Research Article

ϕ-δ-Primary Hyperideals in Krasner Hyperrings

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In this paper, we study commutative Krasner hyperrings with nonzero identity. ϕ-prime, ϕ-primary and ϕ-δ-primary hyperideals are introduced. The concept of δ-primary hyperideals is extended to ϕ-δ-primary hyperideals. Some characterizations of hyperideals are provided to classify them. The relation between ϕ-δ-primary hyperideals and other hyperideals is discussed.

1. Introduction

In commutative ring theory, prime and primary ideals have a significant place. The importance of prime ideals encourages researchers to expand these concepts and find applications. Many different types of generalizations have been investigated by several authors, some of them [1–5]. Prime and primary ideals are generalized to ϕ -prime and ϕ-primary ideals. Let (R, +, ·) be a commutative ring with nonzero identity. Denote the set of all ideals of R by L(R) (proper ideals of R by L*(R). Let ϕ be a function such that ϕ: L(R) → L(R) ∪ {∅}. Let N be a proper ideal of R. N is called a ϕ-primary ideal [1], if ab ∈ N − ϕ(N), then either a ∈ N or b ∈ N for some a, b ∈ R. By the way, N is called a ϕ-primary ideal when, if ab ∈ N − ϕ(N), then either a ∈ N or b ∈ N for some a, b ∈ R, k ∈ N [3, 4]. The image of ideal, ϕ(N), can be equal 0, ∅, N, N2, Nω, Nωω (ω denotes the intersection of ideals of N). A proper ideal N of R is called weakly prime (primary) ideal respectively in [6] ([7]) if 0 ≠ ab ∈ N, for some a, b ∈ R, then either a ∈ N or b ∈ N (bk ∈ N for some k ∈ N). Anderson generalized it in [1], where N is weakly ϕ-prime ideal when ϕ(N) = 0. Zhao [8] introduced δ-primary ideal as an expansion of an ideal, δ: L(R) → L(R) is a function that meets the following requirements: i) Nδ ∩ (N), for all ideals N of R, ii) If N ⊆ M, where N and M are ideals of R, then δ(N)δ(M). (ii) δ(K ∩ L) = δ(K)∩δ(L) for all ideals K, L of R. Entire of the δ ideal expansions provides the property δ3 = δ, which is δ(δ(N)) = δ(N) for all ideal N of R [8]. A. Jaber chose ϕ such a reduction function in [9], which satisfies the following requirements: i) ϕ(N)⊆N, for all ideals N of R, ii) If N ⊆ M, where N and M are ideals of R, then ϕ(N)⊆ϕ(M). He obtained generalization of ϕ-δ-primary ideal by combining these two concepts. Let N be an ideal of R, δ be an ideal expansion and ϕ be an ideal reduction [9]. N is called ϕ-δ-primary if ab ∈ N − ϕ(N), then either a ∈ N or b ∈ δ(N), for all a, b ∈ R. Some results on ϕ − δ-primary ideals can be found in [10, 11].

The theory of hyperstructures was innovated by Marty in 1934 [12]. He defined hypergroupoid (G, *) for G # ∅, P*(G) represents family of nonempty subsets of G and *: G × G → P*(G) is a binary hyperoperation. Let (G, *) be a hypergroupoid. G is a semihypergroup, if ∀a, b, c ∈ G, a ° (b ° c) = (a ° b) ° c, which means \( \cup_{a \in a, b \in b, c \in c} = \cup_{a \in a, b \in b, c \in c} ° c \).
∀a ∈ G, there exists e ∈ G such that a ∈ (e’a) ∩ (a’e) in another phrase {a|e’a∈(a’e)} then e is called identity element. An identity element e is called a scalar identity if |a| = (e’a) ∩ (a’e), for all a ∈ G. Let (G’,·) be a semihypergroup. For ∀a ∈ G, if a’G = G'a = G, then (G’,·) is called hypergroup. Let (G’,·) be a hypergroup and ∅ ≠ K be a subset of G. If a’K = K’a = K, for ∀a ∈ K, then (K’,·) is called subhypergroup of (G’,·) where e is a binary hyperoperation on G. Let (G’,·) be a hypergroup. If a’b = b’a, for ∀a,b ∈ G, then (G’,·) is commutative hypergroup [12]. Mittas pioneered the theory of canonical hypergroups in [13–15]; Let R ≠ ∅, (R,+) is called a canonical hypergroup (+ is a hyperoperation) if the following axioms are satisfied: i) + (b + c) = (a + b) + c, for a, b, c ∈ R; ii) a + b = b + a, for a, b ∈ R; iii) ∅ ∩ R ≠ ∅, such that a + 0 = [a], for any a ∈ R; iv) for any a ∈ R, there exists a unique element a ∈ R such that 0 ∈ a + a' (a is called as the opposite of a and it is denoted by a); v) a ∈ a + b implies that b = a - a + c, and a ∈ c - b, which means (R,+) is reversible. Hyperrings and hyperfields were introduced by Krasner in [16] using the canonical hypergroups. (R,+) is called Krasner hyperring if the following statements hold: i) (R,+) is a canonical hypergroup; ii) (R,·) is a semigroup having 0 as a · 0 = 0, a = 0, for all a ∈ R; iii) (b + c) · a = (b · a) + (c · a) and a · (c + b) = (a · b) + (b · c), for all a, b, c ∈ R. Also, Ameri and Norouzi [17] studied general commutative hyperrings. Corsini and Leoreanu [18] presented some of the most important results on hyperrings and illustrated some of the most recent and interesting applications, that is, to geometry, graphs and hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras and C-algebras, artificial intelligence and probability. Dasgupta [19] investigated extensively the prime and primary hyperideals of multiplicative hyperrings with absorbing zero. Davvaz and Leoreanu-Foata, in their monograph [20], presented the main results obtained in hyperring theory till the publication of it, but also an outline of applications of hyperstructures. Recently, Davvaz [21] published the first monograph on semihypergroup theory which covers most of the mathematical ideas and techniques required in the study of semihypergroups. Recently, in [22], r-hyperideals were mentioned as a generalization of r-ideals in commutative rings, and in [23], properties of r-hyperideals and some generalizations of them are investigated in the commutative Krasner hyperrings. Yeşilot et al. defined a hyperideal expansion and δ-primary hyperideal in [24]. Let (R,ϕ,·) be a commutative Krasner hyperring, with nonzero unit, δ: L(R) → L(R) function is defined as a hyperideal expansion function that meets the following requirements where L(R) denotes all hyperideals of R: i) N ≤ δ(N), for all N ∈ L(R), ii) If N ⊈ M, where N, M ∈ L(R), then δ(N) ≤ δ(M). Given an expansion δ of a hyperideal N of R is called δ-primary if a’b ∈ N, then a ∈ N or b ∈ δ(N), for all a, b ∈ R. Ulucak [25] obtained some results on δ-primary and 2-absorbing δ-primary hyperideals.

In this paper, our goal is to extend the concept of δ-primary hyperideals to φ-δ-primary hyperideals in Krasner hyperrings and we intend to give some generalizations of hyperideals. Throughout this paper, (R,ϕ,·) will be a commutative Krasner hyperring with nonzero identity. We denote the set of all hyperideals of R by L(R) and the set of all proper hyperideals of R by L∗(R). We take φ as a function φ: L(R) → L(R) \cup {∅} at the sections of generalizations of prime and primary hyperideals. Firstly, we define φ-prime and φ-primary hyperideals in Krasner hyperrings and we give some characterizations for prime and primary hyperideals. Let φ be a function such that: L(R) → L(R) \cup {∅} and N is a hyperideal of R. N is said to be φ-prime (φ-primary) hyperideal of R if a’b ∈ N → φ(N), then a ∈ N or b ∈ N (resp., b’ ∈ N, for some n ∈ N), for a, b ∈ R. Among other things, we give some characterizations in Theorem 2 and Theorem 5 as main theorems. Then we define φ-δ-primary hyperideals in Krasner hyperrings and also give several characterizations (See; Theorem 7 and Theorem 11). In this case, difference is that, ϕ: L(R) → L(R) \cup {∅} is a reduction function if φ(N)∩N and N \subseteq M implies φ(N)⊆φ(M) for each N, M ∈ L(R). We investigate the attitude of φ-δ-primary hyperideal under homomorphism, in quotient ring, in Cartesian product and other cases (See; Theorem 10, Proposition 10, Proposition 9, Theorem 12).

2. Generalizations of Prime Hyperideals in Krasner Hyperrings

Throughout this section, (R,ϕ,·) is a commutative Krasner hyperring with nonzero identity. We denote the set of all hyperideals of R by L(R). Initially, we give the definition of φ-prime hyperideal and some examples.

**Definition 1.** Let φ be a function such that:
ϕ: L(R) → L(R) \cup {∅} and N be a hyperideal of R. N is said to be aφ-prime hyperideal of R if a’b ∈ N → φ(N), then a ∈ N or b ∈ N (resp., b’ ∈ N, for some n ∈ N), for a, b ∈ R.

**Example 1.** Let R be a commutative Krasner hyperring. Consider the following functions:
ϕ: L(R) → L(R) \cup {∅} defined as follows:
(i) ϕ₀(N) = ∅,
(ii) ϕ₀(N) = 0,
(iii) ϕ₂(N) = N²,
(iv) ϕₙ(N) = Nⁿ, (for any n ≥ 2),
(v) ϕₙ(N) = \capₙ⁻¹ Nⁿ,
(vi) ϕ₁(N) = N.

It is obvious that ϕ₀ ≤ ϕ₀ ≤ ϕ₁ ≤ ⋯ ≤ ϕ₁ ≤ ϕ₁ ≤ ⋯ ≤ ϕ₂ ≤ φ₁.

**Definition 2.** Let R be a hyperring and N be a proper hyperideal of R.

(i) N is prime hyperideal if and only if it is ϕ₀-prime hyperideal.
(ii) N is weakly prime hyperideal if and only if it is ϕ₀-prime hyperideal.
(iii) Nis almost prime hyperideal if and only if it is $\phi_3$-prime hyperideal.
(iv) Nis almost prime hyperideal if and only if it is $\phi_n$-prime hyperideal.
(v) Nis $w$-prime hyperideal if and only if it is $\phi_w$-prime hyperideal.

Proposition 1. Let $N$ be a proper hyperideal of $\mathcal{R}$. Then $N$ is almost prime if and only if $N$ is $\phi_3$-prime hyperideal of $\mathcal{R}$.

Proof. Assume that $a \in \mathcal{R}$ and $a^2 \in N$, for $a, b \in \mathcal{R}$. It follows $a^2 \notin N$ because of $a \leq a_2$. Therefore $a^2 \notin N$ and $a \notin N$. Hence $N$ is almost prime hyperideal of $\mathcal{R}$.

Theorem 1. Let $\phi: L(\mathcal{R}) \to L(\mathcal{R}) \cup \{\emptyset\}$ be a function and $T$ be a proper hyperideal of $\mathcal{R}$ such that $T$ is $\phi$-prime hyperideal of $\mathcal{R}$. If $T$ is not prime, then $T^2 \subseteq \phi(T)$. Hence at the same time it means $\text{iff} T \subseteq \phi(T)$, then $T$ is prime.

Proof. Assume that $T^2 \subseteq \phi(T)$. We need to prove that $T$ is prime. Take $x, y \in T$ for $x, y \in \mathcal{R}$. If $x^2 \notin T$, then $x \notin T$. Since $T$ is $\phi$-prime, then $x \notin T$. Let us suppose $x^2 \notin T$. We can choose $x^2 \notin T$ for some $m \in T$. Then $x^2 \in T \supseteq \phi(T)$. Hence $x \in T$ and $y \in T$. Now on we can assume that $x^2 \subseteq \phi(T)$.

Corollary 1. If $T$ is $\phi$-prime hyperideal of $\mathcal{R}$ with $\phi \leq \phi_3$, then $T$ is almost prime hyperideal.

Proof. We know that $T$ is $\phi$-prime hyperideal while $T$ is prime, for every $\phi$. Therefore $T$ is $\phi$-prime hyperideal of $\mathcal{R}$. Assume that $T$ is not prime. By Theorem 1, $T^2 \subseteq \phi(T)$ for every $n \geq 2$. Hence $T$ is $\phi$-prime hyperideal of $\mathcal{R}$.

Corollary 2. Let $T$ be $\phi$-prime hyperideal that is not prime. Then $T \in \phi(T)$.

Proof. Let $a \in \sqrt{\phi(T)}$. If $a \in T$, then $a^2 \subset T \subseteq \phi(T)$ from Theorem 1. Let us suppose that $a \notin T$. From the main
Theorem 2, \((T; a) = T\) or \((T; a) = (\phi(T); a)\) as \(T \subseteq (T; a)\) gives \(a'T \subseteq \phi(T)\). Let us suppose \((T; a) = T\). Assume that 

\(a^m \in \phi(T)\), but \(a^{m-1} \notin \phi(T)\). Then \(a^{m-1} \in (T; a)\). Hence \(a^{m-1} \in T - \phi(T)\), so \(a \in T\), which is a contradiction.

In the following, we give a proposition about quotient and localization of a hyperring. Let \(S\) be a multiplicatively closed subset of a Krasner hyperring \(R\). Define \(\phi: L(R)\to L(R)\cup \{\emptyset\}\) and \(\psi_i: L(R^i)\to L(R^i)\cup \{\emptyset\}\) with \(\phi_s(M) = (\phi(M \cap R))^s\). Also \(\phi_s(M) = \emptyset\), where \(\phi(M \cap R) = \emptyset\).

Let \(N, M\) be hyperideals of \(R\) and \(N \subseteq M\). Define \(\phi(M)\) with \(\phi(M \cap R) = \emptyset\). Where \(\phi(N) = \emptyset\).

\[\text{Proposition 2. Suppose that } \phi: L(R)\to L(R)\cup \{\emptyset\} \text{ is a function and } T \text{ is a } \phi\text{-prime hyperideal of } R.\]

(i) If \(M\) is a hyperideal of \(R\) with \(M \subseteq T\), then \(T/M\) is a \(\phi\text{-prime hyperideal of } R/M\).

(ii) Let \(S\) be a multiplicatively closed subset of \(R\) with \(T \cap S = \emptyset\) and \(\phi(M) \subseteq S\). Then \(T/S\) is a \(\phi\text{-prime hyperideal of } R/S\).

Proof

(i) Let \(x, y \in R\). Suppose that \((x \otimes b)M \subseteq (T/M) \otimes (T/M)\). So \(x \otimes b \subseteq \phi(T/M)\). Then \(x' \otimes y' \subseteq \phi(T/M)\). We find that \(x' \otimes y' \subseteq \phi(T/M)\) and \(x \in T\) or \(y \in T\). Therefore \(x \otimes b \subseteq \phi(T/M)\).

(ii) Let \((a/s) \otimes (b/t) \subseteq \phi_s(T_s)\), for some \(a \in R; b \in R; s \in S\). Then we have \(p' \otimes a' \otimes b \subseteq T\) for some \(p \in S\) but \(q' \otimes a' \otimes b \subseteq \phi(T)\) for every \(q \in S\). Now if \(q' \otimes a' \otimes b \subseteq \phi(T)\), then \((a/s) \otimes (b/t) \subseteq \phi(T)\), which is a contradiction. So \(p' \otimes a' \otimes b \subseteq T\) and since \(T\) is \(\phi\text{-prime hyperideal of } R\), then we get either \(p' \otimes a \subseteq T\) or \(b \subseteq T\). Hence \(a \in T\) or \(b \subseteq T\), since \(T \cap S = \emptyset\). Thus \(T/S\) is a \(\phi\text{-prime hyperideal of } R/S\).

Theorem 3

(i) Let \(X\) and \(Y\) be commutative Krasner hyperrings and \(N\) be a weakly prime hyperideal of \(X\). Then \(M = N \otimes Y\) is \(\phi\text{-prime hyperideal of } R\) and \(\phi\text{-prime hyperideal of } X\).

(ii) Suppose \(R\) is a commutative Krasner hyperring and \(M\) and \(M\) is a finitely generated proper hyperideal of \(R\). Then \(M\) is a \(\phi\text{-prime hyperideal of } R\).

Assuming \(M\) is a \(\phi\text{-prime hyperideal with } \phi \subseteq \phi_s\). Then \(M\) is either weakly prime or \(M\) is an idempotent and \(R\) decomposes as \(X \otimes Y\) where \(Y = M\) and \(M = N \otimes Y\) with \(N\) weakly prime. As a result \(M\) is a \(\phi\text{-prime hyperideal of } R\).

Proof

(i) Let \(N\) be a weakly prime hyperideal of \(X\). Then \(M = N \otimes Y\) is a weakly prime hyperideal of \(X\) and \(M\) is weakly prime hyperideal if and only if \(M\) is prime hyperideal.

(ii) Let \(N\) be a \(\phi\text{-prime hyperideal of } X\) and \(\phi\text{-prime hyperideal of } X\).

Assuming \(\phi\text{-prime hyperideal of } R\). Then \(\phi\text{-prime hyperideal of } R\) and \(\phi\text{-prime hyperideal of } R\).

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Assuming \(\phi\text{-prime hyperideal of } R\). Then \(\phi\text{-prime hyperideal of } R\) and \(\phi\text{-prime hyperideal of } R\).
Definition 3. Let $\mathbf{R}$ be a hyperring and $\mathbf{N}$ a proper hyperideal of $\mathbf{R}$. Let $\phi: L(\mathbf{R}) \rightarrow L(\mathbf{R}) \cup \{\emptyset\}$ be a function such that $\phi: L(\mathbf{R}) \rightarrow L(\mathbf{R}) \cup \{\emptyset\}$. $\mathbf{N}$ is called $\phi$-primary hyperideal of $\mathbf{R}$ if $a \cdot b \in \mathbf{N} - \phi(N)$, then either $a \in \mathbf{N}$ or $b \in \mathbf{N}$, for some $a, b \in \mathbf{R}$.

Example 2. Let $\mathbf{R}$ be a commutative Krassner hyperring. Then we define $\phi: L(\mathbf{R}) \rightarrow L(\mathbf{R}) \cup \{\emptyset\}$ such as in Example 1. Also we have the same order, $\phi_0 \leq \phi_1 \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \phi_{n-1} \leq \cdots \leq \phi_2 \leq \phi_1$. $\phi$-primary hyperideal of $\mathbf{R}$.

Definition 4. Let $\mathbf{R}$ be a hyperring and $\mathbf{N}$ be a proper hyperideal of $\mathbf{R}$.

(i) $\mathbf{N}$ is primary hyperideal if and only if it is $\phi$-$\phi$-primary hyperideal.

(ii) $\mathbf{N}$ is weakly primary hyperideal if and only if it is $\phi_0$-primary hyperideal.

(iii) $\mathbf{N}$ is almost primary hyperideal if and only if it is $\phi_2$-primary hyperideal.

(iv) $\mathbf{N}$ is almost $\phi$-primary hyperideal if and only if it is $\phi_n$-primary hyperideal.

(v) $\mathbf{N}$ is primary hyperideal if and only if it is $\phi_n$-$\phi$-primary hyperideal.

3. Generalizations of Primary Hyperideals in Krassner Hyperrings

Similar to the previous section, we consider $(\mathbf{R}, \oplus, \otimes)$ to be a commutative Krassner hyperring with nonzero unit. We denote the set of all hyperideals of $\mathbf{R}$ by $L(\mathbf{R})$. Let us define $\phi$-primary hyperideal.

Proposition 4. Let $\mathbf{N}$ be a proper hyperideal of $\mathbf{R}$.

1. If $\sigma_1, \sigma_2$ are two functions with $\sigma_1 \leq \sigma_2$, such that $\sigma_1, \sigma_2: L(\mathbf{R}) \rightarrow L(\mathbf{R}) \cup \{\emptyset\}$ and $\mathbf{N}$ is $\sigma_1$-primary, then $\mathbf{N}$ is $\sigma_2$-primary.

2. (i) $\mathbf{N}$ is primary hyperideal $\Rightarrow$ $\mathbf{N}$ is weakly primary hyperideal $\Rightarrow$ $\mathbf{N}$ is $\phi_1$-primary hyperideal $\Rightarrow$ $\mathbf{N}$ is almost primary hyperideal.

Proof. Assume that $T^{2}\phi(T)$. We need to see that $T$ is primary. Take some $x \cdot y \in T$ for $x, y \in \mathbf{R}$. If $x \cdot y \not\in \phi(T)$, then since $T^{2}$ is $\phi$-primary, $x \in T$ or $y \not\in \phi(T)$, for some $k \in \mathbf{N}$. If $x \cdot y \in \phi(T)$, assuming that $x^{T}\phi(T)$, we get $x \cdot p_0 \not\in \phi(T)$, where $p_0 \in T$. We have $x \cdot (y \circ p_0) \in T - \phi(T)$. Thus $x \in T$ or $(x \circ p_0) T$ for some $k \in \mathbf{N}$. It follows $x \cdot T \in T$. Now we can assume that $y^{T}\phi(T)$. Because of $T^{2}\phi(T)$, there exist $p_{1, q_{1}} \in T$ with $p_{1} q_{1} \not\in \phi(T)$. Then $(x \circ p_{1} q_{1}) \not\in T - \phi(T)$. As $T$ is $\phi$-primary, so $(x \circ p_{1} q_{1})^{T} \in T$ or $(x \circ p_{1} q_{1})^{T} \in T$ for some $m \in \mathbf{N}$. Consequently $T$ is primary hyperideal of $\mathbf{R}$.

Corollary 3. If $\mathbf{N}$ is $\phi_1$-primary hyperideal of $\mathbf{R}$, where $\phi \leq \phi_1$, then $\mathbf{N}$ is $\phi$-primary hyperideal.

Proof. The proof is similar to Corollary 1.

In the following, some characterizations of $\phi$-primary hyperideals are provided.
Theorem 5. Let $N$ be a proper hyperideal of the commutative Krasner hyperring $\mathcal{R}$. Let $I : L(\mathcal{R}) \to L(\mathcal{R}) \cup \{0\}$ be a function. The following statements hold:

(i) $I$ is a $\phi$-primary hyperideal of $\mathcal{R}$.

(ii) For any $a \in \mathcal{R}$,

(iii) For any $a \in \mathcal{R}$,

(iv) For any hyperideals $I \subseteq \mathcal{R}$, $J \subseteq \mathcal{R}$, if $I \cap J \subseteq \mathcal{R}$, then $I \cap J \subseteq \mathcal{R}$.

Proof

(i) $I$ is a $\phi$-primary hyperideal of $\mathcal{R}$.

(ii) $I$ is an $\phi$-primary hyperideal of $\mathcal{R}$.

(iii) $I$ is an $\phi$-primary hyperideal of $\mathcal{R}$.

(iv) $I$ is an $\phi$-primary hyperideal of $\mathcal{R}$.

Theorem 6

(i) Let $X$, $Y$ be commutative Krasner hyperrings and $N$ be a $\phi$-primary hyperideal of $\mathcal{R}$. Then $M = N \otimes Y$ is a $\phi$-primary hyperideal of $\mathcal{R} = X \otimes Y$, for all $\phi$-primary ideals $N$.

(ii) Let $\mathcal{R}$ be a commutative Krasner hyperring and $N$ be a $\phi$-primary hyperideal of $\mathcal{R}$. Then $M$ is a $\phi$-primary hyperideal of $\mathcal{R}$.

Proof

(i) Let $N$ be a weakly primary hyperideal of $\mathcal{R}$. Assume that $I \subseteq \mathcal{R}$. Then $I \subseteq \mathcal{R}$.

(ii) Let $\mathcal{R}$ be a commutative Krasner hyperring and $N$ be a weakly primary hyperideal of $\mathcal{R}$. Assume that $I \subseteq \mathcal{R}$.

(iii) Let $\mathcal{R}$ be a commutative Krasner hyperring and $N$ be a weakly primary hyperideal of $\mathcal{R}$. Assume that $I \subseteq \mathcal{R}$.

(iv) Let $\mathcal{R}$ be a commutative Krasner hyperring and $N$ be a weakly primary hyperideal of $\mathcal{R}$. Assume that $I \subseteq \mathcal{R}$.

Proposition 5. Let $I : L(\mathcal{R}) \to L(\mathcal{R}) \cup \{0\}$ be a function and $\mathcal{R}$ be a $\phi$-primary hyperideal of $\mathcal{R}$.

(i) If $I$ is an hyperideal of $\mathcal{R}$ with $M \subseteq T$, then $T/I \otimes \mathcal{R}$ is a $\phi$-primary hyperideal of $\mathcal{R}$.

(ii) Let $S$ be a multiplicatively closed subset of $\mathcal{R}$ with $T \cap S = \emptyset$ and $I \subseteq \mathcal{R}$. Then $T/S$ is a $\phi$-primary hyperideal of $\mathcal{R}$.

Proof

(i) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$.

(ii) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$.

(iii) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$.

(x) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$.

(y) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$. We find that $x \circ y \in T - \phi(T)$.

(z) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$.

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(y) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$.

(z) Let $x, y \in \mathcal{R}$. We assume that $(\alpha \circ \phi) \otimes (\beta \circ \phi) \subseteq T/M$.
Proposition 6. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be commutative Krasner hyperrings and let $\omega_i: L(\mathcal{R}_i) \rightarrow L(\mathcal{R}_i) \cup \{\emptyset\}$ be a function, for $i = 1, 2$. Take $\omega = \omega_1 \times \omega_2$ and $\mathcal{R}_1 = \mathcal{R}_1 \times \mathcal{R}_2$. Then $N$ is a $\mathcal{R}_1$-primary hyperideal of $\mathcal{R}_1$ if and only if $N$ is one of the following types:

(i) $N = N_1 \times N_2$, where $N_1$ is a proper hyperideal of $\mathcal{R}_1$, with $\omega(N_1) = N_1$.

(ii) $N = N_1 \times N_2$, where $N_1$, is $\omega_1$-primary hyperideal of $\mathcal{R}_1$, that should be primary if $\omega(N_1) \neq N_1$.

(iii) $N = N_1 \times N_2$, where $N_2$, is $\omega_2$-primary hyperideal of $\mathcal{R}_2$, that should be primary if $\omega(N_2) \neq N_2$.

Proof. ($\Rightarrow$) (i) Obviously $N$ is $\phi$-primary hyperideal, since $N_1 \times N_2 - \phi((N_1 \times N_2) = \emptyset$.

(ii) Let $N_1$ be $\phi_1$-primary hyperideal of $\mathcal{R}_1$ and $\phi_2(\mathcal{R}_2) \neq \mathcal{R}_2$. Assume that $(a_1, b_1) \in N_1 \times N_2 - \phi_1((N_1 \times N_2) = (N_1 - \phi_1(N_1)) \times (\mathcal{R}_2 - \phi_2(\mathcal{R}_2)))$.

(iii) The proof is similar to (ii).

($\Leftarrow$) Suppose that $N$ is $\phi$-primary hyperideal of $\mathcal{R}_1$, where $\phi(N_1) \neq N_1$. Let $a \circ b \in \mathcal{R}_1 - \phi_1(N_1)$, for some $a, b \in \mathcal{R}_1$. Thus $(a \circ b, 0) = (a \circ b, 0) \in N - \phi(N)$. Since $N$ is $\phi$-primary hyperideal of $\mathcal{R}_1$, then $(a, 0) \in N$ or $(b, 0)^k \in N$ for some $K \in \mathbb{N}$. So $a \in N_1$ or $b^k \in N_1$.

4. $\phi$-$\delta$-Primary Hyperideals in Krasner Hyperrings

Let $\mathcal{R}$ be a proper hyperideal of hyperring $\mathcal{R}$. Denote the set of all hyperideals of $\mathcal{R}$, by $L(\mathcal{R})$ and denote the set of all proper hyperideals of $\mathcal{R}$, by $L^*(\mathcal{R})$. The function $\phi: L(\mathcal{R}) \rightarrow L(\mathcal{R}) \cup \{\emptyset\}$ is said to be reduction function if $\phi(N) \cap N \subseteq M$ implies $\phi(N) \subseteq \phi(M)$, for each $N, M \in L(\mathcal{R})$ and $\delta$ be an expansion function such that $\delta: L(\mathcal{R}) \rightarrow L(\mathcal{R})$. Now, we give some examples related to reduction and expansion functions.

Example 3. Let $\mathcal{R}$ be a commutative Krasner hyperring with a nonzero identity. Let us consider the following functions $\omega_l$ on $L(\mathcal{R})$, for any $N \in L(\mathcal{R})$:

(i) $\delta(N) = N$, i.e., $\delta$ is the identity function.

(ii) $\delta_i(N) = \sqrt{N}$, i.e., $\delta$ is the radical operation.

(iii) $\delta_{\text{res}}(N) = (N: M)$ for a fixed $M \in L(\mathcal{R})$.

(iv) $\delta_{\text{ann}}(N) = \text{ann}(\text{ann}(N))$.

(v) $\delta_{M}(N) = N \circ M$ for a fixed $M \in L(\mathcal{R})$.

All the above functions are examples of expansion on $L(\mathcal{R})$.

Example 4. Let $\mathcal{R}$ be a commutative Krasner hyperring with a nonzero identity. Consider the following functions $\phi: L(\mathcal{R}) \rightarrow L(\mathcal{R}) \cup \{\emptyset\}$ defined as follows: for any $N \in L(\mathcal{R})$:

(i) $\phi_0(N) = \emptyset$.

(ii) $\phi_0(N) = 0$.

(iii) $\phi_1(N) = N$.

(iv) $\phi_2(N) = N^2$.

(v) $\phi_3(N) = N^k$.

(vi) $\phi_4(N) = \cap_{i \in \mathbb{N}} N^i$.

All the above functions are reduction on $L(\mathcal{R})$. Remember that $\phi_0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq \phi_{n+1} \leq \phi_n \leq \phi_{n+1} \leq \ldots \leq \phi_2 \leq \phi_1$.

Definition 5. Let $\phi$ be a hyperideal expansion, $\omega$ be a hyperideal reduction and $\mathcal{N}$ be a proper hyperideal of $\mathcal{R}$. $\omega$ is said to be $\omega$-$\delta$-primary hyperideal if $a \circ b \in N - \phi(N)$, then either $a \in N_1$ or $b \in N_1$.

Remark 1. [24] If $\delta_0, \delta_2, \ldots, \delta_4$ are hyperideal expansions, then $\omega_1 \circ \delta_1$ is also a hyperideal expansion.

Definition 6. Let $\mathcal{R}$ be a hyperring and $\mathcal{N}$ be a proper hyperideal of $\mathcal{R}$.

(i) $\mathcal{N}$ is prime hyperideal if and only if it is $\phi_2$-$\delta_0$-primary hyperideal [24].
Proposition 7. Let $R$ be a hyperring and $N$ be a proper hyperideal of $R$. The following statements hold:

(i) If $N$ is a weakly $\phi$-primary hyperideal, then $N$ is a weakly $\phi$-primary hyperideal of $R$. 
(ii) If $N$ is a weakly $\phi$-primary hyperideal, then $N$ is an almost $\phi$-primary hyperideal of $R$. 
(iii) Every $\phi$-prime hyperideal is an almost $\phi$-primary hyperideal. 
(iv) Every $\phi$-prime hyperideal is an almost $\phi$-primary hyperideal.

Proof. (i) If $N$ is a weakly $\phi$-primary hyperideal, then $N$ is a weakly $\phi$-primary hyperideal of $R$. 
(ii) If $N$ is a weakly $\phi$-primary hyperideal, then $N$ is an almost $\phi$-primary hyperideal of $R$. 
(iii) Every $\phi$-prime hyperideal is an almost $\phi$-primary hyperideal.

Proposition 8. Let $\phi$ be a hyperideal reduction, $\delta$ be a hyperideal expansion and $\{m_i : i \in \Delta\}$ be a directed family of $\phi$-primary hyperideals of $R$. Then $M = \cup_{i \in \Delta} M_i$ is $\alpha \phi \delta$-primary hyperideal.

Proof. Let $\{m_i : i \in \Delta\}$ be a directed family of $\phi \delta$-primary hyperideals of $R$. Assume that $a \ast m \in M - \phi (\cup_{i \in \Delta} M_i)$. This implies that $a \ast m \in M_i - \phi (M_i)$ for some $i \in \Delta$. We get either $a \ast m \in M_i$ or $m \in \delta (M_i)$, because of $M_i$ is a $\phi \delta$-primary. If $a \in M_i$, then clearly we have $a \in \cup_{i \in \Delta} M_i$. If $m \in \delta (M_i)$, then we have $m \in \delta (\cup_{i \in \Delta} M_i)$, since $M_i \subseteq \cup_{i \in \Delta} M_i$. Hence $M = \cup_{i \in \Delta} M_i$ is a $\phi \delta$-primary hyperideal.

In the following, we give a characterization for $\phi \delta$-primary hyperideals such that $\phi$ is a hyperideal reduction and $\delta$ is a hyperideal expansion. 

Theorem 7. Let $R$ be a Krasner hyperring and $N$ be a proper hyperideal of $R$. Then the following statements hold:

(i) If $N$ is a weakly $\phi$-primary hyperideal; 
(ii) For each $a \in R - \phi (N)$, $(N : a) = N \cup (\phi (N) : a)$; 
(iii) For each $N \subseteq R - \phi \delta (N)$, $N = \phi (N) \cup (\phi (N) : a)$; 
(iv) For each hyperideal $K$ of $R$, $K \subseteq N \cup (\phi (N) : a)$ implies that $K \subseteq N \cup (\phi (N) : a)$.

Proof. (i) If $N$ is a weakly $\phi$-primary hyperideal and $a \in R - \phi (N)$. It is clear that $N \cup (\phi (N) : a) \subseteq (N : a)$. Let $m \in (N : a)$. Then we have $a \ast m \in N$. If $a \ast m \in \phi (N)$, then we obtain $m \in (\phi (N) : a) \subseteq N \cup (\phi (N) : a)$. Assume that $a \ast m \notin \phi (N)$. Since $a \ast m \ast n \notin \phi (N)$ and $a \notin \delta (N)$, then we get $m \in N \subseteq N \cup (\phi (N) : a)$. Hence, $(N : a) = N \cup (\phi (N) : a)$.

(ii) If $N$ is a weakly $\phi$-primary hyperideal, then it follows from the fact that a hyperideal is a union of two hyperideals. Then it must be equal to one of them.

(iii) If $N$ is a weakly $\phi$-primary hyperideal and $a \in R - \phi (N)$. Assume that $L \subseteq \phi (N)$ and $K \subseteq \phi (N)$. Then there exists $m \in L - \phi (N)$. We have to show $K \subseteq \phi (N)$.

(iii) If $N$ is a weakly $\phi$-primary hyperideal, then it follows from (ii) and Definition 6 (iii), since $\delta \subseteq \delta$. 
(iv) If $N$ is a weakly $\phi$-primary hyperideal, then it follows from (ii) and Definition 6 (iii), since $\phi \subseteq \phi$. 
(v) If $N$ is a weakly $\phi$-primary hyperideal, then it follows from (i) and the fact that $\phi \subseteq \phi \subseteq \phi \subseteq \phi$. 

Proposition 7. Let $\phi$ be a hyperideal reduction, $\delta$ be a hyperideal expansion and $\{m_i : i \in \Delta\}$ be a directed family of $\phi$-primary hyperideals of $R$. Then $M = \cup_{i \in \Delta} M_i$ is $\alpha \phi \delta$-primary hyperideal.
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Theorem 8

(i) Let $\mathfrak{R}$ be an $\alpha\delta$-primary hyperideal of $\mathfrak{R}$ such that $\phi(T): a = \phi(T): a$ \ for \ each \ $a \in \mathfrak{R}$. Then $\phi(T): a$ is an $\alpha\delta$-primary hyperideal of $\mathfrak{R}$.

(ii) Suppose that $\delta \leq \delta$, then $\delta(T) \subseteq \delta(T)$. Then $\delta(T) = \delta_1(T)$.

Proof

(i) Let $T$ be a $\phi\delta$-primary hyperideal of $\mathfrak{R}$ such that $\phi(T): a = \phi(T): a$, for each $a \in \mathfrak{R}$. We show that $\phi(T): a$ is a $\phi\delta$-primary hyperideal of $\mathfrak{R}$. For this, let we take $x, m \in \mathfrak{R}$ such that $x \cdot m \in (T: a) - \phi(T): a$. Then we have $x \cdot a \cdot m \in T$. Since $\phi(T): a = \phi(T): a$, then we also have $x \cdot a \cdot m \notin \phi(T)$. Therefore, $T$ is a $\phi\delta$-primary hyperideal of $\mathfrak{R}$.

Theorem 9

Let $T$ be a $\phi\delta$-primary hyperideal of $\mathfrak{R}$. Then $\delta(T) \subseteq \delta(T)$. Then $\delta(T) = \delta_1(T)$.

Proof

Let $a \cdot m \in T$, for some $a, m \in \mathfrak{R}$. If $a \cdot m \notin \phi(T)$, then we conclude either $a \in \delta(T)$ or $m \in \delta(T)$ as an $\alpha\delta$-primary hyperideal of $\mathfrak{R}$. Assume that $a \cdot m \notin \phi(T)$. If $a \cdot T \subseteq \phi(T)$, then there exists $n \in T$ such that $a \cdot n \notin \phi(T)$. Thus we have $a \cdot n \cdot m \in (T: \phi(T))$, which implies either $a \in \delta(T)$ or $m \in \delta(T)$. Then we get $a \in \delta(T)$ or $m \in T$, which completes the proof. Assume that $a \cdot T \subseteq \phi(T)$. Similarly, we may assume that $T \subseteq \phi(T)$, and $\alpha$-primary hyperideal of $\mathfrak{R}$. As $\delta(T) \subseteq \delta(T)$, we can find $b \in \delta(T)$ and $m \in T$ such that $b \cdot m \subseteq \phi(T)$. Then we conclude that $\alpha_{\phi}(b) \subseteq \delta(T)$ and $\phi(T)$. Since $T$ is a $\phi\delta$-primary hyperideal of $\mathfrak{R}$, then we have either $\alpha_{\phi}(b) \subseteq \delta(T)$ or $\phi(T)$. Therefore, $T$ is a $\phi\delta$-primary hyperideal of $\mathfrak{R}$. □

Definition 8

(i) [24] A hyperideal expansion is said to be global if for any hyperring good homomorphism $\mathfrak{R} \rightarrow \mathfrak{S}$, $\delta(\mu^{-1}(M)) = \mu^{-1}(\delta(M))$, for each $M \in L(\mathfrak{S})$.

(ii) A hyperideal reduction is said to be a global if for any homomorphism $\mathfrak{R} \rightarrow \mathfrak{S}$, $\phi(\mu^{-1}(M)) = \mu^{-1}(\delta(M))$, for each $M \in L(\mathfrak{S})$.

For instance, the hyperideal reductions $\phi_0, \phi_1$ and the hyperideal expansions $\delta_0, \delta_1$ are both global.

Theorem 10

Let $\mu$ be a good homomorphism from Kranser hyperrings $\mathfrak{R}$ into a Kranser hyperring $(\mathfrak{S}, +, \cdot)$. The following statements hold:

(i) $\mu: \mathfrak{R} \rightarrow \mathfrak{S}$ is a $\alpha\delta$-primary hyperideal of $\mathfrak{R}$ if and only if $\mu$ is a $\alpha\delta$-primary hyperideal of $\mathfrak{S}$.

Proof

Let $\mu: \mathfrak{R} \rightarrow \mathfrak{S}$ be a hyperideal expansion of $\mathfrak{R}$ such that $\delta(\mu^{-1}(M)) = \mu^{-1}(\delta(M))$, for each $M \in L(\mathfrak{S})$.

(iii) $\mu: \mathfrak{R} \rightarrow \mathfrak{S}$ be a $\alpha\delta$-primary hyperideal of $\mathfrak{R}$ if and only if $\mu$ is a $\alpha\delta$-primary hyperideal of $\mathfrak{S}$.

Proof

Let $\mu: \mathfrak{R} \rightarrow \mathfrak{S}$ be a hyperideal expansion of $\mathfrak{R}$ such that $\delta(\mu^{-1}(M)) = \mu^{-1}(\delta(M))$, for each $M \in L(\mathfrak{S})$.
Since $N$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$, then we have either $a \in \delta(N)$ or $m \in N$. Since $N \cap \mu(N)$, then we have $a \in \mu(a) \subseteq \delta(\mu(N))$ or $m \in \mu(m) \subseteq \mu(N)$. Therefore, $\mu(N)$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{S}$.

As an instant consequence of the previous theorem, we get the following explicit results.

Corollary 4

(i) Let $N$ be $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$ and $M$ be a hyperideal of $\mathfrak{R}$ with $N \subseteq M$. Suppose that $\phi$ is global reduction function and $\delta$ is global expansion function. Then $N \cap M$ is $\phi$-$\delta$-primary hyperideal of $\mathfrak{M}$.

(ii) Let $N$ and $M$ be two hyperideals of $\mathfrak{R}$ with $N \subseteq M$. Suppose that $\phi$ is global reduction function and $\delta$ is a global expansion function. Then the intersection $N \cap M$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$ if and only if $N \cap M$ is $\phi$-$\delta$-primary hyperideal of $\mathfrak{M}$.

Theorem 11. Let $N$ be a hyperideal of $\mathfrak{R}$. Then $N$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$ if and only if $N \cap \phi(N)$ is a weakly $\delta$-primary hyperideal of $\mathfrak{R}/\phi(N)$.

Proof. Assume that $N$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$. Let $(a, \phi(N)) = (m, \phi(N)) \subseteq N \cap \phi(N)$. Then we have $a \in N \cap \phi(N)$. Since $N$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$, then we get $a \in \delta(N) \cap m \in N$. Hence, $\phi(N)$ is a weakly $\delta$-primary hyperideal of $\mathfrak{R}/\phi(N)$. For the converse, let $a, m \in N \cap \phi(N)$. Then we get $a \in \delta(N \cap \phi(N))$ or $m \in N \cap \phi(N)$ implying $a \in N \cap \phi(N)$. Therefore, $N$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$.

Let $\mathfrak{S}$ be a commutative Krasner hyperring, $\mathfrak{R} \subseteq \mathfrak{S}$ be a multiplicatively closed subset of $\mathfrak{R}$ and $T$ be a proper hyperideal of $\mathfrak{R}$.

Proposition 9. Let $\phi_3: L(\mathfrak{R}_3) \to L(\mathfrak{R}_3) \cup \{\emptyset\}$ be a hyperideal reduction function and $\delta_3: L(\mathfrak{R}_3) \to L(\mathfrak{R}_3)$ be a hyperideal expansion function such that $\delta_3(N_3) = \delta(N_3)$, for each $N_3 \in L(\mathfrak{R}_3)$. If $T$ is a $\phi$-$\delta$-primary hyperideal such that $T \cap S = \emptyset$ and $T \cap \phi(T_3) \subseteq \phi(T_3)$, then $T_3$ is a $\phi_3$-$\delta_3$-primary hyperideal of $\mathfrak{R}_3$.

Proof. Let $a \in \phi_3(t) \subseteq \phi_3(T_3)$, for some $a, m \in \mathfrak{R}_3, t \in S$. Then we have $p \in \phi_3(T_3) \subseteq \phi_3(T_3)$. Since $T$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}_3$, then we have either $a \in \delta(T)$ or $m \in T$. If $a \in \delta(T)$, then $a \in \delta(T) = \delta(T_3)$. If $p \in T$, then we have $m \in T$ and $m \in T$. Therefore, $T_3$ is a $\phi_3$-$\delta_3$-primary hyperideal of $\mathfrak{R}_3$.

Let $\psi_1: L(\mathfrak{R}_1) \to L(\mathfrak{R}_1) \cup \{\emptyset\}$ be hyperideal reduction functions and $\gamma_1: L(\mathfrak{R}_1) \to L(\mathfrak{R}_1)$ be hyperideal expansion functions for $i = 1, 2$. Suppose that $\mathfrak{R}_1 = \mathfrak{R}_1 \times \mathfrak{R}_1$. Also, each hyperideal $N$ of $\mathfrak{R}$ has the form $N = N_1 \times N_2$, where $N_1$ is a hyperideal of $\mathfrak{R}_1$. Furthermore, $\phi: L(\mathfrak{R}) \to L(\mathfrak{R}) \cup \{\emptyset\}$ defined by $\phi(N_1 \times N_2) = \phi_1(N_1) \times \phi_2(N_2)$ is a hyperideal reduction function and also the function $\delta: L(\mathfrak{R}) \to L(\mathfrak{R})$ defined by $\delta(N_1 \times N_2) = \gamma_1(N_1) \times \gamma_2(N_2)$ is a hyperideal expansion function.

Proposition 10. Let the notation be as in the above paragraph and $N = N_1 \times N_2$. Then each of the types of $\mathfrak{R}$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2$.

Proof.

(i) $N = N_1 \times N_2$, where $N$ is a proper hyperideal of $\mathfrak{R}_1 \times \mathfrak{R}_2$, where $N_1$ is $\gamma_1$-primary hyperideal of $\mathfrak{R}_1$.

(ii) $N = N_1 \times \mathfrak{R}_2$, where $N_1$ is $\gamma_1$-primary hyperideal of $\mathfrak{R}_1$.

(iii) $N = \mathfrak{R}_1 \times N_2$, where $N_2$ is $\gamma_2$-primary hyperideal of $\mathfrak{R}_2$.

(iv) $N = N_1 \times N_2$, where $N_1 \phi_1 \gamma_1$-primary hyperideal of $\mathfrak{R}_1 \times \mathfrak{R}_2$.

(v) $N = \mathfrak{R}_1 \times N_2$, where $N_2$ is $\phi_2 \gamma_2$-primary hyperideal of $\mathfrak{R}_1 \times \mathfrak{R}_2$.

Proof. (⇒) Suppose that $N$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$, where $\phi_1(N_1) \neq \phi$. Let $a \neq m = N_1 \neq \phi_1(N_1)$, for some $a, m \in \mathfrak{R}_1$. Thus $(a, 0) = (m, 0) = N \in \phi$ $\neq \phi$. Since $N$ is a $\phi$-$\delta$-primary hyperideal of $\mathfrak{R}$, then $(a, 0) \in N$ or $(m, 0) \in \delta(N)$. So $a \in N_1$, or $m \in \gamma_1(N_1)$. Therefore $N_1$ is $\phi_1$, $\gamma_1$-primary hyperideal of $\mathfrak{R}_1$. Similarly, we can
find \( N_2 \) is \( \phi_2 \)-\( \gamma_2 \)-primary hyperideal of \( \mathcal{R}_2 \). We have to show now that \( N_1 = \mathcal{R}_1 \) or \( N_2 = \mathcal{R}_2 \). Suppose that \( N_1 \neq \mathcal{R}_1 \). Let we take \( m_1 \in N_1 - \phi_1(N_1) \), \( m_2 \in -\mathcal{R}_2 - N_2 \). Notice that \((1,0) \circ \delta(m_1,m_2) = (m_1,0) \in N - \phi(N)\). This implies that \((1,0) \in \delta(N)\), so we find \( 1 \in \gamma_1(N_1) \). Then \( N_1 = \mathcal{R}_1 \). Similarly, one can easily find \( N_2 = \mathcal{R}_2 \), if \( N_2 \neq \mathcal{R}_2 \). Without loss of generality \( N_1 \neq \mathcal{R}_1 \). Let we show now that \( N_1 \) is \( \gamma_1 \)-primary hyperideal with \( \phi_2(\mathcal{R}_2) \neq \mathcal{R} \). For \( m \in \mathcal{R}_2 - \phi_2(\mathcal{R}_2) \), let \( x \circ m \in N \), for some \( x, m \in \mathcal{R}_1 \).

We have that \((x,1) \circ (m,m') = (x \circ m, m') \in N - \phi(N)\). Since \( N \) is a \( \phi \)-primary hyperideal often we find \((x,1) \in \delta(N) = \gamma_1(N_1) \times \gamma_2(N_2) \) or \((m,m') \in N \) which implies that \( x \in \gamma_1(N_1) \) or \( m \in N_1 \). Therefore \( N_1 \) is a \( \gamma_1 \)-primary hyperideal of \( \mathcal{R}_1 \). If \( \phi(\mathcal{R}_1) \neq \mathcal{R}_1 \) and \( N_1 \neq \mathcal{R}_1 \), then similarly one can prove that \( N_2 \) is a \( \gamma_2 \)-primary hyperideal of \( \mathcal{R}_2 \).

\[ \Box \]

5. Conclusion

In this paper, generalizations of prime and primary hyperideals were provided using the function \( \phi \). We introduced \( \phi \)-prime, \( \phi \) primary and \( \phi \)-\( \delta \) primary hyperideals and several characterizations to classify them were provided. Many properties of \( \phi \) prime, \( \phi \)-primary and \( \phi \)-\( \delta \)-primary hyperideals under particular cases were investigated. In the future work, one can develop the study of \( \phi \)-\( \delta \)-primary hyperideals.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Disclosure

A preliminary version of this manuscript was submitted as a preprint in arxiv.org [26].

Conflicts of Interest

All authors declare that they have no conflicts of interest.

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