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Research Article

Exact Wiener Index of the Direct Product of a Path and a Wheel Graph

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The direct product is one of the most important methods to construct large-scale graphs using existing small-scale graphs, and the topological structure parameters of the constructed large-scale graphs can be derived from small-scale graphs. For a simple undirected graph G, its Wiener index $W(G)$ is defined as the sum of the distances between all different unordered pairs of vertices in the graph. Path is one of the most common and useful graphs, and it is found in almost all virtual and real networks; wheel graph is a kind of graph with good properties and convenient construction. In this paper, the exact value of the Wiener index of the direct product of a path and a wheel graph is given, and the obtained Wiener index is only derived from the orders of the two factor graphs.

1. Introduction

In this paper, a simple graph G refers to the set of vertices $V(G)$ and the set of different unordered vertex pairs $\{a, b\}$ in $V(G)$, i.e., the edge set $E(G)$ of G. For an edge $e = \{a, b\}$, we say that e connects a, b, the vertices a and b are the endpoints of e, and a and b are adjacent to each other. The number of the elements in $V(G)$ is the order of the graph G, and size of the graph G is the number of edges in the graph. A graph G is called connected, if given any two vertices $v_i, v_j \in V(G)$, there is a path from v_i to v_j.

The direct product of two given graphs G and H is denoted as $G \times H$, with vertex set

$$\begin{aligned} V(G \times H) &= V(G) \times V(H) \\ &= \{(a_i, b_j): a_i \in V(G), b_j \in V(H)\}. \end{aligned} \tag{1}$$

Two different vertices (a_i, b_j) and (a_k, b_l) of the direct product graph $G \times H$ are adjacent to each other, if and only if $(a_i, a_k) \in E(G)$ and $(b_j, b_l) \in E(H)$. We call G and H the factor graphs of the direct product graph $G \times H$, and for convenience, vertex (a_i, b_j) is usually represented by $a_i b_j$.

A path is the topological structure of bus network, linear array network, and other important networks. Wheel graph has good connectivity, transitivity, stability, and other characteristics. By combining the two graphs in the way of direct product, the resulting graph will not only inherit their excellent properties but also perform better than the factor graphs in terms of various properties. In this paper, we mainly research the Wiener index of the direct product of a path and a wheel graph, so we review the definitions of paths and wheel graphs next.

Let a finite sequence $a_0 e_1 a_1 e_2 a_2 \ldots a_{n-1} e_n a_n$ be composed of vertices and edges, and in this sequence, e_i connects a_{i-1} and a_i $(1 \leq i \leq n)$. When all the vertices and edges in the sequence are different from each other, the graph composed of vertices and edges in the sequence is called a $a_0 a_n$-path, and the length of the path is n, denoted by P_{n+1}. The graph obtained by connecting two vertices a_0 and a_n of a $a_0 a_n$-path with an edge is a cycle C_{n+1} with order $n + 1$. Connect all vertices of C_n to a new isolated vertex (wheel center), and the resulting graph is a wheel graph of order $n + 1$, denoted by W_{n+1}; it is obvious that $n + 1 > 3$. The direct product graph $P_2 \times W_6$ of a path P_2 and a wheel graph W_6 is shown in Figure 1.

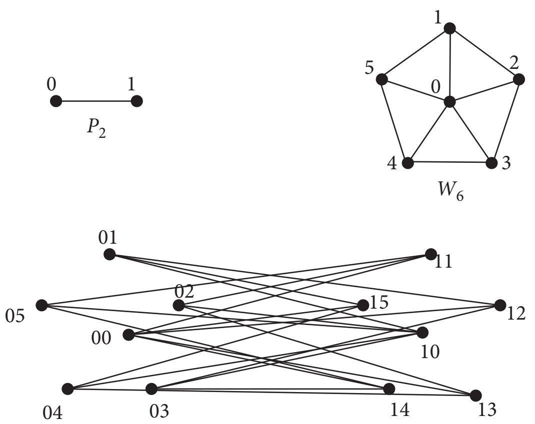

Figure 1: $G = P_2 \times W_6$.

Wiener index is widely used in many fields and disciplines, such as physical chemistry, biology, and computer science [1–5], giving credit to the pioneering work of the chemist Wiener [6]. In graph theory, Wiener index of a connected graph G is defined as $W(G) = 1/2\sum_{a,b\in V(G)\times V(G)} d_G(a,b)$, denoted by $W(G)$, where $d_G(a,b)$ is the distance between vertices a and b in G. Since the Wiener index was successfully introduced into graph theory, many scholars have done a lot of research on it. This paper is mainly interested in the Wiener index of direct product graphs (to know further about Wiener index of product graphs and other graphs, refer to [7–10]).

In 1959, Sabidussi proposed the definition of Cartesian product, strong product, and direct product [11]. Product operations of graphs are effective tools to construct large-scale graphs by using small-scale factor graphs and provide important models for computer connection. By using the graph product methods, large networks can be constructed by factor networks, and all factor networks are subnetworks of the large networks, so the large networks retain the good characteristics of factor networks, and these properties can be well derived from the properties of factor networks. Because of its simple way of connection, direct product has attracted extensive attention of many scholars. Soon after the direct product graph was put forward, Weichsel observed that the direct product graph of two factor graphs is connected while the two factor graphs are connected, and at least one of them is not bipartite [12]. For more than half a century, direct product has been studied by many scholars with different names (such as weak product, Kronecker product, tensor product, cardinal product, and category product), including its matching problem, Hamiltonicity, and connectivity [13–15].

The general result of Wiener index of direct product of graphs has not been obtained yet, but many scholars try to make breakthroughs from different types of factor graphs. For example, Hoji et al. [16] researched the Wiener index of the direct product of a connected graph G and a complete graph K_n of order $n(n \geq 3)$. Pattabiraman et al. gave the Wiener index of the direct product of a path and a cycle [17]. In this paper, by designing the minimum routing, we use the transitivity and symmetry of $P_m \times W_n$ to calculate the exact Wiener index of it. For more literatures on the product graphs, refer to [18] and [19].

2. Main Results

Reviewing Figure 1, we find that arranging the vertices of $P_m \times W_n$ according to Figure 1 is not conducive to observing the topology of $P_m \times W_n$. So, we rearrange the vertices of $P_m \times W_n$ and mark the vertices similar to Figure 1, mark the vertices of P_m as $0, 1, 2, 3, \ldots, m-2, m-1$, and mark the vertices of W_n as $0, 1, 2, 3, \ldots, n-2, n-1$. Here 0 is the wheel center of W_n; on this premise, the set of vertices of $P_m \times W_n$ is labeled $\cup_{i=0}^{m-1} \cup_{j=0}^{n-1} \{ij\}$, and if there is ambiguity, we use (i, j) instead of ij. After rearranging the vertices, we find that $P_m \times W_n$ is actually a m-partite graph.

In order to observe $P_m \times W_n$ better, we first give an example of $P_3 \times W_5$ (see Figure 2). In addition, we calculate the Wiener index of the direct product graph in Figure 1 and the Wiener index of $P_3 \times W_6$ manually and list the results in Table 1.

Next, we will give the main theorems by combining Figure 2 and the properties of the direct product graph (for undefined symbols, refer to Hammack et al. [18]).

Theorem 1. *Let P_{2m} be a path of order $2m (m > 1)$, with vertices $v_0, v_1, v_2, \ldots, v_{2m-1}$, and W_n is a wheel graph of order $n (n > 3)$, with vertices $0, 1, 2, \ldots, n-2, n-1$. The Wiener index of their direct product $G = P_{2m} \times W_n$ is*

$$W(G) = \frac{\left(4m^3 + 17m - 6\right)n^2 - 54mn + 48m - 24}{3}. \quad (2)$$

Proof. Let S_{ij} be the sum of the distances from ij to every vertex $kl \in V(G)$, i.e.,

$$S_{ij} = \sum_{kl\in V(G)} d_G(ij, kl), \quad (3)$$

where $d_G(ij, kl)$ refers to the distance from ij to kl, and note that the distance from ij to itself is 0. By using the symmetry of G, we can get the following formula:

$$\begin{aligned} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{ij} &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{2m-(i+1),j} \\ \Rightarrow \sum_{i=0}^{2m-1}\sum_{j=0}^{n-1} S_{ij} &= 2\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{ij} \\ &= 2\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{2m-1-i,j} = 2W(G) \\ \Rightarrow W(G) &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{ij}. \end{aligned} \quad (4)$$

According to formula (4), we just need to calculate $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{ij}$; next, we use the characteristics of wheel graph to simplify the calculation process again.

According to the definition of W_n, if we do not consider the wheel center and the edges associated with wheel center of W_n, then the graph composed of other vertices and edges is a cycle C_{n-1}. From the transitivity of vertices on a cycle, there is an automorphism of G that maps vertex ij to vertex

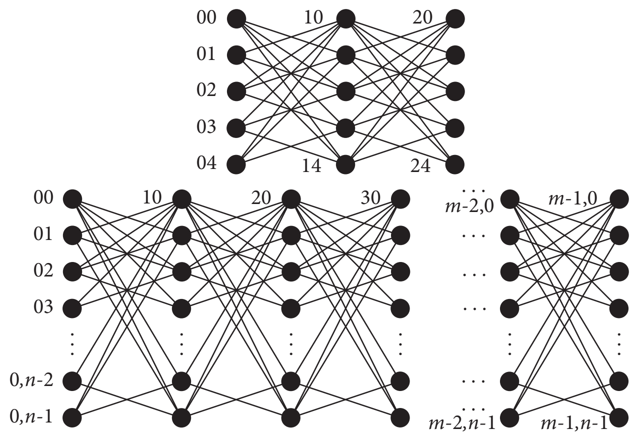

Figure 2: $G' = P_3 \times W_5$ and $G = P_m \times W_n$.

il, where $S_{ij} = S_{il}$, $j \neq l$, and $j, l \neq 0$. Therefore, when $j \neq 0$, we just need to calculate each S_{i1} and multiply it by $n-1$ instead of finding all S_{ij}, $j = 1, 2, 3 \ldots, n-1$. For the case of $j = 0$, we will deal with it separately.

Table 1: Wiener indices of $P_2 \times W_6$ and $P_3 \times W_6$.

Graph	Index		
	$\lvert V(P_m)\rvert$	$\lvert V(W_n)\rvert$	WI
$P_2 \times W_6$	2	6	128
$P_3 \times W_6$	3	6	298

We divide the proof into two cases: in Case 1, we calculate the sum of the distances from vertex $\sum_{i=0}^{m-1} i1$ to all vertices in $V(G)$, and in Case 2, we calculate the sum of the distances from vertex $\sum_{i=0}^{m-1} i0$ to all vertices in $V(G)$. □

Case 1. The $(n-1)$ times of the sum of the distances from vertex $01, 11, \ldots, (m-1, 1)$ to all vertices in $V(G)$, i.e., $(n-1)\sum_{i=0}^{m-1} S_{i1}$. Because when $i = 0, 1$, the distance rule of the distances from vertex $0i$ to all vertices in $V(G)$ is a bit special, for the vertices 01 and 11, we cannot share the same distance rule with other vertices, so we give S_{01} and S_{11} separately.

$$\begin{aligned}
S_{01} &= 2(n-1) + 3 + 3(n-3) + n[2 + \cdots + (2m-1)], \\
S_{11} &= 3 + 3(n-3) + 2(n-1) + 3 + 3(n-3) + n[2 + 3 + \cdots + (2m-2)], \\
\sum_{i=2}^{m-1} S_{i1} &= \sum_{i=2}^{m-1} \{n(2 + 3 + \cdots + i) + 3(n-2) + 2(n-1) + 3(n-2) + n[2 + 3 + \cdots + (2m-1-i)]\} \\
&= \sum_{i=2}^{m-1} \left\{ \frac{i(i+1)n - 2n}{2} + 8n - 14 + \frac{(2m-1-i)(2m-i)n - 2n}{2} \right\}.
\end{aligned} \tag{5}$$

Explanations for the terms appearing in (5) are as follows.

As shown in Figure 3, the distances from 01 to vertices $00, 02, 03, 04 \cdots, (0, n-1)$ are all 2, the distances to vertices $10, 12, (1, n-1)$ are all 1, the distances to vertices $13, 14, \ldots, (1, n-2)$ are all 3, the distances to vertices $20, 21, 22, \ldots, (2, n-1)$ are all 2, the distances to vertices $30, 31, 32, \ldots, (3, n-1)$ are all 3, and so on; the distances from 01 to its farthest vertices $(2m-1, 1), (2m-1, 2), (2m-1, 3), \ldots, (2m-1, n-1)$ are all $2m-1$.

Now let us discuss vertex 11, namely, the second equation of formula (5). For vertex 11, there is a column of vertices that lie on its left side, which are vertices $00, 01, 02, \ldots, (0, n-1)$. The sum of the distances from vertex 11 to these vertices is equal to the sum of the distance from vertex 01 to vertices $10, 11, 12, \ldots, (1, n-2)$. The same rule applies to $\sum_{i=2}^{m-1} S_{i1}$; therefore, we give only the shortest path from 01 to $V(G)$ (Figure 3). By using the structural properties of G and the properties of direct product and formula (5), we give the final result of Case 1 as follows:

$$\begin{aligned}
(n-1)\sum_{i=0}^{m-1} S_{i1} &= (n-1)\left(S_{01} + S_{11} + \sum_{i=2}^{m-1} S_{i1} \right) \\
&= \frac{\left[\left(4m^3 + 17m - 6\right)n^2 + (-4m - 59m)n + 42m - 18\right]}{3}.
\end{aligned} \tag{6}$$

Formula (6) is the final result of Case 1, and we will deal with Case 2 in a similar way.

Case 2. Calculate $\sum_{i=0}^{m-1} S_{i0}$, the sum of the distances from vertices $00, 10, 20, \ldots, (m-1, 0)$ to all vertices in $V(G)$. The

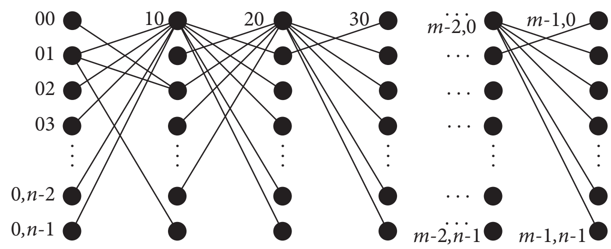

Figure 3: The shortest paths from vertex 01 to all vertices in $V(G)$.

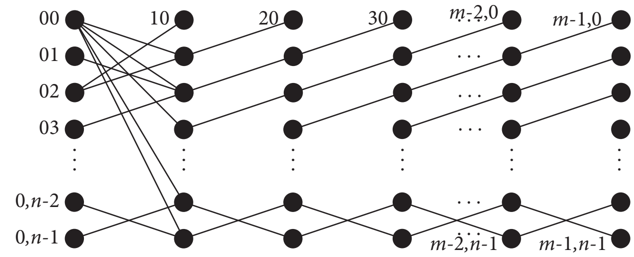

Figure 4: The shortest paths from vertex 00 to all vertices in $V(G)$.

method of computation is similar to Case 1; we first give the calculation process and then explain the formula by making use of Figure 4, namely, the shortest path graph from vertex 00 to all vertices in $V(G)$.

$$\begin{aligned} S_{00} &= 2(n-1)+3+(n-1)+n[2+\cdots+(2m-1)], \\ S_{10} &= 3+(n-1)+2(n-1)+3+(n-1)+n[2+3+\cdots+(2m-2)], \\ \sum_{i=2}^{m-1} S_{i0} &= \sum_{i=2}^{m-1}\{n(2+3+\cdots i)+3+(n-1)+2(n-1) = +3+(n-1)+n[2+3+\cdots+(2m-1-i)]\} \\ &= \sum_{i=2}^{m-1}\left\{\frac{i(i+1)n-2n}{2}+4n+2+\frac{(2m-i-1)(2m-i)n-2n}{2}\right\}. \end{aligned} \tag{7}$$

Explanations for the terms appearing in formula (7) are as follows.

As shown in Figure 4, for vertex 00, the distances from it to vertices $10, 20, 30, \ldots (n-1, 0)$ are all 2, those to the vertex 10 is 3, and those to vertices $11, 12, 13, \ldots, (1, n-1)$ are all 1, and the distance rule to the other vertices is same as Case 1. The explanations of vertex 10 and vertices $\sum_{i=2}^{m-1} S_{i0}$ are similar to Case 1; due to space limitation, they will not be repeated here. Now we summarize each term of formula (7), and by using the properties of the direct product, we get the final result of Case 2 as follows:

$$\begin{aligned} \sum_{i=0}^{m-1} S_{i0} &= \left(S_{00}+S_{10}+\sum_{i=2}^{m-1} S_{i0}\right) \\ &= \frac{\left(4m^3+5m\right)n+6m-6}{3}, \end{aligned} \tag{8}$$

Formula (8) is the final result of Case 2.

By combining the above two cases, the Wiener index of the direct product graph $G = P_{2m} \times W_n$ is

$$\begin{aligned} W(G) &= (n-1)\sum_{i=0}^{m-1} S_{i1} + \sum_{i=0}^{m-1} S_{i0} \\ &= \frac{\left(4m^3+17m-6\right)n^2-54mn+48m-24}{3}. \end{aligned} \tag{9}$$

The following theorem gives the Wiener index of the direct product of an odd order path P_{2m+1} and a wheel graph W_n. The calculation method of Theorem 2 is roughly the same as Theorem 1, and difference is the order of the path in the two theorems.

Theorem 2. *Let P_{2m+1} be a path of order $2m+1\,(m>1)$, with vertices $v_0, v_1, v_2, \ldots, v_{2m}$, and W_n is a wheel graph of order $n(n>3)$, with vertices $0, 1, 2, \ldots, n-2, n-1$. The Wiener index of their direct product $G = P_{2m+1} \times W_n$ is*

$$W(G) = \frac{\left(4m^3+6m^2+20m+6\right)n^2+(-54m-6)n+48m}{3}. \tag{10}$$

Proof. Since the order of the path is $2m+1$, we use the symmetry of G to get the following formula:

$$\begin{aligned} &\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{ij} = \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{2m-i,j} \\ &\Longrightarrow \sum_{j=0}^{n-1}\sum_{i=0}^{2m} S_{ij} = 2\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{ij} + \sum_{j=0}^{n-1} S_{mj} \\ &= 2\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{2m-i,j} + \sum_{j=0}^{n-1} S_{mj} \\ &= 2W(G) \Longrightarrow W(G) = \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{ij} + \frac{1}{2}\sum_{j=0}^{n-1} S_{mj}. \end{aligned} \tag{11}$$

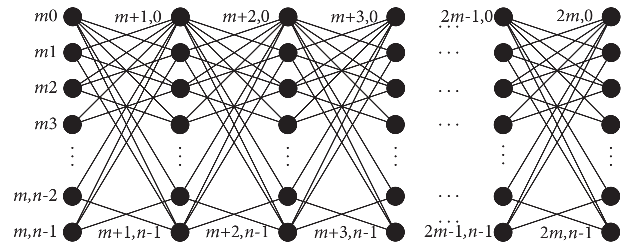

Figure 5: The graph after deleting all vertices and their associated edges on the left of vertices $mj\,(0 \le j \le n-1)$.

According to formula (11) , we need to solve only $\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{ij}$and $1/2\sum_{j=0}^{n-1} S_{mj}$, and the calculation method and process of $\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{ij}$are the same as formula (4) of Theorem 1; therefore, by using formulas (3)–(6), we can give the following result directly:

$$\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{ij} = \frac{\left(8m^3 + 9m^2 + 37m - 12\right)n + (48 - 108m)n + 96m - 48}{6}. \tag{12}$$

Now let us calculate $1/2\sum_{j=0}^{n-1} S_{mj}$. Since the graph G is left-right symmetric with respect to the vertices $\sum_{j=0}^{n-1} mj$, the sum of the distances from $\sum_{j=0}^{n-1} mj$ to all the vertices on their left side is equal to that on their right side, so we can ignore all the vertices on the left side of $\sum_{j=0}^{n-1} mj$. Figure 5 shows the graph after we delete all the vertices and their associated edges on the left side of $\sum_{j=0}^{n-1} mj$. We use $T_{ij}\,(m \le i \le 2m,\ 0 \le j \le n-1)$ to represent the sum of the distances from vertex ij to all vertices in Figure 5. By calculating the sum of the distances from $\sum_{j=0}^{n-1} mj$ to all the vertices in Figure 5, namely, $\sum_{j=0}^{n-1} T_{mj}$, we find that the distance from mi to $mj\,(0 \le i, j \le n-1, i \ne j)$ is calculated twice, i.e., $2n(n-1)$. So, we need to subtract $n(n-1)$ from $\sum_{j=0}^{n-1} T_{mj}$ to get $1/2\sum_{j=0}^{n-1} S_{mj}$.

By using the definition of W_n, if the wheel center and its associated edges are not considered, the remaining vertices and edges are C_{n-1}. Because of the transitivity of the vertices on circle, we can get T_{m1}; then, multiply the result by $(n-1)$ instead of calculating all $\sum_{j=0}^{n-1} T_{mj}$. For T_{m0}, we will calculate it separately. So, we have

$$\begin{aligned}
T_{m0} &= 2(n-1) + 3 + (n-1) + n[2 + 3 + \cdots + (2m - m)],\\
T_{m1} &= 2(n-1) + 3 + 3(n-3) + n[2 + 3 + \cdots + (2m - m)],\\
\frac{1}{2}\sum_{j=0}^{n-1} S_{mj} &= \sum_{j=0}^{n-1} T_{mj} - n(n-1)\\
&= T_{m0} + (n-1)T_{m1} - n(n-1) = \frac{\left(m^2 + m + 6\right)n^2 - 18n + 16}{2}.
\end{aligned} \tag{13}$$

According to the previous explanation, by combining formulas (12) and (13), we obtain the Wiener index of the direct product graph $G = P_{2m+1} \times W_n$ as follows:

$$\begin{aligned}
W(G) &= \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} S_{ij} + \frac{1}{2}\sum_{j=0}^{n-1} S_{mj}\\
&= \frac{\left(4m^3 + 6m^2 + 20m + 3\right)n^2 + (-54m - 3)n + 48m}{3}.
\end{aligned} \tag{14}$$

Recalling the two theorems in the previous paper, it is not difficult to find that the two theorems do not include the Wiener indices of the direct product of paths P_2, P_3 and wheel graph W_n, but we can directly derive the results from the proof processes. Therefore, we give them directly in the following corollary. □

Corollary 1. *The Wiener indices of the direct product of paths P_2, P_3 and a wheel graph W_n are*

(1) $W(P_2 \times W_n) = 5n^2 - 10n + 8$.

(2) $W(P_3 \times W_n) = 11n^2 - 19n + 16.$

Substitute $n = 6$ *into the two formulas of Corollary 1, i.e.,*

$$W(P_2 \times W_6) = 5 \times 6^2 - 10 \times 6 + 8 = 128,$$
$$W(P_3 \times W_6) = 11 \times 6^2 - 10 \times 6 + 16 = 298. \quad (15)$$

The results of the two equations in formula (15) are consistent with the results we calculated manually in Table 1, which proves the correctness of Corollary 1.

3. Conclusion

Direct product is the simplest standard product for the adjacency between vertices, and it is one of the important methods of network design. The direct product of a path and a wheel graph also has good topological properties. Exact Wiener index of the direct product of a path and a wheel graph describes the overall condition of the distances between each pair of vertices in its vertex set, and further we can get the average distance by using the Wiener index. Therefore, it has theoretical and practical significance to research the Wiener index of the direct product of a path and a wheel graph.

Data Availability

All the relevant data are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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