Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 1039222, 14 pages
https://doi.org/10.1155/2022/1039222

Research Article

Extended Residual Power Series Algorithm for Boundary Value Problems

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Received 20 February 2022; Revised 19 August 2022; Accepted 23 August 2022; Published 15 September 2022

Academic Editor: Angel Manuel Ramos

In this article, modification of the residual power series method (RPSM) is proposed for higher order boundary value problems (BVPs). The proposed algorithm is tested against various linear and nonlinear BVPs of orders nine up to thirteen. For the efficiency check of RPSM, obtained series solutions are compared with other available results in the literature. Analysis indicates that RPSM is better in terms of accuracy as compared to other mentioned schemes. As RPSM is applicable to BVPs without linearization, discretization, and perturbation, hence practically it is the best suitable solution tool for more complex BVPs in science and engineering.

1. Introduction

Higher order BVPs can be seen in different fields of engineering. Mostly they appear in fluid dynamics with rotations and instabilities. When the instability is set as ordinary (normal), it provides ninth, tenth, and eleventh-order models. On the other hand, if the instability is set as overstable, it leads to twelfth and thirteen-order models [1, 2]. Moreover, ninth and tenth-order BVPs also arise in astrophysics and hydrodynamics [3].

Solution of initial and boundary value problems for large-scale nonlinear equations is often required in engineering and scientific applications. These problems can be seen in elasticity, cosmology, material science, and engineering [4]. For the analysis and predictions of these problems, obtaining an accurate solution is the major focus of the scientific community [5]. As it is difficult to compute analytical solutions of higher order and nonlinear BVPs, there is a need for approximation techniques.

In the past few decades, various numerical techniques have been developed by many researchers for higher order BVPs. In 1992, Liao proposed the homotopy analysis method for BVPs [6]. After that, the authors proposed a combination of homotopy with perturbation for the solutions of BVPs in [7]. Wazwaz proposed a modified decomposition for higher order BVPs [8, 9]. Qayyum et al. proposed an extension of homotopy perturbation with Laplace transformation for BVPs in [10]. Siddiqi and Zulfiqar [11] applied the variational iteration method (VIM) to eleventh-order BVPs in [11]. In 2008, Noor and Mohyud-Din [12] also used VIM with polynomials to higher order BVPs [12]. Boutayeb and Twizell used the finite difference method for twelfth-order BVPs in [13]. Iftikhar et al. used the differential transform method (DTM) for thirteenth-order problems in [14]. Anderson et al. also attempted thirteenth-order BVPs using VIM in [15]. Ali et al. solved higher order BVPs using the optimal homotopy asymptotic method (OHAM) in [16]. Other scholars who tried different methodologies for the solution of higher order BVPs can be seen in [17–22].

Numerical methods have different limitations, that may include linearization, discretization, or perturbation. To overcome these restrictions, Abu Arqub et al. proposed the residual power series method (RPSM) for initial value problems (IVPs) in 2013 [23]. Since its introduction, RPSM has been applied by various researchers in different fields of science and engineering for IVPs [24, 25]. Kumar used RPSM to fractional Burgers' equations [26]. Mahmood and Yousif [27] applied RPSM to Boussinesq–Burgers equations [27]. Kumar and Yan extended RPSM to fractional diffusion equations [28]. Arqub applied RPSM to fuzzy differential equations [29]. Besides the above-mentioned studies, other scholars have also used RPSM in different scenarios [30–32]. In this discourse, modification of the residual power series method (RPSM) is proposed for ordinary differential equations with boundary conditions in the present manuscript. The proposed extension is tested extensively against different linear and nonlinear BVPs of higher order (nine up to thirteen) without linearization, discretization, and perturbation. This proposed modification is free from round-off errors and has less computational cost. For the rest of the manuscript, a basic idea of the proposed methodology is presented in Section 2, and a numerical illustration of RPSM is given in Section 3, while Section 4 contains the conclusion.

2. Basic Idea of the Modified Residual Power Series Method for BVPs

In this section, RPSM is explained for n^{th} order BVPs. This method comprises the power series expansion about the initial point $r = r_0$. In the case of BVPs, dummy initial conditions need to be taken for initializing the solution process.

$$\Phi^{(n)}(r) = f\left(r, \Phi^{(m)}(r)\right), \quad r_0 \le r \le r_1,\ m = 0(1)n-1, \tag{1}$$

with,

$$\begin{aligned} &\Phi^{(i)}(r_0) = \beta_i, \\ &\Phi^{(j)}(r_1) = \beta_j, \quad i, j = 0(1)n-1, \\ &i + j = n. \end{aligned} \tag{2}$$

Now, assuming the following k^{th} truncated power series as a solution for the given problem, we get

$$\Phi(r) = \sum_{i=0}^{k} A_i (r - r_0)^i, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where $A_i' s, i = 0(1)n-1$ are computed using (3) along with initial conditions, and hence, Equation (3) becomes

$$\Phi(r) = \Phi_{\text{initial}}(r) + \sum_{i=n}^{k} A_i (r - r_0)^i, \tag{4}$$

where

$$\Phi_{\text{initial}} = \sum_{i=0}^{n-1} A_i (r - r_0)^i. \tag{5}$$

Rewriting (5) in the following form:

$$\Phi^{(n)}(r) - f\left(r, \Phi^{(m)}(r)\right) = 0, \tag{6}$$

and using (4) in (5) gives the following k^{th} residual function:

$$\begin{aligned} \mathrm{Res}^k(r) = {} & \left(\sum_{i=n}^{k} i(i-1)\ldots(i-(n-1)) A_i (r - r_0)^{i-n} \right) \\ & - f\left(r, \sum_{i=m}^{k} i(i-1)\ldots(i-m+1) A_i (r - r_0)^{i-m} \right), \\ & m = 0(1)n-1. \end{aligned} \tag{7}$$

To obtain A_k for $k = n, n+1, \ldots$, we use (7) in the following equation [29]:

$$\frac{d^{k-n}}{\mathrm{d}r^{k-n}} \mathrm{Res}^k(r_0) = 0. \tag{8}$$

In the case of BVP, the series solution will contain dummy constants introduced at the start of the solution process. Optimal values of dummy constants can be obtained using the right boundary conditions. Higher accuracy can be achieved by increasing the order of the solution.

3. Numerical Illustration of MRPSM

Test Problem 1. Consider the following ninth-order linear ODE [8]:

$$\Phi^{(ix)}(r) = -9e^r + \Phi(r), \quad 0 \le r \le 1, \tag{9}$$

with boundary conditions

$$\begin{aligned} \Phi(0) &= 1, \\ \Phi'(0) &= 0, \\ \Phi''(0) &= -1, \\ \Phi'''(0) &= -2, \\ \Phi^{(iv)}(0) &= -3, \\ \Phi(1) &= 0, \\ \Phi'(1) &= -e, \\ \Phi''(1) &= -2e, \\ \Phi'''(1) &= -3e. \end{aligned} \tag{10}$$

The exact solution is as follows:

$$\Phi(r) = (1 - r)e^r. \tag{11}$$

After using RPSM, we obtain

$$\begin{aligned} \alpha_1 &= -3.99999078, \\ \alpha_2 &= -5.000189174, \\ \alpha_3 &= -5.998459, \\ \alpha_4 &= -7.00491325, \end{aligned} \tag{12}$$

and hence, the RPS solution becomes

Table 1: Comparison of RPSM with HPM [2], MVIM [33], and MDM [8] in Test Problem 1.

r	Exact sol	RPSM sol	E^* RPSM	E^* HPM [2]	E^* MVIM [33]	E^* MDM [8]
0.0	1	1	0	0	0	0
0.1	0.994654	0.994654	5.3×10^{-13}	-2.0×10^{-10}	-2.0×10^{-10}	-2.0×10^{-10}
0.2	0.977122	0.977122	1.1×10^{-11}	-2.0×10^{-10}	-2.0×10^{-10}	-2.0×10^{-10}
0.3	0.944901	0.944901	5.3×10^{-11}	-2.0×10^{-10}	-2.0×10^{-10}	-2.0×10^{-10}
0.4	0.895095	0.895095	1.3×10^{-10}	-2.0×10^{-10}	-2.0×10^{-10}	-2.0×10^{-10}
0.5	0.824361	0.824361	2.0×10^{-10}	-2.0×10^{-10}	-2.0×10^{-10}	-2.0×10^{-10}
0.6	0.728848	0.728848	2.2×10^{-10}	-6.0×10^{-10}	-6.0×10^{-10}	-6.0×10^{-10}
0.7	0.604126	0.604126	1.6×10^{-10}	-1.0×10^{-9}	-1.0×10^{-9}	-1.0×10^{-9}
0.8	0.445108	0.445108	6.9×10^{-11}	-2.0×10^{-9}	-2.0×10^{-9}	-2.0×10^{-9}
0.9	0.24596	0.24596	8.4×10^{-12}	-3.4×10^{-9}	-3.4×10^{-9}	-3.4×10^{-9}
1	0	2.7×10^{-17}	7.5×10^{-17}	0	0	0

E^* = exact − approx.

$$\Phi(r) = 1 - \frac{1}{2}r^2 - \frac{1}{3}r^3 - \frac{1}{8} - 0.0333333r^5 - 0.00694471r^6 - 0.00119017r^7 - 0.000173733r^8 - \frac{1}{45360}r^9 - \frac{1}{403200}r^{10} - \frac{1}{3991680}r^{11} - \frac{1}{43545600}r^{12}. \tag{13}$$

Results related to Test Problem 1 are shown in Table 1 and Figure 1.

Test Problem 2. Consider the following ninth-order nonlinear ODE [34]:

$$\Phi^{(ix)}(r) - \left(1 + e^{e^r} + e^r\right)e^r + e^{\Phi(r)}\Phi'''(r) + \Phi'(r)\Phi(r) = 0, \tag{14}$$

with boundary conditions

$$\begin{aligned} \Phi(0) &= 1, \\ \Phi'(0) &= 1, \\ \Phi''(0) &= 1, \\ \Phi'''(0) &= 1, \\ \Phi^{(iv)}(0) &= 1 \\ \Phi(1) &= e, \\ \Phi'(1) &= e, \\ \Phi''(1) &= e, \\ \Phi'''(1) &= e. \end{aligned} \tag{15}$$

The exact solution to this problem is as follows:

$$\Phi(r) = e^r. \tag{16}$$

Using RPSM, optimal values of dummies are as follows:

$$\begin{aligned} \alpha_1 &= 0.999964, \\ \alpha_2 &= 1.0007701, \\ \alpha_3 &= 0.9933072, \\ \alpha_4 &= 1.0235536, \end{aligned} \tag{17}$$

and hence, up-to-ten term solution is as follows:

$$\Phi(r) = 1 + r + \frac{1}{2!}r^2 + \frac{1}{3!}r^3 + \frac{1}{4!}r^4 + 0.00833303r^5 + 0.00138996r^6 + 0.000197085r^7 + 0.0000253858r^8 + \frac{1}{362880}r^9 + \frac{1}{3628800}r^{10}. \tag{18}$$

Results related to Test Problem 2 are shown in Table 2 and Figure 2.

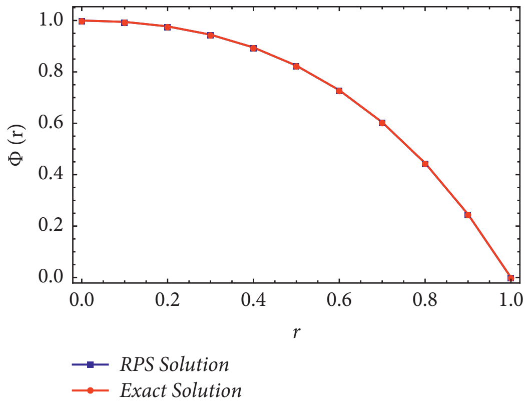
Test Problem 3. Consider the following ninth-order nonlinear ODE [34]:

$$\Phi^{(ix)}(r) = \cos^3 + \Phi'(r)(\Phi(r))^2, \tag{19}$$

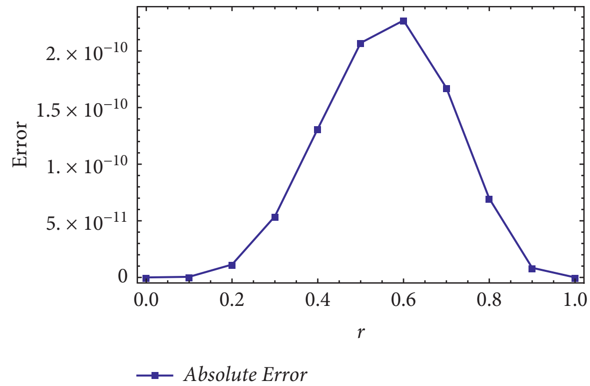
with boundary conditions

$$\begin{aligned} \Phi(0) &= 0, \\ \Phi'(0) &= 1, \\ \Phi''(0) &= 0, \\ \Phi'''(0) &= -1, \\ \Phi^{(iv)}(0) &= 0, \\ \Phi(1) &= \sin(1), \\ \Phi'(1) &= \cos(1), \\ \Phi''(1) &= -\sin(1), \\ \Phi'''(1) &= -\cos(1). \end{aligned} \tag{20}$$

The exact solution to the problem is as follows:



(a)



(b)

Figure 1: Graphical analysis of Test Problem 1. (a) Comparison of exact and RPS solutions. (b) RPSM error.

Table 2: Comparison of exact solution with RPSM and B-spline solutions [34] in Test Problem 2.

r	Exact sol	RPSM sol	E^* RPSM	E^* B-spline [34]
0.1	1.10517	1.10517	2.0×10^{-12}	1.7×10^{-6}
0.2	1.2214	1.2214	4.2×10^{-11}	5.9×10^{-6}
0.3	1.34986	1.34986	1.9×10^{-10}	1.0×10^{-5}
0.4	1.49182	1.49182	4.7×10^{-10}	1.5×10^{-5}
0.5	1.64872	1.64872	7.2×10^{-10}	9.8×10^{-6}
0.6	1.82212	1.82212	7.6×10^{-10}	3.4×10^{-6}
0.7	2.01375	2.01375	5.5×10^{-10}	1.5×10^{-5}
0.8	2.22554	2.22554	2.2×10^{-10}	1.7×10^{-5}
0.9	2.4596	2.4596	2.6×10^{-11}	1.1×10^{-5}

$$\Phi(r) = \sin(r), \quad (21)$$

and we obtain

$$\begin{aligned}\alpha_1 &= 1.000029395,\\ \alpha_2 &= -0.0006356287,\\ \alpha_3 &= -0.994430288,\\ \alpha_4 &= -0.01984309509,\end{aligned} \quad (22)$$

and hence, ten term RPS solution is as follows:

$$\begin{aligned}\Phi(r) &= r - \frac{1}{6}r^3 + 0.00833358r^5 - 8.82818\times10^{-7}r^6\\ &\quad - 0.000197308r^7 - 4.9214\times10^{-7}r^8 - \frac{1}{362880}r^9.\end{aligned} \quad (23)$$

Results related to Test Problem 3 are shown in Table 3 and Figure 3.

Test Problem 4. Consider the following tenth-order linear ODE [2]:

$$\Phi^{(x)}(r) = -8e^r + \Phi''(r), \quad (24)$$

with boundary conditions

$$\begin{aligned}\Phi(0) &= 1,\\ \Phi'(0) &= 0,\\ \Phi''(0) &= -1,\\ \Phi'''(0) &= -2,\\ \Phi^{(iv)}(0) &= -3,\\ \Phi(1) &= 0,\\ \Phi'(1) &= -e,\\ \Phi''(1) &= -2e,\\ \Phi'''(1) &= -3e,\\ \Phi^{(iv)}(1) &= -4e.\end{aligned} \quad (25)$$

The exact solution is as follows:

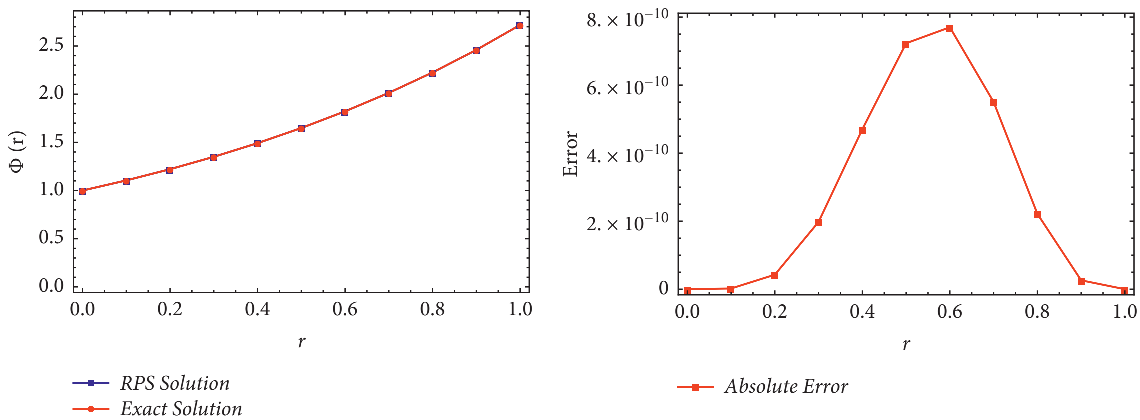
$$\Phi(r) = (1-r)e^r, \quad (26)$$

and we obtain

$$\begin{aligned}\alpha_1 &= -4.00000952,\\ \alpha_2 &= -4.999739,\\ \alpha_3 &= -6.00318249,\\ \alpha_4 &= -6.979722252,\\ \alpha_5 &= -8.056679,\end{aligned} \quad (27)$$

and hence, up to twelve term RPS solution is as follows:

$$\begin{aligned}\Phi(r) &= r - \frac{1}{2}r^3 - \frac{1}{3}r^3 - \frac{1}{8}r^4 - 0.0333334r^5 - 0.00694408r^6\\ &\quad - 0.00119111r^7 - 0.000173108r^8\\ &\quad - 0.000022202r^9 - \frac{1}{403200}r^{10} - \frac{1}{3991680}r^{11}\\ &\quad - \frac{1}{43545600}r^{12}.\end{aligned} \quad (28)$$

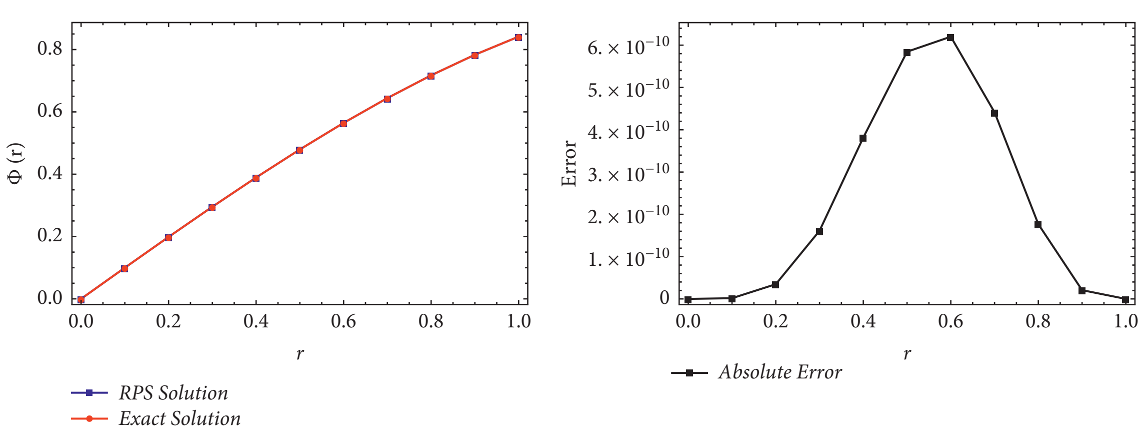


(a) (b)

FIGURE 2: Graphical analysis of Test Problem 2. (a) Comparison of exact and RPS solutions. (b) RPSM error.

TABLE 3: Comparison of RPSM with B-spline [34] in Test Problem 3.

r	Exact sol	RPSM sol	E^* RPSM	E^* B-spline [34]
0.1	0.0998334	0.0998334	1.6×10^{-12}	1.8×10^{-7}
0.2	0.198669	0.198669	3.4×10^{-11}	7.3×10^{-7}
0.3	0.29552	0.29552	1.6×10^{-10}	9.8×10^{-7}
0.4	0.389418	0.389418	3.8×10^{-10}	1.2×10^{-6}
0.5	0.479426	0.479426	5.8×10^{-10}	8.3×10^{-7}
0.6	0.564642	0.564642	6.2×10^{-10}	3.8×10^{-6}
0.7	0.644218	0.644218	4.4×10^{-10}	5.6×10^{-6}
0.8	0.717356	0.717356	1.7×10^{-10}	4.8×10^{-6}
0.9	0.783327	0.783327	2.0×10^{-11}	2.8×10^{-6}

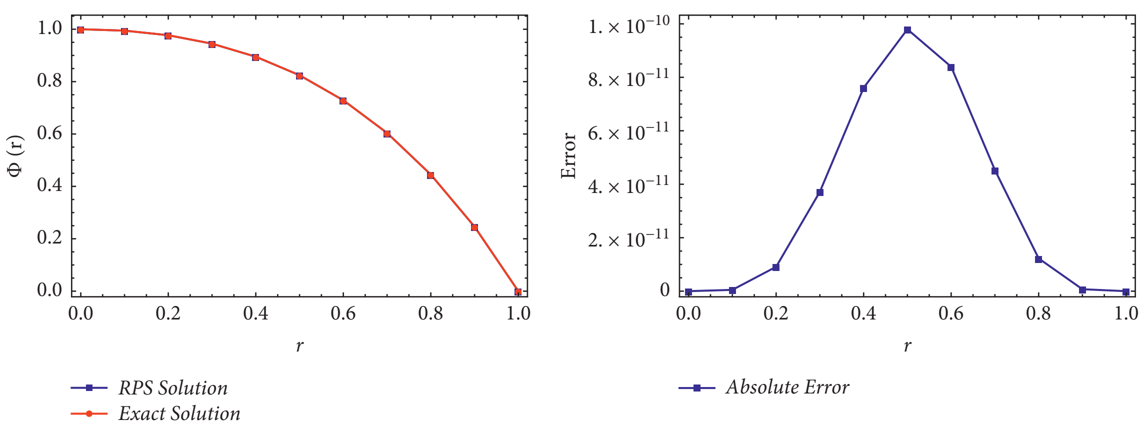


(a) (b)

FIGURE 3: Graphical analysis of Test Problem 3. (a) Comparison of exact and RPS solutions. (b) RPSM error.

Table 4: Comparison of RPSM with HPM [2] and MVIM [33] for Test Problem 4.

r	Exact sol	RPSM sol	E^* RPSM	E^* HPM [2]	E^* MVIM [33]
0	1	1	0	0	0
0.1	0.994654	0.994654	4.9×10^{-13}	0	0
0.2	0.977122	0.977122	9.1×10^{-12}	0	0
0.3	0.944901	0.944901	3.7×10^{-11}	5.0×10^{-10}	5.0×10^{-10}
0.4	0.895095	0.895095	7.6×10^{-11}	6.1×10^{-9}	6.1×10^{-9}
0.5	0.824361	0.824361	9.7×10^{-11}	4.4×10^{-8}	4.4×10^{-8}
0.6	0.728848	0.728848	8.3×10^{-11}	2.2×10^{-7}	2.2×10^{-7}
0.7	0.604126	0.604126	4.5×10^{-11}	9.0×10^{-7}	9.0×10^{-7}
0.8	0.445108	0.445108	1.2×10^{-11}	3.0×10^{-6}	3.0×10^{-6}
0.9	0.24596	0.24596	7.2×10^{-13}	8.6×10^{-6}	8.6×10^{-6}
1	0.	2.1×10^{-17}	7.6×10^{-17}	2.2×10^{-5}	2.2×10^{-5}



(a) (b)

Figure 4: Graphical analysis of Test Problem 4. (a) Comparison of exact and RPS solutions. (b) RPSM error.

Results related to Test Problem 4 are shown in Table 4 and Figure 4.

Test Problem 5. Consider the following tenth-order nonlinear ODE [8]:

$$\Phi^{(x)}(r) = e^{-r}\Phi^2(r), \tag{29}$$

with boundary conditions

$$\begin{aligned} \Phi(0) &= 1, \\ \Phi'(0) &= 1, \\ \Phi''(0) &= 1, \\ \Phi'''(0) &= 1, \\ \Phi^{(iv)}(0) &= 1, \\ \Phi(1) &= e, \\ \Phi'(1) &= e, \\ \Phi''(1) &= e, \\ \Phi'''(1) &= e, \\ \Phi^{(iv)}(1) &= e. \end{aligned} \tag{30}$$

The exact solution is as follows:

$$\Phi(r) = e^r. \tag{31}$$

Here, we obtain

$$\begin{aligned} \alpha_1 &= 1.00000078, \\ \alpha_2 &= 0.999978, \\ \alpha_3 &= 1.00026189, \\ \alpha_4 &= 0.99833070, \\ \alpha_5 &= 1.004668, \end{aligned} \tag{32}$$

and hence, up to twelve term RPS solution is as follows:

$$\begin{aligned} \Phi(r) = {} & 1 + 1.r + \frac{1}{2}r^2 + \frac{1}{6}r^3 + \frac{1}{24}r^4 + 0.00833334r^5 \\ & + 0.00138886r^6 + 0.000198465r^7 + 0.0000247602r^8 \\ & + 2.7686\times10^{-6}r^9 + \frac{1}{3628800}r^{10} + \frac{1}{39916800}r^{11} \\ & + \frac{1}{479001600}r^{12}. \end{aligned} \tag{33}$$

Table 5: Comparison of RPSM with HPM [2], MVIM [33], and MDM [8] in Test Problem 5.

r	Exact sol	RPSM sol	E^* RPSM	E^* HPM [2]	E^* MVIM [33]	E^* MDM [8]
0	1	1	0	0	0	0
0.1	1.10517	1.10517	4.0×10^{-14}	-1.4×10^{-6}	-1.4×10^{-6}	-1.4×10^{-6}
0.2	1.2214	1.2214	7.4×10^{-13}	-2.6×10^{-6}	-2.6×10^{-6}	-2.6×10^{-6}
0.3	1.34986	1.34986	3.0×10^{-12}	-3.7×10^{-6}	-3.7×10^{-6}	-3.7×10^{-6}
0.4	1.49182	1.49182	6.2×10^{-12}	-4.3×10^{-6}	-4.3×10^{-6}	-4.3×10^{-6}
0.5	1.64872	1.64872	8.0×10^{-12}	-4.5×10^{-6}	-4.5×10^{-6}	-4.5×10^{-6}
0.6	1.82212	1.82212	6.9×10^{-12}	-4.3×10^{-6}	-4.3×10^{-6}	-4.3×10^{-6}
0.7	2.01375	2.01375	3.7×10^{-12}	-3.7×10^{-6}	-3.7×10^{-6}	-3.7×10^{-6}
0.8	2.22554	2.22554	1.0×10^{-12}	-2.6×10^{-6}	-2.6×10^{-6}	-2.6×10^{-6}
0.9	2.4596	2.4596	5.9×10^{-14}	-2.6×10^{-6}	-2.6×10^{-6}	-2.6×10^{-6}
1	2.71828	2.71828	1.4×10^{-16}	2.0×10^{-9}	2.0×10^{-9}	2.0×10^{-9}

Results related to Test Problem 5 are shown in Table 5 and Figure 5.

Test Problem 6. Consider the following eleventh-order linear ODE [11]:

$$\Phi^{(xi)}(r) = -22(5+r)e^r + \Phi(r), \quad 0 \le r \le 1. \tag{34}$$

with boundary conditions

$$\begin{aligned}
\Phi(0) &= 1,\\
\Phi'(0) &= 1,\\
\Phi''(0) &= -1,\\
\Phi'''(0) &= -5,\\
\Phi^{(iv)}(0) &= -11,\\
\Phi^{(v)}(0) &= -19\\
\Phi(1) &= 0,\\
\Phi'(1) &= -2e,\\
\Phi''(1) &= -6e,\\
\Phi'''(1) &= -12e,\\
\Phi^{(iv)}(1) &= -20e.
\end{aligned} \tag{35}$$

The exact solution is as follows:

$$\Phi(r) = \left(1-r^2\right)e^r. \tag{36}$$

After applying RPSM and using the right boundary conditions, optimal values of $\alpha_i's$ are as follows:

$$\begin{aligned}
\alpha_1 &= -29.000001164,\\
\alpha_2 &= -40.99996384,\\
\alpha_3 &= -55.00048755,\\
\alpha_4 &= -70.996663,\\
\alpha_5 &= -89.00971073,
\end{aligned} \tag{37}$$

and hence the series solution is as follows:

$$\begin{aligned}
\Phi(r) = {} & 1 - \frac{1}{2!}r^2 - \frac{5}{3!}r^3 - \frac{11}{4!} - \frac{19}{5!}r^5 - 0.0402778r^6\\
& - 0.00813491r^7 - 0.0013641r^8 - 0.000195648r^9\\
& - 0.0000245287r^{10} - \frac{109}{39916800}r^{11} - \frac{131}{479001600}r^{12}\\
& - \frac{155}{6227020800}r^{13} - \frac{181}{87178291200}r^{14} - \cdots.
\end{aligned} \tag{38}$$

Results related to Test Problem 6 are shown in Table 6 and Figure 6.

Test Problem 7. Consider the following twelfth-order linear ODE [35]:

$$\Phi^{(xii)}(r) + r\Phi(r) + \left(120 + 23r + r^3\right)e^r = 0, \tag{39}$$

with boundary conditions

$$\begin{aligned}
\Phi(0) &= 0,\\
\Phi'(0) &= 1,\\
\Phi''(0) &= 0,\\
\Phi'''(0) &= -3,\\
\Phi^{(iv)}(0) &= -8,\\
\Phi^{(v)}(0) &= -15,\\
\Phi(1) &= 0,\\
\Phi'(1) &= -e,\\
\Phi''(1) &= -4e,\\
\Phi'''(1) &= -9e,\\
\Phi^{(iv)}(1) &= -16e,\\
\Phi^{(v)}(1) &= -25e.
\end{aligned} \tag{40}$$

The exact solution is as follows:

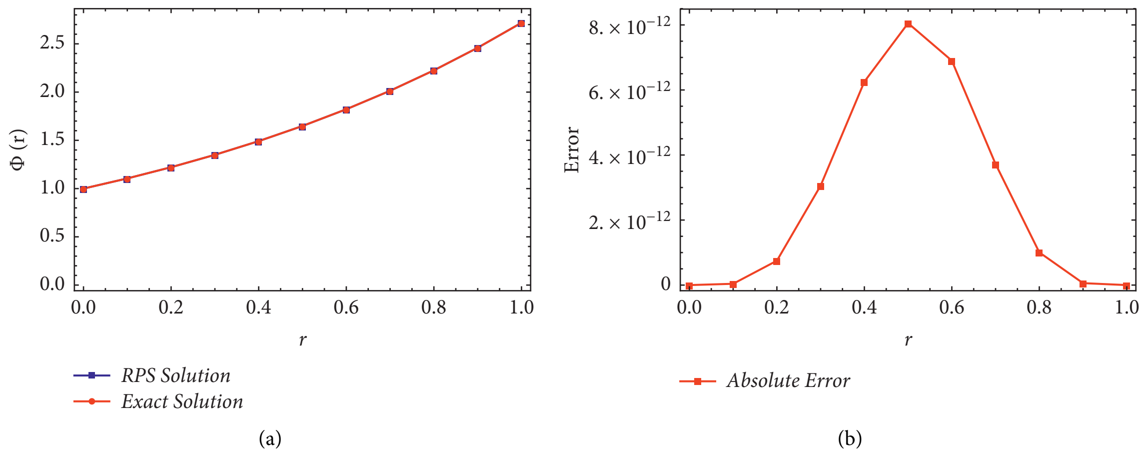


FIGURE 5: Graphical analysis of Test Problem 5. (a) Comparison of exact and RPS solutions. (b) RPSM error.

TABLE 6: Comparison of RPSM and VIM in Test Problem 6.

r	Exact sol	RPSM sol	E^* RPSM	E^* VIM [11]
0.0	1	1	0	0
0.1	1.09412	1.09412	8.8×10^{-16}	6.4×10^{-15}
0.2	1.17255	1.17255	3.8×10^{-14}	2.4×10^{-13}
0.3	1.22837	1.22837	2.3×10^{-13}	1.4×10^{-12}
0.4	1.25313	1.25313	6.5×10^{-13}	3.8×10^{-12}
0.5	1.23654	1.23654	1.0×10^{-12}	6.0×10^{-12}
0.6	1.16616	1.16616	1.1×10^{-12}	6.0×10^{-12}
0.7	1.02701	1.02701	7.3×10^{-13}	3.7×10^{-12}
0.8	0.801195	0.801195	2.3×10^{-13}	1.1×10^{-12}
0.9	0.467325	0.467325	1.6×10^{-14}	9.9×10^{-14}
1	0	-1.3×10^{-16}	7.9×10^{-17}	6.3×10^{-14}

$$\Phi(r) = r(1-r)e^r. \tag{41}$$

Using RPSM, optimal values of $\alpha_i's$ are as follows:

$$\begin{aligned} \alpha_1 &= -24.00000000, \\ \alpha_2 &= -34.9999999989, \\ \alpha_3 &= -48.0000000141, \\ \alpha_4 &= -62.99999988556, \\ \alpha_5 &= -80.000000517, \\ \alpha_6 &= -98.99999896, \end{aligned} \tag{42}$$

and hence, we obtain the following solution:

$$\begin{aligned} \Phi(r) = r &- \frac{1}{2}r^3 - \frac{1}{3}r^4 - \frac{1}{8}r^5 - 0.0333333r^6 - 0.00694444r^7 \\ &- 0.00119048r^8 - 0.000173611r^9 \\ &- 0.0000220459r^{10} - 2.48016\times10^{-6}r^{11} \\ &- \frac{1}{3991680}r^{12} - \frac{1}{43545600}r^{13} - \cdots. \end{aligned} \tag{43}$$

Results related to Test Problem 7 are shown in Table 7 and Figure 7.

Test Problem 8. Consider the following twelfth-order nonlinear ODE [36]:

$$\Phi^{(xii)}(r) = \frac{1}{2}e^{-r}\Phi^2(r), \tag{44}$$

with

$$\begin{aligned} \Phi''(0) &= 2, \\ \Phi(0) &= 2, \\ \Phi^{(iv)}(0) &= 2, \\ \Phi^{(vi)}(0) &= 2, \\ \Phi^{(viii)}(0) &= 2, \\ \Phi^{(x)}(0) &= 2 \\ \Phi(1) &= 2e, \\ \Phi''(1) &= 2e, \\ \Phi^{(iv)}(1) &= 2e, \\ \Phi^{(vi)}(1) &= 2e, \\ \Phi^{(viii)}(1) &= 2e, \\ \Phi^{(x)}(1) &= 2e. \end{aligned} \tag{45}$$

The exact solution is

$$\Phi(r) = 2e^r. \tag{46}$$

After using RPSM, we obtain

$$\begin{aligned} \alpha_1 &= 1.99999999, \\ \alpha_2 &= 2.000000001, \\ \alpha_3 &= 1.99999998, \\ \alpha_4 &= 2.00000011, \\ \alpha_5 &= 1.99999903, \\ \alpha_6 &= 2.00000611, \end{aligned} \tag{47}$$

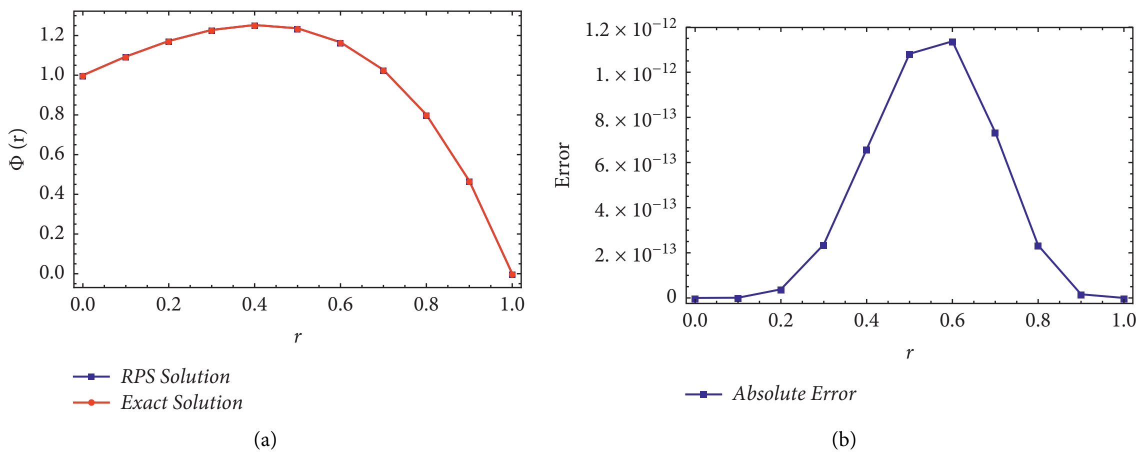


FIGURE 6: Graphical analysis of Test Problem 6. (a) Comparison of exact and RPS solutions. (b) RPSM error.

TABLE 7: Error comparison of RPSM and VIM in Test Problem 7.

r	Exact sol	RPSM sol	E^* RPSM	E^* VIM [35]
0	0	0	0	0
0.1	0.0994654	0.0994654	1.3×10^{-17}	9.5×10^{-13}
0.2	0.195424	0.195424	1.2×10^{-17}	1.2×10^{-13}
0.3	0.28347	0.28347	2.9×10^{-18}	3.3×10^{-13}
0.4	0.358038	0.358038	2.6×10^{-17}	5.3×10^{-13}
0.5	0.41218	0.41218	5.0×10^{-17}	8.0×10^{-13}
0.6	0.437309	0.437309	4.4×10^{-18}	1.1×10^{-12}
0.7	0.422888	0.422888	3.1×10^{-20}	3.9×10^{-13}
0.8	0.356087	0.356087	1.2×10^{-16}	1.2×10^{-13}
0.9	0.221364	0.221364	6.5×10^{-17}	8.2×10^{-13}
1.0	0	2.7×10^{-17}	6.0×10^{-17}	3.2×10^{-13}

and hence, we obtain the following solution:

$$\Phi(r) = 2 + 2.r + r^2 + 0.333333r^3 + \frac{1}{12}r^4 + 0.0166667r^5 + \frac{1}{360}r^6 + 0.000396825r^7 + \frac{1}{20160}r^8 + 5.51146 \times 10^{-6}r^9 + \frac{1}{1814400}r^{10} + 5.01044 \times 10^{-8}r^{11} + \frac{1}{239500800}r^{12} + 3.21181 \times 10^{-10}r^{13} + \cdots. \quad (48)$$

Results related to Test Problem 8 are shown in Table 8 and Figure 8.

Test Problem 9. Consider the following thirteenth-order linear ODE [15]:

$$\Phi^{(xiii)}(r) - \mathrm{Cos}(r) + \mathrm{Sin}(r) = 0, \quad (49)$$

with boundary conditions

$$\begin{aligned} \Phi(0) &= 1, \\ \Phi'(0) &= 1, \\ \Phi''(0) &= -1, \\ \Phi'''(0) &= -1, \\ \Phi^{(iv)}(0) &= 1, \\ \Phi^{(v)}(0) &= 1, \\ \Phi^{(vi)}(0) &= -1, \\ \Phi(1) &= \mathrm{Cos}(1) + \mathrm{Sin}(1), \\ \Phi'(1) &= \mathrm{Cos}(1) - \mathrm{Sin}(1), \\ \Phi''(1) &= -\mathrm{Cos}(1) - \mathrm{Sin}(1), \\ \Phi'''(1) &= -\mathrm{Cos}(1) + \mathrm{Sin}(1), \\ \Phi^{(iv)}(1) &= \mathrm{Cos}(1) + \mathrm{Sin}(1), \\ \Phi^{(v)}(1) &= \mathrm{Cos}(r) - \mathrm{Sin}(r). \end{aligned} \quad (50)$$

The exact solution is

$$\Phi(r) = \sin(r) + \cos(r). \quad (51)$$

Using the RPSM procedure, we obtaint

$$\begin{aligned} \alpha_1 &= -1.0000000005, \\ \alpha_2 &= 1.000000019, \\ \alpha_3 &= 0.99999968, \\ \alpha_4 &= -0.99999713, \\ \alpha_5 &= -1.00001432, \\ \alpha_6 &= 1.0000315, \end{aligned} \quad (52)$$

and hence, we obtain the following solution:

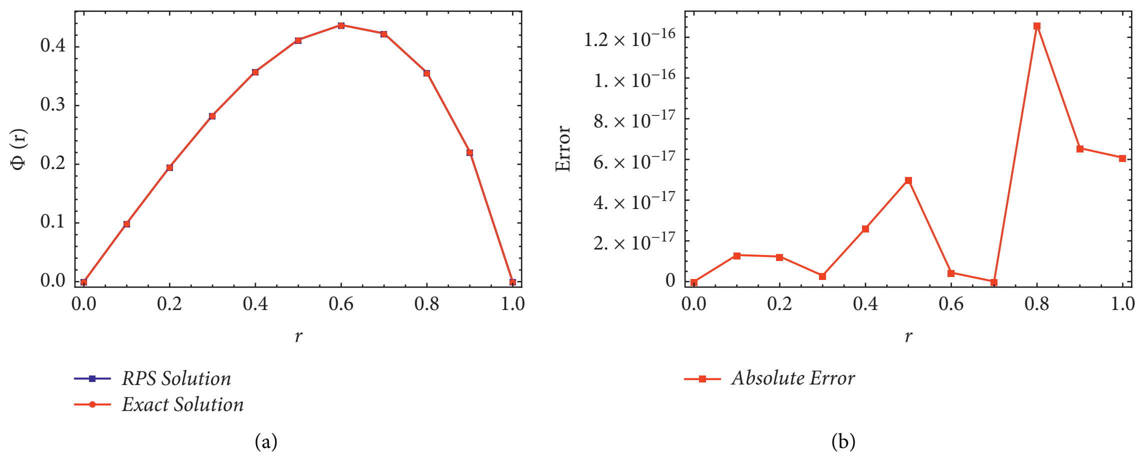


Figure 7: Graphical analysis of Test Problem 7. (a) Comparison of exact and RPS solutions. (b) RPSM error.

Table 8: Error comparison of RPSM and VIM in Test Problem 8.

r	Exact sol	RPSM sol	E^* RPSM	E^* VIM [36]
0	2	2	0	0
0.1	2.21034	2.21034	1.1×10^{-11}	2.0×10^{-4}
0.2	2.44281	2.44281	2.2×10^{-11}	3.9×10^{-4}
0.3	2.69972	2.69972	3.0×10^{-11}	5.4×10^{-4}
0.4	2.98365	2.98365	3.6×10^{-11}	6.3×10^{-4}
0.5	3.29744	3.29744	3.8×10^{-11}	6.6×10^{-4}
0.6	3.64424	3.64424	3.6×10^{-11}	6.3×10^{-4}
0.7	4.02751	4.02751	3.0×10^{-11}	5.3×10^{-4}
0.8	4.45108	4.45108	2.2×10^{-11}	3.8×10^{-4}
0.9	4.91921	4.91921	1.1×10^{-11}	2.0×10^{-4}
1	5.43656	5.43656	3.4×10^{-16}	2.0×10^{-4}

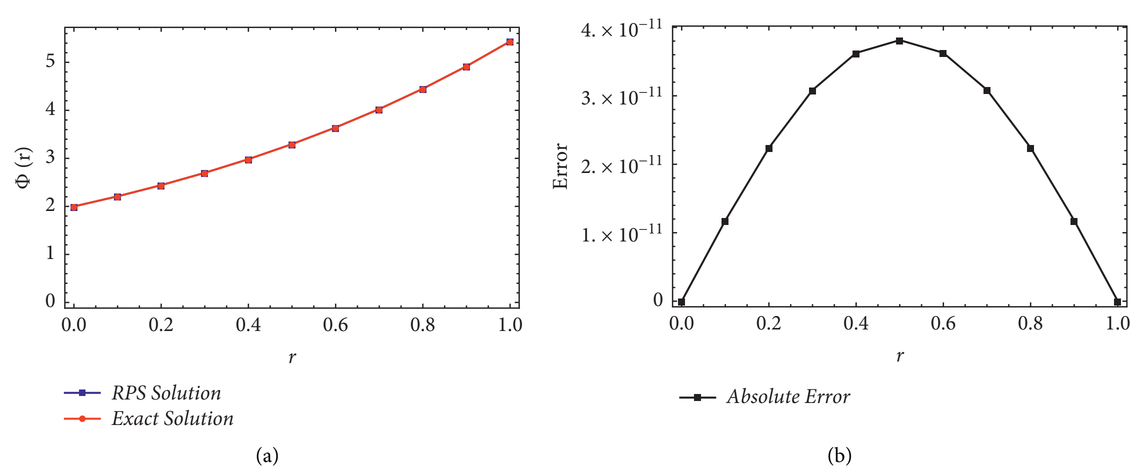


Figure 8: Graphical analysis of Test Problem 8. (a) Comparison of exact and RPS solutions. (b) RPSM error.

Table 9: Comparison of RPSM with DTM and VIM in Test Problem 9.

r	Exact sol	RPSM sol	E^* RPSM	E^* DTM [14]	E^* VIM [15]
0	1	1	0	0	0
0.1	1.09484	1.09484	0	2.2×10^{-16}	3.8×10^{-15}
0.2	1.17874	1.17874	2.2×10^{-16}	0	1.4×10^{-13}
0.3	1.25086	1.25086	0	2.2×10^{-15}	8.8×10^{-13}
0.4	1.31048	1.31048	2.2×10^{-16}	6.6×10^{-15}	2.3×10^{-12}
0.5	1.35701	1.35701	0	1.1×10^{-14}	3.8×10^{-12}
0.6	1.38998	1.38998	0	1.0×10^{-14}	5.1×10^{-11}
0.7	1.40906	1.40906	0	5.3×10^{-15}	1.5×10^{-11}
0.8	1.41406	1.41406	0	8.8×10^{-16}	8.9×10^{-11}
0.9	1.40494	1.40494	0	0	4.7×10^{-10}
1.0	1.38177	1.38177	4.4×10^{-16}	0	2.0×10^{-9}

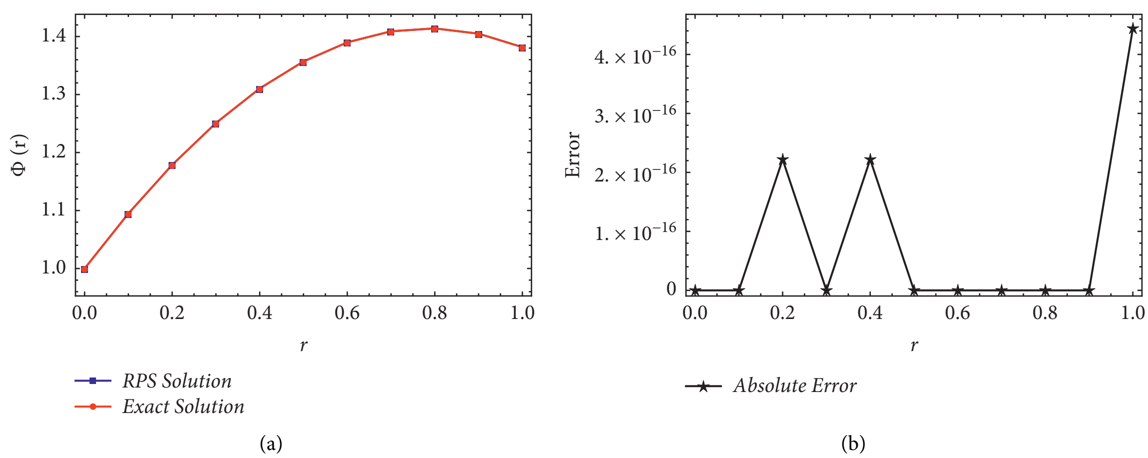


Figure 9: Graphical analysis of Test Problem 9. (a) Comparison of exact and RPS solutions. (b) RPSM error.

$$\begin{aligned}\Phi(r) = {} & 1 + r - \frac{1}{2}r^2 - \frac{1}{6}r^3 + \frac{1}{24}r^4 + \frac{1}{120}r^5 - \frac{1}{720}r^6 \\ & - 0.000198413r^7 + 0.0000248016r^8 \\ & + 2.75573\times10^{-6}r^9 \\ & - 2.75572\times10^{-7}r^{10} - 2.50525\times10^{-8}r^{11} \\ & + 2.08774\times10^{-9}r^{12} + \frac{1}{6227020800}r^{13} - \cdots .\end{aligned} \tag{53}$$

Results related to Test Problem 9 are shown in Table 9 and Figure 9.

Test Problem 10. Consider the following thirteenth-order nonlinear ODE [15]:

$$\Phi^{(xiii)}(r) = (\Phi(r))^2 e^{-r}, \tag{54}$$

with boundary conditions

$$\begin{aligned} \Phi(0) &= 1, \\ \Phi'(0) &= 1, \\ \Phi''(0) &= 1, \\ \Phi'''(0) &= 1, \\ \Phi^{(iv)}(0) &= 1, \\ \Phi^{(v)}(0) &= 1, \\ \Phi^{(vi)}(0) &= 1, \\ \Phi(1) &= e, \\ \Phi'(1) &= e, \\ \Phi''(1) &= e, \\ \Phi'''(1) &= e, \\ \Phi^{(iv)}(1) &= e, \\ \Phi^{(v)}(1) &= e. \end{aligned} \tag{55}$$

Table 10: Comparison of RPSM with DTM and VIM in Test Problem 10.

r	Exact sol	RPSMa sol	E^* RPSM	E^* DTM [14]	E^* VIM [15]
0	1	1	0	0	0
0.1	1.10517	1.10517	7.7×10^{-17}	4.4×10^{-16}	4.1×10^{-14}
0.2	1.2214	1.2214	7.6×10^{-18}	4.4×10^{-16}	2.6×10^{-12}
0.3	1.34986	1.34986	1.1×10^{-19}	2.4×10^{-15}	2.9×10^{-11}
0.4	1.49182	1.49182	1.7×10^{-17}	7.3×10^{-15}	1.6×10^{-10}
0.5	1.64872	1.64872	1.1×10^{-16}	1.2×10^{-14}	6.3×10^{-10}
0.6	1.82212	1.82212	9.3×10^{-17}	1.1×10^{-14}	1.8×10^{-9}
0.7	2.01375	2.01375	1.9×10^{-16}	5.7×10^{-15}	4.4×10^{-9}
0.8	2.22554	2.22554	3.1×10^{-16}	1.7×10^{-15}	9.2×10^{-9}
0.9	2.4596	2.4596	1.9×10^{-16}	8.8×10^{-16}	1.5×10^{-8}
1.0	2.71828	2.71828	6.4×10^{-17}	0.	2.0×10^{-8}

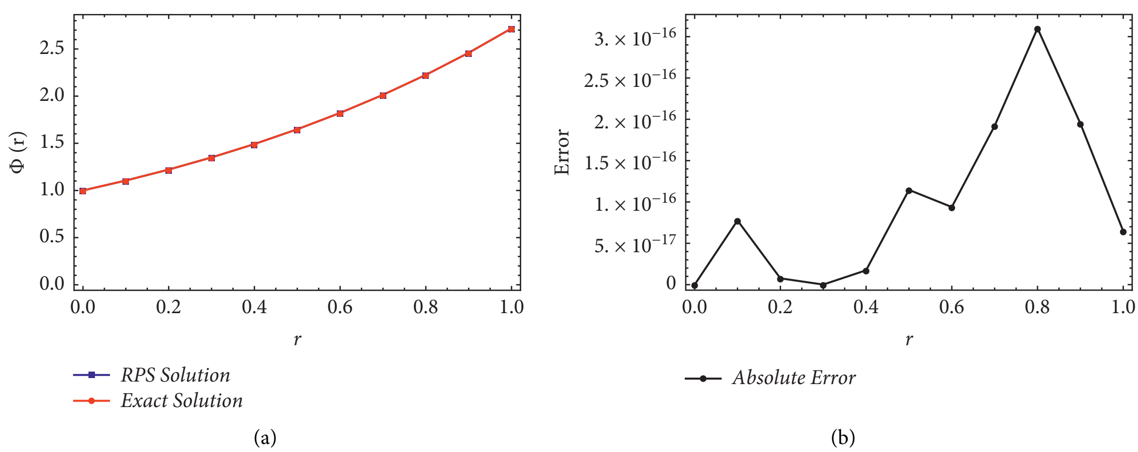


Figure 10: Graphical analysis of Test Problem 10. (a) Comparison of exact and RPS solutions. (b) RPSM error.

The exact solution is

$$\Phi(r) = e^r. \tag{56}$$

After using RPSM, we obtain

$$\begin{aligned} \alpha_1 &= 0.999999999, \\ \alpha_2 &= 1.00000002, \\ \alpha_3 &= 0.9999996, \\ \alpha_4 &= 1.00000344, \\ \alpha_5 &= 0.9999827, \\ \alpha_6 &= 1.000037, \end{aligned} \tag{57}$$

and hence, we obtain the following solution:

$$\begin{aligned} \Phi^{13}(r) = 1 + r + \frac{1}{2}r^2 + \frac{1}{6}r^3 + \frac{1}{24}r^4 + \frac{1}{120}r^5 + \frac{1}{720}r^6 \\ + 0.000198413r^7 + 0.0000248016r^8 \\ + 2.75573 \times 10^{-6}r^9 - 2.75574 \times 10^{-7}r^{10} \\ + 2.50517 \times 10^{-8}r^{11} + 2.08775 \times 10^{-9}r^{12} \\ + \frac{1}{6227020800}r^{13} + \cdots . \end{aligned} \tag{58}$$

Results related to Test Problem 10 are shown in Table 10 and Figure 10.

4. Results and Discussion

In this article, modification of the residual power series method (RPSM) is tested for higher orders BVPs, and obtained series solutions are compared with other available results in the literature in a tabular form. First, RPSM is applied to ninth-order linear BVP, and the results are numerically compared with HPM, MVIM, and MDM in Table 1. The analysis of the table shows that RPSM is better than the rest of the mentioned schemes. A graphical analysis of Test Problem 1 is presented in Figure 1. The exact and RPS solutions are illustrated in Figure 1(a), and this plot shows the validity of approximate solutions. Besides this, the absolute error in Test Problem 1 using RPSM is given in Figure 1(b). The error plot indicates that the error is bounded between $(0, 2.2 \times 10^{-10})$.

Results related to ninth-order nonlinear BVPs (Test Problems 2 and 3) are shown in Table [2, 3] and Figure [2, 3]. A numerical comparison of RPSM with B-spline results is presented in Tables 2 and 3. These tables clearly show the supremacy of RPSM in terms of accuracy. Error bounds in Test Problems 2 and 3 are between $(2.0 \times 10^{-12}, 7.6 \times 10^{-10})$

and $(1.6 \times 10^{-12}, 6.2 \times 10^{-10})$, respectively. Figures 2(a) and 3(a) present the comparison of exact and RPS solutions, while Figures 2(b) and 3(b) show the absolute error in RPSM solution.

Results related to Test Problem 4 (tenth-order linear BVP) are shown in Table 4 and Figure 4. A numerical comparison of RPSM with HPM and MVIM is presented in Table 4, which clearly indicates that RPSM has better accuracy. Error bounds in this problem are $(0, 9.7 \times 10^{-11})$. Figure 4(a) presents the comparison of exact and RPS solutions, while Figure 4(b) shows the absolute error in Test Problem 4.

The numerical results of tenth-order nonlinear BVP (Test Problem 5) are shown in Table 5 and Figure 5. A numerical comparison of RPSM with HPM, MVIM, and MDM is in Table 5. Error bounds in this problem are $(0, 8.0 \times 10^{-12})$. Figure 5(a) presents the comparison of exact and RPS solutions, while Figure 5(b) shows the absolute error in Test Problem 5.

A numerical analysis of eleventh-order linear BVP (Test Problem 6) is shown in Table 6 and Figure 6. Results of RPSM are compared with VIM numerically in Table 6. Error bounds in this problem are $(0, 1.1 \times 10^{-12})$. Figure 6(a) shows the comparison of exact and RPS solutions, while Figure 6(b) presents the absolute error in Test Problem 6.

Numerical analysis of twelfth-order linear BVP (Test Problem 7) is shown in Table 7 and Figure 7. RPSM results are compared with VIM numerically in Table 7. Error bounds in this problem are $(0, 1.2 \times 10^{-16})$. Figure 7(a) shows the comparison of exact and RPS solutions and hence confirms the validity of RPS solution. Figure 7(b) presents the absolute error in Test Problem 7.

Analysis of twelfth-order nonlinear BVP (Test Problem 8) is shown in Table 8 and Figure 8. Here, RPSM and VIM results are compared numerically in Table 8. Error bounds in this case are $(0, 1.23.8 \times 10^{-11})$. Figure 8(a) shows the comparison of exact and RPS solutions and hence confirms the validity of the obtained approximate solution. Figure 8(b) illustrates the absolute error in Test Problem 8.

Results related to Test Problem 9 (thirteenth-order linear BVP) are shown in Table 9 and Figure 9. A numerical comparison of RPSM with DTM and VIM is presented in Table 9. Analysis of the table shows the dominance of RPSM over the other mentioned schemes. Error bounds in this problem are $(0, 4.4 \times 10^{-16})$. Figure 9(a) presents the comparison of exact and RPS solutions, while Figure 9(b) shows the absolute error in Test Problem 9.

Analysis of thirteenth-order nonlinear BVP (Test Problem 10) is shown in Table 10 and Figure 10. Here, RPSM results are compared with DTM and VIM numerically in Table 10. Error bounds in this case are $(0, 3.1 \times 10^{-16})$. Figure 10(a) shows the comparison of exact and RPS solutions and hence confirms the validity of obtained approximate solutions. Figure 10(b) illustrates the absolute error in Test Problem 10.

Numerical and graphical analyses indicate that RPSM is better in terms of accuracy than other mentioned techniques. Hence, RPSM is applicable to different classes of BVPs without linearization, discretization, and perturbation in practical situations.

5. Conclusion

In this article, an extension of RPSM is proposed for BVPs. This algorithm is directly applied to different linear and nonlinear standard problems without linearization, discretization, and perturbation. Numerical and graphical analyses reveal that RPSM is an efficient and effective technique as compared to other mentioned schemes (HPM, VIM, DTM, MVIM, MDM, and B-spline). Also, RPSM provides fast convergent series solutions without being affected by round-off errors and hence can be utilized for more complex problems in different areas of science and engineering.

Data Availability

All the data are available in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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