Research Article

Exact Solutions of Three-Dimensional Max-Type System of Difference Equations

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In this paper, we deal with the form and the periodicity of the solutions of the max-type system of difference equations

\[ u_{n+1} = \max\{A_n/v_n - 1, u_n - 1\}, \quad v_{n+1} = \max\{B_n/w_{n-1}, v_n - 1\}, \quad w_{n+1} = \max\{C_n/u_{n-1}, w_{n-1}\} \]

where the initial conditions \( u_0, v_0, w_0 \in (0, \infty) \) and \( (A_n)_{n \in \mathbb{N}_0}, (B_n)_{n \in \mathbb{N}_0}, (C_n)_{n \in \mathbb{N}_0} \) are positive two-periodic sequences.

1. Introduction

Difference equations and systems which do not stem from the differential ones have attracted some attention in the last few decades (see, e.g., [1–26]). Some of the systems that are of interest are symmetric or those retrieved from symmetric equations stemming from, for example, certain automatic control theory (see [27]). In 2020, Balibrea et al. [7] obtained in an elegant way the general solution of the following max-type system of difference equations:

\[ x_{n+1} = \max\left\{A_n, B_n \right\}, \quad n \in \mathbb{N}_0, \]

where \( A_n \) and \( B_n \) are periodic parameters. In 2020, Balibrea et al. [7] obtained in an elegant way the general solution of the following max-type system of difference equations:

\[ x_n = \max\left\{A_{n-s}x_{n-s}^{y_{n-s}}, B_{n-s}y_{n-s}\right\}, y_n = \max\left\{x_{n-t}^{y_{n-t}}, B_ny_{n-t}\right\}, n \in \mathbb{N}_0, \]

where \( A_n, B_n \in (0, +\infty) \) and \( d = \max\{t, s\} \).

In [8], Cunningham et al. evaluated the solution of the following max-type difference equation with period 2:

\[ x_n = \max\left\{A_{n-2}x_{n-2}^{y_{n-2}}, B_ny_{n-2}\right\}, n \in \mathbb{N}_0, \]

where \( A_n, B_n \in (0, +\infty) \) and \( d = \max\{t, s\} \).

In [3], Berenhaut et al. explained the boundedness nature of positive solutions of the following max-type difference equation system:

\[ x_{n+1} = \max\left\{A_n, B_n \right\}, \quad n \in \mathbb{N}_0, \]

In the beginning, on the study of these equations, experts have been focused on the investigation of the behavior of some particular cases.
In 2015, Grove et al. [13] obtained the solution of the following max-type difference equation system:

\[ x_{n+1} = \max \left\{ \frac{1}{x_n}, A_n x_n \right\}, \quad n \in N_0, \]  

\[ y_{n+1} = \max \left\{ \frac{1}{y_n}, 1, A_n y_n \right\}, \quad n \in N_0, \]  

\[ z_{n+1} = \max \left\{ \frac{1}{z_n}, B_n z_n \right\}, \quad n \in N_0, \]  

where \( N_0 = N \cup \{0\} \). The parameter \( A \) is positive real number.

In [15], Su et al. obtained the solution of the following max-type difference equation system with a period 3 parameter:

\[ x_{n+1} = \max \left\{ \frac{1}{x_n}, A_n, B_n \right\}, \quad n \in N_0, \]  

where \( A_n \) and \( B_n \) are periodic parameters.

Motivated by the above study, our purpose in this paper is to evaluate the eventual periodicity of the following max-type 3-D system of difference equations:

\[ u_{n+1} = \max \left\{ \frac{1}{u_{n-1}}, A_n u_n \right\}, \quad n \in N_0, \]  

\[ v_{n+1} = \max \left\{ \frac{1}{v_{n-1}}, B_n v_n \right\}, \quad n \in N_0, \]  

\[ w_{n+1} = \max \left\{ \frac{1}{w_{n-1}}, C_n w_n \right\}, \quad n \in N_0, \]  

where \( n \in N_0, N_0 = N \cup \{0\}, (A_n)_n \in N_0, (B_n)_n \in N_0, \) and \( (C_n)_n \in N_0 \) are positive periodic sequences and initial conditions \( u_0, u_{-1}, v_0, v_{-1}, w_0, w_{-1} \in (0, +\infty) \).

**Theorem 1.** Suppose that \((u_n, v_n, w_n)\) is a solution of system (1)–(3) such that \( A_0/v_{-1} > u_{-1}, B_0/w_{-1} > v_{-1}, \) and \( C_0/u_{-1} > w_{-1} \). Then, the following statements hold:

1. If \( A_1/v_0 \geq u_0, B_1/w_0 \geq v_0, \) and \( w_0 \geq C_1/u_0 \), then

\[
\begin{align*}
u_{2n+1} &= \frac{A_0}{v_{-1}} v_{2n+1}, \\
v_{2n+1} &= \frac{B_0}{w_{-1}} w_{2n+1}, \\
w_{2n+1} &= \frac{C_0}{u_{-1}} u_{2n+1},
\end{align*}
\]

2. If \( A_1/v_0 \geq u_0, B_1/w_0 \geq v_0, \) and \( w_0 \geq C_1/u_0 \), then

\[
\begin{align*}
u_{2n+1} &= \frac{A_0}{v_{-1}} v_{2n+1}, \\
v_{2n+1} &= \frac{B_0}{w_{-1}} w_{2n+1}, \\
w_{2n+1} &= \frac{C_0}{u_{-1}} u_{2n+1},
\end{align*}
\]

3. If \( A_1/v_0 \geq u_0, B_1/w_0 \geq v_0, \) and \( w_0 \geq C_1/u_0 \), then

\[
\begin{align*}
u_{2n+1} &= \frac{A_0}{v_{-1}} v_{2n+1}, \\
v_{2n+1} &= \frac{B_0}{w_{-1}} w_{2n+1}, \\
w_{2n+1} &= \frac{C_0}{u_{-1}} u_{2n+1},
\end{align*}
\]

**Proof.** (1) From the following conditions:

\[
\begin{align*}
A_0/v_{-1} &\geq u_{-1}, \quad B_0/w_{-1} \geq v_{-1}, \quad C_0/u_{-1} \geq w_{-1},
\end{align*}
\]

we have

\[
\begin{align*}
\frac{A_0}{v_{-1}} v_{2n+1} &\geq u_0, \\
\frac{B_0}{w_{-1}} w_{2n+1} &\geq v_0, \\
\frac{C_0}{u_{-1}} u_{2n+1} &\geq w_0.
\end{align*}
\]
\( u_1 = \max \left\{ \frac{A_0}{v_{-1}}, u_{-1} \right\}, \)
\( = \frac{A_0}{v_{-1}}, \)
\( = \frac{A_0}{v_{-1}}, \)
\( v_1 = \max \left\{ \frac{B_0}{w_{-1}}, v_{-1} \right\}, \)
\( = \frac{B_0}{w_{-1}}, \)
\( = \frac{B_0}{w_{-1}}, \)
\( w_1 = \max \left\{ \frac{C_0}{u_{-1}}, w_{-1} \right\}, \)
\( = \frac{C_0}{u_{-1}}, \)
\( = \frac{C_0}{u_{-1}}, \)
\( u_2 = \max \left\{ \frac{A_0}{v_1}, u_{-1} \right\}, \)
\( = \frac{A_0}{v_1}, \)
\( = \frac{A_0}{v_1}, \)
\( v_2 = \max \left\{ \frac{B_0}{w_1}, v_{-1} \right\}, \)
\( = \frac{B_0}{w_1}, \)
\( = \frac{B_0}{w_1}, \)
\( w_2 = \max \left\{ \frac{C_0}{u_1}, w_{-1} \right\}, \)
\( = \frac{C_0}{u_1}, \)
\( = \frac{C_0}{u_1}, \)
\[ \phantom{=} u_2 = \max \left\{ \frac{A_1}{v_0}, u_0 \right\}, \]
\( = u_0, \)
\( = u_0, \)
\( v_2 = \max \left\{ \frac{B_1}{w_0}, v_0 \right\}, \]
\( = v_0, \)
\( = v_0, \)
\( w_2 = \max \left\{ \frac{C_1}{u_0}, w_0 \right\}, \]
\( = w_0, \)
\( = w_0, \)
\[ \phantom{=} u_4 = \max \left\{ \frac{A_1}{v_2}, u_2 \right\}, \]
\( = \frac{A_1}{v_2}, \)
\( = \frac{A_1}{v_2}, \)
\( v_4 = \max \left\{ \frac{B_1}{w_2}, v_2 \right\}, \]
\( = v_2, \)
\( = v_2, \)
\( w_4 = \max \left\{ \frac{C_1}{u_2}, w_2 \right\}, \]
\( = w_2, \)
\( = w_2, \)
\( \text{By induction, we obtained formula as follows:} \)
\( u_{2n+1} = \frac{A_0}{v_{2n+1}}, \)
\( = \frac{B_0}{w_{2n+1}}, \)
\( = \frac{C_0}{u_{2n+1}},\)
\[ \phantom{=} u_n = u_0, v_2, \]
\( = v_0, w_2, \)
\( = w_0, n \in \mathbb{N}_0, \)
\( \text{which are formulas of odd terms in (4)–(6). Hence, it remains only to prove the formulas for even terms in (4)–(6).} \)
\( \text{Similarly, we can find for even terms.} \)
\[ \phantom{=} u_2 = \max \left\{ \frac{A_1}{v_0}, u_0 \right\}, \]
\( = u_0, \)
\( = u_0, \)
\( v_2 = \max \left\{ \frac{B_1}{w_0}, v_0 \right\}, \]
\( = v_0, \)
\( = v_0, \)
\( w_2 = \max \left\{ \frac{C_1}{u_0}, w_0 \right\}, \]
\( = w_0, \)
\( = w_0, \)
\[ \phantom{=} u_4 = \max \left\{ \frac{A_1}{v_2}, u_2 \right\}, \]
\( = \frac{A_1}{v_2}, \)
\( = \frac{A_1}{v_2}, \)
\( v_4 = \max \left\{ \frac{B_1}{w_2}, v_2 \right\}, \]
\( = v_2, \)
\( = v_2, \)
\( w_4 = \max \left\{ \frac{C_1}{u_2}, w_2 \right\}, \]
\( = w_2, \)
\( = w_2, \)
\( \text{By induction,} \)
\( u_{2n} = u_0, v_2, \)
\( = v_0, w_2, \)
\( = w_0, n \in \mathbb{N}_0, \)
\[ \phantom{=} (2) \text{Because } u_0 \geq A_1/v_0, v_0 \geq B_1/w_0, \text{ and } w_0 \leq C_1/u_0, \text{ we have} \]
\frac{1}{w_0} \geq \frac{u_0}{C_1} \quad (18)

Then,
\begin{align*}
v_0 & \geq \frac{B_1}{w_0} \geq \frac{B_1 u_0}{C_1} \\
u_2 &= \max \left\{ \frac{A_1}{v_0}, u_0 \right\}, \\
&= u_0, \\
v_2 &= \max \left\{ \frac{B_1}{w_0}, v_0 \right\}, \\
&= v_0, \\
w_2 &= \max \left\{ \frac{C_1}{u_0}, w_0 \right\}, \\
&= \frac{C_1}{u_0}
\end{align*}

(19)

\begin{align*}
u_4 &= \max \left\{ \frac{A_1}{v_0}, u_2 \right\}, \\
&= \max \left\{ \frac{A_1}{v_0}, u_0 \right\}, \\
&= u_0, \\
v_4 &= \max \left\{ \frac{B_1}{w_2}, v_2 \right\}, \\
&= \max \left\{ \frac{B_1}{u_0}, v_0 \right\}, \\
&= v_0, \\
w_4 &= \max \left\{ \frac{C_1}{u_2}, w_2 \right\}, \\
&= \max \left\{ \frac{C_1}{u_0}, w_0 \right\}, \\
&= \frac{C_1}{u_0}
\end{align*}

(20)

\begin{align*}
u_6 &= \max \left\{ \frac{A_1}{v_0}, u_4 \right\}, \\
&= \max \left\{ \frac{A_1}{v_0}, u_0 \right\}, \\
v_6 &= \max \left\{ \frac{B_1}{w_4}, v_4 \right\}, \\
&= \max \left\{ \frac{B_1 u_0}{C_1}, v_0 \right\}, \\
w_6 &= \max \left\{ \frac{C_1}{u_4}, w_4 \right\}, \\
&= \max \left\{ \frac{C_1}{u_0}, w_0 \right\} = \frac{C_1}{u_0}
\end{align*}

By induction, we obtain formulas for even terms as given in (11).

(3) Because \(u_0 \geq A_1/v_0, v_0 \leq B_1/w_0\), and \(w_0 \geq C_1/u_0\), then
\begin{align*}
\frac{1}{v_0} & \geq \frac{w_0}{B_1}, \\
\frac{u_0}{v_0} & \geq \frac{A_1}{B_1}, \\
u_0 & \geq A_1 \times \frac{w_0}{B_1}, \\
& \geq A_1 \frac{w_0}{B_1}, \\
u_2 & \geq \frac{A_1}{v_0}, \\
& \leq \frac{A_1}{v_0}, \\
u_0 & \geq A_1 \frac{w_0}{B_1}, \\
v_2 & \leq \frac{B_1}{w_0}, \\
w_2 & \geq \frac{C_1}{u_0}, \\
& \leq \frac{C_1}{u_0}, \\
w_0 & \geq C_1/u_0, \\
u_6 & \geq \frac{A_1}{v_0}, \\
& \leq \frac{A_1}{v_0}, \\
w_6 & = \frac{A_1}{v_0}, \\
& = \frac{A_1}{v_0}
\end{align*}

So, we have
\[
\begin{align*}
    u_4 &= \max \left\{ \frac{A_1}{v_2}, u_2 \right\}, \\
    &= \max \left\{ \frac{A_1}{B_1} w_0, u_0 \right\}, \\
    &= u_0, \\
    v_4 &= \max \left\{ \frac{B_1}{u_2}, v_2 \right\}, \\
    &= \max \left\{ \frac{B_1}{u_0} \frac{B_1}{v_0} \right\}, \\
    &= \frac{B_1}{w_0}, \\
    w_4 &= \max \left\{ \frac{C_1}{u_2}, w_2 \right\}, \\
    &= \max \left\{ \frac{C_1}{u_0} w_0 \right\}, \\
    &= w_0, \\
    u_6 &= \max \left\{ \frac{A_1}{v_4}, u_4 \right\}, \\
    &= \max \left\{ \frac{A_1}{B_1} w_0, u_0 \right\}, \\
    &= u_0, \\
    v_6 &= \max \left\{ \frac{B_1}{w_4}, v_4 \right\}, \\
    &= \max \left\{ \frac{B_1}{w_0} \frac{B_1}{v_0} \right\}, \\
    &= \frac{B_1}{w_0}, \\
    w_6 &= \max \left\{ \frac{C_1}{u_4}, w_4 \right\}, \\
    &= \max \left\{ \frac{C_1}{u_0} w_0 \right\}, \\
    &= w_0.
\end{align*}
\]

By induction, we obtain formulas as given in (26):

\[
\begin{align*}
    u_{2n+2} &= u_0, v_{2n+2}, \\
    &= \frac{B_1}{w_0}, w_{2n}, \\
    &= w_0.
\end{align*}
\]

The proof is completed. \[\square\]

Theorem 2. Suppose that \((u_n, v_n, w_n)\) is a solution of systems (1)–(3) such that \(A_0/v_{-1} \leq u_{-1}, B_0/w_{-1} \leq v_{-1}, \) and \(C_0/u_{-1} \leq w_{-1}\). Then, the following statements hold:

1. If \(A_1/v_0 \leq u_0, B_1/w_0 \leq v_0, \) and \(w_0 \geq C_1/u_0\), then

\[
\begin{align*}
    u_{2n+1} &= u_{-1}, v_{2n+1}, \\
    &= v_{-1}, w_{2n+1}, \\
    &= w_{-1}, u_{2n+1}, \\
    &= u_0, v_{2n+2}, \\
    &= v_0, w_{2n+2}. \\
\end{align*}
\]

2. If \(A_1/v_0 \geq u_0, B_1/w_0 \geq v_0, \) and \(w_0 \leq C_1/u_0\), then

\[
\begin{align*}
    u_{2n+1} &= u_{-1}, v_{2n+1}, \\
    &= v_{-1}, w_{2n+1}, \\
    &= w_{-1}, u_{2n+1}, \\
    &= u_0, v_{2n}, \\
    &= v_0, w_{2n}, \\
    &= w_0. \\
\end{align*}
\]

3. If \(A_1/v_0 \leq u_0, B_1/w_0 \leq v_0, \) and \(w_0 \geq C_1/u_0\), then

\[
\begin{align*}
    u_{n+1} &= u_{-1}, v_{2n+1}, \\
    &= v_{-1}, w_{2n+1}, \\
    &= w_{-1}, u_{2n+1}, \\
    &= u_0, v_{2n}, \\
    &= v_0, w_{2n}. \\
\end{align*}
\]

4. If \(A_1/v_0 \geq u_0, B_1/w_0 \leq v_0, \) and \(w_0 \geq C_1/u_0\), then

\[
\begin{align*}
    u_{n+1} &= u_{-1}, v_{2n+1}, \\
    &= v_{-1}, w_{2n+1}, \\
    &= w_{-1}, u_{2n+1}, \\
    &= u_0, v_{2n}, \\
    &= v_0, w_{2n}. \\
\end{align*}
\]
\[ u_{2n+1} = u_{-1}, v_{2n+1}, \]
\[ = v_{-1}, w_{2n+1}, \]
\[ = w_{-1}, u_{2n}, \]
\[ = u_0, v_{2n+2}, \]
\[ = \frac{B_1}{w_0} w_{2n}, \]
\[ = w_0. \]

(27)

Proof.

(1) From the conditions \( A_0/v_{-1} \leq u_{-1}, B_0/w_{-1} \leq v_{-1}, \) and \( C_0/u_{-1} \leq w_{-1}, \) we have

\[ u_1 = \max \left\{ \frac{A_0}{v_{-1}} u_{-1} \right\}, \]
\[ = u_{-1}, \]
\[ v_1 = \max \left\{ \frac{B_0}{w_{-1}} v_{-1} \right\}, \]
\[ = v_{-1}, \]
\[ w_1 = \max \left\{ \frac{C_0}{u_{-1}} w_{-1} \right\}, \]
\[ = w_{-1}, \]
\[ u_3 = \max \left\{ \frac{A_0}{v_1} u_1 \right\}, \]
\[ = \max \left\{ \frac{A_0}{v_{-1}} u_{-1} \right\}, \]
\[ = u_{-1}, \]
\[ v_3 = \max \left\{ \frac{B_0}{w_1} v_1 \right\}, \]
\[ = \max \left\{ \frac{B_0}{w_{-1}} v_{-1} \right\}, \]
\[ = v_{-1}, \]
\[ w_3 = \max \left\{ \frac{C_0}{u_1} w_1 \right\}, \]
\[ = \max \left\{ \frac{C_0}{u_{-1}} w_{-1} \right\}, \]
\[ = w_{-1}. \]

(28)

Similarly, we can find the proof for even terms.

\[ u_2 = \max \left\{ \frac{A_1}{v_0} u_0 \right\}, \]
\[ = u_0, \]
\[ v_2 = \max \left\{ \frac{B_1}{w_0} v_0 \right\}, \]
\[ = v_0, \]
\[ w_2 = \max \left\{ \frac{C_1}{u_0} w_0 \right\}, \]
\[ = w_0. \]

(30)

By induction,

\[ u_{2n} = u_0, v_{2n}, \]
\[ = v_0, w_{2n}, \]
\[ = w_0. \]

(31)

(2) Because \( u_0 \geq A_1/v_0, v_0 \geq B_1/w_0, \) and \( w_0 \leq C_1/u_0, \) then we have

\[ \frac{1}{w_0} \geq \frac{u_0}{C_1}, \]
\[ v_0 \geq B_1 \times \frac{u_0}{C_1}, \]

as given in (24).
By induction, we obtain the formulas as stated in (25).

(3) Because $u_0 \leq A_1/v_0$, $v_0 \geq B_1/w_0$, and $w_0 \geq C_1/u_0$, then

$$u_2 = \max \left\{ \frac{A_1}{v_0}, u_0 \right\},$$
$$v_2 = \max \left\{ \frac{B_1}{w_0}, v_0 \right\},$$
$$w_2 = \max \left\{ \frac{C_1}{u_0}, w_0 \right\},$$

$$u_4 = \max \left\{ \frac{A_1}{v_2}, u_0 \right\},$$
$$v_4 = \max \left\{ \frac{B_1}{w_2}, v_0 \right\},$$
$$w_4 = \max \left\{ \frac{C_1}{u_2}, w_0 \right\},$$

$$u_6 = \max \left\{ \frac{A_1}{v_4}, u_0 \right\},$$
$$v_6 = \max \left\{ \frac{B_1}{w_4}, v_0 \right\},$$
$$w_6 = \max \left\{ \frac{C_1}{u_4}, w_0 \right\},$$

$$u_0 = \max \left\{ \frac{1}{v_0}, u_0 \right\},$$
$$v_0 = \max \left\{ \frac{B_1}{w_0}, v_0 \right\},$$
$$w_0 = \max \left\{ \frac{C_1}{u_0}, w_0 \right\}.$$
\[ A_1 v_0 - 1 \leq u_{-1}, \quad B_0 w_{-1} \geq v_{-1}, \quad C_0 u_{-1} \geq w_{-1}, \] \tag{35}

then we have
\[ \frac{1}{v_{-1}} \geq \frac{w_{-1}}{B_0}, \] \tag{36}

Then, the following system holds for odd solutions:
\[ u_{2n+1} = u_{-1}, v_{2n+1}, \]
\[ = \frac{B_0}{w_{-1}}, \quad w_{2n+1}, \] \tag{37}
\textbf{Theorem 4.} Suppose that \( \{u_n, v_n, w_n\} \) is a solution of systems (1)-(3) such that

\[
\frac{A_0}{v_{-1}} \geq u_{-1}, \quad \frac{B_0}{w_{-1}} \leq v_{-1}, \quad \frac{C_0}{u_{-1}} \geq w_{-1},
\]

then the following statement holds:

\[
u_{2n+1} = \frac{A_0}{v_{-1}} v_{2n+1},
\]

\[= v_{-1}, w_1,
\]

\[= \frac{C_0}{u_{-1}} w_{2n+1},
\]

\[= \frac{C_0}{A_0} v_{-1}.
\]

\textbf{Theorem 5.} Suppose that \( \{u_n, v_n, w_n\} \) is a solution of systems (1)-(3) such that

\[
\frac{A_0}{v_{-1}} \geq u_{-1}, \quad \frac{B_0}{w_{-1}} \leq v_{-1}, \quad \frac{C_0}{u_{-1}} \leq w_{-1},
\]

then the following statement holds:

\[
u_{2n+1} = \frac{A_0}{v_{-1}} v_{2n+1},
\]

\[= v_{-1}, w_{2n+1},
\]

\[= w_{-1}.
\]

\textbf{Theorem 6.} Suppose that \( \{u_n, v_n, w_n\} \) is a solution of systems (1)-(3) such that

\[
\frac{A_0}{v_{-1}} \leq u_{-1}, \quad \frac{B_0}{w_{-1}} \geq v_{-1}, \quad \frac{C_0}{u_{-1}} \leq w_{-1},
\]

then the following statement holds:

\[
u_{2n+1} = u_{-1} v_{2n+1},
\]

\[= \frac{B_0}{w_{-1}} w_{2n+1},
\]

\[= w_{-1}.
\]

\textbf{2. Conclusion}

We investigate the closed form solutions of an important type of difference equation. The eventual periodicity of the following max-type 3 \( D \)-system of difference equations is

\[
u_{n+1} = \max \left\{ \frac{A_n}{v_n}, \frac{B_n}{u_n}, \frac{C_n}{w_n} \right\} v_{n+1},
\]

\[= \max \left\{ \frac{B_n}{u_n}, \frac{C_n}{w_n} \right\} w_{n+1},
\]

\[= \max \left\{ \frac{C_n}{u_n}, \frac{w_n}{u_n} \right\},
\]

where \( n \in \mathbb{N}, N_o = \mathbb{N} \cup \{0\}, (A_n)_n \in N_o, (B_n)_n \in N_o, \) and \( (C_n)_n \in N_o \) are positive periodic sequences and initial conditions \( u_0, u_{-1}, v_0, v_{-1}, w_0, w_{-1} \in (0, +\infty) \).

\textbf{Data Availability}

All data utilized in this article have been included and the sources where they were adopted were cited accordingly.

\textbf{Conflicts of Interest}

The authors declare that they have no conflicts of interest regarding the publication of this paper.

\textbf{References}


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