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Research Article

Exact Solutions of Three-Dimensional Max-Type System of Difference Equations

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In this paper, we deal with the form and the periodicity of the solutions of the max-type system of difference equations $u_{n+1} = \max\{(A_n/v_n - 1), u_n - 1\}, v_{n+1} = \max\{(B_n/w_{n-1})\}, w_{n+1} = \max\{(C_n/u_{n-1}), w_{n-1}\}$ where the initial conditions $u_{-1}, u_0 \in (0, \infty), v_{-1}, v_0 \in (0, \infty), w_{-1}, w_0 \in (0, \infty)$ and $(A_n)_{n\in N_0}$, $(B_n)_{n\in N_0}$, $(C_n)_{n\in N_0}$ are positive two-periodic sequences.

1. Introduction

Difference equations and systems which do not stem from the differential ones have attracted some attention in the last few decades (see, e.g., [1–26]). Some of the systems that are of interest are symmetric or those retrieved from symmetric by alteration of their parameters. Recently, there has been enormous interest in various types of nonlinear difference equations and the references cited therein. One of the reasons for this is a necessity for some techniques which can be used in inspecting equations arising in mathematical models illustrating real life situations in biology, control theory, economics, physics, sociology, and so on (see, e.g., [21, 23, 24]). Among these equations, the so-called max-type difference equations also attracted some attention.

The study of the system of max-type difference equations attracted recently a considerable attention; see, for example, [1–16], and the references listed therein. This type of difference equations stemming from, for example, certain models are useful in automatic control theory (see [27]). In the beginning, on the study of these equations, experts have been focused on the investigation of the behavior of some particular cases.

In [3], Berenhaut et al. explained the boundedness nature of positive solutions of the following max-type difference equation system:

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \tag{1}$$

where A_n and B_n are periodic parameters. In 2020, Balibrea et al. [7] obtained in an elegant way the general solution of the following max-type system of difference equations:

$$x_n = \max\left\{A, \frac{y_{n-t}}{x_{n-s}}\right\}, y_n = \max\left\{B, \frac{x_{n-t}}{y_{n-s}}\right\}, n \in N_0, \tag{2}$$

where $A_n, B_n \in (0, +\infty), t, s \in \{1, 2, \ldots\}$ with $\gcd(s, t) = 1$. The initial conditions $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1} \in (0, +\infty)$ and $d = \max\{t, s\}$

In [8], Cunningham et al. evaluated the solution of the following max-type difference equation with period 2:

$$x_n = \max\left\{A, \frac{y_{n-1}}{x_{n-2}}\right\}, y_n = \max\left\{B, \frac{x_{n-1}}{y_{n-2}}\right\}, n \in N_0, \tag{3}$$

where $A_n, B_n \in (0, +\infty)$ are periodic sequences with period 2.

In 2015, Grove et al. [13] obtained the solution of the following max-type difference equation system:

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \min\left\{1, \frac{A}{y_n}\right\}\right\}$$
$$y_{n+1} = \max\left\{\frac{1}{y_n}, \min\left\{1, \frac{A}{x_n}\right\}\right\}, n \in N_0, \tag{4}$$

where $N_0 = N \cup \{0\}$. The parameter A is positive real number.

In [15], Su et al. obtained the solution of the following max-type difference equation system with a period 3 parameter:

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}. \tag{5}$$

In [26], C.M. Kent and M.A. Radin explained the boundedness nature of positive solutions of the following difference equation:

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \tag{6}$$

where A_n and B_n are periodic parameters.

Motivated by the above study, our purpose in this paper is to evaluate the eventual periodicity of the following max-type 3 D-system of difference equations:

$$u_{n+1} = \max\left\{\frac{A_n}{v_{n-1}}, u_{n-1}\right\}, \tag{7}$$

$$v_{n+1} = \max\left\{\frac{B_n}{w_{n-1}}, v_{n-1}\right\}, \tag{8}$$

$$w_{n+1} = \max\left\{\frac{C_n}{u_{n-1}}, w_{n-1}\right\}, \tag{9}$$

where $n \in N_0, N_0 = N \cup \{0\}$, $(A_n)_n \in N_0$, $(B_n)_n \in N_0$, and $(C_n)_n \in N_0$ are positive periodic sequences and initial conditions $u_0, u_{-1}, v_0, v_{-1}, w_0, w_{-1} \in (0, +\infty)$.

Theorem 1. *Suppose that (u_n, v_n, w_n) is a solution of system (1)–(3) such that $A_0/v_{-1} > u_{-1}$, $B_0/w_{-1} > v_{-1}$, and $C_0/u_{-1} > w_{-1}$. Then, the following statements hold:*

(1) If $A_1/v_0 > u_0$, $B_1/w_0 \geq v_0$, and $w_0 \geq C_1/u_0$, then

$$\begin{aligned} u_{2n+1} &= \frac{A_0}{v_{-1}}, v_{2n+1}, \\ &= \frac{B_0}{w_{-1}}, w_{2n+1}, \\ &= \frac{C_0}{u_{-1}}, u_{2n}, \\ &= u_0, v_{2n}, \\ &= v_0, w_{2n}, \\ &= w_0, n \in N_0. \end{aligned} \tag{10}$$

2If $A_1/v_0 \geq u_0$, $B_1/w_0 \geq v_0$, and $w_0 \geq C_1/u_0$, then

$$\begin{aligned} u_{2n+1} &= \frac{A_0}{v_{-1}}, v_{2n+1}, \\ &= \frac{B_0}{w_{-1}}, w_{2n+1}, \\ &= \frac{C_0}{u_{-1}}, \\ & u_{2n} \\ &= u_0, v_{2n}, \\ &= v_0, w_{2n+2}, \\ &= \frac{c_1}{u_0}. \end{aligned} \tag{11}$$

3If $A_1/v_0 \geq u_0$, $B_1/w_0 \geq v_0$, and $w_0 \geq C_1/u_0$, then

$$\begin{aligned} & u_{2n+1}, \\ &= \frac{A_0}{v_{-1}}, v_{2n+1}, \\ &= \frac{B_0}{w_{-1}}, w_{2n+1}, \\ &= \frac{C_0}{u_{-1}}, u_{2n+2}, \\ &= \frac{A_1 w_0}{B_1}, v_{2n+2}, \\ &= \frac{B_1}{w_0}, w_{2n}, \\ &= w_0. \end{aligned} \tag{12}$$

Proof. (1) From the following conditions:

$$\frac{A_0}{v_{-1}} \geq u_{-1}, \frac{B_0}{w_{-1}} \geq v_{-1}, \frac{C_0}{u_{-1}} \geq w_{-1}, \tag{13}$$

we have

$$\begin{aligned}
u_1 &= \max\left\{\frac{A_0}{v_{-1}}, u_{-1}\right\}, \\
&= \frac{A_0}{v_{-1}}, \\
v_1 &= \max\left\{\frac{B_0}{w_{-1}}, v_{-1}\right\}, \\
&= \frac{B_0}{w_{-1}}, \\
w_1 &= \max\left\{\frac{C_0}{u_{-1}}, w_{-1}\right\}, \\
&= \frac{C_0}{u_{-1}}, \\
u_3 &= \max\left\{\frac{A_0}{v_1}, u_1\right\}, \\
&= \max\left\{\frac{A_0 w_{-1}}{B_0}, \frac{A_0}{v_{-1}}\right\}, \\
&= \frac{A_0}{v_{-1}}, \\
v_3 &= \max\left\{\frac{B_0}{w_1}, v_1\right\}, \\
&= \max\left\{\frac{B_0 u_{-1}}{C_0}, \frac{B_0}{w_{-1}}\right\}, \\
&= \frac{B_0}{w_{-1}}, \\
w_3 &= \max\left\{\frac{C_0}{u_1}, w_1\right\}, \\
&= \max\left\{\frac{C_0 v_{-1}}{A_0}, \frac{C_0}{u_{-1}}\right\}, \\
&= \frac{C_0}{u_{-1}}.
\end{aligned} \tag{14}$$

By induction, we obtained formula as follows:

$$\begin{aligned}
u_{2n+1} &= \frac{A_0}{v_{-1}}, v_{2n+1}, \\
&= \frac{B_0}{w_{-1}}, w_{2n+1}, \\
&= \frac{C_0}{u_{-1}}, u_{2n}, \\
&= u_0, v_{2n}, \\
&= v_0, w_{2n}, \\
&= w_0, n \in \mathrm{N}_0,
\end{aligned} \tag{15}$$

which are formulas of odd terms in (4)–(6). Hence, it remains only to prove the formulas for even terms in (4)–(6).

Similarly, we can find for even terms.

$$\begin{aligned}
u_2 &= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= u_0, \\
v_2 &= \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&= v_0, \\
w_2 &= \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&= w_0, \\
u_4 &= \max\left\{\frac{A_1}{v_2}, u_2\right\}, \\
&= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= u_0, \\
v_4 &= \max\left\{\frac{B_1}{w_2}, v_2\right\}, \\
&= \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&= v_0, \\
w_4 &= \max\left\{\frac{C_1}{u_2}, w_2\right\}, \\
&= \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&= w_0.
\end{aligned} \tag{16}$$

By induction,

$$\begin{aligned}
u_{2n} &= u_0, v_{2n}, \\
&= v_0, w_{2n}, \\
&= w_0, n \in \mathbb{N}_o.
\end{aligned} \tag{17}$$

(2) Because $u_0 \geq A_1/v_0$, $v_0 \geq B_1/w_0$, and $w_0 \leq C_1/u_0$, we have

$$\frac{1}{w_0} \geq \frac{u_0}{C_1}. \tag{18}$$

Then,

$$\begin{aligned}
v_0 &\geq \frac{B_1}{w_0} \geq \frac{B_1 u_0}{C_1}, \\
u_2 &= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= u_0, \\
v_2 &= \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&= v_0, \\
w_2 &= \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&= \frac{C_1}{u_0}.
\end{aligned} \tag{19}$$

$$\begin{aligned}
u_4 &= \max\left\{\frac{A_1}{v_2}, u_2\right\}, \\
&= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= u_0, \\
v_4 &= \max\left\{\frac{B_1}{w_2}, v_2\right\}, \\
&= \max\left\{\frac{B_1}{C_1} u_0, v_0\right\}, \\
&= v_0, \\
w_4 &= \max\left\{\frac{C_1}{u_2}, w_2\right\}, \\
&= \max\left\{\frac{C_1}{u_0}, \frac{C_1}{u_0}\right\}, \\
&= \frac{C_1}{u_0}, \\
u_6 &= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= u_0, \\
v_6 &= \max\left\{\frac{B_1}{w_4}, v_4\right\}, \\
&= \max\left\{\frac{B_1 u_0}{C_1}, v_0\right\}, \\
&= v_0, \\
w_6 &= \max\left\{\frac{C_1}{u_4}, w_4\right\}, \\
&= \max\left\{\frac{C_1}{u_0}, \frac{C_1}{u_0}\right\} = \frac{C_1}{u_0}.
\end{aligned} \tag{20}$$

By induction, we obtain formulas for even terms as given in (11).

(3) Because $u_0 \geq A_1/v_0$, $v_0 \leq B_1/w_0$, and $w_0 \geq C_1/u_0$, then

$$\begin{aligned}
\frac{1}{v_0} &\geq \frac{w_0}{B_1}, \\
u_0 &\geq A_1 \times \frac{w_0}{B_1}, \\
u_0 &\geq A_1 \frac{w_0}{B_1}, \\
u_2 &= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= u_0, \\
v_2 &= \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&= \frac{B_1}{w_0}, \\
w_2 &= \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&= w_0.
\end{aligned} \tag{21}$$

So, we have

$$\begin{aligned}
u_4 &= \max\left\{\frac{A_1}{v_2}, u_2\right\}, \\
&= \max\left\{\frac{A_1}{B_1}w_0, u_0\right\}, \\
&= u_0, \\
v_4 &= \max\left\{\frac{B_1}{w_2}, v_2\right\}, \\
&= \max\left\{\frac{B_1}{w_0}, \frac{B_1}{v_0}\right\}, \\
&= \frac{B_1}{w_0}, \\
w_4 &= \max\left\{\frac{C_1}{u_2}, w_2\right\}, \\
&= \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&= w_0, \\
u_6 &= \max\left\{\frac{A_1}{v_4}, u_4\right\}, \\
&= \max\left\{\frac{A_1 w_0}{B_1}, u_0\right\}, \\
&= u_0, \\
v_6 &= \max\left\{\frac{B_1}{w_4}, v_4\right\}, \\
&= \max\left\{\frac{B_1}{w_0}, \frac{B_1}{w_0}\right\}, \\
&= \frac{B_1}{w_0}, \\
w_6 &= \max\left\{\frac{C_1}{u_4}, w_4\right\}, \\
&= \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&= w_0.
\end{aligned} \tag{22}$$

By induction, we obtain formulas as given in (26):

$$\begin{aligned}
u_{2n+2} &= u_0, v_{2n+2}, \\
&= \frac{B_1}{w_0}, w_{2n}, \\
&= w_0.
\end{aligned} \tag{23}$$

The proof is completed. □

Theorem 2. *Suppose that (u_n, v_n, w_n) is a solution of systems (1)–(3) such that $A_0/v_{-1} \le u_{-1}$, $B_0/w_{-1} \le v_{-1}$, and $C_0/u_{-1} \le w_{-1}$. Then, the following statements hold:*

(1) If $A_1/v_0 \le u_0$, $B_1/w_0 \le v_0$, and $w_0 \ge C_1/u_0$, then

$$\begin{aligned}
u_{2n+1} &= u_{-1}, v_{2n+1}, \\
&= v_{-1}, w_{2n+1}, \\
&= w_{-1}, u_{2n}, \\
&= u_0, v_{2n}, \\
&= v_0, w_{2n}, \\
&= w_0.
\end{aligned} \tag{24}$$

2If $A_1/v_0 \ge u_0$, $B_1/w_0 \ge v_0$, and $w_0 \le C_1/u_0$,

$$\begin{aligned}
u_{2n+1} &= u_{-1}, v_{2n+1}, \\
&= v_{-1}, w_{2n+1}, \\
&= w_{-1}, u_{2n}, \\
&= u_0, v_{2n}, \\
&= v_0, w_{2n+2}, \\
&= \frac{C_1}{u_0}.
\end{aligned} \tag{25}$$

3If $A_1/v_0 \le u_0$, $B_1/w_0 \le v_0$, and $w_0 \ge C_1/u_0$,

$$\begin{aligned}
u_{n+1} &= u_{-1}, v_{2n+1}, \\
&= v_{-1}, w_{2n+1}, \\
&= w_{-1}, u_{2n+2}, \\
&= \frac{A_1}{v_0}, v_{2n}, \\
&= v_0, w_{2n}, \\
&= w_0.
\end{aligned} \tag{26}$$

4If $A_1/v_0 \ge u_0$, $B_1/w_0 \le v_0$, and $w_0 \ge C_1/u_0$,

$$\begin{aligned} u_{2n+1} &= u_{-1}, v_{2n+1}, \\ &= v_{-1}, w_{2n+1}, \\ &= w_{-1}, u_{2n}, \\ &= u_0, v_{2n+2}, \\ &= \frac{B_1}{w_0}, w_{2n}, \\ &= w_0. \end{aligned} \tag{27}$$

Proof.

(1) From the conditions $A_0, /v_{-1} \le u_{-1}$, $B_0/w_{-1} \le v_{-1}$, and $C_0/u_{-1}, \le w_{-1}$, we have

$$\begin{aligned} u_1 &= \max\left\{\frac{A_0}{v_{-1}}, u_{-1}\right\}, \\ &= u_{-1}, \\ v_1 &= \max\left\{\frac{B_0}{w_{-1}}, v_{-1}\right\}, \\ &= v_{-1}, \\ w_1 &= \max\left\{\frac{C_0}{u_{-1}}, w_{-1}\right\}, \\ &= w_{-1}, \\ u_3 &= \max\left\{\frac{A_0}{v_1}, u_1\right\}, \\ &= \max\left\{\frac{A_0}{v_{-1}}, u_{-1}\right\}, \\ &= u_{-1}, \\ v_3 &= \max\left\{\frac{B_0}{w_1}, v_1\right\}, \\ &= \max\left\{\frac{B_0}{w_{-1}}, v_{-1}\right\}, \\ &= v_{-1}, \\ w_3 &= \max\left\{\frac{C_0}{u_1}, w_1\right\}, \\ &= \max\left\{\frac{C_0}{u_{-1}}, w_{-1}\right\}, \\ &= w_{-1}. \end{aligned} \tag{28}$$

By induction, we obtained formula as follows:

$$\begin{aligned} u_{2n+1} &= u_{-1}, v_{2n+1}, \\ &= v_{-1}, w_{2n+1}, \\ &= w_{-1}. \end{aligned} \tag{29}$$

Similarly, we can find the proof for even terms.

$$\begin{aligned} u_2 &= \max\left\{\frac{A_1}{v_0}, u_0\right\} \\ &= u_0, \\ v_2 &= \max\left\{\frac{B_1}{w_0}, v_0\right\} \\ &= v_0, \\ w_2 &= \max\left\{\frac{C_1}{u_0}, w_0\right\} \\ &= w_0, \\ u_4 &= \max\left\{\frac{A_1}{v_2}, u_2\right\} \\ &= \max\left\{\frac{A_1}{v_0}, u_0\right\} = u_0, \\ v_4 &= \max\left\{\frac{B_1}{w_2}, v_2\right\} \\ &= \max\left\{\frac{B_1}{w_0}, v_0\right\} = v_0, \\ w_4 &= \max\left\{\frac{C_1}{u_2}, w_2\right\} \\ &= \max\left\{\frac{C_1}{u_0}, w_0\right\} = w_0. \end{aligned} \tag{30}$$

By induction,

$$\begin{aligned} u_{2n} &= u_0, v_{2n}, \\ &= v_0, w_{2n}, \\ &= w_0, \end{aligned} \tag{31}$$

as given in (24).

(2) Because $u_0 \ge A_1/v_0$, $v_0 \ge B_1/w_0$, and $w_0 \le C_1/u_0$, then we have

$$\frac{1}{w_0} \ge \frac{u_0}{C_1},$$

$$v_0 \ge B_1 \times \frac{u_0}{C_1},$$

$$
\begin{aligned}
u_2 &= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= u_0, \\
v_2 &= \max\left\{\frac{B_1}{w_0}, v_0\right\} \\
&= v_0, \\
w_2 &= \max\left\{\frac{C_1}{u_0}, w_0\right\} \\
&= \frac{C_1}{u_0}, \\
u_4 &= \max\left\{\frac{A_1}{v_2}, u_2\right\} \\
&= \max\left\{\frac{A_1}{v_0}, u_0\right\} \\
&= u_0, \\
v_4 &= \max\left\{\frac{B_1}{w_2}, v_2\right\} \\
&= \max\left\{\frac{B_1 u_0}{C_1}, v_0\right\} \\
&= v_0, \\
w_4 &= \max\left\{\frac{C_1}{u_2}, w_2\right\} \\
&= \max\left\{\frac{C_1}{u_0}, \frac{C_1}{u_0}\right\} \\
&= \frac{C_1}{u_0}, \\
u_6 &= \max\left\{\frac{A_1}{v_4}, u_4\right\} \\
&= \max\left\{\frac{A_1}{u_0}, u_0\right\} \\
&= u_0, \\
v_6 &= \max\left\{\frac{B_1}{w_4}, v_4\right\} \\
&= \max\left\{\frac{B_1 u_0}{C_1}, v_0\right\} \\
&= v_0, \\
w_6 &= \max\left\{\frac{C_1}{u_4}, w_4\right\} \\
&= \max\left\{\frac{C_1}{u_0}, \frac{C_1}{u_0}\right\} \\
&= \frac{C_1}{u_0}.
\end{aligned}
\tag{32}
$$

By induction, we obtain the formulas as stated in (25).

(3) Because $u_0 \le A_1/v_0$, $v_0 \ge B_1/w_0$, and $w_0 \ge C_1/u_0$, then

$$
\begin{aligned}
\frac{1}{u_0} &\ge \frac{v_0}{A_1}, \\
w_0 &\ge C_1 \times \frac{1}{u_0}, \\
w_0 &\ge \frac{C_1 v_0}{A_1}, \\
u_2 &= \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&= \frac{A_1}{v_0}, \\
v_2 &= \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&= v_0, \\
w_2 &= \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&= w_0, \\
u_4 &= \max\left\{\frac{A_1}{v_2}, u_2\right\}, \\
&= \max\left\{\frac{A_1}{v_0}, \frac{A_1}{v_0}\right\}, \\
&= \frac{A_1}{v_0}, \\
v_4 &= \max\left\{\frac{B_1}{w_2}, v_2\right\}, \\
&= \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&= v_0, \\
w_4 &= \max\left\{\frac{C_1}{u_2}, w_2\right\}, \\
&= \max\left\{\frac{C_1 v_0}{A_1}, w_0\right\}, \\
&= w_0, \\
u_6 &= \max\left\{\frac{A_1}{v_4}, u_4\right\},
\end{aligned}
$$

$$
\begin{aligned}
&= \max\left\{\frac{A_1}{v_0}, \frac{A_1}{v_0}\right\}, \\
&= \frac{A_1}{v_0}, \\
v_6 &= \max\left\{\frac{B_1}{w_4}, v_4\right\}, \\
&= \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&= v_0, \\
w_6 &= \max\left\{\frac{C_1}{u_4}, w_4\right\}, \\
&= \max\left\{\frac{C_1 v_0}{A_1}, w_0\right\}, \\
&= w_0.
\end{aligned} \tag{33}
$$

By induction, we obtain formula as in (26).

(4) Because $u_0 \geq A_1/v_0$, $v_0 \leq B_1/w_0$, and $w_0 \geq C_1/u_0$, then

$$
\begin{aligned}
&\frac{1}{v_0} \geq \frac{w_0}{B_1}, \\
&u_0 \geq \frac{A_1 w_0}{B_1}, \\
&u_2 = \max\left\{\frac{A_1}{v_0}, u_0\right\}, \\
&\quad = u_0, \\
&v_2 = \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&\quad = \frac{B_1}{w_0}, \\
&v_2 = \max\left\{\frac{B_1}{w_0}, v_0\right\}, \\
&\quad = \frac{B_1}{w_0}, \\
&u_4 = \max\left\{\frac{A_1}{v_2}, u_2\right\}, \\
&\quad = \max\left\{\frac{A_1 w_0}{B_1}, u_0\right\}, \\
&\quad = u_0, \\
&v_4 = \max\left\{\frac{B_1}{w_2}, v_2\right\}, \\
&\quad = \max\left\{\frac{B_1}{w_0}, \frac{B_1}{w_0}\right\}, \\
&\quad = \frac{B_1}{w_0}, \\
&w_4 = \max\left\{\frac{C_1}{u_2}, w_2\right\}, \\
&\quad = \max\left\{\frac{C_1}{u_0}, w_0\right\} m \\
&\quad = w_0, \\
&u_6 = \max\left\{\frac{A_1}{v_4}, u_4\right\}, \\
&\quad = \max\left\{\frac{A_1 w_0}{B_1}, u_0\right\}, \\
&\quad = u_0, \\
&v_6 = \max\left\{\frac{B_1}{w_4}, v_4\right\}, \\
&\quad = \max\left\{\frac{B_1}{w_0}, \frac{B_1}{w_0}\right\}, \\
&\quad = \frac{B_1}{w_0}, \\
&w_6 = \max\left\{\frac{C_1}{u_4}, w_4\right\}, \\
&\quad = \max\left\{\frac{C_1}{u_0}, w_0\right\}, \\
&\quad = w_0.
\end{aligned} \tag{34}
$$

Then, by induction, we get formulas as given in (27). □

Theorem 3. *Suppose that $\{u_n, v_n, w_n\}$ is a solution of systems (1)–(3) such that*

$$
\frac{A_0}{v_{-1}} \leq u_{-1}, \quad \frac{B_0}{w_{-1}} \geq v_{-1}, \quad \frac{C_0}{u_{-1}} \geq w_{-1}, \tag{35}
$$

then we have

$$
\begin{aligned}
&\frac{1}{v_{-1}} \geq \frac{w_{-1}}{B_0}, \\
&u_{-1} \geq \frac{A_0 w_{-1}}{B_0}.
\end{aligned} \tag{36}
$$

Then, the following system holds for odd solutions:

$$
\begin{aligned}
u_{2n+1} &= u_{-1}, v_{2n+1}, \\
&= \frac{B_0}{w_{-1}}, w_{2n+1}, \\
&= \frac{C_0}{u_{-1}}.
\end{aligned} \tag{37}
$$

Theorem 4. *Suppose that $\{u_n, v_n, w_n\}$ is a solution of systems (1)–(3) such that*

$$\frac{A_0}{v_{-1}} \geq u_{-1}, \frac{B_0}{w_{-1}} \leq v_{-1}, \frac{C_0}{u_{-1}} \geq w_{-1}, \tag{38}$$

then the following statement holds:

$$\begin{aligned} u_{2n+1} &= \frac{A_0}{v_{-1}}, v_{2n+1}, \\ &= v_{-1}, w_1, \\ &= \frac{C_0}{u_{-1}}, w_{2n+3}, \\ &= \frac{C_0}{A_0} v_{-1}. \end{aligned} \tag{39}$$

Theorem 5. *Suppose that $\{u_n, v_n, w_n\}$ is a solution of systems (1)–(3) such that*

$$\frac{A_0}{v_{-1}} \geq u_{-1}, \frac{B_0}{w_{-1}} \leq v_{-1}, \frac{C_0}{u_{-1}} \leq w_{-1}, \tag{40}$$

then the following statement holds:

$$\begin{aligned} u_{2n+1} &= \frac{A_0}{v_{-1}}, v_{2n+1}, \\ &= v_{-1}, w_{2n+1}, \\ &= w_{-1}. \end{aligned} \tag{41}$$

Theorem 6. *Suppose that $\{u_n, v_n, w_n\}$ is a solution of systems (1)–(3) such that*

$$\frac{A_0}{v_{-1}} \leq u_{-1}, \frac{B_0}{w_{-1}} \geq v_{-1}, \frac{C_0}{u_{-1}} \leq w_{-1}, \tag{42}$$

then the following statement holds:

$$\begin{aligned} u_{2n+1} &= u_{-1}, v_{2n+1}, \\ &= \frac{B_0}{w_{-1}}, w_{2n+1}, \\ &= w_{-1}. \end{aligned} \tag{43}$$

2. Conclusion

We investigate the closed form solutions of an important type of difference equation. The eventual periodicity of the following max-type 3 D-system of difference equations is

$$\begin{aligned} u_{n+1} &= \max\left\{\frac{A_n}{v_{n-1}}, u_{n-1}\right\} v_{n+1}, \\ &= \max\left\{\frac{B_n}{w_{n-1}}, v_{n-1}\right\} w_{n+1}, \\ &= \max\left\{\frac{C_n}{u_{n-1}}, w_{n-1}\right\}, \end{aligned} \tag{44}$$

where $n \in N_o, N_o = N \cup \{0\}$, $(A_n)_n \in N_o$, $(B_n)_n \in N_o$, and $(C_n)_n \in N_o$ are positive periodic sequences and initial conditions $u_0, u_{-1}, v_0, v_{-1}, w_0, w_{-1} \in (0, +\infty)$.

Data Availability

All data utilized in this article have been included and the sources where they were adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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