Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 1241303, 21 pages
https://doi.org/10.1155/2022/1241303

Research Article

Analysis with Applications of the Generalized Type-II Progressive Hybrid Censoring Sample from Burr Type-XII Model

M. Nagy, M. E. Bakr, and Adel Fahad Alrasheedi

Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to M. Nagy; mnaji@ksu.edu.sa

Received 31 October 2021; Revised 20 December 2021; Accepted 6 January 2022; Published 18 February 2022

Academic Editor: Firdous Khan

In this article, based on the generalized Type-II progressive hybrid censoring sample from the Burr Type-XII distribution, maximum likelihood and Bayesian inference are discussed. Point and interval estimates of unknown parameters, reliability, and hazard functions are developed. We employed several loss functions, such as squared error, LINEX, and general entropy, as symmetric and asymmetric loss functions and various prior distributions as informative and non-informative priors for Bayesian inference of unknown parameters. Under a generalized Type-II progressive hybrid censoring sample, we also propose a Bayesian one-sample prediction for unobserved failures. We conduct simulation study using the MCMC algorithm for the Bayesian approach based on several prior distributions. Finally, we apply the results of the theoretical research to real data.

1. Introduction

In reliability and survival analysis, the progressive Type-II censoring scheme is the most commonly used scheme. It is preferable to the classical Type-II censoring scheme. Progressive censoring is beneficial in various real-life applications such as industry, life research, and clinical applications. It allows the removal of surviving experimental units until the end of the test. Suppose that a life test is conducted with n units, and due to cost and time constraints, it is not desirable to capture all failures. Therefore, only a subset of the unit failures is observed. Such a sample is called a censored sample. Assume that one of the units was broken by accident after the test began, but before all of the units had burned out. If the experiment is still ongoing, this unit must be removed from the life test. The progressive censoring scheme gives a methodology for analyzing this type of data in this case. Some primary referred works are [1–5].

The disadvantages of the Type-II progressive censoring method are that the experiment time can be quite long if the units are highly reliable. As a result, the works of [6, 7] address this issue by suggesting a new type of censoring in which the experiment's stopping time is $\min\{X_{m:m:n}, T\}$, where $T \in (0, \infty)$ is fixed before the starting of the life test. This type of censoring is known as progressive hybrid censoring (PHCS). Under PHCS, the total duration of the experiment will not exceed T. Several authors have looked into the PHCS, including [8, 9].

However, the disadvantage of PHCS is that it cannot be implemented when so few failures can be detected before T. For this reason, Lee et al. [10] proposed a general type of censoring, called generalized Type-II PHCS, in which the lower number of failures is specified. The experiment of lifetime test would save time and cost of failures based on this censoring scheme. Moreover, the estimates of the statistical efficiency of the experiment are improved by more failures. In the following section, the comprehensive notion of generalized Type-II PHCS and its advantages are explained. For recent work on PHCS, see, for example, [11–14].

The contribution in this paper is that we use the generalized Type-II PHCS data from Burr Type-XII distribution to discuss statistical inference for unknown distribution's parameters and prediction for the removed units in each phase of the generalized Type-II PHCS.

Using the exponential form, the Burr Type-XII distribution has the following density function (PDF), distribution function (CDF), and hazard rate function, respectively, given by

$$f(x|\lambda,\gamma) = \lambda\gamma \frac{x^{\gamma-1}}{1+x^{\gamma}} e^{-\lambda \ln(1+x^{\gamma})}, \tag{1}$$

$$F(x|\lambda,\gamma) = 1 - e^{-\lambda \ln(1+x^{\gamma})}, x > 0, \lambda > 0, \gamma > 0. \tag{2}$$

The survival and hazard rate functions are given, respectively, by

$$\begin{aligned} S(x|\lambda,\gamma) &= e^{-\lambda \ln(1+x^{\gamma})}, \\ h(x|\lambda,\gamma) &= \lambda\gamma \frac{x^{\gamma-1}}{(1+x^{\gamma})}. \end{aligned} \tag{3}$$

The Burr Type-XII distribution has been extensively investigated by numerous researchers due to its applications in various disciplines such as biological, industrial, reliability and life testing, clinical studies, and so on (see [15, 16]). Recently, Nagy et al. in [17] have considered generalized Type-I PHCS with one specific time and two numbers of failures (k, m). This is one of the methods' drawbacks, especially when the units are highly reliable. In this paper, we employed a generalized Type-II PHCS to control the duration of the experiment which involves some additional complications.

The rest of the article is structured as follows. Section 2 gives a summary of the generalized Type-II PHCS. Section 3 calculates maximum likelihood (ML) estimates, while Section 4 calculates Bayesian estimates for unknown parameters and survival and hazard functions using three loss functions. The Bayesian prediction for the removed units in each phaseof the generalized Type-II PHCS is derived in Section 5. Simulation studies are carried out in Section 6 to compare the efficacy of the different inference methodologies. Real data are utilized to demonstrate the theoretical findings in Section 7. Finally, the article is concluded in Section 8.

2. The Model Explanation

Consider a life test in which n identical items are put on test. The generalized Type-II PHCS may be described as follows. Let $T_1, T_2 \in (0,\infty)$ and integer $m \in \{1, 2, \ldots, n\}$ be prefixed such that $T_1 < T_2$ with $R = (R_1, R_2, \ldots, R_m)$ is also prefixed integer satisfying $n = m + R_1 + \cdots + R_m$. At the time of first failure, R_1 of the remaining units are randomly eliminated. Similarly at the time of the second failure,R_2 of the remaining units are removed, and so on. This process repeats until the termination time $T^* = \max\{T_1, \min\{X_{m:m:n}, T_2\}\}$; at this time, all the remaining units are removed from the experiment. Let D_1 and D_2 denote the numbers of observed failures up to time T_1 and T_2, respectively. Also, let d_1 and d_2 be the observed values of D_1 and D_2, respectively. A schematic representation of the generalized Type-II PHCS can be found in Figure 1.

Under the generalized Type-II PHCS described above, we have one of the following types of observations:

(1) If the m^{th} failure time occurs before T_1, then instead of terminating the test by withdrawing the remaining R_m items after the m^{th} failure, we continue to observe failures (without any further withdrawals) up to time T_1. Therefore, the observed failure times are $\{X_{1:m:n} < \ldots < X_{m:m:n} < \ldots < X_{d_1:n}\}$.

(2) If the m^{th} failure occurs between T_1 and T_2, then the termination time is $X_{m:m:n}$, and the observed failure times are $\{X_{1:m:n} < \ldots < X_{d_1:m:n} < \ldots < X_{m:m:n}\}$.

(3) If the m^{th} failure occurs after T_2, then the experiment terminates at T_2 and the observed failure times are $\{X_{1:m:n} < \ldots < X_{d_1:m:n} < \ldots < X_{d_2:m:n}\}$.

Based on the generalized Type-II PHCS, the likelihood function is given by

$$L(\theta|\underline{x}) = \begin{cases} \left[\prod_{i=1}^{d_1}\sum_{j=i}^{m}(R_j+1)\right]\prod_{i=1}^{d_1} f(x_{i:m:n})[\overline{F}(x_{i:m:n})]^{R_i}[\overline{F}(T_1)]^{R^*_{d_1+1}}, d_1 = m, m+1, \ldots, n, \\ \left[\prod_{i=1}^{d_1}\sum_{j=i}^{m}(R_j+1)\right]\prod_{i=1}^{m} f(x_{i:m:n})[\overline{F}(x_{i:m:n})]^{R^*_i}, d_1 = 0, 1, \ldots, m-1, d_2 = m, \\ \left[\prod_{i=1}^{d_2}\sum_{j=i}^{m}(R_j+1)\right]\prod_{i=1}^{d_2} f(x_{i:m:n})[\overline{F}(x_{i:m:n})]^{R_i}[\overline{F}(T_2)]^{R^*_{d_2+1}}, d_2 = 0, 1, \ldots, m-1. \end{cases} \tag{4}$$

Therefore, these cases can be combined and obtained as

$$L(\theta|\underline{x}) = \left[\prod_{i=1}^{d^*}\sum_{j=i}^{m}(R_j+1)\right]\prod_{i=1}^{d^*} f(x_{i:m:n})[\overline{F}(x_{i:m:n})]^{R^*_i}[\overline{F}(T_1)]^{R^*_{d_1+1}}[\overline{F}(T_2)]^{R^*_{d_2+1}}, \tag{5}$$

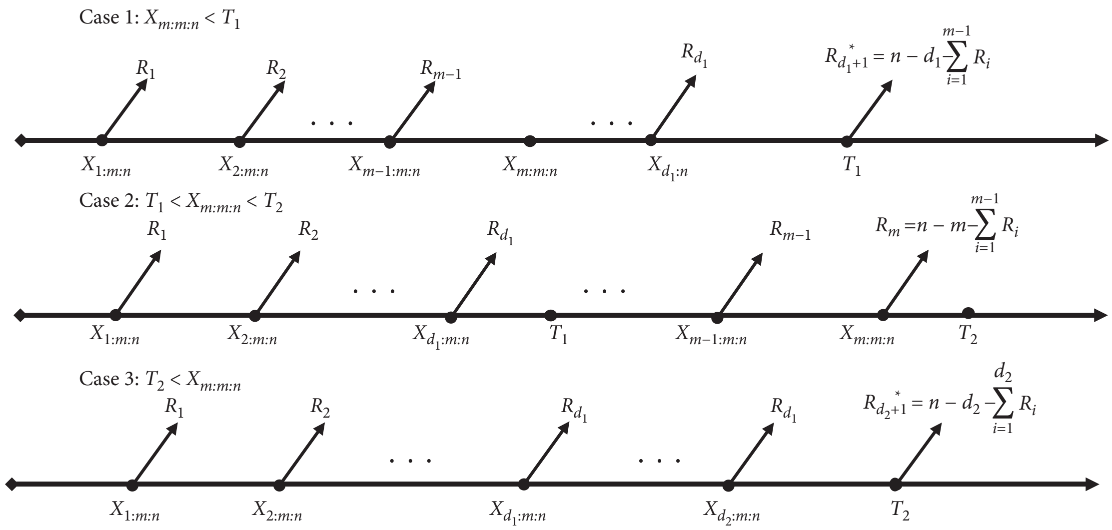


FIGURE 1: Schematic representation of generalized Type-II progressive hybrid censoring scheme.

where

$$d^* = \begin{cases} d_1, & \text{if } X_{m:m:n} < T_1, \\ m, & \text{if } T_1 < X_{m:m:n} < T_2, \\ d_2, & \text{if } T_2 < X_{m:m:n}. \end{cases}$$

$$R^* = \begin{cases} \left(R_1, \ldots, R_{m-1}, 0, \ldots, 0, R^*_{d_1+1}\right) & \text{if } X_{m:m:n} < T_1, \\ \left(R_1, \ldots, R_m\right), & \text{if } T_1 < X_{m:m:n} < T_2, \\ \left(R_1, \ldots, R_{d_2}, R^*_{d_2+1}\right), & \text{if } T_2 < X_{m:m:n}, \end{cases} \tag{6}$$

where $R^*_{d_1+1}$ and $R^*_{d_2+1}$ represent the number of surviving units that are removed at T_1 and T_2 and are given by

$$R^*_{d_1+1} = \begin{cases} n - d_1 - \sum_{j=1}^{m-1} R_j, & \text{if } X_{m:m:n} < T_1, \\ 0 & \text{if } T_1 < X_{m:m:n} < T_2, \\ 0 & \text{if } T_2 < X_{m:m:n}, \end{cases}$$
$$R^*_{d_2+1} = \begin{cases} 0, & \text{if } X_{m:m:n} < T_1, \\ 0, & \text{if } T_1 < X_{m:m:n} < T_2, \\ n - d_2 - \sum_{j=1}^{d_2} R_j, & \text{if } T_2 < X_{m:m:n}. \end{cases} \tag{7}$$

The likelihood function of λ, γ under the generalized Type-II PHCS can be derived using (1) and (2) in (5), as

$$\begin{aligned} L(\lambda, \gamma \mid \underline{x}) &= \left[\prod_{i=1}^{d^*} \sum_{j=i}^{m} \left(R_j + 1\right)\right] \lambda^{d^*} \gamma^{d^*} \prod_{i=1}^{d^*} \frac{x_i^{\gamma-1}}{1 + x_i^{\gamma}} \\ &\quad \times \exp\left\{-\lambda\left[\sum_{i=1}^{d^*} \left(R_i^* + 1\right) \ln\left(1 + x_i^{\gamma}\right) + R^*_{d_1+1} \ln\left(1 + T_1^{\gamma}\right) + R^*_{d_2+1} \ln\left(1 + T_2^{\gamma}\right)\right]\right\} \\ &= \left[\prod_{i=1}^{d^*} \sum_{j=i}^{m} \left(R_j + 1\right)\right] \lambda^{d^*} \gamma^{d^*} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1 + x_i^{\gamma}}\right)\right] \\ &\quad \times \exp\left\{-\lambda\left[\sum_{i=1}^{d^*} \left(R_i^* + 1\right) \ln\left(1 + x_i^{\gamma}\right) + R^*_{d_1+1} \ln\left(1 + T_1^{\gamma}\right) + R^*_{d_2+1} \ln\left(1 + T_2^{\gamma}\right)\right]\right\}, \end{aligned} \tag{8}$$

where $x_i = x_{i:\, d^*:\, n}$ for simplicity of notation.

3. Estimation Using ML Technique

In this section, using the maximum likelihood technique, we compute the ML estimate of the unknown parameters of the Burr Type-XII model with generalized Type-II HPCS. Since the critical points of any function $f(x)$ are identical to the critical points of $\ln(f(x))$, (8) gives the corresponding log-likelihood function as

$$\begin{aligned}\ln L(\lambda,\gamma|\underline{x}) &= \text{const.} + d^*(\ln\lambda + \ln\gamma) + (\gamma-1)\sum_{i=1}^{d^*}\ln x_i - \sum_{i=1}^{d^*}\ln\left(1+x_i^{\gamma}\right)\\ &\quad -\lambda\left[\sum_{i=1}^{d^*}\left(R_i^*+1\right)\ln\left(1+x_i^{\gamma}\right) + R_{d_1+1}^*\ln\left(1+T_1^{\gamma}\right) + R_{d_2+1}^*\ln\left(1+T_2^{\gamma}\right)\right].\end{aligned} \tag{9}$$

To determine the critical points, set the first derivatives of (9) with respect to λ and γ equal to zero:

$$\begin{aligned}\frac{\partial\ln L(\lambda,\gamma|\underline{x})}{\partial\lambda} &= \frac{d^*}{\lambda} - \left[\sum_{i=1}^{d^*}\left(R_i^*+1\right)\ln\left(1+x_i^{\gamma}\right) + R_{d_1+1}^*\ln\left(1+T_1^{\gamma}\right) + R_{d_2+1}^*\ln\left(1+T_2^{\gamma}\right)\right]\\ &= 0,\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial\ln L(\lambda,\gamma|\underline{x})}{\partial\gamma} &= \frac{d^*}{\gamma} + \sum_{i=1}^{d^*}\ln x_i - \sum_{i=1}^{d^*}\left\{\lambda\left(R_i^*+1\right)+1\right\}\frac{x_i^{\gamma}\ln x_i}{\left(1+x_i^{\gamma}\right)}\\ &\quad -\lambda\left[R_{d_1+1}^*\frac{T_1^{\gamma}\ln T_1^{\gamma}}{\left(1+T_1^{\gamma}\right)} + R_{d_2+1}^*\frac{T_2^{\gamma}\ln T_2^{\gamma}}{\left(1+T_2^{\gamma}\right)}\right] = 0.\end{aligned} \tag{11}$$

Using numerical technique, we can compute ML estimates of the parameters λ and γ, are $\widehat{\lambda}_{ML}$ and $\widehat{\gamma}_{ML}$ respectively, by solving equations (9) and (11). Byusing the invariance property of the ML estimates, we can calculate the ML estimates of the survival and the hazard functions, respectively, as

$$\begin{aligned}\widehat{S}_{ML}(t) &= \left(1+t^{\widehat{\gamma}_{ML}}\right)^{-\widehat{\lambda}_{ML}},\\ \widehat{h}_{ML}(t) &= \widehat{\lambda}_{ML}\widehat{\gamma}_{ML}t^{\widehat{\gamma}_{ML}-1}\left(1+t^{\widehat{\gamma}_{ML}}\right)^{-1}.\end{aligned} \tag{12}$$

The conditions necessary for the existence and uniqueness of ML estimates of the parameters λ and γ are discussed by Nagy et al. [17].

3.1. Approximate Confidence Intervals.

3.1. Approximate Confidence Intervals. The $100(1-\alpha)\%$ confidence intervals for any parameters θ are given by

$$\left(\widehat{\theta} - z_{\alpha/2}SE\left(\widehat{\theta}_{ML}\right), \widehat{\theta} + z_{\alpha/2}SE\left(\widehat{\theta}_{ML}\right)\right), \tag{13}$$

where $z_{\alpha/2}$ is the right $(\alpha/2)$ percentile of standard normal distribution and $SE(\widehat{\theta}_{ML}) = \sqrt{\operatorname{var}(\widehat{\theta}_{ML})}$ (standard deviation) is the square root of the estimated variance of $\widehat{\theta}_{ML}$. Now, the estimated variances $\operatorname{var}(\widehat{\lambda}_{ML})$ and $\operatorname{var}(\widehat{\gamma}_{ML})$ of $\widehat{\lambda}_{ML}$ and $\widehat{\gamma}_{ML}$ are given by the first and the second diagonal elements of the Fisher information matrix of the parameters λ and γ. For large d^*, the observed Fisher information matrix of the parameters λ and γ is given by

$$FI(\widehat{\lambda},\widehat{\gamma}) = \begin{bmatrix}\dfrac{d^*}{\lambda^2}, & \displaystyle\sum_{i=1}^{d^*}\left(R_i^*+1\right)\frac{x_i^{\gamma}\ln x_i}{\left(1+x_i^{\gamma}\right)},\\ \displaystyle\sum_{i=1}^{d^*}\left(R_i^*+1\right)\frac{x_i^{\gamma}\ln x_i}{\left(1+x_i^{\gamma}\right)}, & \dfrac{d^*}{\gamma^2} + \displaystyle\sum_{i=1}^{d^*}\left[\lambda\left(R_i^*+1\right)+1\right]\left[\frac{\left(\ln x_i\right)^2x_i^{\gamma}}{\left(1+x_i^{\gamma}\right)^2}\right].\end{bmatrix}_{\left(\widehat{\lambda}_{ML},\widehat{\gamma}_{ML}\right)} \tag{14}$$

The approximate estimates of $\mathrm{var}(\widehat{S}_{ML}(t))$ and $\mathrm{var}(\widehat{h}_{ML}(t))$ can be constructed by using the delta method (see [18, 19]) as

$$\begin{aligned}\mathrm{var}(\widehat{S}(t)) &\simeq \left[M_1^t I^{-1}(\lambda,\gamma) M_1\right]_{(\widehat{\lambda}_{ML},\widehat{\gamma}_{ML})},\\ \mathrm{var}(\widehat{h}(t)) &\simeq \left[M_2^t I^{-1}(\lambda,\gamma) M_2\right]_{(\widehat{\lambda}_{ML},\widehat{\gamma}_{ML})},\end{aligned} \tag{15}$$

where M^t is the transpose of the matrix M, with

$$\begin{aligned}M_1 &= \left[\frac{\partial S(t)}{\partial\lambda}\ \frac{\partial S(t)}{\partial\gamma}\right],\\ &= \left[-(1+t^\gamma)^{-\lambda}\ln(1+t^\gamma)\ \ -\lambda t^\gamma (1+t^\gamma)^{-(\lambda+1)}\ln(t)\right],\\ M_2 &= \left[\frac{\partial h(t)}{\partial\lambda}\ \frac{\partial h(t)}{\partial\gamma}\right],\\ &= \left[\gamma t^{\gamma-1}(1+t^\gamma)^{-1}\ \ \frac{\lambda\left\{(1+t^\gamma)\left[t^{\gamma-1}+\gamma t^{\gamma-1}\ln(t)\right]-\gamma t^{2\gamma-1}\ln(t)\right\}}{(1+t^\gamma)^2}\right].\end{aligned} \tag{16}$$

4. Bayesian Estimations

In this study, we investigate three forms of loss functions for Bayesian estimation. The first is the squared error loss function (SELF), a symmetric function that in parameter estimation assigns equal weight to overestimates and underestimates. The second is the LINEX loss function (LLF), which is asymmetric and provides overestimation and underestimation distinct weights. The generalization of the entropy loss function is the third loss function (GELF). Assume that both λ and γ are unknown and are separately distributed as $\pi_\lambda(a_1,b_1)$ and $\pi_\gamma(a_2,b_2)$ priors, respectively. The prior density function for joining is

$$\pi(\lambda,\gamma) \propto \lambda^{a_1-1}\gamma^{a_2-1}\exp(-\lambda b_1)\exp(-\gamma b_2), \tag{17}$$

where a_1, b_1, a_2, b_2 are positive constants.

The posterior density function of λ,γ given by the generalized Type-II PHCS data is obtained as

$$\pi^*(\lambda,\gamma|\underline{x}) = \frac{L(\lambda,\gamma|\underline{x})\pi(\lambda,\gamma)}{\int_0^\infty\int_0^\infty L(\lambda,\gamma|\underline{x})\pi(\lambda,\gamma)d\lambda d\gamma}. \tag{18}$$

From (8) and (17),

$$\begin{aligned}\pi^*(\lambda,\gamma|\underline{x}) &= I^{-1}\lambda^{d^*+a_1-1}\gamma^{d^*+a_2-1}\exp\left[\sum_{i=1}^{d^*}\ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right)-\gamma b_2\right]\\ &\quad\times\exp\left\{-\lambda\left[\sum_{i=1}^{d^*}(R_i^*+1)\ln(1+x_i^\gamma)+R_{d_1+1}^*\ln(1+T_1^\gamma)+R_{d_2+1}^*\ln(1+T_2^\gamma)+b_1\right]\right\},\end{aligned} \tag{19}$$

where the normalization constant I is given by

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty L(\lambda, \gamma | \underline{x}) \pi(\lambda, \gamma) \mathrm{d}\lambda \mathrm{d}\gamma \\ &= \int_0^\infty \int_0^\infty \lambda^{d^*+a_1-1} \gamma^{d^*+a_2-1} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \\ &\quad \times \exp\left\{-\lambda\left[\sum_{i=1}^{d^*} (R_i^*+1)\ln(1+x_i^\gamma) + R_{d_1+1}^*\ln(1+T_1^\gamma) + R_{d_2+1}^*\ln(1+T_2^\gamma) + b_1\right]\right\} d\lambda\, d\gamma \\ &= \Gamma(d^*+a_1) \int_0^\infty \gamma^{d^*+a_2-1} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \\ &\quad \times \left[\sum_{i=1}^{d^*} (R_i^*+1)\ln(1+x_i^\gamma) + R_{d_1+1}^*\ln(1+T_1^\gamma) + R_{d_2+1}^*\ln(1+T_2^\gamma) + b_1\right]^{-(d^*+a_1)} d\gamma . \end{aligned} \tag{20}$$

4.1. Bayesian Estimations under SELF. A commonly used loss function is SELF, and the Bayesian estimate $\widehat{\theta}_{BS}$ relative to SELF is given by

$$\widehat{\theta}_{BS} = E_{\pi^*}[\theta]. \tag{21}$$

Hence, from (19) and (21), the Bayesian estimates of λ and γ under SELF are obtained, respectively, as

$$\begin{aligned} \widehat{\lambda}_{BS} &= I^{-1}\Gamma(d^*+a_1+1) \int_0^\infty \gamma^{d^*+a_2-1} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \\ &\quad \times \left[\sum_{i=1}^{d^*} (R_i^*+1)\ln(1+x_i^\gamma) + R_{d_1+1}^*\ln(1+T_1^\gamma) + R_{d_2+1}^*\ln(1+T_2^\gamma) + b_1\right]^{-(d^*+a_1+1)} d\gamma , \\ \widehat{\gamma}_{BS} &= I^{-1}\Gamma(d^*+a_1) \int_0^\infty \gamma^{d^*+a_2} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \\ &\quad \times \left[\sum_{i=1}^{d^*} (R_i^*+1)\ln(1+x_i^\gamma) + R_{d_1+1}^*\ln(1+T_1^\gamma) + R_{d_2+1}^*\ln(1+T_2^\gamma) + b_1\right]^{-(d^*+a_1)} d\gamma . \end{aligned} \tag{22}$$

4.2. Bayesian Estimations under LLF. Under the assumption that the minimal loss occurs at $\widehat{\theta} = \theta$, LLF can be expressed as

$$L_{BL}(\widehat{\theta}, \theta) = \exp[\upsilon(\widehat{\theta} - \theta)] - \upsilon(\widehat{\theta} - \theta) - 1, \tag{23}$$

where $\upsilon \neq 0$. The value $\widehat{\theta}_{BL}$ which minimizes $E_{\pi^*}[L_L(\widehat{\theta}, \theta)]$ is given by

$$\widehat{\theta}_{BL} = \frac{-1}{\upsilon} \ln\{E_{\pi^*}[\exp(-\upsilon\theta)]\}, \tag{24}$$

provided that the involved expectation $E_{\pi^*}[\exp(-\upsilon\theta)]$ is finite. The problem of choosing the value of the parameter ν has been discussed by Calabria and Pulcini [20].

From (19) and (24), the Bayesian estimator of λ and γ based on the LLF is as follows:

$$\widehat{\lambda}_{BL} = -\frac{1}{v}\ln\left\{ I^{-1}\Gamma\left(d^* + a_1\right) \int_0^\infty \gamma^{d^*+a_2-1} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \times \left[\sum_{i=1}^{d^*} \left(R_i^* + 1\right)\ln\left(1 + x_i^\gamma\right) + R_{d_1+1}^* \ln\left(1 + T_1^\gamma\right) + R_{d_2+1}^* \ln\left(1 + T_2^\gamma\right) + v + b_1\right]^{-\left(d^*+a_1\right)} d\gamma \right\},$$

$$\widehat{\gamma}_{BL} = -\frac{1}{v}\ln\left\{ I^{-1}\Gamma\left(d^* + a_1\right) \int_0^\infty \gamma^{d^*+a_2-1} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma\left(b_2 + v\right)\right] \times \left[\sum_{i=1}^{d^*} \left(R_i^* + 1\right)\ln\left(1 + x_i^\gamma\right) + R_{d_1+1}^* \ln\left(1 + T_1^\gamma\right) + R_{d_2+1}^* \ln\left(1 + T_2^\gamma\right) + b_1\right]^{-\left(d^*+a_1\right)} d\gamma \right\}. \quad (25)$$

4.3. Bayesian Estimations under GELF. Another commonly used asymmetric loss function is GELF which is given by

$$L_{BE}\left(\widehat{\theta}, \theta\right) \propto \left(\frac{\widehat{\theta}}{\theta}\right)^\kappa - \kappa \ln\left(\frac{\widehat{\theta}}{\theta}\right) - 1. \quad (26)$$

For $\kappa > 0$, the Bayesian estimate $\widehat{\theta}_{BE}$ relative to the GELF is given by

$$\widehat{\theta}_{BE} = \left\{E_{\pi^*}\left[\theta\right]^{-\kappa}\right\}^{-1/\kappa}, \quad (27)$$

provided that the involved expectation $E_{\pi^*}[\theta]^{-\kappa}$ is finite. From (19) and (27), one obtains the Bayesian estimator of λ and γ using GELF as follows:

$$\widehat{\lambda}_{BE} = \left\{ I^{-1}\Gamma\left(d^* + a_1 - \kappa\right) \int_0^\infty \gamma^{d^*+a_2-1} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \times \left[\sum_{i=1}^{d^*} \left(R_i^* + 1\right)\ln\left(1 + x_i^\gamma\right) + R_{d_1+1}^* \ln\left(1 + T_1^\gamma\right) + R_{d_2+1}^* \ln\left(1 + T_2^\gamma\right) + b_1\right]^{-\left(d^*+a_1-\kappa\right)} d\gamma \right\}^{-1/\kappa},$$

$$\widehat{\gamma}_{BE} = \left\{ I^{-1}\Gamma\left(d^* + a_1\right) \int_0^\infty \gamma^{d^*+a_2-\kappa-1} \exp\left[\sum_{i=1}^{d^*} \ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \times \left[\sum_{i=1}^{d^*} \left(R_i^* + 1\right)\ln\left(1 + x_i^\gamma\right) + R_{d_1+1}^* \ln\left(1 + T_1^\gamma\right) + R_{d_2+1}^* \ln\left(1 + T_2^\gamma\right) + b_1\right]^{-\left(d^*+a_1\right)} d\gamma \right\}^{-1/\kappa}. \quad (28)$$

Because all of the above integrals in the Bayesian estimate of unknown parameters λ and γ cannot be computed analytically, the Markov chain Monte Carlo (MCMC) algorithm is employed to evaluate them. The conditional posterior distributions $\pi_1^*(\lambda|\gamma; \underline{x})$ and $\pi_2^*(\gamma|\lambda; \underline{x})$ of the parameters λ and γ can now be computed, respectively, based on the posterior distribution in (19).

$$\pi_1^*(\lambda|\gamma;\underline{x}) = \frac{[W(\gamma;\underline{x}) + \gamma\, d]^{-(d^*+a_1)}}{\Gamma(d^*+a_1)}\lambda^{d^*+a_1-1} \times \exp\left\{-\lambda\left[\sum_{i=1}^{d^*}(R_i^*+1)\ln(1+x_i^\gamma) + R_{d_1+1}^*\ln(1+T_1^\gamma) + R_{d_2+1}^*\ln(1+T_2^\gamma) + b_1\right]\right\}$$
$$= G\left[d^*+a_1, \sum_{i=1}^{d^*}(R_i^*+1)\ln(1+x_i^\gamma) + R_{d_1+1}^*\ln(1+T_1^\gamma) + R_{d_2+1}^*\ln(1+T_2^\gamma) + b_1\right], \tag{29}$$

$$\pi_2^*(\gamma|\lambda;\underline{x}) = \gamma^{d^*+a_2-1}\exp\left[\sum_{i=1}^{d^*}\ln\left(\frac{x_i^{\gamma-1}}{1+x_i^\gamma}\right) - \gamma b_2\right] \exp\left\{-\lambda\left[\sum_{i=1}^{d^*}(R_i^*+1)\ln(1+x_i^\gamma) + R_{d_1+1}^*\ln(1+T_1^\gamma) + R_{d_2+1}^*\ln(1+T_2^\gamma) + b_1\right]\right\}. \tag{30}$$

The Metropolis–Hastings sampler is employed to create samples of γ inside the MCMC algorithm because the conditional distribution of γ in (30) is not a well-known distribution (for further details, see [21–23]). The MCMC algorithm (1) in [17] is used to create λ and γ samples from conditional posterior distributions, which will be utilized to approximate Bayes estimates.

Assume that $g(\lambda,\gamma)$ is any function in λ and γ; then, the Bayesian estimates of g using the MCMC values are obtained as follows.

Under SELF, the Bayesian estimate of g is given by

$$\widehat{g(\lambda,\gamma)}_{BS} = \frac{1}{N-B}\sum_{i=1}^{N-B} g\left(\lambda^{(i)},\gamma^{(i)}\right). \tag{31}$$

Based on the LLF,

$$\widehat{g(\lambda,\gamma)}_{BL} = \frac{1}{v}\ln\left[\frac{1}{N-B}\sum_{i=1}^{N-B} e^{vg\left(\lambda^{(i)},\gamma^{(i)}\right)}\right]. \tag{32}$$

By using GELF,

$$\widehat{g(\lambda,\gamma)}_{BE} = \left[\frac{1}{N-B}\sum_{i=1}^{N-B}\left[g\left(\lambda^{(i)},\gamma^{(i)}\right)\right]^{-\kappa}\right]^{-1/\kappa}. \tag{33}$$

The $100(1-\alpha)\%$ Bayesian confidence interval or credible interval (L,U) for any parameter θ, if

$$\int_L^U \pi^*(\theta|\underline{x})d\theta = 1-\alpha. \tag{34}$$

Since the integration in (34) cannot be solved analytically, the $100(1-\alpha)$ MCMC approximate credible intervals for λ and γ using the $(N-B)$ generated values after sorting it in an ascending order, $(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(N-B)})$ and $(\gamma^{(1)},\gamma^{(2)},\ldots,\gamma^{(N-B)})$, are given as follows:

$$\left(\lambda_{[(N-B)\alpha/2]},\lambda_{[(N-B)(1-\alpha)/2]}\right), \quad \left(\gamma_{[(N-B)\alpha/2]},\gamma_{[(N-B)(1-\alpha)/2]}\right), \tag{35}$$

and the lengths of the credible intervals are the absolute difference between the upper and the lower bounds.

5. One-Sample Bayesian Prediction

In this section, the Bayesian point and interval predictions are made for the unobserved or remote units in each phase given the generalized Type-II PHCS. The prediction interval is obtained by using equi-tailed (ET) or two-tailed interval and the highest posterior density (HPD) interval.

Let $Y_{i:R_j^*}$ denote the i^{th} order statistic out of R_j^* removed units at stage j, (for $i = 1,2,\ldots,R_j^*$). Basak et al. in [24] calculated the conditional density function of $Y_{i:R_j^*}$, given the observed generalized Type-II PHCS, by

$$f\left(Y_{i:R_j^*}|\underline{x}\right) = f(y|\underline{x}) = \frac{R_j^*!}{(i-1)!\left(R_j^*-i\right)!}\frac{\left[F(y)-F\left(y_j\right)\right]^{i-1}[1-F(y)]^{R_j^*-i}f(y)}{\left[1-F\left(y_j\right)\right]^{R_j^*}}, y > y_j, \tag{36}$$

where

$$j = \begin{cases} 1, \ldots, d_1, d_1 + 1, & \text{if } X_{m:m:n} < T_1 < T_2 \\ 1, \ldots, d_1, \ldots, m, & \text{if } T_1 < X_{m:m:n} < T_2, \\ 1, \ldots, d_1, \ldots, d_2, d_2 + 1, & \text{if } T_1 < T_2 < X_{m:m:n}. \end{cases} \tag{37}$$

By compensation in (36) by using (1) and (2), the conditional density function of $Y_{i:\, R_j^*}$ given generalized Type-II PHCS is then given as follows:

$$f(y|\underline{x}) = \sum_{k=0}^{i-1} C_k \frac{\lambda\gamma y^{\gamma-1}}{1+y^{\gamma}} \exp\left\{-\lambda\left[\varpi_k \ln\left(\frac{1+y^{\gamma}}{1+y_j^{\gamma}}\right)\right]\right\}, y > y_j, \tag{38}$$

where $C_k = (-1)^k \binom{i-1}{k} R_j^*!/(i-1)!(R_j^*-i)!$ and $\varpi_k = k + R_j^* - i + 1$ for $k = 0, \ldots, i-1$; therefore, the Bayesian predictive density function of $Y_{i:\, R_j^*}$, given generalized Type-II PHCS, is obtained as

$$\begin{aligned} f^*(y|\mathbf{x}) &= \int_0^{\infty}\int_0^{\infty} f(y|\mathbf{x})\pi^*(\lambda, \gamma|\mathbf{x})d\lambda d\gamma \\ &= \frac{1}{N-B}\sum_{h=1}^{N-B}\sum_{k=0}^{i-1} C_k \frac{\lambda^{(h)}\gamma^{(h)} y^{\gamma^{(h)}-1}}{1+y^{\gamma^{(h)}}} \exp\left\{-\lambda^{(h)}\left[\varpi_k \ln\left(\frac{1+y^{\gamma^{(h)}}}{1+y_j^{\gamma^{(h)}}}\right)\right]\right\}. \end{aligned} \tag{39}$$

The Bayesian predictive survival function of $Y_{i:\, R_j^*}$, given generalized Type-II PHCS, is given as

$$\begin{aligned} \overline{F}^*(t|\underline{x}) &= \int_t^{\infty} f^*(y|\underline{x})\mathrm{d}x \\ &= \frac{1}{N-B}\sum_{h=1}^{N-B}\sum_{k=0}^{i-1} \frac{C_k}{\varpi_k}\left(\frac{1+t^{\gamma^{(h)}}}{1+y_j^{\gamma^{(h)}}}\right)^{-\lambda^{(h)}\varpi_k}. \end{aligned} \tag{40}$$

The Bayesian point predictor of $Y_{i:R_j^*}$ under the squared error loss function is the mean of the predictive density, given by

$$\begin{aligned} \widehat{Y}_{i:\, R_j^*} &= \int_0^{\infty} x f^*(y|\underline{x})\mathrm{d}x, \\ &= \frac{1}{N-B}\sum_{h=1}^{N-B}\sum_{k=0}^{i-1} C_k \lambda^{(h)}\left(1+y_j^{\gamma^{(h)}}\right)^{\lambda^{(h)}\varpi_k} \\ &\quad \frac{\Gamma\left(1+\left(1/\gamma^{(h)}\right)\right)\Gamma\left(\lambda^{(h)}\varpi_k - \left(1/\gamma^{(h)}\right)\right)}{\Gamma\left(1+\lambda^{(h)}\varpi_k\right)}. \end{aligned} \tag{41}$$

The $100(1-\alpha)\%$ equi-tailed (ET) interval Bayesian predictive bounds for $Y_{i:\, R_j^*}$ can be obtained by solving the following two equations:

$$\begin{aligned} \overline{F}^*(U_{ET}|\underline{x}) &= 1 - \frac{\alpha}{2}, \\ \overline{F}^*(L_{ET}|\underline{x}) &= \frac{\alpha}{2}, \end{aligned} \tag{42}$$

but the $100(1-\alpha)\%$ highest posterior density (HPD) interval can be obtained by solving the following two equations:

$$\begin{aligned} f^*(L_{HPD}|\underline{x}) - f^*(U_{HPD}|\underline{x}) &= 0, \\ \overline{F}^*(L_{HPD}|\underline{x}) - \overline{F}^*(U_{HPD}|\underline{x}) &= 1 - \alpha, \end{aligned} \tag{43}$$

where U_{ET}, $U_{HP\,D}$, L_{ET}, and $L_{HP\,D}$ denote the upper and lower bounds of interval estimation for ET and HPD methods, respectively.

6. Simulation Study

For passing through and understanding statistical inferential approaches, and combination of theoretical and applied skills, sothe analytical technique was discussed in previous sections, and this section will focus on the numerical technique through a simulation study. We simulated random samples from the Burr Type-XII distribution in equation (2) and evaluated the coverage probabilities of the resulting CIs. We used parameter values $\lambda = 2$ and $\gamma = 1$, for sample of size $n = 50$, with different values of m, T_1, and T_2. For the point estimate, we computed the ML and Bayesian estimates of λ, γ, $S(t)$, and $h(t)$. For Bayesian estimates, we used SELF, LLF (with $\upsilon = 0.5$), and GELF (with $\kappa = 0.5$) using informative (Prior1) and non-informative priors (Prior2); for each estimate, the mean square error (MSE) and estimated expected bias (EB) have been determined. We also construct the average confidence (ACL) and the coverage probabilities (CP) of the 90% and 95% asymptotic confidence intervals and Bayesian credible intervals for $\widehat{\lambda}$, $\widehat{\gamma}$, $\widehat{S(t)}$, and $\widehat{h(t)}$ using 10,000 simulations. We used three different progressive Type-II censored sampling schemes, namely,

(1) Scheme 1: $R_m = n - m, R_i = 0$ if $i \neq m$.

(2) Scheme 2: $R_1 = n - m, R_i = 0$ if $i \neq 1$.

(3) Scheme 3: $R_1 = R_m = n - m/2, R_i = 0$ if $i \neq 1$ and m.

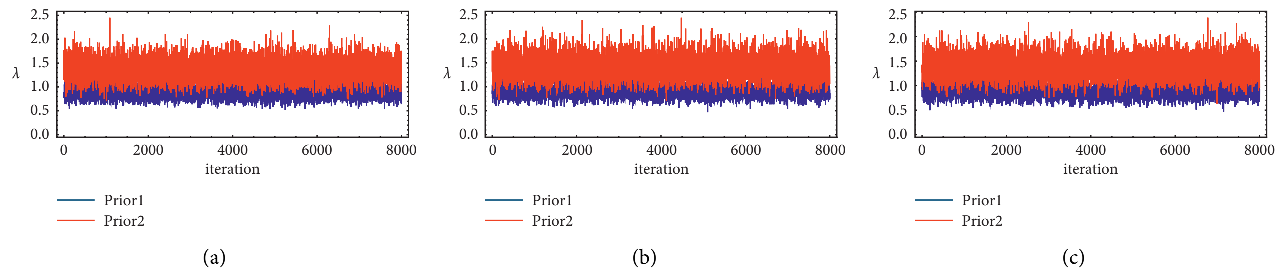


Figure 2: Trace plots of 10,000 iterations of λ with Prior1 and Prior2. (a) Scheme I. (b) Scheme II. (c) Scheme III.

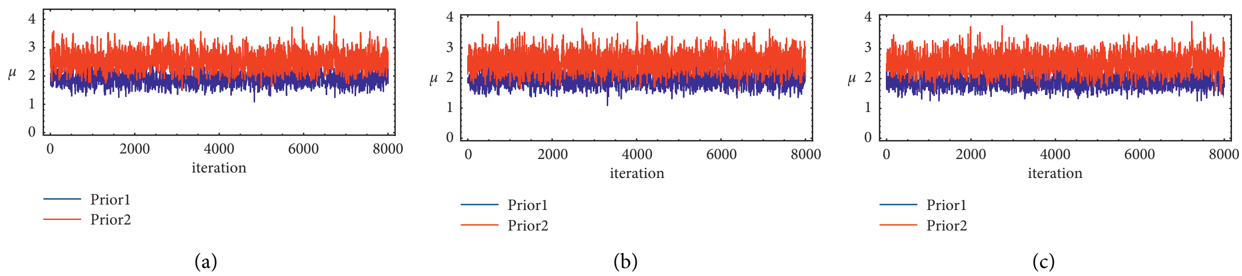


Figure 3: Trace plots of 10,000 iterations of γ with Prior1 and Prior2. (a) Scheme I. (b) Scheme II. (c) Scheme III.

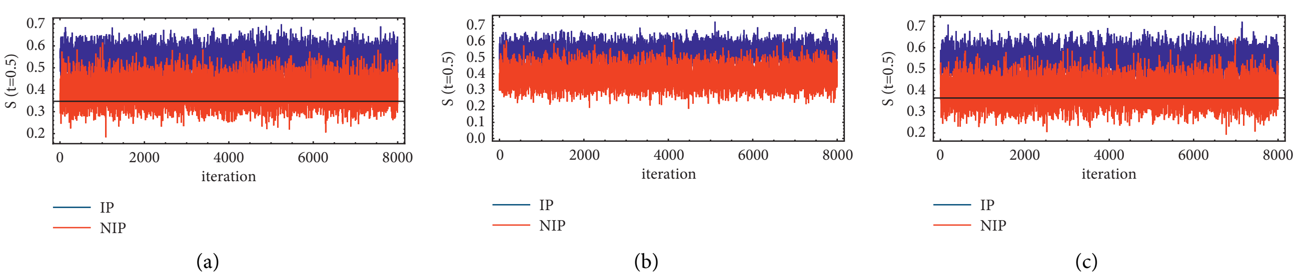


Figure 4: Trace plots of 10,000 iterations of $S(t)$ with Prior1 and Prior2. (a) Scheme I. (b) Scheme II. (c) Scheme III.

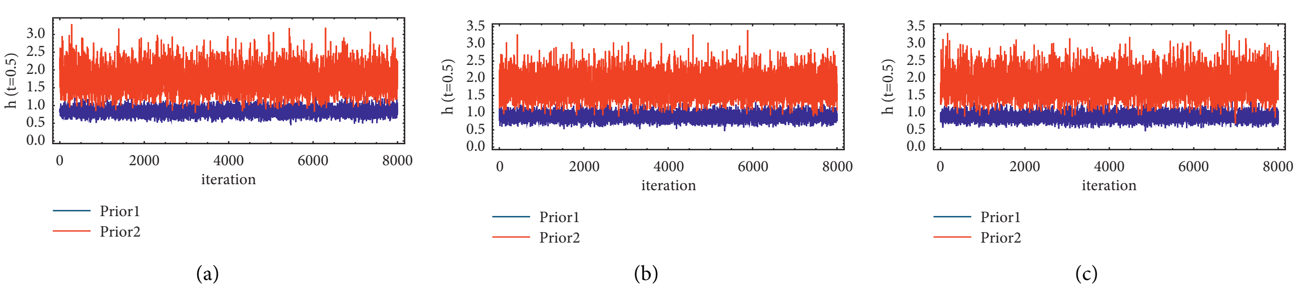
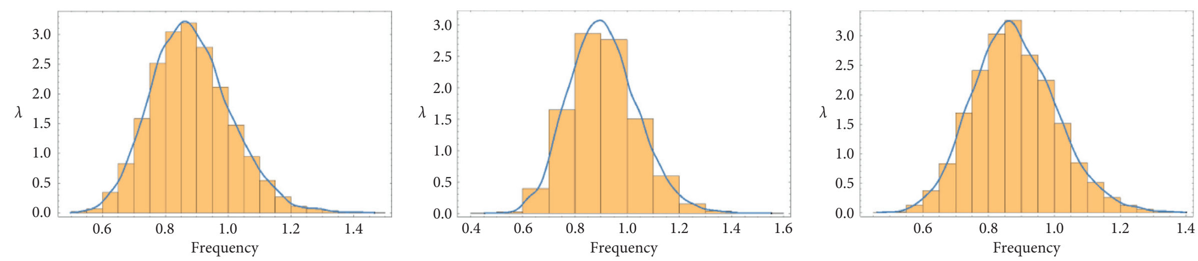


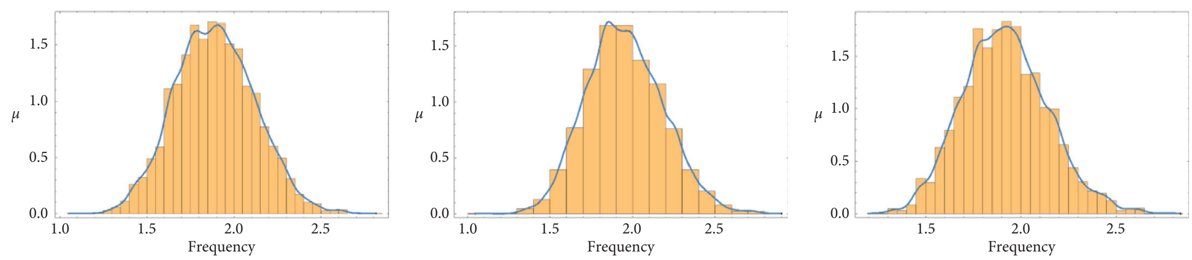
Figure 5: Trace plots of 10,000 iterations of $h(t)$ with Prior1 and Prior2. (a) Scheme I. (b) Scheme II. (c) Scheme III.

For computing the Bayes estimates, we generate the MCMC algorithm suggested in Section 4 with $N = 10,000$ observations. The ML estimates for unknown parameters λ and γ have been used as initial values for running the MCMC algorithm. The generated sequences' starting values may be different from converged sequences, so the first $B = 2,000$ values are removed here to avoid the effects of the initial values. To check the convergence of MCMC samples and determine the B-burn-in, we provide the key diagnostic test, trace plots, and posterior density plots for different parameters and censoring schemes. Figures 2–5 show the trace plots of



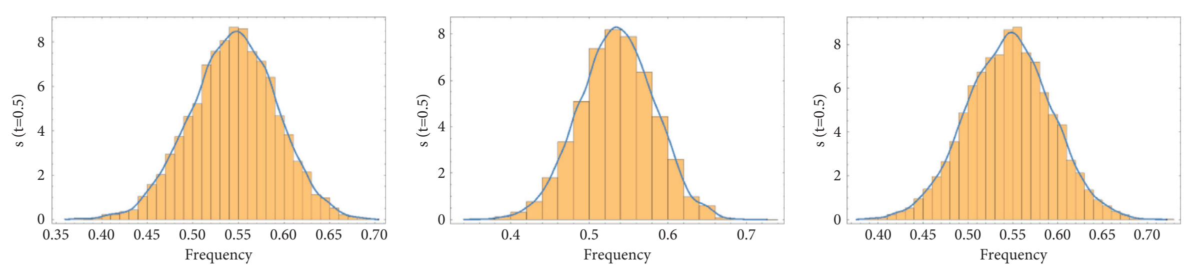
(a) (b) (c)

FIGURE 6: The marginal posterior density with histograms for λ. (a) Scheme I. (b) Scheme II. (c) Scheme III.



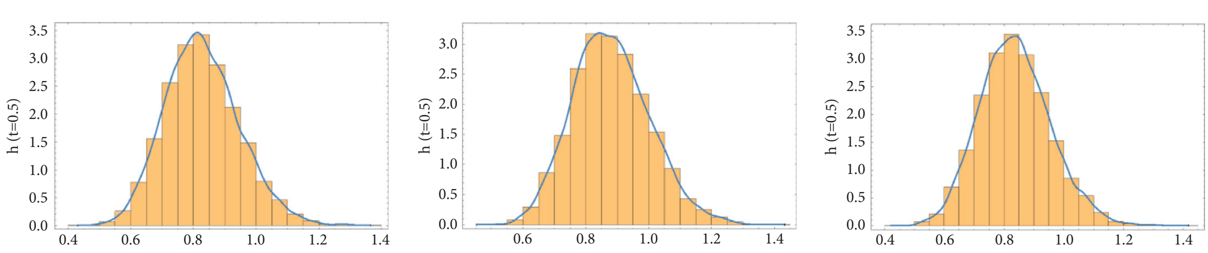
(a) (b) (c)

FIGURE 7: The marginal posterior density with histograms for γ. (a) Scheme I. (b) Scheme II. (c) Scheme III.



(a) (b) (c)

FIGURE 8: The marginal posterior density with histograms for $S(t = 0.5)$. (a) Scheme I. (b) Scheme II. (c) Scheme III.



(a) (b) (c)

FIGURE 9: The marginal posterior density with histograms for $h(t = 0.5)$. (a) Scheme I. (b) Scheme II. (c) Scheme III.

iterations for posterior densities of λ, γ, $S(t)$, and $h(t)$ using the three above schemes at $m = 40$, $T_1 = 0.5$, and $T_2 = 1$ with Prior1 and Prior2. All censoring schemes are plotted in the same way, and it has been found that the trace plots of all censoring schemes converge very well. Furthermore, the approximate marginal posterior density with histograms for λ, γ, $S(t = 0.5)$, and $h(t = 0.5)$ is shown in Figures 6–9, respectively.

Table 1: MSE and EB of the estimates for λ.

Sch .	(T_1, T_2)	(n, m)	$\hat{\lambda}_{ML}$	Bayesian $\hat{\lambda}_{BS}$ Prior2	Prior2	$\hat{\lambda}_{BL}$ Prior2	Prior2	$\hat{\lambda}_{BE}$ Prior2	Prior2
				MSE					
Sch.1	(0.5, 1)	(50, 20)	1.2785	0.0182	1.8465	0.0177	0.5710	0.0179	0.9019
		(50, 30)	0.5571	0.0215	0.5340	0.0208	0.3140	0.0210	0.3705
		(50, 40)	0.5760	0.0214	0.4992	0.0209	0.3008	0.0212	0.3563
	(1, 1.5)	(50, 20)	0.5301	0.0238	0.4778	0.0233	0.2981	0.0236	0.3538
		(50, 30)	0.4940	0.0207	0.4618	0.0203	0.2915	0.0207	0.3327
		(50, 40)	0.4544	0.0258	0.4554	0.0250	0.2632	0.0253	0.3276
	(1.5, 2)	(50, 20)	0.4668	0.0201	0.4555	0.0197	0.2813	0.0199	0.3221
		(50, 30)	0.4649	0.0264	0.4227	0.0257	0.2789	0.0262	0.3095
		(50, 40)	0.4649	0.0264	0.4227	0.0257	0.2789	0.0262	0.3095
Sch.2	(0.5, 1)	(50, 20)	1.3343	0.0191	1.6631	0.0188	0.5555	0.0190	0.8658
		(50, 30)	0.4303	0.0258	0.4226	0.0250	0.2623	0.0253	0.3044
		(50, 40)	0.4699	0.0201	0.4246	0.0197	0.2666	0.0199	0.3023
	(1, 1.5)	(50, 20)	0.4574	0.0257	0.4244	0.0250	0.2625	0.0255	0.2991
		(50, 30)	0.3714	0.0182	0.3484	0.0178	0.2140	0.0179	0.2723
		(50, 40)	0.3449	0.0172	0.3235	0.0169	0.2189	0.0171	0.2410
	(1.5, 2)	(50, 20)	0.3212	0.0160	0.2964	0.0156	0.2061	0.0158	0.2233
		(50, 30)	0.3074	0.0170	0.2922	0.0167	0.2069	0.0169	0.2218
		(50, 40)	0.3140	0.0268	0.2651	0.0259	0.1979	0.0262	0.2139
Sch.3	(0.5, 1)	(50, 20)	1.1223	0.0192	1.3739	0.0188	0.5006	0.0190	0.7552
		(50, 30)	0.3032	0.0278	0.2551	0.0269	0.1961	0.0272	0.2067
		(50, 40)	0.2772	0.0183	0.2401	0.0181	0.1828	0.0183	0.1967
	(1, 1.5)	(50, 20)	0.2591	0.0272	0.2133	0.0264	0.1691	0.0268	0.1795
		(50, 30)	0.2559	0.0262	0.2137	0.0254	0.1668	0.0257	0.1759
		(50, 40)	0.2275	0.0184	0.1952	0.0179	0.1537	0.0181	0.1629
	(1.5, 2)	(50, 20)	0.2042	0.0172	0.1708	0.0168	0.1387	0.0170	0.1435
		(50, 30)	0.1917	0.0186	0.1670	0.0182	0.1326	0.0183	0.1390
		(50, 40)	0.1899	0.0181	0.1606	0.0178	0.1317	0.0180	0.1367
				EB					
Sch.1	(0.5, 1)	(50, 20)	0.3474	0.0033	0.3786	0.0051	0.1921	0.0093	0.2113
		(50, 30)	0.2304	0.0121	0.1981	0.0031	0.1226	0.0012	0.1163
		(50, 40)	0.1988	0.0043	0.1638	0.0048	0.0946	0.0094	0.0868
	(1, 1.5)	(50, 20)	0.1875	0.0038	0.1622	0.0051	0.0945	0.0096	0.0854
		(50, 30)	0.1819	0.0036	0.1612	0.0074	0.0856	0.0128	0.0715
		(50, 40)	0.1735	0.0036	0.1612	0.0074	0.0856	0.0128	0.0715
	(1.5, 2)	(50, 20)	0.1766	0.0038	0.1586	0.0072	0.0820	0.0127	0.0684
		(50, 30)	0.1634	0.0055	0.1560	0.0038	0.0720	0.0083	0.0572
		(50, 40)	0.1784	0.0112	0.1560	0.0008	0.0979	0.0043	0.0912
Sch.2	(0.5, 1)	(50, 20)	0.3293	0.0018	0.3494	0.0066	0.1820	0.0108	0.1954
		(50, 30)	0.1689	0.0113	0.1477	0.0009	0.0904	0.0042	0.0842
		(50, 40)	0.1595	0.0006	0.1472	0.0083	0.0662	0.0128	0.0519
	(1, 1.5)	(50, 20)	0.1755	0.0010	0.1472	0.0081	0.0830	0.0125	0.0748
		(50, 30)	0.1545	0.0001	0.1430	0.0072	0.0933	0.0108	0.0861
		(50, 40)	0.1437	0.0023	0.1368	0.0048	0.0885	0.0085	0.0801
	(1.5, 2)	(50, 20)	0.1449	0.0041	0.1362	0.0031	0.0885	0.0067	0.0807
		(50, 30)	0.1437	0.0109	0.1128	0.0001	0.0711	0.0054	0.0606
		(50, 40)	0.1365	0.0115	0.1066	0.0007	0.0628	0.0046	0.0516
Sch.3	(0.5, 1)	(50, 20)	0.3035	0.0015	0.3292	0.0069	0.1715	0.0111	0.1814
		(50, 30)	0.1097	0.0060	0.1014	0.0020	0.0519	0.0059	0.0402
		(50, 40)	0.1206	0.0096	0.0920	0.0012	0.0529	0.0066	0.0413
	(1, 1.5)	(50, 20)	0.1111	0.0055	0.0900	0.0023	0.0598	0.0061	0.0517
		(50, 30)	0.0997	0.0002	0.0894	0.0076	0.0434	0.0115	0.0303
		(50, 40)	0.1084	0.0054	0.0848	0.0023	0.0547	0.0060	0.0458
	(1.5, 2)	(50, 20)	0.0991	0.0094	0.0801	0.0017	0.0509	0.0020	0.0423
		(50, 30)	0.1002	0.0079	0.0749	0.0030	0.0348	0.0084	0.0219
		(50, 40)	0.0886	0.0020	0.0707	0.0057	0.0415	0.0095	0.0322

Table 2: MSE and EB of the estimates for γ.

				Bayesian					
				$\widehat{\gamma}_{BS}$		$\widehat{\gamma}_{BL}$		$\widehat{\gamma}_{BE}$	
Sch.	(T_1, T_2)	(n, m)	$\widehat{\lambda}_{ML}$	Prior2	Prior2	Prior2	Prior2	Prior2	Prior2
				MSE					
		(50, 20)	0.0416	0.0049	0.0338	0.0049	0.0314	0.0048	0.0296
	(0.5, 1)	(50, 30)	0.0422	0.0053	0.0336	0.0052	0.0314	0.0051	0.0297
		(50, 40)	0.0413	0.0072	0.0327	0.0071	0.0306	0.0069	0.0291
		(50, 20)	0.0413	0.0072	0.0327	0.0071	0.0306	0.0069	0.0291
Sch.1	(1, 1.5)	(50, 30)	0.0385	0.0054	0.0315	0.0053	0.0295	0.0051	0.0279
		(50, 40)	0.0369	0.0070	0.0289	0.0068	0.0270	0.0067	0.0256
		(50, 20)	0.0360	0.0071	0.0283	0.0069	0.0266	0.0068	0.0254
	(1.5, 2)	(50, 30)	0.0315	0.0066	0.0243	0.0065	0.0229	0.0064	0.0218
		(50, 40)	0.0279	0.0057	0.0217	0.0056	0.0207	0.0055	0.0202
		(50, 20)	0.0276	0.0054	0.0216	0.0054	0.0206	0.0053	0.0200
	(0.5, 1)	(50, 30)	0.0277	0.0072	0.0214	0.0071	0.0204	0.0070	0.0197
		(50, 40)	0.0273	0.0056	0.0207	0.0055	0.0198	0.0054	0.0191
		(50, 20)	0.0269	0.0063	0.0207	0.0062	0.0198	0.0061	0.0192
Sch.2	(1, 1.5)	(50, 30)	0.0271	0.0072	0.0206	0.0071	0.0197	0.0070	0.0190
		(50, 40)	0.0258	0.0070	0.0199	0.0069	0.0189	0.0067	0.0182
		(50, 20)	0.0259	0.0073	0.0197	0.0071	0.0188	0.0070	0.0184
	(1.5, 2)	(50, 30)	0.0251	0.0055	0.0194	0.0055	0.0185	0.0055	0.0177
		(50, 40)	0.0223	0.0045	0.0180	0.0045	0.0173	0.0044	0.0166
		(50, 20)	0.0228	0.0047	0.0179	0.0046	0.0171	0.0045	0.0164
	(0.5, 1)	(50, 30)	0.0230	0.0055	0.0176	0.0054	0.0168	0.0053	0.0161
		(50, 40)	0.0213	0.0049	0.0171	0.0048	0.0165	0.0048	0.0162
		(50, 20)	0.0209	0.0043	0.0166	0.0042	0.0159	0.0042	0.0153
Sch.3	(1, 1.5)	(50, 30)	0.0193	0.0048	0.0152	0.0047	0.0147	0.0046	0.0144
		(50, 40)	0.0168	0.0050	0.0130	0.0049	0.0126	0.0048	0.0124
		(50, 20)	0.0166	0.0050	0.0129	0.0049	0.0125	0.0048	0.0122
	(1.5, 2)	(50, 30)	0.0159	0.0044	0.0123	0.0043	0.0119	0.0043	0.0115
		(50, 40)	0.0144	0.0048	0.0112	0.0047	0.0109	0.0046	0.0107
				EB					
		(50, 20)	0.0416	0.0049	0.0338	0.0049	0.0314	0.0048	0.0296
	(0.5, 1)	(50, 30)	0.0385	0.0054	0.0315	0.0053	0.0295	0.0051	0.0279
		(50, 40)	0.0369	0.0070	0.0289	0.0068	0.0270	0.0067	0.0256
		(50, 20)	0.0360	0.0071	0.0283	0.0069	0.0266	0.0068	0.0254
Sch.1	(1, 1.5)	(50, 30)	0.0315	0.0066	0.0243	0.0065	0.0229	0.0064	0.0218
		(50, 40)	0.0279	0.0057	0.0217	0.0056	0.0207	0.0055	0.0202
		(50, 20)	0.0276	0.0054	0.0216	0.0054	0.0206	0.0053	0.0200
	(1.5, 2)	(50, 30)	0.0422	0.0053	0.0336	0.0052	0.0314	0.0051	0.0297
		(50, 40)	0.0277	0.0072	0.0214	0.0071	0.0204	0.0070	0.0197
		(50, 20)	0.0413	0.0072	0.0327	0.0071	0.0306	0.0069	0.0291
	(0.5, 1)	(50, 30)	0.0273	0.0056	0.0207	0.0055	0.0198	0.0054	0.0191
		(50, 40)	0.0269	0.0063	0.0207	0.0062	0.0198	0.0061	0.0192
		(50, 20)	0.0271	0.0072	0.0206	0.0071	0.0197	0.0070	0.0190
Sch.2	(1, 1.5)	(50, 30)	0.0258	0.0070	0.0199	0.0069	0.0189	0.0067	0.0182
		(50, 40)	0.0259	0.0073	0.0197	0.0071	0.0188	0.0070	0.0184
		(50, 20)	0.0251	0.0055	0.0194	0.0055	0.0185	0.0055	0.0177
	(1.5, 2)	(50, 30)	0.0223	0.0045	0.0180	0.0045	0.0173	0.0044	0.0166
		(50, 40)	0.0228	0.0047	0.0179	0.0046	0.0171	0.0045	0.0164
		(50, 20)	0.0413	0.0072	0.0327	0.0071	0.0306	0.0069	0.0291
	(0.5, 1)	(50, 30)	0.0230	0.0055	0.0176	0.0054	0.0168	0.0053	0.0161
		(50, 40)	0.0213	0.0049	0.0171	0.0048	0.0165	0.0048	0.0162
		(50, 20)	0.0209	0.0043	0.0166	0.0042	0.0159	0.0042	0.0153
Sch.3	(1, 1.5)	(50, 30)	0.0193	0.0048	0.0152	0.0047	0.0147	0.0046	0.0144
		(50, 40)	0.0168	0.0050	0.0130	0.0049	0.0126	0.0048	0.0124
		(50, 20)	0.0166	0.0050	0.0129	0.0049	0.0125	0.0048	0.0122
	(1.5, 2)	(50, 30)	0.0159	0.0044	0.0123	0.0043	0.0119	0.0043	0.0115
		(50, 40)	0.0144	0.0048	0.0112	0.0047	0.0109	0.0046	0.0107

Table 3: MSE and EB of the estimates for $S(t)$.

Sch.	(T_1, T_2)	(n, m)	$\widehat{\lambda}_{ML}$	Bayesian $\widehat{S(t)}_{BS}$ Prior2	Bayesian $\widehat{S(t)}_{BS}$ Prior2	Bayesian $\widehat{S(t)}_{BL}$ Prior2	Bayesian $\widehat{S(t)}_{BL}$ Prior2	Bayesian $\widehat{S(t)}_{BE}$ Prior2	Bayesian $\widehat{S(t)}_{BE}$ Prior2
					MSE				
		(50, 20)	4.37×10^{-4}	4.3×10^{-6}	9.98×10^{-4}	4.3×10^{-6}	9.15×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
	(0.5, 1)	(50, 30)	7.65×10^{-4}	4.3×10^{-6}	0.002	4.3×10^{-6}	0.002	1.4×10^{-6}	2.50×10^{-4}
		(50, 40)	5.47×10^{-4}	4.3×10^{-6}	9.98×10^{-4}	4.3×10^{-6}	9.15×10^{-4}	2.9×10^{-6}	2.50×10^{-4}
		(50, 20)	4.37×10^{-4}	5.7×10^{-6}	7.49×10^{-4}	5.7×10^{-6}	6.66×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
Sch – I	(1, 1.5)	(50, 30)	5.47×10^{-4}	4.3×10^{-6}	9.15×10^{-4}	4.3×10^{-6}	9.15×10^{-4}	2.9×10^{-6}	2.50×10^{-4}
		(50, 40)	2.19×10^{-4}	5.7×10^{-6}	4.16×10^{-4}	5.7×10^{-6}	4.16×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
		(50, 20)	2.19×10^{-4}	4.3×10^{-6}	4.99×10^{-4}	4.3×10^{-6}	4.99×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
	(1.5, 2)	(50, 30)	4.37×10^{-4}	5.7×10^{-6}	7.49×10^{-4}	5.7×10^{-6}	7.49×10^{-4}	2.9×10^{-6}	2.50×10^{-4}
		(50, 40)	2.19×10^{-4}	5.7×10^{-6}	3.33×10^{-4}	5.7×10^{-6}	3.33×10^{-4}	2.9×10^{-6}	8.32×10^{-5}
		(50, 20)	5.47×10^{-4}	4.3×10^{-6}	0.001	4.3×10^{-6}	0.001	2.9×10^{-6}	2.50×10^{-4}
	(0.5, 1)	(50, 30)	9.84×10^{-4}	4.3×10^{-6}	0.002	4.3×10^{-6}	0.002	2.9×10^{-6}	3.33×10^{-4}
		(50, 40)	6.56×10^{-4}	4.3×10^{-6}	0.001	4.3×10^{-6}	9.98×10^{-4}	2.9×10^{-6}	3.33×10^{-4}
		(50, 20)	4.37×10^{-4}	5.7×10^{-6}	6.66×10^{-4}	5.7×10^{-6}	6.66×10^{-4}	2.9×10^{-6}	2.50×10^{-4}
Sch – II	(1, 1.5)	(50, 30)	4.37×10^{-4}	4.3×10^{-6}	7.49×10^{-4}	4.3×10^{-6}	7.49×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
		(50, 40)	2.19×10^{-4}	5.7×10^{-6}	3.33×10^{-4}	5.7×10^{-6}	3.33×10^{-4}	2.9×10^{-6}	8.32×10^{-5}
		(50, 20)	3.28×10^{-4}	5.7×10^{-6}	4.99×10^{-4}	5.7×10^{-6}	4.99×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
	(1.5, 2)	(50, 30)	3.28×10^{-4}	4.3×10^{-6}	5.82×10^{-4}	4.3×10^{-6}	5.82×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
		(50, 40)	2.19×10^{-4}	5.7×10^{-6}	2.50×10^{-4}	5.7×10^{-6}	2.50×10^{-4}	2.9×10^{-6}	8.32×10^{-5}
		(50, 20)	5.47×10^{-4}	4.3×10^{-6}	0.001	4.3×10^{-6}	$1.08E-03$	2.9×10^{-6}	2.50×10^{-4}
	(0.5, 1)	(50, 30)	5.47×10^{-4}	2.9×10^{-6}	0.001	2.9×10^{-6}	9.98×10^{-4}	1.4×10^{-6}	2.50×10^{-4}
		(50, 40)	3.28×10^{-4}	4.3×10^{-6}	4.99×10^{-4}	4.3×10^{-6}	4.99×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
		(50, 20)	2.19×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	2.9×10^{-6}	1.66×10^{-4}
Sch – III	(1, 1.5)	(50, 30)	5.47×10^{-4}	4.3×10^{-6}	0.001	4.3×10^{-6}	9.98×10^{-4}	1.4×10^{-6}	2.50×10^{-4}
		(50, 40)	2.19×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	2.9×10^{-6}	8.32×10^{-5}
		(50, 20)	2.19×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	2.9×10^{-6}	8.32×10^{-5}
	(1.5, 2)	(50, 30)	6.56×10^{-4}	4.3×10^{-6}	0.001	4.3×10^{-6}	$1.08E-03$	1.4×10^{-6}	2.50×10^{-4}
		(50, 40)	2.19×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	4.3×10^{-6}	4.16×10^{-4}	2.9×10^{-6}	8.32×10^{-5}
					EB				
		(50, 20)	0.006	0.002	0.021	0.002	0.021	6.83×10^{-4}	$1.14E-03$
	(0.5, 1)	(50, 30)	0.011	0.002	0.028	0.002	0.027	6.83×10^{-4}	1.30×10^{-4}
		(50, 40)	0.008	0.001	0.020	0.001	0.020	6.83×10^{-4}	3.12×10^{-4}
		(50, 20)	0.006	0.002	0.017	0.001	0.017	6.83×10^{-4}	7.28×10^{-4}
Sch – I	(1, 1.5)	(50, 30)	0.007	0.001	0.019	0.001	0.018	6.83×10^{-4}	5.20×10^{-4}
		(50, 40)	0.004	0.002	0.012	0.001	0.012	5.46×10^{-4}	2.08×10^{-4}
		(50, 20)	0.004	0.001	0.014	0.001	0.014	8.19×10^{-4}	3.12×10^{-4}
	(1.5, 2)	(50, 30)	0.006	0.002	0.018	0.002	0.018	6.83×10^{-4}	1.04×10^{-4}
		(50, 40)	0.003	0.001	0.010	0.001	0.010	6.83×10^{-4}	6.24×10^{-4}
		(50, 20)	0.007	0.002	0.023	0.002	0.022	6.83×10^{-4}	2.08×10^{-4}
	(0.5, 1)	(50, 30)	0.012	0.002	0.029	0.002	0.028	6.83×10^{-4}	8.32×10^{-4}
		(50, 40)	0.009	0.002	0.020	0.002	0.020	6.83×10^{-4}	8.32×10^{-4}
		(50, 20)	0.005	0.001	0.015	0.001	0.015	8.19×10^{-4}	8.32×10^{-4}
Sch – II	(1, 1.5)	(50, 30)	0.006	0.001	0.017	0.001	0.017	6.83×10^{-4}	6.24×10^{-4}
		(50, 40)	0.004	0.001	0.011	0.001	0.011	6.83×10^{-4}	2.08×10^{-4}
		(50, 20)	0.004	0.001	0.013	0.001	0.013	8.19×10^{-4}	2.08×10^{-4}
	(1.5, 2)	(50, 30)	0.003	0.001	0.015	0.001	0.015	8.19×10^{-4}	$1.66E-03$
		(50, 40)	0.003	0.001	0.010	0.001	0.010	6.83×10^{-4}	6.24×10^{-4}
		(50, 20)	0.007	0.002	0.023	0.002	0.022	6.83×10^{-4}	2.08×10^{-4}
	(0.5, 1)	(50, 30)	0.007	0.001	0.021	0.001	0.020	8.19×10^{-4}	$1.98E-03$
		(50, 40)	0.005	0.001	0.013	0.001	0.013	6.83×10^{-4}	8.32×10^{-4}
		(50, 20)	0.003	0.001	0.013	0.001	0.013	8.19×10^{-4}	$1.56E-03$
Sch – III	(1, 1.5)	(50, 30)	0.006	0.001	0.021	0.001	0.020	8.19×10^{-4}	$2.18E-03$
		(50, 40)	0.004	0.001	0.012	0.001	0.012	6.83×10^{-4}	$1.46E-03$
		(50, 20)	0.004	0.001	0.013	0.001	0.013	8.19×10^{-4}	$1.35E-03$
	(1.5, 2)	(50, 30)	0.007	0.002	0.021	0.001	0.021	6.83×10^{-4}	$1.56E-03$
		(50, 40)	0.004	0.001	0.012	0.001	0.012	6.83×10^{-4}	$1.46E-03$

Table 4: MSE and EB of the estimates for $h(t)$.

				Bayesian					
				$\widehat{h(t)}_{BS}$		$\widehat{h(t)}_{BL}$		$\widehat{h(t)}_{BE}$	
Sch.	(T_1, T_2)	(n, m)	$\widehat{\lambda}_{ML}$	Prior2	Prior2	Prior2	Prior2	Prior2	Prior2
				MSE					
Sch − I	(0.5, 1)	(50, 20)	0.023	4.4×10^{-5}	0.031	4.4×10^{-5}	0.026	4.4×10^{-5}	0.016
		(50, 30)	0.020	4.4×10^{-5}	0.028	4.4×10^{-5}	0.024	4.4×10^{-5}	0.015
		(50, 40)	0.022	5.9×10^{-5}	0.030	5.9×10^{-5}	0.021	5.9×10^{-5}	0.019
	(1, 1.5)	(50, 20)	0.010	5.9×10^{-5}	0.011	5.9×10^{-5}	0.010	7.4×10^{-5}	0.008
		(50, 30)	0.026	5.9×10^{-5}	0.032	5.9×10^{-5}	0.027	5.9×10^{-5}	0.018
		(50, 40)	0.008	8.8×10^{-5}	0.009	8.8×10^{-5}	0.008	8.8×10^{-5}	0.006
	(1.5, 2)	(50, 20)	0.011	7.4×10^{-5}	0.012	7.4×10^{-5}	0.011	7.4×10^{-5}	0.009
		(50, 30)	0.022	5.9×10^{-5}	0.028	5.9×10^{-5}	0.024	5.9×10^{-5}	0.017
		(50, 40)	0.009	7.4×10^{-5}	0.011	8.8×10^{-5}	0.010	8.8×10^{-5}	0.008
Sch − II	(0.5, 1)	(50, 20)	0.024	4.4×10^{-5}	0.031	4.4×10^{-5}	0.027	5.9×10^{-5}	0.017
		(50, 30)	0.020	4.4×10^{-5}	0.027	4.4×10^{-5}	0.024	4.4×10^{-5}	0.015
		(50, 40)	0.011	5.9×10^{-5}	0.014	5.9×10^{-5}	0.012	5.9×10^{-5}	0.009
	(1, 1.5)	(50, 20)	0.012	5.9×10^{-5}	0.014	5.9×10^{-5}	0.013	7.4×10^{-5}	0.009
		(50, 30)	0.017	5.9×10^{-5}	0.023	5.9×10^{-5}	0.020	5.9×10^{-5}	0.013
		(50, 40)	0.008	7.4×10^{-5}	0.009	7.4×10^{-5}	0.008	8.8×10^{-5}	0.006
	(1.5, 2)	(50, 20)	0.012	7.4×10^{-5}	0.014	7.4×10^{-5}	0.013	7.4×10^{-5}	0.009
		(50, 30)	0.024	4.4×10^{-5}	0.034	4.4×10^{-5}	0.028	5.9×10^{-5}	0.018
		(50, 40)	0.007	7.4×10^{-5}	0.008	7.4×10^{-5}	0.008	7.4×10^{-5}	0.006
Sch − III	(0.5, 1)	(50, 20)	0.024	4.4×10^{-5}	0.031	4.4×10^{-5}	0.027	5.9×10^{-5}	0.017
		(50, 30)	0.067	4.4×10^{-5}	0.176	4.4×10^{-5}	0.090	4.4×10^{-5}	0.054
		(50, 40)	0.014	5.9×10^{-5}	0.019	5.9×10^{-5}	0.017	5.9×10^{-5}	0.012
	(1, 1.5)	(50, 20)	0.024	5.9×10^{-5}	0.034	5.9×10^{-5}	0.027	5.9×10^{-5}	0.019
		(50, 30)	0.060	4.4×10^{-5}	0.116	4.4×10^{-5}	0.073	4.4×10^{-5}	0.046
		(50, 40)	0.014	5.9×10^{-5}	0.018	5.9×10^{-5}	0.017	5.9×10^{-5}	0.011
	(1.5, 2)	(50, 20)	0.023	5.9×10^{-5}	0.034	5.9×10^{-5}	0.027	5.9×10^{-5}	0.018
		(50, 30)	0.078	4.4×10^{-5}	0.159	4.4×10^{-5}	0.082	4.4×10^{-5}	0.056
		(50, 40)	0.015	5.9×10^{-5}	0.020	5.9×10^{-5}	0.018	5.9×10^{-5}	0.012
				EB					
Sch − I	(0.5, 1)	(50, 20)	0.054	1.8×10^{-4}	0.071	3.4×10^{-4}	0.064	0.0028	0.026
		(50, 30)	0.046	3.8×10^{-4}	0.065	5.4×10^{-4}	0.059	0.0030	0.019
		(50, 40)	0.032	2.9×10^{-5}	0.043	1.2×10^{-4}	0.039	0.0025	0.015
	(1, 1.5)	(50, 20)	0.030	4.0×10^{-4}	0.032	5.4×10^{-4}	0.030	0.0030	0.009
		(50, 30)	0.055	1.0×10^{-4}	0.065	5.9×10^{-5}	0.059	0.0025	0.029
		(50, 40)	0.026	1.2×10^{-5}	0.030	1.6×10^{-4}	0.028	0.0025	0.013
	(1.5, 2)	(50, 20)	0.036	2.5×10^{-4}	0.038	1.0×10^{-4}	0.036	0.0022	0.016
		(50, 30)	0.049	1.6×10^{-4}	0.059	3.2×10^{-4}	0.055	0.0027	0.026
		(50, 40)	0.031	6.2×10^{-4}	0.035	4.7×10^{-4}	0.034	0.0018	0.018
Sch − II	(0.5, 1)	(50, 20)	0.057	3.5×10^{-4}	0.074	5.1×10^{-4}	0.067	0.0030	0.028
		(50, 30)	0.046	3.7×10^{-4}	0.062	5.3×10^{-4}	0.056	0.0030	0.018
		(50, 40)	0.030	1.5×10^{-5}	0.039	1.5×10^{-4}	0.036	0.0025	0.012
	(1, 1.5)	(50, 20)	0.039	4.0×10^{-4}	0.044	2.4×10^{-4}	0.041	0.0022	0.019
		(50, 30)	0.050	4.4×10^{-5}	0.062	1.2×10^{-4}	0.057	0.0025	0.027
		(50, 40)	0.030	5.1×10^{-4}	0.033	3.7×10^{-4}	0.031	0.0019	0.016
	(1.5, 2)	(50, 20)	0.041	2.6×10^{-4}	0.045	1.2×10^{-4}	0.043	0.0022	0.022
		(50, 30)	0.060	4.4×10^{-4}	0.074	2.9×10^{-4}	0.068	0.0021	0.036
		(50, 40)	0.026	1.3×10^{-4}	0.029	1.2×10^{-5}	0.028	0.0024	0.013
Sch − III	(0.5, 1)	(50, 20)	0.057	3.5×10^{-4}	0.074	5.1×10^{-4}	0.067	0.0030	0.028
		(50, 30)	0.100	3.5×10^{-4}	0.158	1.9×10^{-4}	0.129	0.0022	0.069
		(50, 40)	0.042	4.1×10^{-4}	0.056	2.5×10^{-4}	0.052	0.0021	0.027
	(1, 1.5)	(50, 20)	0.050	1.3×10^{-4}	0.062	1.2×10^{-5}	0.056	0.0025	0.029
		(50, 30)	0.088	3.1×10^{-4}	0.135	1.5×10^{-4}	0.115	0.0024	0.060
		(50, 40)	0.041	3.8×10^{-4}	0.055	2.2×10^{-4}	0.051	0.0022	0.027
	(1.5, 2)	(50, 20)	0.054	1.2×10^{-4}	0.066	4.4×10^{-5}	0.061	0.0025	0.033
		(50, 30)	0.096	8.8×10^{-5}	0.146	7.4×10^{-5}	0.121	0.0025	0.065
		(50, 40)	0.045	6.0×10^{-4}	0.058	4.4×10^{-4}	0.054	0.0019	0.029

Table 5: The ACL of 90% and 95% CIs and corresponding CP for $\hat{\lambda}_{ML}$ and $\hat{\lambda}_B$.

		$\hat{\lambda}_{ML}$				$\hat{\lambda}_B$ Prior2				$\hat{\lambda}_B$ Prior2			
		90%		95%		90%		95%		90%		95%	
(T_1, T_2)	(n, m)	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP
Sch.1													
	(50, 20)	2.870	0.964	3.247	0.964	0.957	0.987	1.107	0.980	2.817	0.907	3.223	0.942
(0.5, 1)	(50, 30)	3.015	0.977	3.399	0.959	0.870	0.994	1.012	0.985	2.950	0.923	3.405	0.939
	(50, 40)	2.242	0.952	2.612	0.955	0.807	0.995	0.944	0.984	2.203	0.911	2.577	0.936
	(50, 20)	1.748	0.917	1.987	0.957	0.778	0.981	0.900	0.980	1.695	0.893	1.915	0.940
(1, 1.5)	(50, 30)	2.112	0.970	2.481	0.949	0.708	0.992	0.824	0.979	2.053	0.921	2.421	0.906
	(50, 40)	1.463	0.922	1.682	0.960	0.654	0.978	0.757	0.984	1.423	0.898	1.638	0.932
	(50, 20)	1.491	0.946	1.737	0.956	0.691	0.992	0.801	0.980	1.446	0.915	1.677	0.941
(1.5, 2)	(50, 30)	1.835	0.978	2.122	0.959	0.632	0.993	0.727	0.980	1.792	0.908	2.069	0.921
	(50, 40)	1.268	0.953	1.499	0.954	0.582	0.984	0.673	0.980	1.232	0.933	1.457	0.927
Sch.2													
	(50, 20)	2.864	0.954	3.267	0.954	0.954	0.977	1.108	0.980	2.786	0.901	3.226	0.926
(0.5, 1)	(50, 30)	3.063	0.966	3.352	0.951	0.865	1.000	0.998	0.980	3.027	0.911	3.334	0.937
	(50, 40)	2.203	0.944	2.565	0.944	0.804	0.993	0.932	0.983	2.150	0.909	2.521	0.920
	(50, 20)	1.730	0.930	2.035	0.956	0.775	0.991	0.899	0.976	1.672	0.900	1.980	0.931
(1, 1.5)	(50, 30)	2.445	0.950	2.482	0.957	0.707	0.991	0.816	0.979	2.411	0.890	2.428	0.927
	(50, 40)	1.436	0.946	1.704	0.955	0.647	0.994	0.754	0.985	1.401	0.919	1.654	0.927
	(50, 20)	1.556	0.923	1.773	0.949	0.687	0.981	0.802	0.974	1.528	0.900	1.729	0.920
(1.5, 2)	(50, 30)	1.960	0.985	2.283	0.965	0.626	0.995	0.728	0.981	1.901	0.907	2.246	0.936
	(50, 40)	1.263	0.945	1.480	0.960	0.574	0.992	0.670	0.980	1.227	0.925	1.444	0.927
Sch.3													
	(50, 20)	2.546	0.961	3.258	0.954	0.931	0.973	1.107	0.980	2.502	0.923	3.218	0.926
(0.5, 1)	(50, 30)	3.859	0.974	4.444	0.965	0.837	0.995	0.970	0.989	4.129	0.899	4.871	0.926
	(50, 40)	2.187	0.959	2.516	0.959	0.779	0.998	0.898	0.979	2.160	0.896	2.523	0.925
	(50, 20)	1.860	0.965	2.198	0.963	0.757	0.987	0.878	0.981	1.813	0.910	2.175	0.934
(1, 1.5)	(50, 30)	3.231	0.970	3.475	0.966	0.688	0.989	0.794	0.983	3.435	0.886	3.703	0.930
	(50, 40)	1.754	0.959	2.045	0.963	0.632	0.996	0.734	0.982	1.752	0.902	2.039	0.931
	(50, 20)	1.621	0.970	1.969	0.965	0.676	0.971	0.782	0.985	1.588	0.930	1.960	0.921
(1.5, 2)	(50, 30)	2.933	0.973	3.184	0.957	0.609	0.995	0.703	0.982	3.056	0.879	3.383	0.909
	(50, 40)	1.593	0.959	1.840	0.969	0.561	0.984	0.650	0.972	1.577	0.886	1.839	0.931

The trace plots, as seen in these diagrams, exhibit delicate mixing of the chains and converge to their distributions. Furthermore, in all situations, histograms and density plots are approximately symmetrical about their means. As a result, the MCMC-generated sample can be used to develop Bayesian estimates and approximate credible intervals, as well as to estimate parameters and their functions. The values of MSE and EB of the ML and Bayesian estimates for λ, γ, $S(t = 0.5)$, and $h(t = 0.5)$ are presented in Tables 1–4, respectively. Tables 5–8 show the ACL of 90% and 95% confidence intervals (CIs), as well as the associated coverage portability (CP).

7. Numerical Example

For this example, we use the real data in ([25], p.105) which is consideredas breakdown times in minutes of an insulating fluid between electrodes at voltage 34 kV.Zimmer et al. in [26] indicated that the Burr Type-XII distribution fits these data and used it to obtain the estimates of the Burr Type-XII parameters. Table 9 shows the 19 breakdown times of an insulating fluid between electrodes. We will use these data to consider the following progressively censored schemes.

For generating generalized Type-II progressive hybrid censored samples, suppose $m = 15$, $R_1 = R_m = 2$; then, we would have the following progressive data: 0.19, 0.78, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 31.75, 32.52, and 33.91, with 0.96, 12.06, 36.71, and 72.89 being randomly selected and removed from the given data. If we consider different values of T_1 and T_2, then we have different kinds of generalized Type-II PHCS, namely,

(1) Scheme I: suppose $T_1 = 50$, $T_2 = 80$; since $X_{15:15:19} < T_1$, then the experiment would have terminated at T_1, with $d^* = 16, R_1 = 2, R_m = 0$, $R'_{d_1+1} = 1, R'_{d_2+1} = 0$, and we would have the following data: 0.19, 0.78, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 31.75, 32.52, 33.91, and 36.71.

(2) Scheme II: suppose $T_1 = 30$, $T_2 = 40$; since $T_1 < X_{15:15:19} < T_2$, then the experiment would have terminated at $X_{15:15:19}$, with $d^* = m = 15, R_1 = R_m = 2$, $R'_{d_1+1} = R'_{d_2+1} = 0$, and we would have the following data: 0.19, 0.78, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 31.75, 32.52, and 33.91.

(3) Scheme III: suppose $T_1 = 20$, $T_2 = 30$; since $T_2 < X_{15:15:19}$, then the experiment would have terminated at T_2, with $d^* = d_2 = 12, R_1 = 2$, $R'_{d_1+1} = 0$, $R^{d_2+1} = 4$, and we would have the following

Table 6: The ACL of 90% and 95% CIs and corresponding CP for $\hat{\gamma}_{ML}$ and $\hat{\gamma}_B$.

						Bayesian							
		$\hat{\gamma}_{ML}$				Prior2				Prior2			
		90%		95%		90%		95%		90%		95%	
(T_1, T_2)	(n, m)	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP
						Sch.1							
	(50, 20)	0.980	0.946	1.130	0.997	0.478	1.000	0.555	1.000	0.955	0.936	1.084	0.975
(0.5, 1)	(50, 30)	0.874	0.949	1.005	0.982	0.413	0.987	0.478	1.000	0.845	0.920	0.963	0.964
	(50, 40)	0.745	0.961	0.859	0.977	0.384	0.999	0.439	1.000	0.730	0.935	0.827	0.945
	(50, 20)	0.762	0.963	0.878	0.976	0.444	1.000	0.510	0.999	0.741	0.942	0.841	0.957
(1, 1.5)	(50, 30)	0.722	0.962	0.836	0.974	0.382	1.000	0.444	1.000	0.704	0.940	0.804	0.945
	(50, 40)	0.583	0.939	0.679	0.959	0.351	0.986	0.406	1.000	0.569	0.926	0.659	0.941
	(50, 20)	0.677	0.964	0.786	0.973	0.407	0.999	0.469	0.998	0.664	0.946	0.759	0.953
(1.5, 2)	(50, 30)	0.644	0.975	0.754	0.981	0.348	1.000	0.403	0.985	0.624	0.937	0.726	0.944
	(50, 40)	0.525	0.963	0.615	0.963	0.321	1.000	0.373	1.000	0.512	0.944	0.594	0.941
						Sch.2							
	(50, 20)	0.978	0.953	1.135	0.987	0.478	1.000	0.552	1.000	0.945	0.937	1.090	0.949
(0.5, 1)	(50, 30)	0.861	0.950	0.997	0.985	0.407	1.000	0.474	0.998	0.837	0.926	0.954	0.962
	(50, 40)	0.735	0.964	0.857	0.990	0.380	1.000	0.438	0.997	0.715	0.950	0.820	0.963
	(50, 20)	0.754	0.957	0.881	0.986	0.437	1.000	0.507	1.000	0.738	0.933	0.856	0.966
(1, 1.5)	(50, 30)	0.795	0.964	0.839	0.984	0.376	1.000	0.434	1.000	0.772	0.944	0.808	0.970
	(50, 40)	0.583	0.952	0.679	0.983	0.351	1.000	0.402	0.998	0.572	0.923	0.654	0.960
	(50, 20)	0.686	0.954	0.788	0.980	0.403	0.999	0.468	0.999	0.672	0.918	0.764	0.947
(1.5, 2)	(50, 30)	0.665	0.952	0.771	0.985	0.347	1.000	0.399	0.995	0.646	0.918	0.740	0.967
	(50, 40)	0.531	0.935	0.609	0.985	0.322	0.986	0.367	1.000	0.515	0.916	0.593	0.966
						Sch.3							
	(50, 20)	0.961	0.972	1.135	0.988	0.469	0.989	0.551	1.000	0.935	0.936	1.089	0.949
(0.5, 1)	(50, 30)	0.964	0.951	1.109	0.984	0.395	0.990	0.457	1.000	0.928	0.905	1.058	0.951
	(50, 40)	0.728	0.971	0.853	0.987	0.369	1.000	0.428	1.000	0.709	0.945	0.822	0.948
	(50, 20)	0.811	0.968	0.938	0.995	0.436	1.000	0.498	1.000	0.787	0.947	0.907	0.978
(1, 1.5)	(50, 30)	0.875	0.971	1.017	0.989	0.362	0.997	0.422	1.000	0.846	0.920	0.967	0.949
	(50, 40)	0.673	0.965	0.780	0.982	0.342	1.000	0.394	1.000	0.655	0.935	0.749	0.961
	(50, 20)	0.739	0.967	0.864	0.986	0.402	0.990	0.458	1.000	0.720	0.940	0.830	0.961
(1.5, 2)	(50, 30)	0.812	0.960	0.936	0.977	0.334	1.000	0.387	0.994	0.776	0.922	0.883	0.937
	(50, 40)	0.618	0.975	0.721	0.985	0.311	0.996	0.363	1.000	0.602	0.949	0.695	0.955

Table 7: The ACL of 90% and 95% CIs and corresponding CP for $\widehat{S(t)}_{ML}$ and $\widehat{S(t)}_B$.

						$\widehat{S(t)}_B$							
		$\widehat{S(t)}_{ML}$				Prior2				Prior2			
		90%		95%		90%		95%		90%		95%	
(T_1, T_2)	(n, m)	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP
						Sch.1							
	(50, 20)	0.092	0.773	0.112	0.809	0.022	1.000	0.027	1.000	0.145	0.969	0.189	0.964
(0.5, 1)	(50, 30)	0.118	0.810	0.144	0.825	0.023	1.000	0.027	1.000	0.172	0.965	0.221	0.963
	(50, 40)	0.095	0.843	0.113	0.862	0.022	1.000	0.026	0.993	0.129	0.952	0.162	0.960
	(50, 20)	0.072	0.828	0.090	0.868	0.021	1.000	0.024	0.978	0.104	0.948	0.136	0.961
(1, 1.5)	(50, 30)	0.073	0.763	0.094	0.810	0.021	1.000	0.025	0.963	0.110	0.972	0.147	0.934
	(50, 40)	0.059	0.852	0.071	0.888	0.020	1.000	0.024	1.000	0.078	0.960	0.100	0.944
	(50, 20)	0.061	0.829	0.070	0.850	0.019	1.000	0.022	0.933	0.088	0.964	0.110	0.954
(1.5, 2)	(50, 30)	0.069	0.796	0.085	0.821	0.019	1.000	0.023	1.000	0.103	0.954	0.135	0.949
	(50, 40)	0.051	0.874	0.060	0.869	0.019	1.000	0.022	0.918	0.069	0.996	0.085	0.944
						Sch.2							
	(50, 20)	0.094	0.780	0.116	0.781	0.023	1.000	0.027	1.000	0.147	0.968	0.193	0.958
(0.5, 1)	(50, 30)	0.117	0.791	0.146	0.814	0.023	1.000	0.027	0.978	0.167	0.949	0.220	0.969
	(50, 40)	0.097	0.855	0.113	0.872	0.022	1.000	0.026	0.963	0.132	0.968	0.160	0.949

Table 7: Continued.

(T_1, T_2)	(n, m)	$\widehat{S(t)}_{ML}$ 90% ACL	CP	95% ACL	CP	$\widehat{S(t)}_B$ Prior2 90% ACL	CP	95% ACL	CP	Prior2 90% ACL	CP	95% ACL	CP
(1, 1.5)	(50, 20)	0.072	0.834	0.082	0.814	0.021	0.990	0.024	0.948	0.103	0.956	0.127	0.959
	(50, 30)	0.110	0.767	0.091	0.813	0.021	0.979	0.025	0.933	0.157	0.958	0.144	0.955
	(50, 40)	0.058	0.862	0.068	0.848	0.020	1.000	0.024	1.000	0.078	0.959	0.096	0.945
(1.5, 2)	(50, 20)	0.057	0.803	0.069	0.805	0.019	0.990	0.022	0.903	0.082	0.950	0.107	0.942
	(50, 30)	0.067	0.772	0.075	0.792	0.019	0.979	0.023	1.000	0.101	0.940	0.126	0.965
	(50, 40)	0.051	0.869	0.060	0.897	0.019	1.000	0.022	0.888	0.068	0.959	0.087	0.949
						Sch.3							
(0.5, 1)	(50, 20)	0.106	0.821	0.116	0.782	0.022	1.000	0.027	1.000	0.151	0.965	0.193	0.958
	(50, 30)	0.094	0.701	0.116	0.741	0.022	0.990	0.027	0.948	0.138	0.933	0.184	0.940
	(50, 40)	0.073	0.817	0.087	0.820	0.022	0.979	0.026	0.933	0.097	0.946	0.122	0.950
(1, 1.5)	(50, 20)	0.068	0.804	0.076	0.827	0.021	0.967	0.024	0.918	0.097	0.953	0.122	0.966
	(50, 30)	0.092	0.718	0.108	0.769	0.021	0.956	0.025	0.902	0.135	0.936	0.174	0.947
	(50, 40)	0.066	0.843	0.076	0.843	0.020	1.000	0.024	1.000	0.087	0.951	0.109	0.952
(1.5, 2)	(50, 20)	0.060	0.827	0.071	0.810	0.019	0.967	0.022	0.873	0.089	0.995	0.111	0.956
	(50, 30)	0.082	0.718	0.102	0.758	0.019	0.956	0.023	0.999	0.122	0.927	0.161	0.941
	(50, 40)	0.060	0.826	0.070	0.815	0.019	1.000	0.022	0.918	0.081	0.943	0.101	0.953

Table 8: The ACL of 90% and 95% CIs and corresponding CP for $\widehat{h(t)}_{ML}$ and $\widehat{h(t)}_B$.

(T_1, T_2)	(n, m)	$\widehat{h(t)}_{ML}$ 90% ACL	CP	95% ACL	CP	$\widehat{h(t)}_B$ Prior2 90% ACL	CP	95% ACL	CP	Prior2 90% ACL	CP	95% ACL	CP
						Sch.1							
(0.5, 1)	(50, 20)	0.495	0.752	0.552	0.780	0.082	1.000	0.095	0.982	0.499	0.991	0.571	0.978
	(50, 30)	0.497	0.788	0.548	0.796	0.082	1.000	0.095	1.000	0.497	0.979	0.574	0.970
	(50, 40)	0.361	0.820	0.419	0.831	0.081	1.000	0.094	1.000	0.362	0.968	0.427	0.973
(1, 1.5)	(50, 20)	0.325	0.806	0.364	0.837	0.078	1.000	0.090	1.000	0.315	0.971	0.352	0.967
	(50, 30)	0.361	0.743	0.428	0.781	0.071	1.000	0.082	0.970	0.355	0.993	0.428	0.940
	(50, 40)	0.269	0.829	0.309	0.856	0.076	1.000	0.088	1.000	0.262	0.973	0.302	0.952
(1.5, 2)	(50, 20)	0.302	0.807	0.352	0.820	0.076	1.000	0.088	1.000	0.294	0.977	0.341	0.953
	(50, 30)	0.349	0.774	0.407	0.791	0.071	1.000	0.081	1.000	0.346	0.976	0.405	0.957
	(50, 40)	0.255	0.850	0.304	0.838	0.074	1.000	0.086	1.000	0.247	1.000	0.298	0.946
						Sch.2							
(0.5, 1)	(50, 20)	0.493	0.758	0.561	0.753	0.082	0.974	0.095	0.990	0.489	0.983	0.576	0.963
	(50, 30)	0.509	0.770	0.542	0.785	0.082	1.000	0.095	1.000	0.518	0.971	0.562	0.969
	(50, 40)	0.350	0.831	0.409	0.841	0.081	1.000	0.094	1.000	0.345	0.984	0.411	0.959
(1, 1.5)	(50, 20)	0.325	0.811	0.385	0.785	0.077	1.000	0.090	1.000	0.313	0.972	0.378	0.969
	(50, 30)	0.450	0.746	0.452	0.784	0.078	1.000	0.090	0.974	0.456	0.982	0.454	0.963
	(50, 40)	0.266	0.838	0.316	0.818	0.076	1.000	0.088	1.000	0.260	0.973	0.308	0.955
(1.5, 2)	(50, 20)	0.330	0.781	0.367	0.776	0.076	1.000	0.087	1.000	0.329	0.972	0.359	0.950
	(50, 30)	0.406	0.750	0.467	0.764	0.076	0.988	0.088	0.980	0.399	0.973	0.473	0.975
	(50, 40)	0.260	0.845	0.298	0.865	0.074	1.000	0.086	1.000	0.251	0.972	0.293	0.956
						Sch.3							
(0.5, 1)	(50, 20)	0.437	0.799	0.559	0.754	0.082	0.974	0.095	1.015	0.441	0.982	0.575	0.963
	(50, 30)	0.696	0.682	0.800	0.714	0.082	0.924	0.096	0.891	0.809	0.952	0.969	0.948
	(50, 40)	0.365	0.794	0.423	0.791	0.081	1.022	0.094	1.015	0.367	0.963	0.438	0.962
(1, 1.5)	(50, 20)	0.380	0.782	0.441	0.798	0.078	1.032	0.090	1.015	0.376	0.976	0.448	0.977
	(50, 30)	0.638	0.699	0.684	0.742	0.078	0.960	0.091	0.913	0.730	0.957	0.780	0.960
	(50, 40)	0.348	0.820	0.405	0.813	0.077	1.050	0.090	1.016	0.355	0.969	0.416	0.966
(1.5, 2)	(50, 20)	0.355	0.804	0.440	0.781	0.076	1.009	0.088	1.015	0.352	1.000	0.454	0.968
	(50, 30)	0.652	0.699	0.709	0.731	0.077	0.945	0.089	0.913	0.720	0.948	0.818	0.951
	(50, 40)	0.348	0.803	0.404	0.786	0.076	1.016	0.088	1.015	0.350	0.969	0.418	0.964

Table 9: The observed breakdown times of an insulating fluid between electrodes.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
x_i	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50	7.35	8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89

Table 10: The ML and Bayesian estimates of $\widehat{\lambda}$, $\widehat{mu}$, $\widehat{S(t)}$, and $\widehat{h(t)}$ at selected censoring schemes from the real dataset.

			Bayesian					
			SELF		LLF		GELF	
Estimator	Scheme	ML	Prior1	Prior2	Prior1	Prior2	Prior1	Prior2
$\widehat{\lambda}$	1	0.277	0.677	0.455	0.670	0.446	0.645	0.397
	2	0.266	0.631	0.266	0.626	0.263	0.608	0.233
	3	0.245	1.039	0.728	1.021	0.695	0.989	0.603
$\widehat{\gamma}$	1	1.692	0.440	0.449	0.439	0.444	0.431	0.422
	2	1.677	0.712	1.866	0.708	1.756	0.697	1.691
	3	1.641	0.585	1.421	0.582	1.339	0.568	1.248
$R(\widehat{t=1})$	1	0.825	0.630	0.736	0.629	0.733	0.623	0.726
	2	0.832	0.649	0.834	0.648	0.833	0.644	0.830
	3	0.844	0.495	0.622	0.493	0.617	0.482	0.593
$h(\widehat{t=1})$	1	0.235	0.144	0.093	0.144	0.093	0.141	0.088
	2	0.223	0.219	0.221	0.219	0.220	0.214	0.209
	3	0.201	0.293	0.438	0.292	0.434	0.286	0.410

Table 11: The CIs and ACL for $\widehat{\lambda}$, $\widehat{\gamma}$, $\widehat{S(t)}$, and $\widehat{h(t)}$ at selected censoring schemes from real dataset.

				Bayesian			
		Asymp.CI		Prior1		Prior2	
Estimator	Sch.	90%	95%	90%	95%	90%	95%
$\widehat{\lambda}$	1	(0.090, 0.465) 0.375	(0.055, 0.500) 0.445	(0.424, 0.992) 0.568	(0.388, 1.037) 0.649	(0.196, 0.809) 0.613	(0.184, 0.954) 0.769
	2	(0.083, 0.448) 0.365	(0.049, 0.483) 0.434	(0.428, 0.880) 0.451	(0.400, 0.944) 0.544	(0.121, 0.482) 0.362	(0.111, 0.547) 0.436
	3	(0.067, 0.423) 0.357	(0.033, 0.457) 0.424	(0.657, 1.542) 0.885	(0.602, 1.650) 1.048	(0.276, 1.446) 1.17	(0.210, 1.664) 1.454
$\widehat{\gamma}$	1	(0.714, 2.669) 1.954	(0.531, 2.852) 2.322	(0.326, 0.571) 0.245	(0.311, 0.596) 0.285	(0.266, 0.686) 0.42	(0.227, 0.700) 0.473
	2	(0.698, 2.656) 1.958	(0.514, 2.840) 2.326	(0.536, 0.924) 0.388	(0.505, 0.959) 0.454	(0.971, 3.294) 2.323	(0.873, 3.385) 2.512
	3	(0.651, 2.632) 1.981	(0.465, 2.818) 2.353	(0.406, 0.789) 0.383	(0.387, 0.841) 0.454	(0.645, 2.566) 1.921	(0.554, 3.281) 2.727
$R(\widehat{t=1})$	1	(0.718, 0.932) 0.214	(0.698, 0.952) 0.255	(0.503, 0.745) 0.242	(0.487, 0.764) 0.277	(0.570, 0.873) 0.303	(0.516, 0.880) 0.364
	2	(0.726, 0.937) 0.211	(0.707, 0.957) 0.25	(0.543, 0.743) 0.2	(0.520, 0.757) 0.238	(0.716, 0.920) 0.204	(0.684, 0.925) 0.242
	3	(0.739, 0.948) 0.209	(0.720, 0.968) 0.248	(0.343, 0.634) 0.291	(0.318, 0.658) 0.34	(0.366, 0.826) 0.46	(0.315, 0.864) 0.549
$h(\widehat{t=1})$	1	(0.137, 0.332) 0.195	(0.119, 0.350) 0.231	(0.107, 0.188) 0.081	(0.100, 0.194) 0.094	(0.057, 0.136) 0.079	(0.053, 0.147) 0.094
	2	(0.127, 0.318) 0.191	(0.109, 0.336) 0.227	(0.162, 0.284) 0.123	(0.152, 0.296) 0.144	(0.136, 0.320) 0.184	(0.122, 0.342) 0.22
	3	(0.105, 0.298) 0.193	(0.087, 0.316) 0.229	(0.213, 0.384) 0.171	(0.202, 0.401) 0.199	(0.249, 0.665) 0.416	(0.223, 0.721) 0.498

data: 0.19, 0.78, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, and 8.27, with 0.96, 12.06, 36.71, and 72.89 being removed from the data and 31.75, 32.52, and 33.91 being removed after T_2.

Table 10 shows the ML and Bayesian estimates for the unknown parameters λ and γ and survival and hazard functions based on the generalized Type-II PHCS and two different selections Prior1 and Prior2. Table 11 also includes

Table 12: Bayesian point predictor and 95% ET and HPD prediction intervals for $Y_{i:R_j^*}$.

Sch.	j	i	Prior2 $\widehat{Y}_{i:R_j^*}$	Prior2 ET interval	Prior2 HPD interval	Prior2 $\widehat{Y}_{i:R_j^*}$	Prior2 ET interval	Prior2 HPD interval
1	1	1	5.958	(1.462, 23.953)	(1.336, 18.976)	7.137	(1.465, 25.126)	(1.336, 19.782)
				22.491	17.64		23.661	18.446
		2	9.780	(5.901, 28.392)	(5.775, 23.415)	11.576	(5.904, 29.565)	(5.775, 24.220)
				22.491	17.64		23.661	18.446
	$d_1 + 1$	1	13.267	(8.339, 31.639)	(7.319, 27.378)	15.852	(8.339, 34.000)	(7.269, 28.999)
				23.299	20.058		25.661	21.73
2	1	1	8.914	(6.880, 19.345)	(6.533, 16.794)	10.546	(6.884, 20.413)	(6.529, 17.515)
				12.465	10.261		13.53	10.986
		2	16.993	(7.195, 51.064)	(5.917, 42.481)	20.278	(7.218, 53.677)	(5.910, 44.332)
				43.868	36.564		46.459	38.422
	15	1	10.922	(7.499, 25.069)	(6.800, 21.527)	12.995	(7.503, 26.732)	(6.678, 22.673)
				17.57	14.726		19.229	15.995
		2	27.598	(15.672, 94.192)	(13.232, 60.735)	33.467	(15.717, 73.947)	(13.176, 63.447)
				78.52	47.503		58.23	50.271
3	1	1	9.945	(5.907, 28.950)	(5.775, 23.927)	11.767	(5.910, 30.059)	(5.775, 24.706)
				23.043	18.152		24.149	18.931
		2	13.583	(2.829, 47.559)	(1.492, 38.983)	16.315	(2.857, 49.985)	(1.486, 40.743)
				44.73	37.491		47.129	39.257
	$d_2 + 1$	1	12.867	(2.659, 47.155)	(1.451, 37.949)	15.622	(2.672, 50.932)	(1.439, 40.491)
				44.496	36.498		48.26	39.052
		2	20.884	(13.339, 49.105)	(13.315, 42.236)	24.765	(13.367, 51.445)	(11.942, 43.930)
				35.766	28.921		38.078	31.988
		3	32.324	(10.856, 100.422)	(7.273, 82.897)	38.728	(10.940, 105.534)	(7.342, 87.613)
				89.565	75.624		94.594	80.271
		4	40.942	(6.162, 70.194)	(5.718, 76.662)	33.243	(6.221, 99.726)	(5.750, 80.644)
				64.033	70.944		93.505	74.895

the asymptotic CIs of 90% and 95%, as well as the credible intervals. Table 12 shows the point predictors and 95% Bayesian prediction boundaries of $Y_{i:R_k^*}$ for three different censoring schemes with Prior1 and Prior2.

8. Conclusions

When the observed sample is a generalized Type-II PHCS sample, Bayesian and ML estimates of the unknown parameters, as well as the survival and hazard functions of the Burr Type-XII distribution, are produced. Squared error, LINEX, and general entropy loss functions based on Prior1 and Prior2 distributions are considered in the Bayesian method. The parameters, as well as the survival and hazard functions, are given asymptotic and credible CIs of 90% and 95%. In the generalized Type-II PHCS, Bayesian point and interval prediction of non-observed failures were also established. The following conclusions can be drawn from the numerical results:

(1) In most cases, the Bayesian estimates with Prior1 are better than the MLEs.

(2) The results of the estimates from ML that can be seen in Tables 1–4 are similar to the Bayesian estimators using Prior2. Thus, when we have no prior knowledge about the unknown parameters, it is often easier to use the ML instead of the Bayesian estimators because the computation of the Bayesian estimators is more complicated.

(3) In most cases, the MSE increases as m decreases.

(4) The ACL of the CIs increases as T_1 and T_2 decrease.

(5) The credible intervals perform well compared to the asymptotic CIs.

(6) In all cases, the ACL of 95% CIs is larger than that of 90% CIs.

(7) The HPD prediction intervals appear to be more accurate than the ET prediction intervals.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

The authors acknowledge the financial support from the Researchers Supporting Project (RSP-2022/323), King Saud University, Riyadh, Saudi Arabia.

References

[1] N. Balakrishnan and R. Aggarwala, *Progressive Censoring: Theory, Methods, and Applications*, Springer Science & Business Media, Berlin, Germany, 2000.

[2] N. Balakrishnan, "Progressive censoring methodology: an appraisal," *Test*, vol. 16, no. 2, pp. 211–259, 2007.
[3] E. Cramer and G. Iliopoulos, "Adaptive progressive Type-II censoring," *Test*, vol. 19, no. 2, pp. 342–358, 2010.
[4] M. M. M. El-Din and A. R. Shafay, "One- and two-sample Bayesian prediction intervals based on progressively Type-II censored data," *Statistical Papers*, vol. 54, no. 2, pp. 287–307, 2013.
[5] N. Balakrishnan and A. C. Cohen, *Order Statistics & Inference: Estimation Methods*, Academic Press, San Diego, CA, USA, 1991.
[6] D. Kundu and A. Joarder, "Analysis of Type-II progressively hybrid censored data," *Computational Statistics & Data Analysis*, vol. 50, no. 10, pp. 2509–2528, 2006.
[7] A. Childs, B. Chandrasekar, and N. Balakrishnan, "Exact likelihood inference for an exponential parameter under progressive hybrid censoring schemes," in *Statistical Models and Methods for Biomedical and Technical Systems*, pp. 319–330, Birkhäuser, Boston, MA, USA, 2008.
[8] C. T. Lin and Y. L. Huang, "On progressive hybrid censored exponential distribution," *Journal of Statistical Computation and Simulation*, vol. 82, no. 5, pp. 689–709, 2012.
[9] F. Hemmati and E. Khorram, "Statistical analysis of the log-normal distribution under type-II progressive hybrid censoring schemes," *Communications in Statistics - Simulation and Computation*, vol. 42, no. 1, pp. 52–75, 2013.
[10] K. Lee, H. Sun, and Y. Cho, "Exact likelihood inference of the exponential parameter under generalized Type II progressive hybrid censoring," *Journal of the Korean Surgical Society*, vol. 45, no. 1, pp. 123–136, 2016.
[11] M. M. Mohie El-Din, A. R. Shafay, and M. Nagy, "Statistical inference under adaptive progressive censoring scheme," *Computational Statistics*, vol. 33, no. 1, pp. 31–74, 2018.
[12] I. Kumar and K. Kumar, "On estimation of $P(V<U)$ for inverse Pareto distribution under progressively censored data," *International Journal of System Assurance Engineering and Management*, vol. 13, pp. 1–14, 2021.
[13] A. Chaturvedi, N. Kumar, and K. Kumar, "Statistical inference for the reliability functions of a family of lifetime distributions based on progressive type II right censoring," *Statistica*, vol. 78, no. 1, pp. 81–101, 2018.
[14] M. M. Mohie El-Din, M. Nagy, and M. H. Abu-Moussa, "Estimation and prediction for Gompertz distribution under the generalized progressive hybrid censored data," *Annals of Data Science*, vol. 6, no. 4, pp. 673–705, 2019.
[15] R. M. El-Sagheer, M. Rashad, Hasaballah, and M. Mustafa, "Inference of process capability index cpy for 3-burr-XII distribution based on progressive type-II censoring," *International Journal of Mathematics and Mathematical Sciences*, vol. 2020, pp. 1–13, 2020.
[16] S. Saini, S. Tomer, and R. Garg, "On the reliability estimation of multicomponent stress-strength model for Burr XII distribution using progressively first-failure censored samples," *Journal of Statistical Computation and Simulation*, vol. 91, pp. 1–38, 2021.
[17] M. Nagy, K. S. Sultan, K. S. Sultan, and M. H. Abu-Moussa, "Analysis of the generalized progressive hybrid censoring from Burr Type-XII lifetime model," *AIMS Mathematics*, vol. 6, no. 9, pp. 9675–9704, 2021.
[18] W. H. Greene, *Econometric Analysis*pp. 201–215, Prentice-Hall, New Jersey, NJ, USA, 4th edition, 2000.
[19] A. Agresti, *Logistic Regression, Categorical Data Analysis*, Wiley & Sons, Inc, Hoboken, NJ, USA, 2002.
[20] R. Calabria and G. Pulcini, "Point estimation under asymmetric loss functions for left-truncated exponential samples," *Communications in Statistics - Theory and Methods*, vol. 25, no. 3, pp. 585–600, 1996.
[21] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, "Equation of state calculations by fast computing machines," *The Journal of Chemical Physics*, vol. 21, no. 6, pp. 1087–1092, 1953.
[22] E. A. Ahmed, "Estimation and prediction for the generalized inverted exponential distribution based on progressively first-failure-censored data with application," *Journal of Applied Statistics*, vol. 44, no. 9, pp. 1576–1608, 2017.
[23] H. Panahi and S. Asadi, "On adaptive progressive hybrid censored Burr type III distribution: application to the nano droplet dispersion data," *Quality Technology & Quantitative Management*, vol. 18, no. 2, pp. 179–201, 2021.
[24] I. Basak, P. Basak, and N. Balakrishnan, "On some predictors of times to failure of censored items in progressively censored samples," *Computational Statistics & Data Analysis*, vol. 50, no. 5, pp. 1313–1337, 2006.
[25] W. Nelson, *Applied Life Data Analysis*, Wiley, New York, NY, USA, 1982.
[26] W. J. Zimmer, J. B. Keats, and F. K. Wang, "The Burr XII distribution in reliability analysis," *Journal of Quality Technology*, vol. 30, no. 4, pp. 386–394, 1998.