde{\nu}^*)}{(\tilde{\nu}_0^* + a_0)(\tilde{\nu}^* + a)\left(\frac{n}{\tilde{\nu}^* + a} + K\right)\left(\frac{1}{\tilde{\nu}_0^* + a_0} + \frac{n-1}{\tilde{\nu}^* + a} + K\right)}. \tag{J.3}$$

Since the denominator on the right-hand side of inequality (J.3) is strictly positive, then, inequality (J.3) can only be true if

$$\tilde{\nu}_0^* > \tilde{\nu}^*. \tag{J.4}$$

Now, substituting the value of $p^*(\beta)$, given by (H.3), we get

$$\frac{p^* - b}{\nu^* + a} = \frac{\left[\dfrac{\dfrac{b_0 + (\nu_0^* + a_0)\dfrac{nb}{\nu^* + a}}{(1-\beta)\nu_0^* + a_0} + T}{\dfrac{1 + (\nu_0^* + a_0)\dfrac{n}{\nu^* + a}}{(1-\beta)\nu_0^* + a_0} + K}\right] - b}{\nu^* + a}$$

$$= \frac{\left[\dfrac{b_0 + (\nu_0^* + a_0)\dfrac{nb}{\nu^* + a}}{(1-\beta)\nu_0^* + a_0} + T\right] - b\left[\dfrac{1 + (\nu_0^* + a_0)\dfrac{n}{\nu^* + a}}{(1-\beta)\nu_0^* + a_0} + K\right]}{(\nu^* + a)\left[\dfrac{1 + (\nu_0^* + a_0)\dfrac{n}{\nu^* + a}}{(1-\beta)\nu_0^* + a_0} + K\right]} \tag{J.5}$$

$$= \frac{G(b) - \dfrac{b - b_0}{(1-\beta)\nu_0^* + a_0}}{(\nu^* + a)\left[\dfrac{1 + (\nu_0^* + a_0)\dfrac{n}{\nu^* + a}}{(1-\beta)\nu_0^* + a_0} + K\right]}$$

$$= \frac{[(1-\beta)\nu_0^* + a_0]G(b) - (b - b_0)}{n(\nu_0^* + a_0) + (\nu^* + a) + [(1-\beta)\nu_0^* + a_0](\nu^* + a)K}$$

$$= \frac{[(1-\beta)\nu_0^* + a_0]G(b) - (b - b_0)}{n(\nu_0^* + a_0) + \{1 + [(1-\beta)\nu_0^* + a_0]K\}(\nu^* + a)}.$$

Substituting the relationship (J.5), in (I.1), we obtain

$$\pi^*(\beta) = \left(\nu^* + \frac{1}{2}a\right)\left(\frac{[(1-\beta)\nu_0^* + a_0]G(b) - (b - b_0)}{n(\nu_0^* + a_0) + \{1 + [(1-\beta)\nu_0^* + a_0]K\}(\nu^* + a)}\right)^2. \tag{J.6}$$

Similarly, we can find that

$$\pi^c(\beta) = \left(\frac{1}{K} + \frac{1}{2}a\right)\left(\frac{[(1-\beta)\frac{1}{K} + a_0]G(b) - (b - b_0)}{n(\frac{1}{K} + a_0) + \{1 + [(1-\beta)\frac{1}{K} + a_0]K\}(\frac{1}{K} + a)}\right)^2. \tag{J.7}$$

To prove the assertion of the theorem, first, we are going to show that

$$\pi^*(1) > \pi^c(1). \tag{J.8}$$

Here, we introduce the notation

$$\overline{\nu}_0^* = \nu_0^*(1), \tag{J.9}$$

$$\overline{\nu}^* = \nu^*(1). \tag{J.10}$$

By the relationships (J.6) and (J.7), proving inequality (J.8) is equivalent to proving the following relationship:

$$\begin{aligned} &\left(\overline{\nu}^* + \frac{1}{2}a\right)\left[\frac{a_0 G(b) - (b - b_0)}{n(\overline{\nu}_0^* + a_0) + (1 + a_0 K)(\overline{\nu}^* + a)}\right]^2 \\ &\quad > \left(\frac{1}{K} + \frac{1}{2}a\right)\left[\frac{a_0 G(b) - (b - b_0)}{n(\frac{1}{K} + a_0) + (1 + a_0 K)(\frac{1}{K} + a)}\right]^2, \end{aligned} \tag{J.11}$$

which in turn is equivalent to showing that

$$\begin{aligned} &\frac{\left(\overline{\nu}^* + \frac{1}{2}a\right)}{\left[n(\overline{\nu}_0^* + a_0) + (1 + a_0 K)(\overline{\nu}^* + a)\right]^2} \\ &\quad > \frac{\left(\frac{1}{K} + \frac{1}{2}a\right)}{\left[n(\frac{1}{K} + a_0) + (1 + a_0 K)(\frac{1}{K} + a)\right]^2}. \end{aligned} \tag{J.12}$$

Next, define the function

$$\begin{aligned} \Psi^1(t) &= \frac{\left[\overline{\nu}^* + t\left(\frac{1}{K} - \overline{\nu}^*\right) + \frac{1}{2}a\right]}{\left\{n[\overline{\nu}_0^* + t\left(\frac{1}{K} - \overline{\nu}_0^*\right) + a_0] + (1 + a_0 K)[\overline{\nu}^* + t\left(\frac{1}{K} - \overline{\nu}^*\right) + a]\right\}^2} \\ &= \frac{P_1(t)}{P_2(t)^2}, \end{aligned} \tag{J.13}$$

where

$$P_1(t) = \overline{\nu}^* + t\left(\frac{1}{K} - \overline{\nu}^*\right) + \frac{1}{2}a > 0, \tag{J.14}$$

$$\begin{aligned} P_2(t) = n &\left[\overline{\nu}_0^* + t\left(\frac{1}{K} - \overline{\nu}_0^*\right) + a_0\right] \\ &+ (1 + a_0 K)\left[\overline{\nu}^* + t\left(\frac{1}{K} - \overline{\nu}^*\right) + a\right] > 0, \end{aligned} \tag{J.15}$$

so proving (J.12) is equivalent to proving

$$\Psi^1(0) > \Psi^1(1). \tag{J.16}$$

We can easily see that $P_1(t)$, $P_2(t)$ and $\Psi^1(t)$, are differentiable for each $t \in [0, 1]$, and $\Psi^{1\prime}(t)$ is given by

$$\Psi^{1\prime}(t) = \frac{P_1'(t)P_2(t)^2 - 2P_1(t)P_2(t)P_2'(t)}{P_2(t)^4} = \frac{P_1'(t)P_2(t) - 2P_1(t)P_2'(t)}{P_2(t)^3}, \tag{J.17}$$

where

$$P_1'(t) = \frac{1}{K} - \overline{\nu}^* > 0, \tag{J.18}$$

$$P_2(t) = n\left(\frac{1}{K} - \overline{\nu}_0^*\right) + (1 + a_0K)\left(\frac{1}{K} - \overline{\nu}^*\right) > 0. \tag{J.19}$$

Now, we are going to analyze the sign of $P_1'(t)P_2(t)^2 - 2P_1(t)P_2(t)P_2'(t)$:

$$\begin{aligned} P_1'(t)P_2(t) - 2P_1(t)P_2'(t) &= \left(\frac{1}{K} - \overline{\nu}^*\right)\left\{n\left[\overline{\nu}_0^* + t\left(\frac{1}{K} - \overline{\nu}_0^*\right) + a_0\right]\right. \\ &\left. +(1 + a_0K)\left[\overline{\nu}^* + t\left(\frac{1}{K} - \overline{\nu}^*\right) + a\right]\right\} \\ &- 2\left[\overline{\nu}^* + t\left(\frac{1}{K} - \overline{\nu}^*\right) + \frac{1}{2}a\right]\left[n\left(\frac{1}{K} - \overline{\nu}_0^*\right)\right. \\ &\left. +(1 + a_0K)\left(\frac{1}{K} - \overline{\nu}^*\right)\right] \\ &= n\left[(\overline{\nu}_0^* + a_0)\left(\frac{1}{K} - \overline{\nu}^*\right) - (2\overline{\nu}^* + a)\left(\frac{1}{K} - \overline{\nu}_0^*\right)\right] \\ &- \overline{\nu}^*(1 + a_0K)\left(\frac{1}{K} - \overline{\nu}^*\right) \\ &- t\left(\frac{1}{K} - \overline{\nu}^*\right)\left[n\left(\frac{1}{K} - \overline{\nu}_0^*\right) + (1 + a_0K)\left(\frac{1}{K} - \overline{\nu}^*\right)\right]. \end{aligned} \tag{J.20}$$

Using the value of $\overline{\nu}_0^*$ given by (30) we find the identity

$$\overline{\nu}_0^* = \frac{1}{\dfrac{n}{\overline{\nu}^* + a} + K} = \frac{\overline{\nu}^* + a}{n + (\overline{\nu}^* + a)K}, \tag{J.21}$$

so we can define a new function $\psi^1(\overline{\nu}^*)$ as follows:

$$\begin{aligned} &n\left[(\overline{\nu}_0^* + a_0)\left(\frac{1}{K} - \overline{\nu}^*\right) - (2\overline{\nu}^* + a)\left(\frac{1}{K} - \overline{\nu}_0^*\right)\right] \\ &- \overline{\nu}^*(1 + a_0K)\left(\frac{1}{K} - \overline{\nu}^*\right) \\ &= n\left\{\left[\frac{\overline{\nu}^* + a}{n + (\overline{\nu}^* + a)K} + a_0\right]\left(\frac{1}{K} - \overline{\nu}^*\right)\right. \\ &\left. -(2\overline{\nu}^* + a)\left[\frac{1}{K} - \frac{\overline{\nu}^* + a}{n + (\overline{\nu}^* + a)K}\right]\right\} - \overline{\nu}^*(1 + a_0K)\left(\frac{1}{K} - \overline{\nu}^*\right) \\ &= \frac{n}{n + (\overline{\nu}^* + a)K}\left(\{(\overline{\nu}^* + a) + a_0[n + (\overline{\nu}^* + a)K]\}\left(\frac{1}{K} - \overline{\nu}^*\right)\right. \\ &\left. -(2\overline{\nu}^* + a)\left\{\frac{1}{K}[n + (\overline{\nu}^* + a)K] - (\overline{\nu}^* + a)\right\}\right) \\ &- \overline{\nu}^*(1 + a_0K)\left(\frac{1}{K} - \overline{\nu}^*\right) = \psi^1(\overline{\nu}^*). \end{aligned} \tag{J.22}$$

Then,

$$
\begin{aligned}
&[n+(\overline{\nu}^*+a)K]\psi^1(\overline{\nu}^*)\\
&= n\left(\{(\overline{\nu}^*+a)+a_0[n+(\overline{\nu}^*+a)K]\}\left(\frac{1}{K}-\overline{\nu}^*\right)\right.\\
&\quad \left.-(2\overline{\nu}^*+a)\left\{\frac{1}{K}[n+(\overline{\nu}^*+a)K]-(\overline{\nu}^*+a)\right\}\right)\\
&\quad -\overline{\nu}^*(1+a_0K)\left(\frac{1}{K}-\overline{\nu}^*\right)[n+(\overline{\nu}^*+a)K]=n\psi_1^1(\overline{\nu}^*)+\psi_2^1(\overline{\nu}^*),
\end{aligned}
\tag{J.23}
$$

where

$$
\begin{aligned}
\psi_1^1(\overline{\nu}^*) &= \{(\overline{\nu}^*+a)+a_0[n+(\overline{\nu}^*+a)K]\}\left(\frac{1}{K}-\overline{\nu}^*\right)\\
&\quad -(2\overline{\nu}^*+a)\left\{\frac{1}{K}[n+(\overline{\nu}^*+a)K]-(\overline{\nu}^*+a)\right\}
\end{aligned}
\tag{J.24}
$$

$$
\psi_2^1(\overline{\nu}^*) = -\overline{\nu}^*(1+a_0K)\left(\frac{1}{K}-\overline{\nu}^*\right)[n+(\overline{\nu}^*+a)K]<0. \tag{J.25}
$$

We can easily see that $\psi_1^1(\overline{\nu}^*)$ is a quadratic polynomial of $\overline{\nu}^*$, whose derivatives are

$$
\begin{aligned}
{\psi_1^1}'(\overline{\nu}^*) &= (1+a_0K)\left(\frac{1}{K}-\overline{\nu}^*\right)-\{(\overline{\nu}^*+a)+a_0[n+(\overline{\nu}^*+a)K]\}\\
&\quad -2\left\{\frac{1}{K}[n+(\overline{\nu}^*+a)K]-(\overline{\nu}^*+a)\right\}
\end{aligned}
\tag{J.26}
$$

$$
{\psi_1^1}''(\overline{\nu}^*) = -2(1+a_0K). \tag{J.27}
$$

Making use of the Taylor series, we obtain

$$
\begin{aligned}
\psi_1^1(\overline{\nu}^*) &= \psi_1^1(0)+{\psi_1^1}'(0)\overline{\nu}^*+\frac{1}{2}{\psi_1^1}''(0){\overline{\nu}^*}^2\\
&= \frac{1}{K}[a_0(n+aK)-(n-1)a]\\
&\quad -\left[(2n-1)\frac{1}{K}+(n-1)a_0+a(1+a_0K)\right]\overline{\nu}^*\\
&\quad -(1+a_0K){\overline{\nu}^*}^2.
\end{aligned}
\tag{J.28}
$$

Now, from the assumption (41), we have that

$$
a_0(n+aK)-(n-1)a\le 0. \tag{J.29}
$$

Then, from the inequality (J.29), the equality (J.28) implies that

$$
\psi_1^1(\overline{\nu}^*)<0. \tag{J.30}
$$

Applying the inequalities (J.25) and (J.30), to identity (J.23), we get the relationship

$$[n+(\overline{\nu}^*+a)K]\psi^1(\overline{\nu}^*)=n\psi^1_1(\overline{\nu}^*)+\psi^1_2(\overline{\nu}^*)<0. \tag{J.31}$$

Moreover, since $[n+(\overline{\nu}^*+a)K]>0$, it must hold that

$$\psi^1(\overline{\nu}^*)<0, \tag{J.32}$$

thus, applying inequality (J.32) to equality (J.22), proves that

$$\begin{aligned} n&\left[(\overline{\nu}^*_0+a_0)\left(\frac{1}{K}-\overline{\nu}^*\right)-(2\overline{\nu}^*+a)\left(\frac{1}{K}-\overline{\nu}^*_0\right)\right]\\ &-\overline{\nu}^*(1+a_0K)\left(\frac{1}{K}-\overline{\nu}^*\right)<0. \end{aligned} \tag{J.33}$$

Therefore, from the relationships (J.20) and (J.33) we see that

$$P_1'(t)P_2(t)-2P_1(t)P_2'(t)<0, \tag{J.34}$$

which, together with (J.17) and (J.19), imply that

$$\Psi^{1'}(t)<0, \quad \forall t\in[0,1], \tag{J.35}$$

so the function $\Psi^1(t)$ is strictly decreasing for all $t\in[0,1]$, then, the inequality (J.16) is valid, which finally proves (J.8).

Second, we are going to prove that

$$\lim_{\beta\downarrow 0}\pi^*(\beta)<\lim_{\beta\downarrow 0}\pi^c(\beta). \tag{J.36}$$

Again, making use of equations (J.6) and (J.7), we see that proving (J.36) is equivalent to showing that

$$\begin{aligned} &\left(\tilde{\nu}^*+\frac{1}{2}a\right)\left\{\frac{(\tilde{\nu}^*_0+a_0)G(b)-(b-b_0)}{n(\tilde{\nu}^*_0+a_0)+[1+(\tilde{\nu}^*_0+a_0)K](\tilde{\nu}^*+a)}\right\}^2\\ &<\left(\frac{1}{K}+\frac{1}{2}a\right)\left\{\frac{(\frac{1}{K}+a_0)G(b)-(b-b_0)}{n(\frac{1}{K}+a_0)+[1+(\frac{1}{K}+a_0)K](\frac{1}{K}+a)}\right\}^2. \end{aligned} \tag{J.37}$$

Now, consider the function

$$\Psi^2(\nu_0,\nu)=\left(\nu+\frac{1}{2}a\right)\left\{\frac{(\nu_0+a_0)G(b)-(b-b_0)}{n(\nu_0+a_0)+[1+(\nu_0+a_0)K](\nu+a)}\right\}^2, \tag{J.38}$$

where $\nu_0, \nu \in [0, \frac{1}{K}]$, so proving (J.37) is equivalent to proving

$$\Psi^2(\tilde{\nu}_0^*, \tilde{\nu}^*) < \Psi^2\left(\frac{1}{K}, \frac{1}{K}\right). \tag{J.39}$$

It is easy to see that $\Psi^2(\nu_0, \nu)$ is differentiable for all $\nu_0, \nu \in [0, \frac{1}{K}]$.

Next, define the differentiable function

$$W_0(\nu_0, \nu) = \frac{(\nu_0 + a_0)G(b) - (b - b_0)}{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)}, \tag{J.40}$$

which, due to inequality (H.16), satisfies

$$W_0(\nu_0, \nu) > 0. \tag{J.41}$$

Using equation (J.40), we can rewrite the expression of $\Psi^2(\nu_0, \nu)$, given by (J.38), as follows:

$$\Psi^2(\nu_0, \nu) = \left(\nu + \frac{1}{2}a\right)[W_0(\nu_0, \nu)]^2. \tag{J.42}$$

Then, by the chain rule

$$\frac{\partial \Psi^2}{\partial \nu_0} = 2\left(\nu + \frac{1}{2}a\right) W_0(\nu_0, \nu)\frac{\partial W_0}{\partial \nu_0}, \tag{J.43}$$

where

$$\begin{aligned} \frac{\partial W_0}{\partial \nu_0} &= \frac{G(b)\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^2} \\ &\quad - \frac{[(\nu_0 + a_0)G(b) - (b - b_0)][n + (\nu + a)K]}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^2} \\ &= \frac{G(b)(\nu + a) + (b - b_0)[n + (\nu + a)K]}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^2} > 0. \end{aligned} \tag{J.44}$$

Applying inequalities (J.41) and (J.44) to equation (J.43) gives us the relationship

$$\frac{\partial \Psi^2}{\partial \nu_0} > 0, \tag{J.45}$$

which proves that $\Psi^2(\nu_0, \nu)$ is strictly increasing with respect to $\nu_0 \in [0, \frac{1}{K}]$ for all $\nu \in [0, \frac{1}{K}]$.

In the same manner, we define the differentiable function

$$W(\nu_0, \nu) = \frac{(\nu + \frac{1}{2}a)}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^2} > 0, \tag{J.46}$$

Using equation (J.46), we can rewrite the expression of $\Psi^2(\nu_0, \nu)$, given by (J.38), as follows:

$$\Psi^2(\nu_0, \nu) = [(\nu_0 + a_0)G(b) - (b - b_0)]^2 W(\nu_0, \nu). \tag{J.47}$$

Again, by the chain rule

$$\frac{\partial \Psi^2}{\partial \nu} = [(\nu_0 + a_0)G(b) - (b - b_0)]^2 \frac{\partial W}{\partial \nu}, \tag{J.48}$$

where

$$\begin{aligned}
\frac{\partial W}{\partial \nu} &= \frac{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^2}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^4} \\
&\quad - \frac{2\left(\nu + \frac{1}{2}a\right)\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}[1 + (\nu_0 + a_0)K]}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^4} \\
&= \frac{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a) - 2\left(\nu + \frac{1}{2}a\right)[1 + (\nu_0 + a_0)K]}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^3} \\
&= \frac{n(\nu_0 + a_0) - \nu - \nu(\nu_0 + a_0)K}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^3} \\
&= \frac{(\frac{1}{K} - \nu)(\nu_0 + a_0)K + (n-1)(\nu_0 + a_0) - \nu}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^3} \\
&> \frac{(\frac{1}{K} - \nu)(\nu_0 + a_0)K + (\nu_0 - \nu)}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^3}.
\end{aligned} \tag{J.49}$$

Then, we can see that

$$\frac{\partial W}{\partial \nu} > \frac{(\frac{1}{K} - \nu)(\nu_0 + a_0)K + (\nu_0 - \nu)}{\{n(\nu_0 + a_0) + [1 + (\nu_0 + a_0)K](\nu + a)\}^3} \geq 0, \quad \forall \nu_0 \geq \nu. \tag{J.50}$$

Applying inequalities (H.16) and (J.50) to equation (J.48) gives us the relationship

$$\frac{\partial \Psi^2}{\partial \nu} > 0, \quad \forall \nu_0 \geq \nu, \tag{J.51}$$

which proves that $\Psi^2(\nu_0, \nu)$ is strictly increasing with respect to $\nu \in [0, \nu_0]$ for all $\nu_0 \in (0, \frac{1}{K}]$.

Finally, from inequalities (H.13), (H.26) and (J.4), we have that $0 < \tilde{\nu}^* < \tilde{\nu}_0^* < \dfrac{1}{K}$, then, by the strictly increasing behavior of $\Psi^2(\nu_0, \nu)$ with respect to $\nu_0 \in (0, \frac{1}{K}]$ and $\nu \in [0, \nu_0]$, we can conclude that

$$\Psi^2(\tilde{\nu}_0^*, \tilde{\nu}^*) < \Psi^2\left(\frac{1}{K}, \tilde{\nu}^*\right) < \Psi^2\left(\frac{1}{K}, \frac{1}{K}\right), \tag{J.52}$$

so inequality (J.39) is valid, which finally proves (J.36).

Therefore, since the functions $\pi^*(\beta)$ and $\pi^c(\beta)$ are continuous for all $\beta \in (0,1]$, and the inequalities $\pi^*(1) > \pi^c(1)$ and $\lim_{\beta \downarrow 0} \pi^*(\beta) < \lim_{\beta \downarrow 0} \pi^c(\beta)$ are valid, by the Intermediate Value Theorem we can grantee the existence of a value $\hat{\beta} \in (0,1)$ such that $\pi^*(\hat{\beta}) = \pi^c(\hat{\beta})$.

Finally by Theorem 8, for this value $\hat{\beta}$, it must hold that $p^*(\hat{\beta}) < p^c(\hat{\beta})$, which finishes the proof■