

Research Article

An Application of Homotopy Perturbation Method to Fractional-Order Thin Film Flow of the Johnson–Segalman Fluid Model

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Thin film flow is an important theme in fluid mechanics and has many industrial applications. These flows can be observed in oil refinement process, laser cutting, and nuclear reactors. In this theoretical study, we explore thin film flow of non-Newtonian Johnson–Segalman fluid on a vertical belt in fractional space in lifting and drainage scenarios. Modelled fractional-order boundary value problems are solved numerically using the homotopy perturbation method along with Caputo definition of fractional derivative. In this study, instantaneous and average velocities and volumetric flux are computed in lifting and drainage cases. Validity and convergence of homotopy-based solutions are confirmed by finding residual errors in each case. Moreover, the consequences of different fractional and fluid parameters are graphically studied on the velocity profile. Analysis shows that fractional parameters have opposite effects of the fluid velocity.

1. Introduction

Modelling and analysis of non-Newtonian fluids is an important and active research theme in industrial engineering. Food processing, paper production, blood flow analysis, and mud drilling are different applications areas of non-Newtonian fluids. These fluids are defined through a nonlinear relationship between rate of deformation and stress tensors, and therefore, it has several models in different scenarios. Johnson–Segalman fluid is one of the very significant fluid models which have numerous engineering and industrial applications.

Thin film flow can be detected in different natural situations, for example, movement of raindrop on window glass, tears in the eyes, and lava flow. Industrial applications

of such flows include oil refining, nuclear reactors, and laser cutting [1–4]. The initial work on thin film flow was carried out in [3] for Newtonian fluids, but this study has limitations and cannot be generalized for non-Newtonian fluids such as melted plastics, gels, pastes, honey, ketchup, and blood [5]. Siddiqui et al. examined thin film of different fluids including PTT (Phan-Thien and Tanner) and third and fourth grade fluids in [6–8]. Landau [9] and Stuart [10] extended these analyses to turbulence. Ullah et al. studied the film flow under slip conditions in generalized Maxwell fluids in [11]. Ruan et al. studied thin film from a distributed source on a vertical wall in [12]. Ahmad and Xu proposed an improved nanofluid model in thin film under the action of gravity in [13].

Fractional calculus has gained significant attention due to its striking application as a new modelling tool in a variety of fields, such as fluid dynamics, hydrology, system control theory, signal processing, physics, biology, and finance [14–20]. These fractional models are more appropriate for describing memory and transmissible properties of different materials than integral-order models. In the past few decades, various numerical techniques have been developed by different researchers for nonlinear BVPs. Wazwaz proposed the modified decomposition method (MDM) for BVPs in [21]. Noor and Mohyud-Din used VIM along with He's polynomials for the solution of higher order BVPs [22]. Liu et al. used the multiscale method for nanoparticle diffusion in sheared cellular blood flow in [23]. Jafari and Gejji used adomian decomposition for the solution of system of FDEs in [24]. Hashim et al. apply OHAM to fractional-order fuzzy differential equations in [22]. Rysak and Gregorczyk proposed DTM for fractional dynamical systems in [25]. Zada et al. apply NIM to fractional PDEs in [26]. Yaghouti used radial basis functions to different families of fractional differential equations in [27]. Al-Kuze et al. used spectral quasi-linearization and irreversibility analysis of magnetized cross-fluid flow [28].

In this study, we extend the theoretical study of thin film flow to fractional space in case of non-Newtonian Johnson–Segalman fluid since exact solutions of highly nonlinear fractional differential equations (FDEs) are not possible. The conventional approach is to use perturbation techniques in such scenarios, which again need small or large parameters. To avoid this, we utilize the well-known homotopy perturbation method (HPM) [29–33] along with fractional calculus for solution purpose. This method is effectively used by many scholars in [29–33]. Validation and convergence of the obtained numerical solutions are confirmed by means of finding residual errors. To the best of the authors' knowledge, the given problem has not been attempted before in fractional space.

2. Preliminaries

2.1. Fractional Calculus. Few basic definitions of fractional calculus are given below.

Definition 1. Let $f(t), t > 0$, be a real function in the space $C_\mu, \mu \in \mathbb{R}$, if \exists a real number $p > \mu$ $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is in the space

$$C_\mu^m \Leftrightarrow f^m \in C_\mu, m \in \mathbb{N}. \quad (1)$$

Definition 2. The Caputo fractional derivative D^α is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^m d\tau, \text{ for } m \quad (2)$$

$$-1 < \alpha < m, m \in \mathbb{N}, t > 0.$$

Definition 3. $\alpha \geq 0$ order Riemann–Liouville fractional integration is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, k-1 < \alpha < k, \text{ and } k \in \mathbb{N}. \quad (3)$$

2.2. Basic Equations. The basic equations of incompressible fluid of Johnson–Segalman fluid are

$$\operatorname{div} \mathcal{V} = 0, \quad (4)$$

$$\rho \frac{D\mathcal{V}}{Dt} = \operatorname{div} \sigma + \rho f, \quad (5)$$

where \mathcal{V} is the velocity vector, ρ is the constant density, f is the body force per unit mass, and σ is the Cauchy stress tensor, where

$$\sigma = -p\mathbf{I} + \mathbf{T}, \quad (6)$$

$$\mathbf{T} = \mathbf{S} + 2\mu\mathbf{D}, \quad (7)$$

$$\mathbf{S} + m \left[(\mathbf{W} - a\mathbf{D})^T \mathbf{S} + \mathbf{S}(\mathbf{W} - a\mathbf{D}) + \frac{D\mathbf{S}}{Dt} \right] = 2\mathbf{D}\eta. \quad (8)$$

3. Mathematical Formulation in Lifting Case

The belt is passing through the container filled with Johnson–Segalman fluid, and it is moving vertically with constant speed U_0 . A uniform thickness δ of the thin film is taken up by the belt, but due to gravity fluid, it is draining down. Let pressure be assumed to be atmospheric, and the flow is uniform, steady, and laminar. Also, the x -axis is considered to be normal, while the y -axis is along the belt.

The boundary conditions are

$$\begin{aligned} x &= 0, \\ v &= U_0, \\ x &= \delta, \\ T_{xy} &= 0. \end{aligned} \quad (9)$$

Velocity is of the form

$$\mathcal{V} = (0, v(x), 0), \quad (10)$$

$$\mathbf{T} = \mathbf{T}(x) \text{ and } \mathbf{T}_{xz}, \quad (11)$$

where the extra and shear stress tensor is $\mathbf{T} = \mathbf{T}(x)$ and \mathbf{T}_{xz} , respectively.

Substitution of (8) in (3) and (4) shows that (3) is satisfied, while (4) has the form

$$0 = \frac{dT_{xx}}{dx} + \rho f_1, \quad (12)$$

$$0 = \frac{dT_{xy}}{dx} + \rho f_2, \quad (13)$$

where f_1 and f_2 are the components of body force.

Since gravitational force is in downward while y -axis is in upward directions, hence, (12) and (13) become

$$\begin{aligned} 0 &= \frac{dT_{xx}}{dx}, \\ 0 &= \frac{dT_{xy}}{dx} - \rho g. \end{aligned} \quad (14)$$

By using equations (8), (10), and (11) in (7), components of \mathbf{S} are

$$S_{xx} = \frac{-\eta m(1-a)(dv/dx)^2}{1+m^2(1-a^2)(dv/dx)^2}, \quad (15)$$

$$S_{xy} = \frac{\eta dv/dx}{1+m^2(1-a^2)(dv/dx)^2} = S_{yx}, \quad (16)$$

$$S_{yy} = \frac{\eta m(1+a)(dv/dx)^2}{1+m^2(1-a^2)(dv/dx)^2}. \quad (17)$$

Using (15)–(17), stress tensor \mathbf{T} in (6) is

$$T_{xx} = \frac{-\eta m(1-a)(dv/dx)^2}{1+m^2(1-a^2)(dv/dx)^2}, \quad (18)$$

$$T_{xy} = \mu \left(\frac{dv}{dx} \right) + \frac{\eta dv/dx}{1+m^2(1-a^2)(dv/dx)^2} = T_{yx}, \quad (19)$$

$$T_{yy} = \frac{\eta m(1+a)(dv/dx)^2}{1+m^2(1-a^2)(dv/dx)^2}. \quad (20)$$

Substituting T_{xy} in (14) gives

$$\frac{d}{dx} \left[\mu \left(\frac{dv}{dx} \right) + \frac{\eta dv/dx}{1+m^2(1-a^2)(dv/dx)^2} \right] = \rho g. \quad (21)$$

Boundary conditions become

$$\frac{dv}{dx} = 0 \text{ at } x = \delta, \quad (22)$$

$$v = U_0 \text{ at } x = 0. \quad (23)$$

Introduce nondimensional parameters as

$$x^* = \frac{x}{\delta}, v^* = \frac{v}{U_0} \text{ and } \varphi = \frac{\mu}{\mu + \eta}, \quad (24)$$

where φ is the ratios of viscosities. Equations (21)–(23) become

$$\frac{d}{dx} \left[\varphi \left(\frac{dv}{dx} \right) + \frac{(1-\varphi)dv/dx}{1+W_e^2(1-a^2)(dv/dx)^2} \right] = S_t. \quad (25)$$

$$\text{At } x = 1, \frac{dv}{dx} = 0. \quad (26)$$

$$\text{At } x = 0, v = 1, \quad (27)$$

where $S_t = \rho g \delta^2 / \mu_{\text{eff}} U_0$ and $W_e = m U_0 / \delta$ are Stokes and Weissenberg numbers, respectively, and $\mu_{\text{eff}} = \mu + \eta$.

Simplification of equation (25) gives

$$\frac{d^2 v}{dx^2} + W_e^4 (1-a^2)^2 \left[\varphi \frac{d^2 v}{dx^2} - S_t \right] \left(\frac{dv}{dx} \right)^4 - W_e^2 (1-a^2) \left(\frac{d^2 v}{dx^2} + \varphi \frac{d^2 v}{dx^2} + 2S_t \right) \left(\frac{dv}{dx} \right)^2 - S_t = 0, \quad (28)$$

with

$$\begin{aligned} \frac{dv}{dx} &= 0 \text{ at } x = 1, \\ v &= 1 \text{ at } x = 0. \end{aligned} \quad (29)$$

Now, using definition of fractional calculus given in Section 2.1, the fractional form of equation (28) is

$$\begin{aligned} &\left[\frac{d^2 v(x)}{dx^2} + W_e^4 (1-a^2)^2 \left[\varphi \frac{d^2 v(x)}{dx^2} - S_t \right] (D^\alpha v(x))^4 \right. \\ &\quad \left. - W_e^2 (1-a^2) \left[\frac{d^2 v(x)}{dx^2} + \varphi \frac{d^2 v(x)}{dx^2} + 2S_t \right] (D^\alpha v(x))^2 - S_t \right] = 0, \end{aligned} \quad (30)$$

with

$$v(0) = 1, v'(1) = 0, 0 < \alpha < 1. \quad (31)$$

4. Homotopy Solution of Johnson–Segalman Fluid in the Lifting Case

Homotopy $\Omega \times [0, 1] \longrightarrow \Re$ for (30) is

$$(1-p) \frac{d^2 v(x)}{dx^2} + p \left[\frac{d^2 v(x)}{dx^2} + W_e^4 (1-a^2)^2 \left[\phi \frac{d^2 v(x)}{dx^2} - S_t \right] (D^\alpha v(x))^4 - W_e^2 (1-a^2) \left[\frac{d^2 v(x)}{dx^2} + \phi \frac{d^2 v(x)}{dx^2} + 2S_t \right] (D^\alpha v(x))^2 - S_t \right] = 0. \quad (32)$$

Using (30) and (31), we obtain the following.
Zeroth-order problem:

First-order problem:

$$v_0''(x) = 0, \quad v_0(0) = 1, v_0'(1) = 0. \quad (33)$$

$$\begin{aligned} & -S_t - 2S_t W_e^2 (D^\alpha v_0(x))^2 + 2a^2 S_t W_e^2 (D^\alpha v_0(x))^2 - S_t W_e^4 (D^\alpha v_0(x))^4 \\ & + 2a^2 S_t W_e^4 (D^\alpha v_0(x))^4 - a^4 S_t W_e^4 (D^\alpha v_0(x))^4 - W_e^2 (D^\alpha v_0(x))^2 v_0''(x) \\ & + a^2 W_e (D^\alpha v_0(x))^2 v_0''(x) - W_e^2 \phi (D^\alpha v_0(x))^2 v_0''(x) + a^2 W_e^2 \phi (D^\alpha v_0(x))^2 v_0''(x) \\ & + W_e^4 \phi (D^\alpha v_0(x))^4 v_0''(x) - 2a^2 W_e^4 \phi (D^\alpha v_0(x))^4 v_0''(x) + a^4 W_e^4 \phi (D^\alpha v_0(x))^4 v_0''(x) \\ & + v_1''(x) = 0, v_1(0) = 0, v_1'(1) = 0. \end{aligned} \quad (34)$$

Second-order problem:

$$\begin{aligned} & W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) + 4a^2 S_t W_e^2 (D^\alpha v_0(x)) - 4S_t (D^\alpha v_1(x)) \\ & - 4S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) 8a^2 S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) \\ & - 4a^4 S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) - 2W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) \\ & + 2a^2 W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) - 2W_e^2 \phi (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) \\ & + 2a^2 W_e^2 \phi (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) + 4W_e^4 \phi (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) v_0''(x) \\ & - 8a^2 W_e^4 \phi (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) v_0''(x) + 4a^4 W_e^4 \phi (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) v_0''(x) \\ & - W_e^2 (D^\alpha v_0(x))^2 v_1''(x) + a^2 W_e^2 (D^\alpha v_0(x))^2 v_1''(x) - W_e^2 \phi (D^\alpha v_0(x))^2 v_1''(x) \\ & + a^2 W_e^2 \phi (D^\alpha v_0(x))^2 v_1''(x) + W_e^4 \phi (D^\alpha v_0(x))^4 v_1''(x) - 2a^2 W_e^4 \phi (D^\alpha v_0(x))^4 v_1''(x) \\ & + a^4 W_e^4 \phi (D^\alpha v_0(x))^4 v_1''(x) + v_2''(x) = 0, v_2(0) = 0, v_2'(1) = 0. \end{aligned} \quad (35)$$

Similarly, we can find higher order problems and their solutions.

Third-order approximate solution, after applying Caputo definition and keeping $\alpha = 0.99$, $\phi = 0.1$, $S_t = 0.01$, $a = 0.01$, and $W_e = 0.1$ fixed, is

$$V(x) = 1 + \frac{1}{2} (-0.02t + 0.01x^2) + (0.0796711 (-8.24342 \times 10^{-8} x^{2.98} + 1.23219 \times 10^{-7} x^4 - 8.24098 \times 10^{-8} x^5 + 2.04999 \times 10^{-8} x^6)) x^{1.98}$$

$$\frac{+ (0.0398355 (-9.06776 \times 10^{-8} x^{2.98} + 1.35541 \times 10^{-7} x^4) - 9.06508 \times 10^{-8} x^5 + 2.25499 \times 10^{-8} x^6))}{x^{1.98}}. \quad (36)$$

The residual is denoted by R and is defined as

$$R = \frac{d^2 V(x)}{dx^2} + W_e^4 (1 - a^2)^2 \left[\varphi \frac{d^2 V(x)}{dx^2} - S_t \right] (D^\alpha V(x))^4 - W_e^2 (1 - a^2) \left[\frac{d^2 V(x)}{dx^2} + \varphi \frac{d^2 V(x)}{dx^2} + 2S_t \right] (D^\alpha V(x))^2 - S_t. \quad (37)$$

5. Flow Rate and Average Velocity in the Lifting Case

Flow rate in this case is

$$Q = \frac{1}{6(15 - 16\alpha + 4\alpha^2)\Gamma(4 - \alpha)^2} (-3 + 2\alpha) \left(-\frac{1}{-7 + 2\alpha} (3(-1 + a^2)S_t^3 W_e^2 (-3 + \alpha)^2 (-16 + \alpha(25 + \alpha(-13 + 2\alpha))) (3 + \phi)) - 2(-3 + S_t)(-5 + 2\alpha)\Gamma(4 - \alpha)^2 \right), \quad (38)$$

and average velocity is

$$\bar{V} = \frac{1}{6(15 - 16\alpha + 4\alpha^2)\Gamma(4 - \alpha)^2} (-3 + 2\alpha) \left(-\frac{1}{-7 + 2\alpha} (3(-1 + a^2)S_t^3 W_e^2 (-3 + \alpha)^2 (-16 + \alpha(25 + \alpha(-13 + 2\alpha))) (3 + \phi)) - 2(-3 + S_t)(-5 + 2\alpha)\Gamma(4 - \alpha)^2 \right). \quad (39)$$

6. Mathematical Formulation in the Drainage Case

Let fluid be draining down due to gravity on the infinite stationary belt. Equations (4)–(8) become

$$0 = \frac{dT_{xy}}{dx} + \rho g. \quad (40)$$

Using equation (19) in (14) gives the form

$$\frac{d}{dx} \left[\mu \left(\frac{dv}{dx} \right) + \frac{\eta dv/dx}{1 + m^2(1 - a^2)(dv/dx)^2} \right] = -\rho g, \quad (41)$$

with

$$\begin{aligned} \frac{dv}{dx} &= 0 \text{ at } x = \delta, \\ v &= 0 \text{ at } x = 0. \end{aligned} \quad (42)$$

Equation (40) in the dimensionless form is

$$\frac{d}{dx} \left[\frac{(1 - \varphi) dv/dx}{1 + W_e^2(1 - a^2)(dv/dx)^2} + \varphi \left(\frac{dv}{dx} \right) \right] = -S_t. \quad (43)$$

$$\begin{aligned} \text{At } x = 1, \quad \frac{dv}{dx} &= 0, \\ \text{At } x = 0, \quad v &= 0. \end{aligned} \quad (44)$$

Simplification of (43) gives

$$\begin{aligned} \frac{d^2 v}{dx^2} + W_e^4 (1 - a^2)^2 \left[\varphi \frac{d^2 v}{dx^2} + S_t \right] \left(\frac{dv}{dx} \right)^4 - W_e^2 (1 - a^2) \left(\frac{d^2 v}{dx^2} + \varphi \frac{d^2 v}{dx^2} - 2S_t \right) \left(\frac{dv}{dx} \right)^2 + S_t &= 0. \end{aligned} \quad (45)$$

Three cases as fractional boundary value problems are obtained.

TABLE 1: Solution and residual error for α keeping $a = 0.1, W_e = 0.01, S_t = 0.001$, and $\phi = 0.1$ fixed in lifting.

x	$\alpha = 0.2$		$\alpha = 0.6$		$\alpha = 0.99$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	0.905	-8.76128×10^{-25}	0.905	-9.08606×10^{-24}	0.905	-6.59422×10^{-23}
0.2	0.82	-2.5981×10^{-24}	0.82	-1.45747×10^{-23}	0.82	-4.249×10^{-23}
0.3	0.745	-4.78151×10^{-24}	0.745	-1.74003×10^{-23}	0.745	-2.54931×10^{-23}
0.4	0.68	-7.11425×10^{-24}	0.68	-1.79924×10^{-23}	0.68	-1.41813×10^{-24}
0.5	0.625	-9.29068×10^{-24}	0.625	-1.69071×10^{-23}	0.625	-6.9968×10^{-24}
0.6	0.58	-1.10441×10^{-23}	0.58	-1.4743×10^{-23}	0.58	-3.0318×10^{-24}
0.7	0.545	-1.2179×10^{-23}	0.545	-1.20434×10^{-23}	0.545	-1.0542×10^{-24}
0.8	0.52	-1.25923×10^{-23}	0.52	-9.26204×10^{-24}	0.52	-2.52839×10^{-25}
0.9	0.505	-1.22773×10^{-23}	0.505	-6.71129×10^{-24}	0.505	-2.91563×10^{-27}
1.	0.5	-1.13159×10^{-23}	0.5	-4.57323×10^{-24}	0.5	1.0211×10^{-26}

TABLE 2: Solution and residual error for ϕ keeping $\alpha = 0.99, W_e = 0.01, S_t = 0.001$, and $a = 0.1$ fixed in lifting.

x	$\phi = 0.1$		$\phi = 0.6$		$\phi = 0.9$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	0.905	-6.59422×10^{-23}	0.905	-9.09923×10^{-23}	0.905	-1.08443×10^{-22}
0.2	0.82	-4.249×10^{-23}	0.82	-5.8628×10^{-23}	0.82	-6.98245×10^{-23}
0.3	0.745	-2.54931×10^{-23}	0.745	-3.51797×10^{-23}	0.745	-4.19438×10^{-23}
0.4	0.68	-1.41813×10^{-23}	0.68	-1.95548×10^{-23}	0.68	-2.32819×10^{-23}
0.5	0.625	-6.9968×10^{-24}	0.625	-9.6702×10^{-24}	0.625	-1.15297×10^{-23}
0.6	0.58	-3.0318×10^{-24}	0.58	-4.18483×10^{-24}	0.58	-4.96745×10^{-24}
0.7	0.545	-1.0542×10^{-24}	0.545	-1.45024×10^{-24}	0.545	-1.7253×10^{-24}
0.8	0.52	-2.52839×10^{-25}	0.52	-3.53713×10^{-25}	0.52	-4.20658×10^{-25}
0.9	0.505	-2.91563×10^{-27}	0.505	-1.76326×10^{-26}	0.505	-2.24163×10^{-26}
1.	0.5	1.0211×10^{-26}	0.5	5.3895×10^{-27}	0.5	9.2139×10^{-27}

TABLE 3: Solution and residual error for S_t keeping $\alpha = 0.98, W_e = 0.001, \phi = 0.1$, and $a = 0.1$ fixed in lifting.

x	$S_t = 0.001$		$S_t = 0.01$		$S_t = 0.1$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	0.999905	-7.23543×10^{-27}	0.99905	-6.29219×10^{-22}	0.9905	-6.30818×10^{-17}
0.2	0.99982	-3.63823×10^{-27}	0.9982	-4.18657×10^{-22}	0.982	-4.18849×10^{-17}
0.3	0.999745	-2.86435×10^{-27}	0.99745	-2.5807×10^{-22}	0.9745	-2.57976×10^{-17}
0.4	0.99968	-1.51698×10^{-27}	0.9968	-1.46438×10^{-22}	0.968	-1.46541×10^{-18}
0.5	0.999625	-8.70705×10^{-28}	0.99625	-7.53253×10^{-23}	0.9625	-7.52476×10^{-18}
0.6	0.99958	-4.61286×10^{-28}	0.9958	-3.40645×10^{-23}	0.958	-3.37798×10^{-18}
0.7	0.999545	-4.34662×10^{-28}	0.99545	-1.28478×10^{-23}	0.9545	-1.25942×10^{-18}
0.8	0.99952	-4.00347×10^{-29}	0.9952	-3.70343×10^{-24}	0.952	-3.64878×10^{-19}
0.9	0.999505	-1.66647×10^{-29}	0.99505	-1.07941×10^{-24}	0.9505	-8.14999×10^{-20}
1.	0.9995	9.95937×10^{-30}	0.995	-8.78476×10^{-26}	0.95	-8.71776×10^{-21}

TABLE 4: Solution and residual error for W_e keeping $\alpha = 0.95, S_t = 0.001, \phi = 0.1$, and $a = 0.1$ fixed in lifting.

x	$W_e = 0.001$		$W_e = 0.01$		$W_e = 0.1$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	0.905	-5.48732×10^{-27}	0.905	-5.51525×10^{-23}	0.905	-5.4982×10^{-19}
0.2	0.82	-3.75182×10^{-27}	0.82	-3.98956×10^{-23}	0.82	-3.98946×10^{-19}
0.3	0.745	-2.5638×10^{-27}	0.745	-2.63606×10^{-23}	0.745	-2.63525×10^{-19}
0.4	0.68	-1.17028×10^{-27}	0.68	-1.60262×10^{-23}	0.68	-1.6049×10^{-19}
0.5	0.625	-7.01495×10^{-28}	0.625	-8.9237×10^{-24}	0.625	-8.93613×10^{-20}
0.6	0.58	-4.80416×10^{-28}	0.58	-4.47664×10^{-24}	0.58	-4.46647×10^{-20}
0.7	0.545	-3.2225×10^{-28}	0.545	-1.95569×10^{-24}	0.545	-1.9563×10^{-20}
0.8	0.52	7.79×10^{-30}	0.52	-7.49229×10^{-25}	0.52	-7.37223×10^{-21}
0.9	0.505	1.3884×10^{-28}	0.505	-2.49356×10^{-25}	0.505	-2.42499×10^{-21}
1.	0.5	9.40717×10^{-29}	0.5	-6.13229×10^{-26}	0.5	-5.63892×10^{-22}

TABLE 5: Solution and residual error for α keeping $a = 0.1$, $W_e = 0.01$, $S_t = 0.001$, and $\phi = 0.1$ fixed in Case 1 of drainage.

x	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	0.000095	1.61655×10^{-24}	0.000095	9.08344×10^{-24}	0.000095	6.30751×10^{-23}
0.2	0.00018	4.16166×10^{-24}	0.00018	1.45689×10^{-23}	0.00018	4.18362×10^{-23}
0.3	0.000255	6.96312×10^{-24}	0.000255	1.73923×10^{-23}	0.000255	2.57623×10^{-23}
0.4	0.00032	9.57269×10^{-24}	0.00032	1.79837×10^{-23}	0.00032	1.46516×10^{-23}
0.5	0.000375	1.16444×10^{-23}	0.000375	1.6899×10^{-23}	0.000375	7.51521×10^{-24}
0.6	0.00042	1.29537×10^{-23}	0.00042	1.47362×10^{-23}	0.00042	3.40636×10^{-24}
0.7	0.000455	1.34057×10^{-23}	0.000455	1.20384×10^{-23}	0.000455	1.25929×10^{-24}
0.8	0.00048	1.30281×10^{-23}	0.00048	9.25865×10^{-24}	0.00048	3.70727×10^{-25}
0.9	0.000495	1.19467×10^{-23}	0.000495	6.70931×10^{-24}	0.000495	8.09182×10^{-26}
1.	0.0005	1.0354×10^{-23}	0.0005	4.57227×10^{-24}	0.0005	1.89895×10^{-26}

TABLE 6: Solution and residual error for ϕ keeping $\alpha = 0.98$, $W_e = 0.01$, $S_t = 0.001$, and $a = 0.1$ fixed for Case 1 of drainage.

x	$\phi = 0.01$		$\phi = 0.1$		$\phi = 0.3$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	0.000095	5.92844×10^{-23}	0.000095	6.30751×10^{-23}	0.000095	8.18541×10^{-23}
0.2	0.00018	3.93158×10^{-23}	0.00018	4.18362×10^{-23}	0.00018	5.42952×10^{-23}
0.3	0.000255	2.42117×10^{-23}	0.000255	2.57623×10^{-23}	0.000255	3.34316×10^{-23}
0.4	0.00032	1.37711×10^{-23}	0.00032	1.46516×10^{-23}	0.00032	1.90143×10^{-23}
0.5	0.000375	7.0636×10^{-24}	0.000375	7.51521×10^{-24}	0.000375	9.74926×10^{-24}
0.6	0.00042	3.20361×10^{-24}	0.00042	3.40636×10^{-24}	0.00042	4.41229×10^{-24}
0.7	0.000455	1.18469×10^{-24}	0.000455	1.25929×10^{-24}	0.000455	1.62912×10^{-24}
0.8	0.00048	3.49409×10^{-25}	0.00048	3.70727×10^{-25}	0.00048	4.82038×10^{-25}
0.9	0.000495	7.66394×10^{-26}	0.000495	8.09182×10^{-26}	0.000495	1.05556×10^{-25}
1.	0.0005	1.87118×10^{-26}	0.0005	1.89895×10^{-26}	0.0005	2.15706×10^{-24}

TABLE 7: Solution and residual error for S_t keeping $\alpha = 0.99$, $W_e = 0.01$, $\phi = 0.01$, and $a = 0.1$ fixed in Case 1 of drainage.

x	$S_t = 0.001$		$S_t = 0.01$		$S_t = 0.1$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	0.000095	5.82872×10^{-23}	0.000095	5.82511×10^{-18}	0.000095	5.8252×10^{-13}
0.2	0.00018	3.7525×10^{-23}	0.00018	3.75136×10^{-18}	0.00018	3.75135×10^{-13}
0.3	0.000255	2.24807×10^{-23}	0.000255	2.25476×10^{-18}	0.000255	2.25477×10^{-13}
0.4	0.00032	1.25172×10^{-23}	0.00032	1.24919×10^{-18}	0.00032	1.2492×10^{-13}
0.5	0.000375	6.177×10^{-24}	0.000375	6.22301×10^{-19}	0.000375	6.22317×10^{-14}
0.6	0.00042	2.69667×10^{-24}	0.00042	2.67581×10^{-19}	0.00042	2.67572×10^{-14}
0.7	0.000455	9.14662×10^{-25}	0.000455	9.26852×10^{-20}	0.000455	9.2681×10^{-15}
0.8	0.00048	2.38577×10^{-25}	0.00048	2.30937×10^{-20}	0.00048	2.3095×10^{-15}
0.9	0.000495	2.25046×10^{-26}	0.000495	3.77223×10^{-21}	0.000495	3.76414×10^{-16}
1.	0.0005	3.98217×10^{-27}	0.0005	1.84694×10^{-22}	0.0005	1.88184×10^{-17}

TABLE 8: Solution and residual error for γ keeping $S_t = 0.01$, $\phi = 0.1$, $W_e = 0.001$, and $a = 0.1$ fixed in Case 2 of drainage.

x	$\gamma = 1.2$		$\gamma = 1.6$		$\gamma = 1.99$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	5.72659×10^{-4}	-1.32727×10^{-18}	1.75703×10^{-4}	3.197×10^{-18}	5.1638×10^{-5}	1.29417×10^{-17}
0.2	1.31563×10^{-3}	-2.02422×10^{-18}	5.3263×10^{-3}	3.56035×10^{-18}	2.05125×10^{-4}	1.22623×10^{-17}
0.3	2.14014×10^{-3}	-1.98946×10^{-18}	1.01899×10^{-3}	3.88346×10^{-18}	4.59664×10^{-4}	1.24389×10^{-17}
0.4	3.02251×10^{-3}	-2.9885×10^{-19}	1.61463×10^{-3}	3.46852×10^{-18}	8.14833×10^{-4}	1.35961×10^{-17}
0.5	3.95057×10^{-3}	-1.84225×10^{-18}	2.30744×10^{-3}	3.68767×10^{-18}	1.27034×10^{-3}	1.31157×10^{-17}
0.6	4.91675×10^{-3}	-2.46976×10^{-18}	3.08901×10^{-3}	3.57668×10^{-18}	1.82595×10^{-3}	1.37891×10^{-17}
0.7	5.91581×10^{-3}	-2.09831×10^{-18}	3.95307×10^{-3}	3.36789×10^{-18}	2.4815×10^{-3}	1.30025×10^{-17}
0.8	6.94391×10^{-3}	-1.35899×10^{-18}	4.89465×10^{-3}	3.06616×10^{-18}	3.23682×10^{-3}	1.40449×10^{-17}
0.9	7.99811×10^{-3}	-5.26745×10^{-19}	5.90971×10^{-3}	4.003×10^{-18}	4.09177×10^{-3}	1.29653×10^{-17}
1.	9.07604×10^{-3}	-1.50223×10^{-18}	6.99484×10^{-3}	2.56045×10^{-18}	5.04625×10^{-3}	1.32103×10^{-17}

TABLE 9: Solution and residual error for α keeping $\gamma = 1.95, S_t = 0.001, \phi = 0.1, W_e = 0.01$, and $a = 0.1$ fixed in Case 3 of drainage.

x	$\alpha = 0.3$		$\alpha = 0.7$		$\alpha = 0.99$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	5.87208×10^{-6}	-9.076×10^{-20}	5.87208×10^{-6}	1.16701×10^{-19}	5.87208×10^{-6}	6.48938×10^{-21}
0.2	2.26882×10^{-5}	6.62623×10^{-20}	2.26882×10^{-5}	-3.50514×10^{-20}	2.26882×10^{-5}	-1.64872×10^{-19}
0.3	5.0024×10^{-5}	7.16925×10^{-20}	5.0024×10^{-5}	-5.38787×10^{-20}	5.0024×10^{-5}	-5.48153×10^{-20}
0.4	8.76616×10^{-5}	6.17061×10^{-20}	8.76616×10^{-5}	4.81294×10^{-20}	8.76616×10^{-5}	-2.07791×10^{-19}
0.5	1.35451×10^{-4}	-7.89358×10^{-20}	1.35451×10^{-4}	8.88535×10^{-20}	1.35451×10^{-4}	6.49314×10^{-20}
0.6	1.9328×10^{-4}	5.62103×10^{-20}	1.9328×10^{-4}	-8.1769×10^{-20}	1.9328×10^{-4}	-1.40755×10^{-20}
0.7	2.61056×10^{-4}	2.13652×10^{-20}	2.61056×10^{-4}	2.46311×10^{-19}	2.61056×10^{-4}	8.01669×10^{-21}
0.8	3.38702×10^{-4}	4.3926×10^{-20}	3.38702×10^{-4}	-3.74249×10^{-20}	3.38702×10^{-4}	-1.55483×10^{-19}
0.9	4.26153×10^{-4}	-1.25871×10^{-19}	4.26153×10^{-4}	-4.78429×10^{-20}	4.26153×10^{-4}	-1.57049×10^{-19}
1.	5.2335×10^{-4}	7.27423×10^{-20}	5.2335×10^{-4}	3.94566×10^{-20}	5.2335×10^{-4}	-5.88772×10^{-20}

TABLE 10: Solution and residual error for γ keeping $\alpha = 0.95, S_t = 0.001, \phi = 0.1, W_e = 0.01$, and $a = 0.1$ fixed in Case 3 of drainage.

x	$\gamma = 1.2$		$\gamma = 1.6$		$\gamma = 1.99$	
	$V(x)$	Res error	$V(x)$	Res error	$V(x)$	Res error
0.1	5.72659×10^{-4}	-8.85316×10^{-20}	1.75703×10^{-5}	-9.8013×10^{-20}	5.1638×10^{-6}	4.95955×10^{-21}
0.2	1.31563×10^{-4}	-1.35853×10^{-19}	5.3263×10^{-5}	-5.72227×10^{-20}	2.0512×10^{-5}	1.06505×10^{-20}
0.3	2.14014×10^{-4}	1.63841×10^{-20}	1.01899×10^{-4}	3.02498×10^{-22}	4.5966×10^{-5}	2.98796×10^{-20}
0.4	3.02251×10^{-4}	-1.77002×10^{-19}	1.61463×10^{-4}	2.74106×10^{-19}	8.1483×10^{-5}	8.85672×10^{-20}
0.5	3.95057×10^{-4}	6.75864×10^{-20}	2.30744×10^{-4}	-2.9077×10^{-20}	1.2703×10^{-4}	-1.25093×10^{-19}
0.6	4.91675×10^{-4}	-1.24341×10^{-19}	3.08901×10^{-4}	2.8962×10^{-19}	1.8259×10^{-4}	7.08777×10^{-21}
0.7	5.91581×10^{-4}	3.33152×10^{-20}	3.95307×10^{-4}	4.4664×10^{-20}	2.4815×10^{-4}	3.73471×10^{-20}
0.8	6.94391×10^{-4}	-5.4743×10^{-20}	4.89465×10^{-4}	-4.42716×10^{-20}	3.2368×10^{-4}	-2.48835×10^{-20}
0.9	7.99811×10^{-4}	-4.85706×10^{-20}	5.90971×10^{-4}	-1.45847×10^{-19}	4.0917×10^{-4}	1.10885×10^{-19}
1.	9.07604×10^{-4}	-2.09174×10^{-19}	6.99484×10^{-4}	1.98754×10^{-20}	5.0462×10^{-4}	2.57162×10^{-19}

Case 1

$$\frac{d^2 v(x)}{dx^2} + W_e^4 (1 - a^2)^2 \left[\phi \frac{d^2 v(x)}{dx^2} + S_t \right] (D^\alpha v(x))^4$$

$$- W_e^2 (1 - a^2) \left[\frac{d^2 v(x)}{dx^2} + \phi \frac{d^2 v(x)}{dx^2} - 2S_t \right] (D^\alpha v(x))^2 + S_t = 0, \quad (46)$$

with

$$v(0) = 0, v'(1) = 0, 0 < \alpha < 1. \quad (47)$$

Case 2

$$D^\gamma v(x) + W_e^4 (1 - a^2)^2 [\phi D^\gamma v(x) + S_t] \left(\frac{dv(x)}{dx} \right)^4$$

$$- W_e^2 (1 - a^2) [D^\gamma v(x) + D^\gamma v(x) - 2S_t] \left(\frac{dv(x)}{dx} \right)^2 + S_t = 0. \quad (48)$$

$$v(0) = 0, v'(1) = 0, 1 < \gamma < 2. \quad (49)$$

Case 3. Here both first- and second-order derivatives are replaced by the noninteger order derivative as follows:

$$D^\gamma v(x) + W_e^4 (1 - a^2)^2 [\phi D^\gamma v(x) + S_t] (D^\alpha v(x))^4$$

$$- W_e^2 (1 - a^2) [D^\gamma v(x) + D^\gamma v(x) - 2S_t] (D^\alpha v(x))^2 + S_t = 0, \quad (50)$$

with

$$v(0) = 0, v(1) = 0, 1 < \gamma < 2, 0 < \alpha < 1. \quad (51)$$

7. Results and Discussion

In this paper, the fractional study of thin film flow of Johnson–Segalman fluid is carried out in lifting and drainage cases. The obtained boundary value problems are solved for different values of involved parameters and results mentioned in Tables 1–4 for lifting while Tables 5–10 in drainage situation. Tables 1 and 5 present solutions and corresponding errors for different α . Tables 2 and 6 show solutions along with errors for ϕ . Tables 3 and 7 show case solutions and errors for various S_t . Table 4 demonstrates solutions and errors for various W_e in the lifting case.

Tables 8–10 exhibit solutions and residual errors for different values of fractional parameters in Cases 2 and 3, respectively. Exploration of these tables confirms that solutions are consistent. Influence of different parameters on $V(x)$ in lifting and drainage cases are seen graphically. Figures 1–8 show the effect of different parameters on $V(x)$

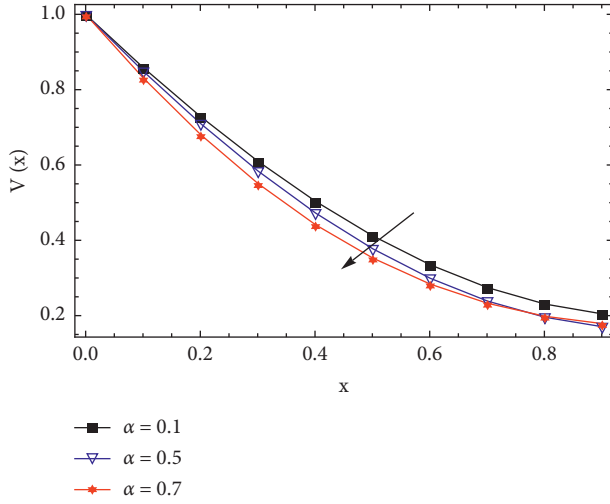


FIGURE 1: Effect of α on $V(x)$ for $a = 0.1, W_e = 1, S_t = 1$, and $\phi = 0.1$ fixed in the lifting case.

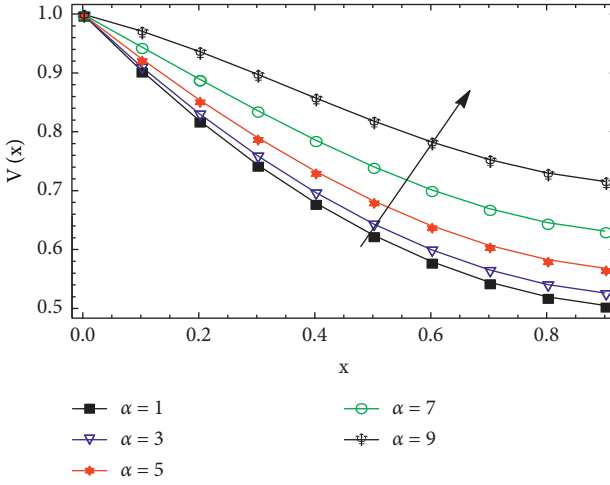


FIGURE 2: Effect of a on $V(x)$ for $\alpha = 0.95, S_t = 1, \phi = 0.01$, and $W_e = 0.1$ fixed in the lifting case.

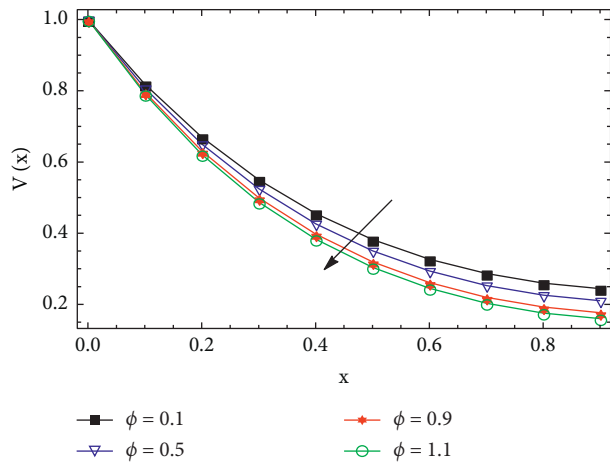


FIGURE 3: Effect of ϕ on $V(x)$ for $\alpha = 0.98, a = 0.1, W_e = 1$, and $S_t = 1$ fixed in the lifting case.

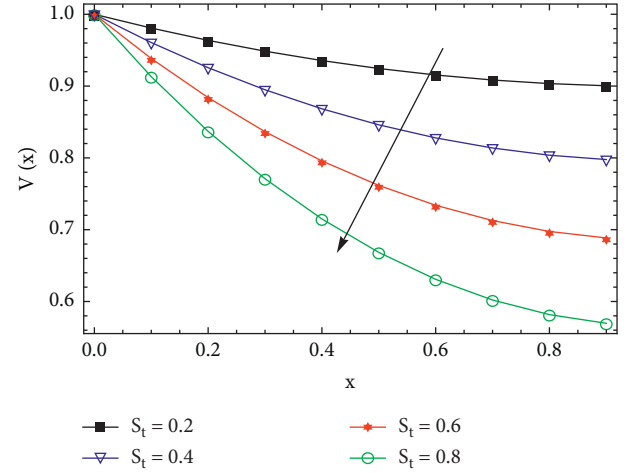


FIGURE 4: The effect of S_t on $V(x)$ for $\alpha = 0.95, a = 0.1, W_e = 1$, and $\phi = 0.1$ fixed in the lifting case.

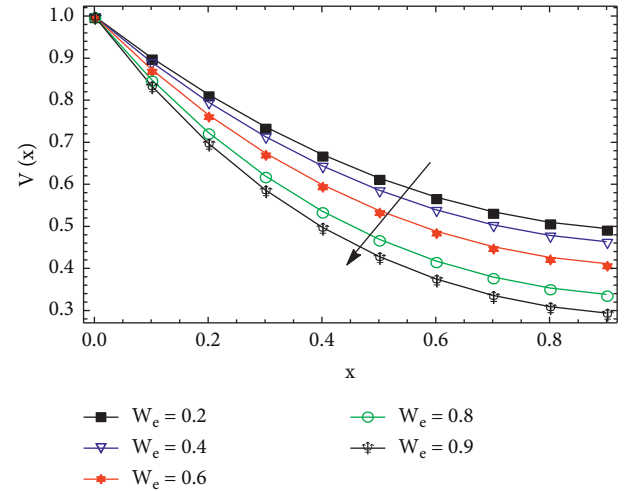


FIGURE 5: Effect of W_e on $V(x)$ for $\alpha = 0.98, a = 0.1, S_t = 1$, and $\phi = 0.1$ fixed in the lifting case.

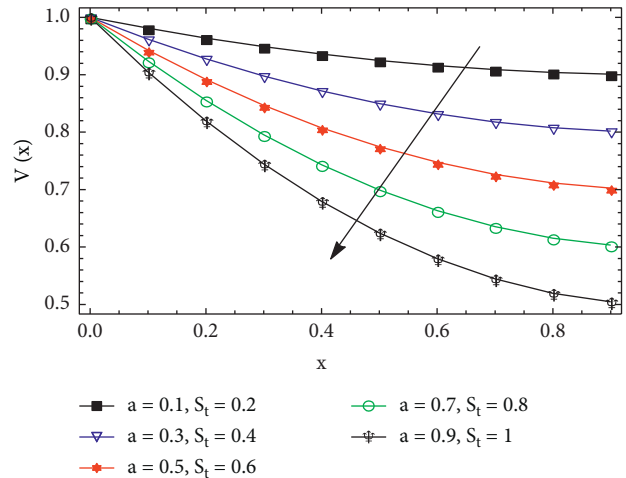


FIGURE 6: Effect of increasing S_t and a simultaneously on $V(x)$ for $\alpha = 0.99, W_e = 0.1$, and $\phi = 0.1$ fixed in the lifting case.

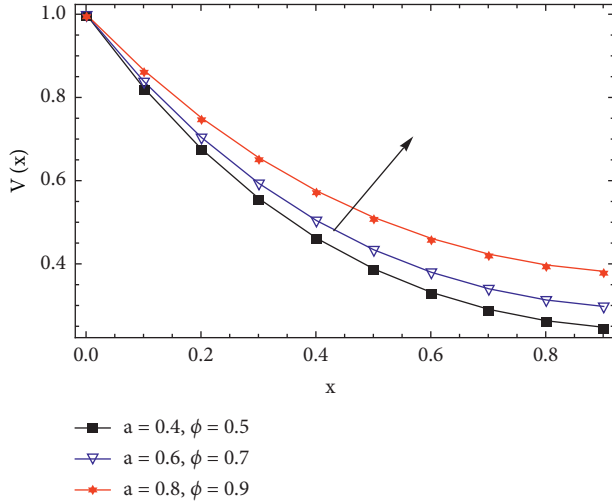


FIGURE 7: Effect of increasing ϕ and a simultaneously on $V(x)$ for $\alpha = 0.95$, $W_e = 1$, $S_t = 0.1$ fixed in the lifting case.

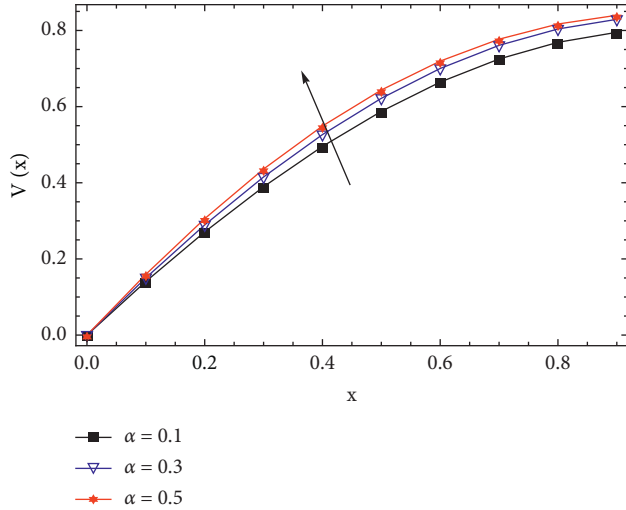


FIGURE 8: Effect of α on $V(x)$ for $a = 0.1$, $W_e = 1$, $S_t = 1$, and $\phi = 0.1$ fixed for Case 1 in the drainage case.

in the lifting case. Figure 1 shows the effect of α on $V(x)$. It is seen that increasing α reduces $V(x)$. Figure 2 shows effect of a on $V(x)$, showing that a has direct relationship with $V(x)$. Figures 3–5 depict the consequence of ϕ , S_t , and W_e on $V(x)$, respectively. It is observed that Weissenberg number is the dimensionless number while this parameter is used to measure the flow behavior into motion regarding particles. It is physically defined as ratio among viscous force and elastic force. Viscous force is enhanced against higher values of Weissenberg number, whereas higher viscous force brings a declination into motion of fluid particles. Momentum layers are also the reducing function versus implanted higher values of the Weissenberg number. It is seen that $V(x)$ has inverse relationship with mentioned parameters in all cases. Figure 6 indicates the impact of increasing a and S_t simultaneously on $V(x)$. It is observed that $V(x)$ decreases with an increase in a and S_t , and hence, S_t is a more

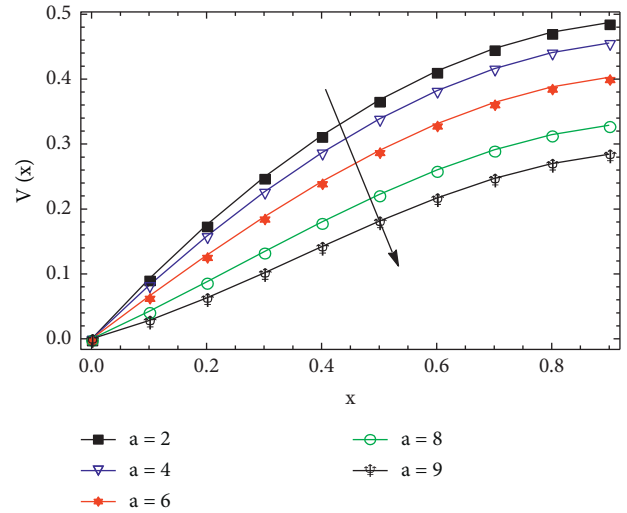


FIGURE 9: Effect of a on $V(x)$ for $\alpha = 0.95$, $W_e = 0.1$, $S_t = 1$, and $\phi = 0.01$ fixed for Case 1 in the drainage case.

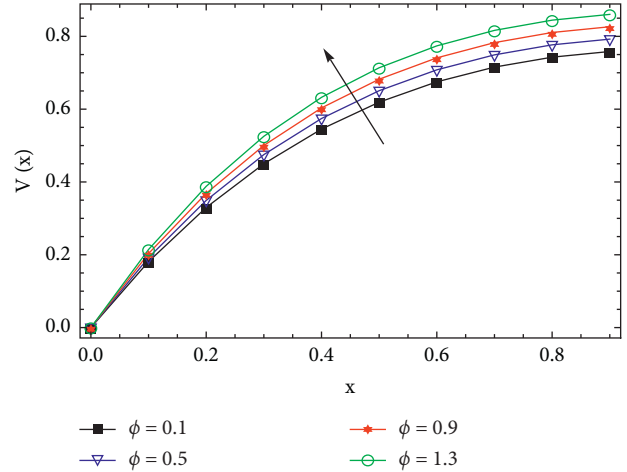


FIGURE 10: The effect of ϕ on $V(x)$ for $\alpha = 0.97$, $a = 0.1$, $W_e = 1$, and $S_t = 1$ fixed for Case 1 in the drainage case.

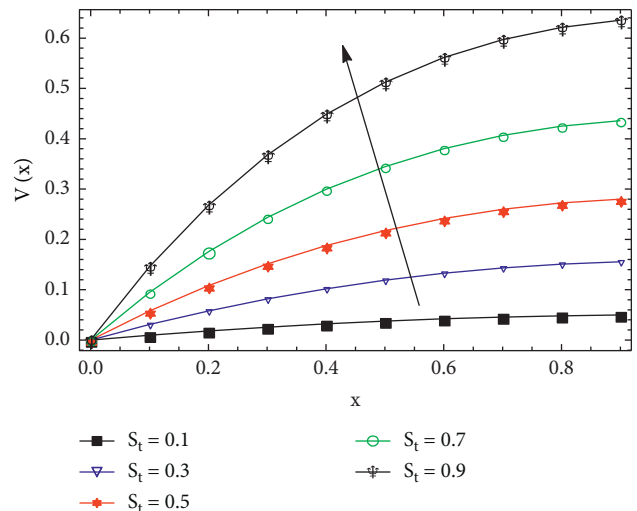


FIGURE 11: The effect of S_t on $V(x)$ for $\alpha = 0.95$, $a = 0.2$, $W_e = 1$, and $\phi = 0.1$ fixed for Case 1 in the drainage case.

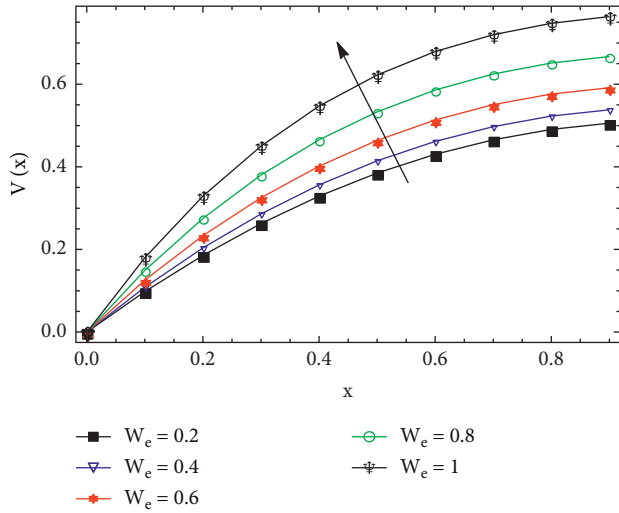


FIGURE 12: The effect of W_e on $V(x)$ for $\alpha = 0.95, a = 0.1, S_t = 1$, and $\phi = 0.1$ fixed for Case 1 in the drainage case.

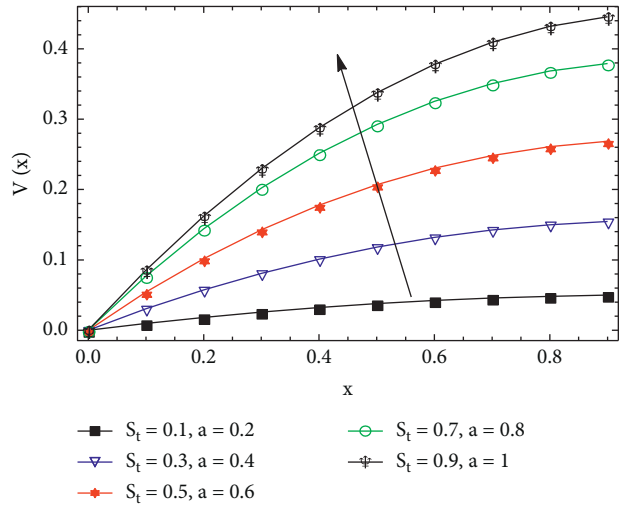


FIGURE 13: Effect of increasing S_t and a simultaneously on $V(x)$ for $\alpha = 0.95, W_e = 1$, and $\phi = 0.01$ fixed for Case 1 in the drainage case.

influential parameter then a . Figure 7 shows the combine effect of a and ϕ . Motion into fluid particles is reduced versus higher impact of ratio of viscosity number. For higher values of ϕ make a declination into motion of fluid. So, fluid becomes more viscous and thick. It is seen that $V(x)$ increases with increase in a and ϕ , and hence, a is more influential than ϕ . Figures 8–14 show the effect of different parameters on $V(x)$ in Case 1 of drainage. Figure 8 depicts the effect of α on $V(x)$. It is seen that $V(x)$ has direct association with α . Figure 9 shows the impact of a on $V(x)$. It is seen that $V(x)$ has inverse association with a . Figures 10–12 show the effect of ϕ , S_t , and W_e on $V(x)$, respectively. It is realized that $V(x)$ has direct connection with these parameters. Figure 13 shows the combined effect of increasing S_t and a . It is observed that $V(x)$ increases with increase in S_t and a , and hence, S_t is more influential than a . Stoke number

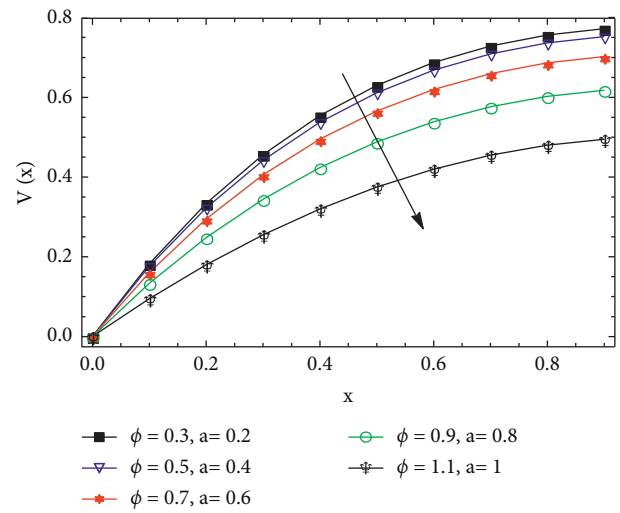


FIGURE 14: Effect of increasing ϕ and a simultaneously on $V(x)$ for $\alpha = 0.95, W_e = 1$, and $S_t = 1$ fixed for Case 1 in the drainage case.

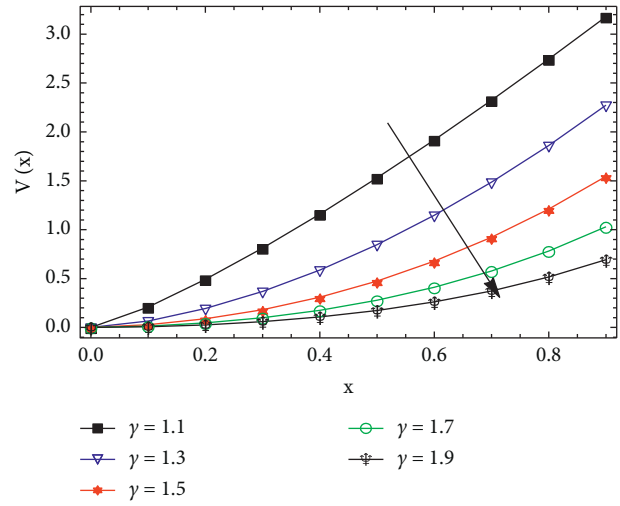


FIGURE 15: The effect of γ on $V(x)$ for $a = 0.1, W_e = 1, S_t = 1$, and $\phi = 0.1$ fixed for Case 2 in the drainage case.

(S_t) is a dimensionless number. Physically, it is used to know characterization of fluid particles during the flow of fluid. Basically, it is defined as the ratio among characteristics time and characteristics time during the flow regarding particle. So, it is noticed that flow of particles is directly proportional against velocity of fluid. Furthermore, Figure 14 shows the effect of increasing ϕ and a on $V(x)$. Investigation shows that a is a more influential than ϕ . Figure 15 shows the effect of γ on $V(x)$ in Case 2 of drainage. It is seen that $V(x)$ has inverse association with γ . Figures 16 and 17 indicate the effect of α and γ on $V(x)$ in Case 3 of drainage. It is observed that α and γ exhibit opposite behaviors on $V(x)$ in this case. In addition to the above results, volumetric flow and average velocities are also computed in lifting and drainage cases.

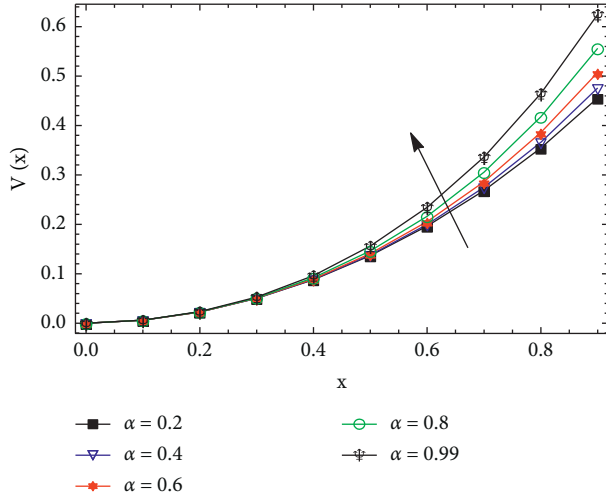


FIGURE 16: The effect of α on $V(x)$ for $\gamma = 1.95, a = 0.1, W_e = 1, S_t = 1$, and $\phi = 0.1$ fixed for Case 3 in the drainage case.

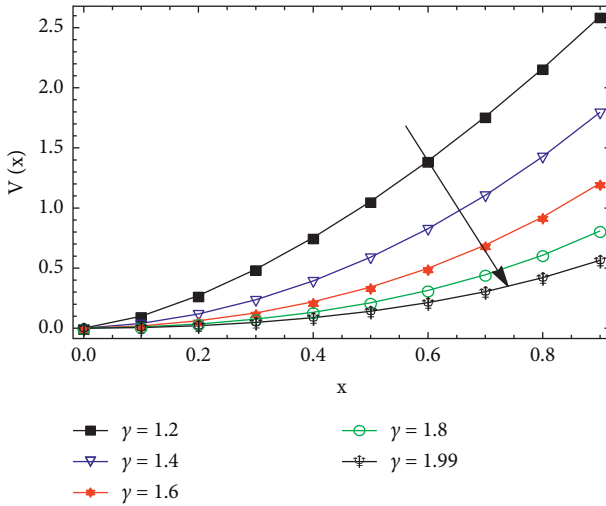


FIGURE 17: The effect of γ on $V(x)$ for $\alpha = 99, a = 0.1, W_e = 1, S_t = 1$, and $\phi = 0.1$ fixed for Case 3 in the drainage case.

8. Conclusions

In this study, thin film of Johnson–Segalman fluid flow is investigated in fractional space for the case of lifting and drainage.

Solutions of boundary value problems are obtained using fractional calculus and HPM. Validity of obtained solutions is established through residual errors' computation. Few important results related to different parameters are established in the fractional space. It is comprehended that fractional parameter α has shown reverse effect in drainage and lifting. The study also discloses that fractional state Stokes and Weissenberg numbers showed similar effects on the fluid velocity in drainage and lifting cases. It is further seen that the Stokes number is the most influential parameter in both lifting and drainage situations. [34–39].

Nomenclature

Symbols

\mathbf{v} :	Vector velocity
μ :	Kinematic viscosity
div :	Divergence
ρ :	Fluid density
σ :	Cauchy stress tensor
p :	Pressure
T :	Tensor
η :	Independent variable
g :	Gravitational force
f :	Body force
f_1, f_2 :	Components of body force
$\alpha\gamma$:	Fractional parameters
C_μ :	Real number
x, y :	Space coordinates
T_{xx}, T_{xy} :	Components of tensor
ϕ :	Ratio of viscosities
S_t :	Stoke number
We :	Weissenberg number.

Appendix

A. Homotopy Solution in Case 1 of Drainage

Homotopy $\Omega \times [0, 1] \longrightarrow \mathfrak{R}$, for equation (46), is

$$\begin{aligned}
 (1-p) \frac{d^2 v(x)}{dx^2} + p \frac{d^2 v(x)}{dx^2} + W_e^4 (1-a^2)^2 \left[\phi \frac{d^2 v(x)}{dx^2} + S_t \right] (D^\alpha v(x))^4 \\
 - W_e^2 (1-a^2) \left[\frac{d^2 v(x)}{dx^2} + \phi \frac{d^2 v(x)}{dx^2} - 2S_t \right] (D^\alpha v(x))^2 + S_t = 0.
 \end{aligned} \tag{A.1}$$

Using (46) and (47), various order problems are obtained.

$$v_0''(x) = 0, v_0(0) = 0, v_0'(1) = 0. \quad (\text{A.2})$$

Zeroth-order problem:

First-order problem:

$$\begin{aligned} & S_t + 2S_t W_e^2 (D^\alpha v_0(x))^2 - 2a^2 S_t W_e^2 (D^\alpha v_0(x))^2 + S_t W_e^4 (D^\alpha v_0(x))^4 \\ & - 2a^2 S_t W_e^4 (D^\alpha v_0(x))^4 + a^4 S_t W_e^4 (D^\alpha v_0(x))^4 - W_e^2 (D^\alpha v_0(x))^2 v_0''(x) \\ & + a^2 W_e (D^\alpha v_0(x))^2 v_0''(x) - W_e^2 \phi (D^\alpha v_0(x))^2 v_0''(x) - 2a^2 W_e^4 \phi (D^\alpha v_0(x))^4 v_0''(x) \\ & + a^2 W_e^2 \phi (D^\alpha v_0(x))^2 v_0''(x) + W_e^4 \phi (D^\alpha v_0(x))^4 v_0''(x) \\ & + a^4 W_e^4 \phi (D^\alpha v_0(x))^4 v_0''(x) + v_1''(x) = 0, v_1(0) = 0, v_1'(1) = 0. \end{aligned} \quad (\text{A.3})$$

Second-order problem:

$$\begin{aligned} & 4S_t W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) - 4a^2 S_t W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) \\ & + 4S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) - 8a^2 S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) \\ & + 4a^4 S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) - 2W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) \\ & + 2a^2 W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) - 2W_e^2 \phi (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) \\ & + 2a^2 W_e^2 \phi (D^\alpha v_0(x)) (D^\alpha v_1(x)) v_0''(x) + 4W_e^4 \phi (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) v_0''(x) \\ & - 8a^2 W_e^4 \phi (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) v_0''(x) + 4a^4 W_e^4 \phi (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) v_0''(x) \\ & - W_e^2 (D^\alpha v_0(x))^2 v_1''(x) + a^2 W_e^2 (D^\alpha v_0(x))^2 v_1''(x) - W_e^2 \phi (D^\alpha v_0(x))^2 v_1''(x) \\ & + a^2 W_e^2 \phi (D^\alpha v_0(x))^2 v_1''(x) + W_e^4 \phi (D^\alpha v_0(x))^4 v_1''(x) - 2a^2 W_e^4 \phi (D^\alpha v_0(x))^4 v_1''(x) \\ & + a^4 W_e^4 \phi (D^\alpha v_0(x))^4 v_1''(x) + v_2''(x) = 0, v_2(0) = 0, v_2'(1) = 0. \end{aligned} \quad (\text{A.4})$$

Similarly, we can find higher order problems and their solutions.

Third-order approximate solution after applying Caputo definition and keeping $\alpha = 0.99$, $a = 0.01$, $\phi = 0.1$, $W_e = 0.1$, and $S_t = 0.01$ fixed is

$$\begin{aligned} V(x) = & \frac{1}{2} (0.2x - 0.1x^2) + (0.0380982 (0.000092534x^{2.96} - 0.000137786x^4 \\ & + 0.000092426x^5 - 0.0000228777x^6)) \\ & \frac{x^{1.96} + (0.0761964 (0.0000841218x^{2.96} \\ & - 0.00012526x^4 + 0.0000840236x^5 - 0.0000207979x^6))}{x^{2.96}}. \end{aligned} \quad (\text{A.5})$$

The residual is

$$\begin{aligned}
 R = & \frac{d^2 v(x)}{dx^2} + W_e^4 (1 - a^2)^2 \left[\varphi \frac{d^2 v(x)}{dx^2} + S_t \right] (D^\alpha v(x))^4 \\
 & - W_e^2 (1 - a^2) \left[\frac{d^2 v(x)}{dx^2} + \varphi \frac{d^2 v(x)}{dx^2} - 2S_t \right] (D^\alpha v(x))^2 + S_t.
 \end{aligned} \tag{A.6}$$

B. Homotopy Solution in Case 2 of Drainage

Homotopy $\Omega \times [0, 1] \longrightarrow R$ for equation (48) is

$$\begin{aligned}
 (1 - p) \frac{d^2 v(x)}{dx^2} + p \left[D^\gamma v(x) + W_e^4 (1 - a^2)^2 [\varphi D^\gamma v(x) + S_t] \left(\frac{dv(x)}{dx} \right)^4 \right. \\
 \left. - W_e^2 (1 - a^2) \left[D^\gamma v(x) + D^\gamma v(x) - 2S_t \right] \left(\frac{dv(x)}{dx} \right)^2 + S_t \right] = 0.
 \end{aligned} \tag{B.1}$$

Using (48) and (49), we obtain the following.
Zeroth-order problem:

$$D^\gamma v_0(x) = 0, v_0(0) = 0, v_0'(1) = 0. \tag{B.2}$$

First-order problem:

$$\begin{aligned}
 S_t + D^\gamma v_1(x) + 2S_t W_e^2 (v_0'(x))^2 - 2a^2 S_t W_e^2 (V_0)' [t]^2 - W_e^2 (D^\gamma v_0(x)) (v_0'(x))^2 \\
 + a^2 W_e^2 (D^\gamma v_0(x)) (v_0'(x))^2 - \phi W_e^2 (D^\gamma v_0(x)) (v_0'(x))^2 + a^2 \phi W_e^2 (D^\gamma v_0(x)) (v_0'(x))^2 \\
 + S_t W_e^4 (v_0'(x))^4 - 2a^2 S_t W_e^4 (v_0'(x))^4 + a^4 S_t W_e^4 (v_0'(x))^4 + \phi W_e^4 (D^\gamma v_0(x)) (v_0'(x))^4 \\
 - 2a^2 \phi W_e^4 (D^\gamma v_0(x)) (v_0'(x))^4 + a^4 \phi W_e^4 D^\gamma v_0(x) (v_0'(x))^4 = 0, v_1(0) = 0, v_1'(1) = 0.
 \end{aligned} \tag{B.3}$$

Second-order problem:

$$\begin{aligned}
 D^\gamma v_2(x) - W_e^2 (D^\gamma v_1(x)) (v_0'(x))^2 + a^2 W_e^2 (D^\gamma v_1(x)) (v_0'(x))^2 \\
 - \phi W_e^2 (D^\gamma v_1(x)) (v_0'(x))^2 + a^2 \phi W_e^2 (D^\gamma v_1(x)) (v_0'(x))^2 + \phi W_e^4 (D^\gamma v_1(x)) (v_0'(x))^4 \\
 - 2a^2 \phi W_e^4 (D^\gamma v_1(x)) (v_0'(x))^4 + a^4 \phi W_e^4 (D^\gamma v_1(x)) (v_0'(x))^4 + 4S_t W_e^2 (v_0'(x)) (v_1'(x)) \\
 - 4a^2 S_t W_e^2 (v_0'(x)) (v_1'(x)) - 2W_e^2 (D^\gamma v_0(x)) (v_0'(x)) (v_1'(x)) + 2a^2 W_e^2 (D^\gamma v_0(x)) (v_0'(x)) (v_1'(x)) \\
 - 2\phi W_e^2 (D^\gamma v_0(x)) (v_0'(x)) (v_1'(x)) + 2a^2 \phi W_e^2 (D^\gamma v_0(x)) (v_0'(x)) (v_1'(x)) \\
 + 4S_t W_e^4 (v_0'(x))^3 (v_1'(x)) - 8a^2 S_t W_e^4 (v_0'(x))^3 (v_1'(x)) + 4a^4 S_t W_e^4 (v_0'(x))^3 (v_1'(x)) + \\
 4\phi W_e^4 (D^\gamma v_0(x)) (v_0'(x))^3 (v_1'(x)) - 8a^2 \phi W_e^4 (D^\gamma v_0(x)) (v_0'(x))^3 (v_1'(x)) \\
 + 4a^4 \phi W_e^4 (D^\gamma v_0(x)) (v_0'(x))^3 (v_1'(x)) = 0, v_2(0) = 0, v_2'(1) = 0.
 \end{aligned} \tag{B.4}$$

Similarly, we can find higher order problems and their solutions.

Keeping $\gamma = 1.7$, $a = 0.1$, $S_t = 0.01$, $W_e = 0.001$, and $\phi = 0.1$ fixed and using Caputo definition, the approximate solution is

$$V(x) = -0.00647381x^{1.7} - 6.77768 \times 10^{-13}x^{3.1}. \quad (\text{B.5})$$

The residual is

$$\begin{aligned} R = & D^\gamma V(x) + W_e^4(1-a^2)^2 [\phi D^\gamma V(x) + S_t] \left(\frac{dV(x)}{dx} \right)^4 \\ & - W_e^2(1-a^2) [D^\gamma V(x) + D^\gamma V(x) - 2S_t] \left(\frac{dV(x)}{dx} \right)^2 + S_t. \end{aligned} \quad (\text{B.6})$$

C. Homotopy Solution in Case 3 of Drainage

Homotopy $\Omega \times [0, 1] \longrightarrow R$ for equation (50) is

$$\begin{aligned} (1-p) \frac{d^2 v(x)}{dx^2} + p \left[D^\gamma v(x) + W_e^4(1-a^2)^2 [\phi D^\gamma v(x) + S_t] (D^\alpha v(x))^4 \right. \\ \left. - W_e^2(1-a^2) [D^\gamma v(x) + D^\gamma v(x) - 2S_t] (D^\alpha v(x))^2 + S_t \right] = 0. \end{aligned} \quad (\text{C.1})$$

Using (50) and (51), we obtain the following.
Zeroth-order problem:

$$D^\gamma v_0(x) = 0, v_0(0) = 0, v_0'(1) = 0. \quad (\text{C.2})$$

First-order problem:

$$\begin{aligned} S_t + 2S_t W_e^2 (D^\alpha v_0(x))^2 - 2a^2 S_t W_e^2 (D^\alpha v_0(x))^2 + S_t W_e^4 (D^\alpha v_0(x))^4 \\ - 2a^2 S_t W_e^4 (D^\alpha v_0(x))^4 + a^4 S_t W_e^4 (D^\alpha v_0(x))^4 - W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_0(x)) \\ + a^2 W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_0(x)) - \phi W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_0(x)) \\ + a^2 \phi W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_0(x)) + \phi W_e^4 (D^\alpha v_0(x))^4 (D^\gamma v_0(x)) \\ - 2a^2 \phi W_e^4 (D^\alpha v_0(x))^4 (D^\gamma v_0(x)) + a^4 \phi W_e^4 (D^\alpha v_0(x))^4 (D^\gamma v_0(x)) \\ + (D^\gamma v_1(x)) = 0, v_1(0) = 0, v_1'(1) = 0. \end{aligned} \quad (\text{C.3})$$

Second-order problem:

$$\begin{aligned} 4S_t W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) - 4a^2 S_t W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) \\ + 4S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) - 8a^2 S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) \\ + 4a^4 S_t W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) - 2W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) (D^\gamma v_0(x)) \\ + 2a^2 W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) (D^\gamma v_0(x)) - 2\phi W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) (D^\gamma v_0(x)) \\ + 2a^2 \phi W_e^2 (D^\alpha v_0(x)) (D^\alpha v_1(x)) (D^\gamma v_0(x)) + 4\phi W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) (D^\gamma v_0(x)) \\ - 8a^2 \phi W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) (D^\gamma v_0(x)) + 4a^4 \phi W_e^4 (D^\alpha v_0(x))^3 (D^\alpha v_1(x)) (D^\gamma v_0(x)) - \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

$$\begin{aligned}
& \cdot W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_1(x)) + a^2 W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_1(x)) - \phi W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_1(x)) \\
& + a^2 \phi W_e^2 (D^\alpha v_0(x))^2 (D^\gamma v_1(x)) + \phi W_e^4 (D^\alpha v_0(x))^4 (D^\gamma v_1(x)) - 2a^2 \phi W_e^4 (D^\alpha v_0(x))^4 (D^\gamma v_1(x)) \\
& + a^4 \phi W_e^4 (D^\alpha v_0(x))^4 (D^\gamma v_1(x)) + (D^\gamma v_2(x)) = 0, v_2(0) = 0, v_2'(1).
\end{aligned} \tag{C.4}$$

Similarly, we can find higher order problems and their solutions.

Approximate solution after applying Caputo definition and keeping $\alpha = 0.95$, $\phi = 0.1$, $W_e = 0.1$, $a = 0.1$, $S_t = 0.1$, and $\gamma = 1.99$ fixed is

$$V(x) = -0.0504625x^{1.99} - 2.39478 \times 10^{-6}x^{4.07}. \tag{C.5}$$

The residual is

$$\begin{aligned}
R = & D^\gamma V(x) + W_e^4(1-a^2)^2 [\phi D^\gamma V(x) + S_t] (D^\alpha V(x))^4 \\
& - W_e^2(1-a^2) [D^\gamma V(x) + D^\gamma V(x) - 2S_t] (D^\alpha V(x))^2 + S_t.
\end{aligned} \tag{C.6}$$

Flow rate and average velocity in the drainage case is

$$\begin{aligned}
Q &= \int_0^1 v(x) dx, \\
Q &= \frac{1}{6(15-16\alpha+4\alpha^2)\Gamma(4-\alpha)^2} \left(S_t(-3+2\alpha) \left(\frac{1}{-7+2\alpha} (3(-1+a^2)S_t^2W_e^2 \right. \right. \\
& \quad \cdot (-3+\alpha)^2(-16+\alpha(25+\alpha(-13+2\alpha)))(3+\phi))2(-5+2\alpha)\Gamma(4-\alpha)^2 \Big).
\end{aligned} \tag{C.7}$$

For drainage problem, the average velocity \bar{V} is

$$\begin{aligned}
\bar{V} &= \frac{1}{6(15-16\alpha+4\alpha^2)\Gamma(4-\alpha)^2} \left(S_t(-3+2\alpha) \left(\frac{1}{-7+2\alpha} (3(-1+a^2)S_t^2W_e^2(-3+\alpha)^2 \right. \right. \\
& \quad \cdot (-16+\alpha(25+\alpha(-13+2\alpha)))(3+\phi)) + 2(-5+2\alpha)\Gamma(4-\alpha)^2 \Big).
\end{aligned} \tag{C.8}$$

Data Availability

All the data used to support the findings of the study are given within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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