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Research Article

Statistical Inference for the Gompertz Distribution Based on Adaptive Type-II Progressive Censoring Scheme

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The topic of estimating the parameters of Gompertz distribution using an adaptive Type-II progressively censored data are described in this paper. The unknown parameters, the reliability, and the hazard functions are estimated using maximum likelihood and Bayesian estimation methods. The approximate confidence intervals of them are then determined. Furthermore, the Markov chain Monte Carlo approach is used to perform a Bayesian estimate procedure and compute the credible intervals. Finally, a Monte Carlo simulation study is done to assess the performance of the two estimating methods, and a numerical example with real data is shown to demonstrate the procedures' utility.

1. Introduction

In life testing and survival analysis, there are numerous scenarios, in which units are withdrawn or lost from the experiment before it fails. The data obtained from such an experiment is referred to as censored data. The most important reason for censoring is to reduce the total length of the test, as well as the expense and labor involved with it. A censoring method that can achieve a balance between the total time spent on the experiment, the number of units used in the experiment, and the efficiency of statistical inference based on the experiment's results is also desirable. Because of time constraints and other data gathering constraints, censoring is prevalent in life experiments.

Naturally, there are different types of censoring schemes. The most common censoring schemes are Type-I censoring, in which the life testing experiment is terminated at a specified time T and in this case, the number of failures (from a sample size n) is a random variable; and Type-II censoring, in which the life testing experiment is terminated upon the rth failure, and in this case, the total time of test is a random variable. One of the disadvantages of the traditional Type-I and Type-II censoring schemes is that they do not have the flexibility to allow units to be removed at times other than the experiment's end.

As a result, we investigate a more general censoring scheme known as progressive Type-II censoring, which offers this feature. In a brief, this is how it works: consider a reliability experiment, in which n units are tested throughout a lifetime experiment. R_1 units of the $(n-1)$ surviving units are randomly eliminated from the experiment when the first failure (X_1) occurs. Similarly, R_2 units of the $(n-2-R_1)$ surviving units are randomly removed from the experiment when the second failure (X_2) occurs. The test continues until the mth failure (X_m) occurs and at that time the remaining $(R_m = n-m-R_1-\cdots-R_{m-1})$ units are removed. Prior to the study, the R_i values are set. The progressive Type-II censoring sample is defined as the ith order observed failure times denoted by $X_{i:\,m:\,n}^{R_1,R_2,\ldots,R_m}$; $i=1,2,\ldots,m$. The censoring scheme in the notation of the $X_{i:\,m:\,n}$'s be suppressed for convenience. We also use the notation $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ to

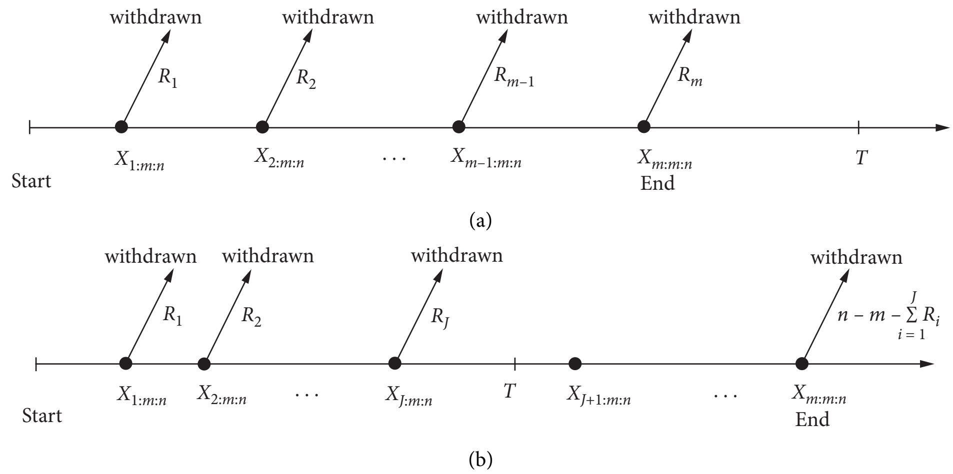

Figure 1: Schematic representation of the adaptive Type-II progressive censoring scheme: (a) case 1: experiment terminates before time T (i.e., $X_m < T$); (b) case 2: experiment terminates after time T (i.e., $T < X_m$).

represent the observed values of a progressively Type-II censored sample.

The adaptive Type-II progressive censoring, abbreviated by (AT2PC), is a mixture of Type-I censoring and Type-II progressive censoring systems (cf. Ng and Chan [1]). In this censoring scheme, we allow $R_1 - R_2 - \cdots - R_m$ to depend on the failure times. As a result, the effective sample size is always m, and it is known in advance. The following are some of the benefits of a well-designed AT2PC life testing experiment: (i) reduce the total test time; (ii) reduce the costs associated with unit failure; and (iii) improve the statistical analysis efficiency.

The following is a description of the censoring scheme: consider n identical units in a life testing experiment, and presume the experimenter selected an ideal aggregate test time T, while the experiment may run throughout time T. The experiment stops at time $X_{m:\,m:\,n}$ if the m^{th} progressively censored failure happens before time T (i.e., $X_{m:\,m:\,n} < T$). Otherwise, the experiment will be rapidly ended if the experimental time T has passed but the number of observed failures has not surpassed m (i.e., $X_{m:\,m:\,n} > T$). As a result, we attempt to include as many items as possible on the test. The number of failures observed before to time T is denoted by J. Hence, we have

$$X_{J:\,m:\,n} \leq T < X_{J+1:\,m:\,n},\ J = 0, 1, \ldots, m, \tag{1}$$

where $X_{0:\,m:\,n} \equiv 0$ and $X_{m+1:\,m:\,n} \equiv \infty$. After the experiment elapsed the time T, we set

$$R_{J+1} = \ldots = R_{m-1} = 0, \quad \text{and} \quad R_m = R_m^* = n - m - \sum_{i=1}^{J} R_i. \tag{2}$$

Use this formula to finish the experiment rapidly if the $(J+1)^{th}$ failure time is more than T for $J+1 < m$. It is worth noting that the value of T influences the values of R_i, as well as providing a compromise between a shorter experimental time and a larger chance of observing outlier failures. The usual progressive Type-II censoring scheme with the prefixed progressive censoring scheme $(R_1, \ldots, R_m)$ is produced when $T \longrightarrow \infty$. The schematic representation of the adaptive Type-II progressive censoring scheme is shown in Figure 1. Cramer and Iliopoulos [2], Mahmoud et al. [3], Balakrishnan and Cramer [4], Ye et al. [5], El-Sayed et al. [6], Mohie El-Din et al. [7], Almetwally et al. [8], Almetwally et al. [9], Nassr et al. [10], Mohan and Chacko [11], Almongy et al. [12], Haj Ahmad et al. [13], Dutta and Kayal [14, 15], and Abo-Kasem et al. [16] provide comprehensive reviews of the literature on the adaptive Type-II progressive censoring scheme.

The Gompertz distribution, which is used as a survival model in reliability and survival analysis and plays a significant role in modeling human mortality and fitting actuarial tables, is the underlying distribution in this work. The investigation of statistical methods and characterization of the Gompertz distribution, which was initially described by Gompertz [17], has been contributed by many researchers. The probability density function (PDF), cumulative distribution function (CDF), reliability function, and hazard rate function of the two-parameter Gompertz distribution are provided, respectively, as follows:

$$f(x; \delta, \beta) = \delta\beta e^{-\beta\left(e^{\delta x}-1\right)+\delta x}, \quad x > 0, \tag{3}$$

$$F(x; \delta, \beta) = 1 - e^{-\beta\left(e^{\delta x}-1\right)}, \quad x > 0, \tag{4}$$

$$s(t) = e^{-\beta\left(e^{\delta t}-1\right)}, \quad t > 0, \tag{5}$$

and

$$h(t) = \beta\delta e^{\delta t}, \quad t > 0, \tag{6}$$

where $\delta > 0$ and $\beta > 0$ are the scale and shape parameters, respectively. The PDF and hazard rate function of the Gompertz distribution are represented in Figures 2 and 3, respectively, for different values of the scale parameter δ and shape parameter β.

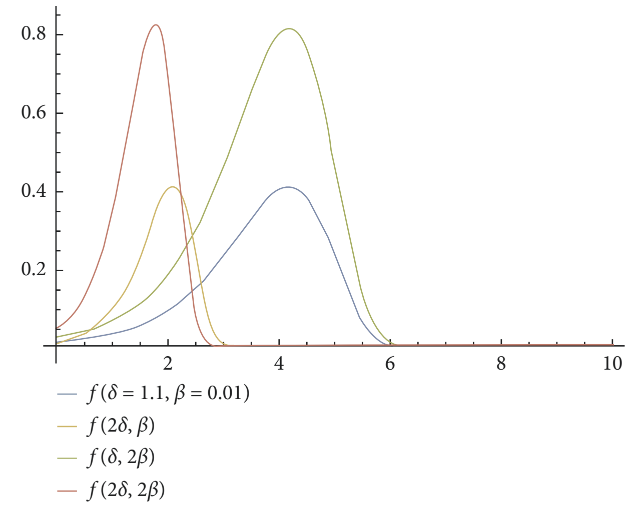

FIGURE 2: PDF of the Gompertz distribution for different values of δ and β.

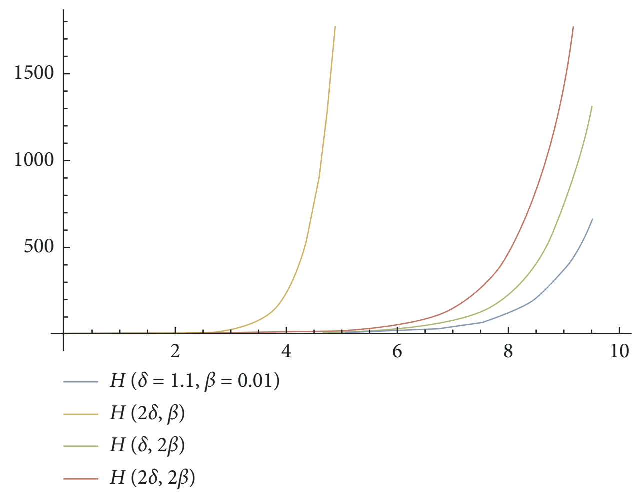

FIGURE 3: Hazard rate function of the Gompertz distribution for different values of δ and β.

The Gompertz distribution is used in a variety of domains, including gerontology, computer science, biology, and marketing science, to describe human mortality and create actuarial tables. Figures 2 and 3 illustrate that the Gompertz distribution has a unimodal PDF and an increasing hazard function, respectively. The features, inferential methods, and a wide range of applications of the Gompertz distribution have been studied by a number of researchers. Willekens [18] looked at the relationships between the Gompertz distribution and other distributions, including the Type-I extreme value and Weibull distributions. Using a progressive Type-II censoring scheme, Chang and Tsai [19] calculated the maximum likelihood (ML) estimates as well as the exact confidence interval for the parameters of the Gompertz distribution. Soliman et al. [20] used progressive first-failure censoring to compute the ML and Bayesian estimates for the unknown parameters, as well as the hazard rate and reliability functions of the Gompertz distribution. For the unknown parameters, they also calculated approximate and exact confidence intervals. Based on a generalized progressively hybrid censoring scheme, Mohie El-Din et al. [21] derived ML and Bayesian estimates for the parameters of the Gompertz distribution, as well as one- and two-sample Bayesian predictions for future observations from the same population.

This is how the rest of the paper is organized. The ML estimators of the unknown parameters, as well as the associated survival and hazard functions, are derived in Section 2. The approximate confidence intervals for δ and β, as well as the associated survival and hazard rate functions, are also provided. In Section 3, the Gibbs sampling process is utilized to generate a sample from the posterior density function, which is then used to compute Bayesian estimates and create credible intervals. A Monte Carlo simulation study is undertaken in Section 4 to assess the performance of the two estimation methods as well as the confidence intervals. In Section 5, some numerical findings utilizing a real data set are presented to demonstrate the inferential processes. Finally, in Section 6, we finish the work with some conclusions.

2. Maximum Likelihood Estimation

The ML estimators of δ, β, $s(t)$, and $h(t)$ are discussed in this section. Also we calculated the approximate confidence intervals δ, β, $s(t)$, and $h(t)$.

Let $X_{1:m:n} < \cdots < X_{J:m:n} < T < X_{J+1:m:n} < \cdots < X_{m:m:n}$ be an adaptive Type-II progressive censored sample from a continuous population with CDF $F(x)$ and PDF $f(x)$ along with a censoring scheme $\underline{R} = (R_1, \ldots, R_j, 0, \ldots, 0, R_m^*)$, where T is prefixed and $R_m^* = n - m - \sum_{i=1}^{J} R_i$. Then, the joint density function of $X_{1:m:n} < \cdots < X_{J:m:n} < T < X_{J+1:m:n} < \cdots < X_{m:m:n}$ is given as (Ng et al. [22])

$$\begin{aligned} & f_{1,2,\ldots,m:m:n}(x_1, x_2, \ldots, x_m) \\ & = d_J \left(\prod_{i=1}^{m} f(x_{i:m:n}) \right) \left(\prod_{i=1}^{J} (1 - F(x_{i:m:n}))^{R_i} \right) \\ & \quad \times (1 - F(x_{m:m:n}))^{R_m^*}, \\ & 0 < x_{1:m:n} < x_{2:m:n} < \cdots < x_{m:m:n} < \infty, \end{aligned} \tag{7}$$

where

$$d_J = \prod_{i=1}^{m} \left[n - i + 1 - \sum_{k=1}^{\max\{i-1, J\}} R_k \right]. \tag{8}$$

From (3), (4), and (7), the likelihood function of δ and β is given by the following equation:

$$\begin{aligned} L(\underline{x}; \delta, \beta) \propto & \left[\prod_{i=1}^{m} \delta\beta e^{-\beta\left(e^{\delta x_i} - 1\right) + \delta x_i} \right] \\ & \times \left[\prod_{i=1}^{J} e^{-\beta R_i \left(e^{\delta x_i} - 1\right)} \right] \left[e^{-\beta R_m^* \left(e^{\delta x_m} - 1\right)} \right], \end{aligned} \tag{9}$$

and then, the log-likelihood function may be written as follows:

$$\ell(\underline{x};\delta,\beta) \propto m\log\delta + m\log\beta + \delta\sum_{i=1}^{m}x_i - \beta\sum_{i=1}^{m}\left(e^{\delta x_i}-1\right) - \beta\sum_{i=1}^{J}R_i\left(e^{\delta x_i}-1\right) - \beta R_m^*\left(e^{\delta x_m}-1\right). \tag{10}$$

The likelihood equations are constructed by differentiating (10) with regard to δ and β and equating to zero:

$$\frac{\partial\ell(\underline{x};\delta,\beta)}{\partial\delta} = \frac{m}{\delta} + \sum_{i=1}^{m}x_i - \beta\sum_{i=1}^{m}x_i e^{\delta x_i} - \beta\sum_{i=1}^{J}R_i x_i e^{\delta x_i} - \beta R_m^* x_m e^{\delta x_m} = 0, \tag{11}$$

$$\frac{\partial\ell(\underline{x};\delta,\beta)}{\partial\beta} = \frac{m}{\beta} - \sum_{i=1}^{m}\left(e^{\delta x_i}-1\right) - \sum_{i=1}^{J}R_i\left(e^{\delta x_i}-1\right) - R_m^*\left(e^{\delta x_m}-1\right) = 0. \tag{12}$$

Based on a progressive Type-II censored sample, Ghitany et al. [23] formulated the necessary and sufficient condition for the existence and uniqueness of the ML estimators of the parameters of the Gompertz distribution. The following theorem introduces the necessary and sufficient condition for the existence and uniqueness of the ML estimators of the parameters of the Gompertz distribution based on an adaptive progressive Type-II censored sample by substituting $k = m$, $R_{J+1} = \ldots = R_{m-1} = 0$, and $R_m = R_m^*$ into Theorem 1, obtained by Ghitany et al. [23].

Theorem 1. *Let $X_{1:m:n} < \cdots < X_{J:m:n} < T < X_{J+1:m:n} < \cdots < X_{m:m:n}$ be an adaptive Type-II progressive censored sample with a censoring scheme $\underline{\mathbf{R}} = (R_1,\ldots,R_j,0,\ldots,0,R_m^*)$. Then, the maximum likelihood estimates $\widehat{\delta}_{ML}$ and $\widehat{\beta}_{ML}$ of the parameters δ and β of the Gompertz distribution exist, and they are unique with*

$$\widehat{\beta}_{ML} = m\left[\sum_{i=1}^{m}\left(e^{\delta x_i}-1\right) + \sum_{i=1}^{J}R_i\left(e^{\delta x_i}-1\right) + R_m^*\left(e^{\delta x_m}-1\right)\right]^{-1}, \tag{13}$$

and $\widehat{\delta}_{ML}$ as the solution of the nonlinear equation,

$$\frac{m}{\delta} + \sum_{i=1}^{m}x_i - \frac{m\left(\sum_{i=1}^{m}x_i e^{\delta x_i} + \sum_{i=1}^{J}R_i x_i e^{\delta x_i} + R_m^* x_m e^{\delta x_m}\right)}{\sum_{i=1}^{m}\left(e^{\delta x_i}-1\right) + \sum_{i=1}^{J}R_i\left(e^{\delta x_i}-1\right) + R_m^*\left(e^{\delta x_m}-1\right)} = 0, \tag{14}$$

if and only if

$$2\sum_{i=1}^{m}x_i\left(\sum_{i=1}^{m}x_i + \sum_{i=1}^{J}R_i x_i + R_m^* x_m\right) > m\left(\sum_{i=1}^{m}x_i^2 + \sum_{i=1}^{J}R_i x_i^2 + R_m^* x_m^2\right). \tag{15}$$

The ML estimators of the reliability function $\widehat{s}(t)$ and hazard rate function $\widehat{h}(t)$ may be produced utilizing the invariance property of the ML estimator by substituting $\widehat{\delta}_{ML}$ and $\widehat{\beta}_{ML}$ in (5) and (6), respectively, and given by the following equation:

$$\widehat{s}(t) = e^{-\widehat{\beta}\left(e^{\widehat{\delta}t}-1\right)}, \quad t > 0, \tag{16}$$

and

$$\widehat{h}(t) = \widehat{\delta}e^{\widehat{\beta}t}, \quad t > 0. \tag{17}$$

2.1. Approximate Confidence Intervals. From the log-likelihood function in (10), we have

$$\frac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\delta^2} = \frac{-m}{\delta^2} - \beta\sum_{i=1}^{m}(x_i)^2 e^{\delta x_i} - \beta\sum_{i=1}^{m}(x_i)^2 e^{\delta x_i} - \beta R_m^*(x_m)^2 e^{\delta x_m}, \tag{18}$$

$$\frac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\delta\,\partial\beta} = \frac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\beta\,\partial\delta} = -\sum_{i=1}^{m}x_i e^{\delta x_i} - \sum_{i=1}^{J}R_i x_i e^{\delta x_i} - R_m^* x_m e^{\delta x_m}, \tag{19}$$

and

$$\frac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\beta^2} = \frac{-m}{\beta^2}. \tag{20}$$

Taking the expectation of minus equations (18), (19), and (20) yields the Fisher information matrix $I(\delta,\beta)$. However, because the expectation is difficult to get, so under some regularity criteria, $(\widehat{\delta},\widehat{\beta})$ is approximately bivariately normal distributed with mean (δ,β) and covariance matrix $I_0^{-1}(\widehat{\delta},\widehat{\beta})$: $(\widehat{\delta},\widehat{\beta}) \sim N((\delta,\beta), I_0^{-1}(\widehat{\delta},\widehat{\beta}))$, where $I_0^{-1}(\widehat{\delta},\widehat{\beta})$ is the inverse of the observed Fisher information matrix given by the following equation:

$$I_0^{-1}(\widehat{\delta},\widehat{\beta}) = \begin{bmatrix} -\dfrac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\delta^2} & -\dfrac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\delta\,\partial\beta} \\ -\dfrac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\delta\,\partial\beta} & -\dfrac{\partial^2\ell(\underline{x};\delta,\beta)}{\partial\beta^2} \end{bmatrix}^{-1}_{(\widehat{\delta},\widehat{\beta})} = \begin{bmatrix} var(\widehat{\delta}) & cov(\widehat{\delta},\widehat{\beta}) \\ cov(\widehat{\delta},\widehat{\beta}) & var(\widehat{\beta}) \end{bmatrix}. \tag{21}$$

Thus, the $100(1-\gamma)\%$ approximate confidence intervals for δ and β are, respectively,

$$\widehat{\delta} \pm z_{\gamma/2}\sqrt{var(\widehat{\delta})} \text{ and } \widehat{\beta} \pm z_{\gamma/2}\sqrt{var(\widehat{\beta})}. \tag{22}$$

where $var(\widehat{\delta})$ and $var(\widehat{\beta})$ are the first and second elements on the main diagonal of the covariance matrix $I_0^{-1}(\widehat{\delta},\widehat{\beta})$ and $z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $r/2$.

2.2. Approximate Confidence Intervals Using the Delta Method. Greene [24] proposed the delta technique as a general way for computing approximate confidence intervals for ML estimator functions. See Agresti [25] for a description of the delta approach, which takes a function that is too complex to compute the variance analytically, generates a linear approximation of it, and then computes the variance of the simpler linear function that can be utilized for large sample inference. The approximate confidence intervals for $s(t)$ and $h(t)$ are calculated using the delta approach in this subsection. Let

$$V_1 = \left[\frac{\partial s(t)}{\partial \delta} \quad \frac{\partial s(t)}{\partial \beta}\right] \text{and } V_2 = \left[\frac{\partial h(t)}{\partial \delta} \quad \frac{\partial h(t)}{\partial \beta}\right], \tag{23}$$

where

$$\begin{aligned} \frac{\partial s(t)}{\partial \delta} &= -\beta t e^{\beta\left(1-e^{\delta t}\right)+\delta t}, \quad \frac{\partial s(t)}{\partial \beta} = e^{\beta\left(1-e^{\delta t}\right)}, \\ \frac{\partial h(t)}{\partial \delta} &= \beta\delta t e^{\delta t}, \quad \frac{\partial h(t)}{\partial \beta} = \delta t e^{\beta t}. \end{aligned} \tag{24}$$

Then, the approximate estimates of $var(\widehat{s}(t))$ and $var(\widehat{h}(t))$ are given, respectively, by the following equation:

$$var(\widehat{s}(t)) \simeq \left[V_1^T I_0^{-1} V_1\right]_{(\widehat{\delta},\widehat{\beta})} \text{and } var(\widehat{h}(t)) \simeq \left[V_2^T I_0^{-1} V_2\right]_{(\widehat{\delta},\widehat{\beta})}, \tag{25}$$

where V_i^T is the transpose of the vector V_i, $i = 1, 2$. So, the $100(1-\gamma)\%$ approximate confidence interval of $s(t)$ and $h(t)$ are

$$\left(\widehat{s}(t) - z_{\gamma/2}\sqrt{var(\widehat{s}(t))}, \widehat{s}(t) + z_{\gamma/2}\sqrt{var(\widehat{s}(t))}\right) \tag{26}$$

and

$$\left(\widehat{h}(t) - z_{\gamma/2}\sqrt{var(\widehat{h}(t))}, \widehat{h}(t) + z_{\gamma/2}\sqrt{var(\widehat{h}(t))}\right). \tag{27}$$

3. Bayesian Estimation

The Bayesian estimation of $\delta, \beta, s(t)$, and $h(t)$ is developed in this section. As Arnold and Press [26] have highlighted, there is no mechanism for selecting adequate priors for Bayesian estimation. Under the assumption that both parameters δ and β are unknown and have independent gamma priors, Bayesian estimation is utilized here, with PDFs provided by the following equation:

$$\phi_1(\delta \mid a, b) = \frac{b^a}{\Gamma(a)}\delta^{a-1}e^{-b\delta} \text{ and } \phi_2(\beta|c, d) = \frac{d^c}{\Gamma(c)}\beta^{c-1}e^{-d\beta}, \tag{28}$$

respectively, where $a, b, c, d > 0$ and $\Gamma(.)$ denotes the complete gamma function. It is worth mentioning that the gamma prior's class is flexible because these distributions may be used to model a wide range of prior information. Furthermore, by setting hyperparameters to zero, the improper priors of δ and β may be derived as special cases of independent gamma priors. Several researchers have employed gamma priors, such as Maiti and Kayal [27] and Dey et al. [28]. The joint prior density of δ and β is given as follows:

$$\phi(\delta, \beta) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)}\delta^{a-1}\beta^{c-1}e^{-b\delta-d\beta}. \tag{29}$$

The joint posterior density of δ and β is generated by using the likelihood function in (9) and the joint prior in (29), which is given by the following equation:

$$\begin{aligned} \phi^*((\delta,\beta)|\underline{x}) &= \frac{L(\underline{x};\delta,\beta)\phi(\delta,\beta)}{\int_0^\infty\int_0^\infty L(\underline{x};\delta,\beta)\phi(\delta,\beta)d\delta\, d\beta} \\ &\propto \delta^{m+a-1}\beta^{m+c-1}e^{-b\delta-d\beta} \\ &\times\left[m\prod_{i=1} e^{-\beta\left(e^{\delta x_i}-1\right)+\delta x_i}\right]\left[J\prod_{i=1} e^{-\beta R_i\left(e^{\delta x_i}-1\right)}\right] \\ &\left[e^{-\beta R_m^*\left(e^{\delta x_m}-1\right)}\right]. \end{aligned} \tag{30}$$

Therefore, the Bayesian estimator of some function of δ and β say $g(\delta, \beta)$, with respect to the squared error loss function, will be the posterior expectation of $g(\delta, \beta)$, i.e.,

$$\begin{aligned} \widehat{g}(\beta,\delta) &= E_{\delta,\beta|\underline{x}}[g(\delta,\beta)] \\ &= \frac{\int_0^\infty\int_0^\infty g(\delta,\beta)\times L(\underline{x};\delta,\beta)\times\phi(\delta,\beta)d\delta\, d\beta}{\int_0^\infty\int_0^\infty L(\underline{x};\delta,\beta)\times\phi(\delta,\beta)d\delta\, d\beta}. \end{aligned} \tag{31}$$

Unfortunately, in most cases, the integrals in (31) cannot be derived in explicit form. Even if this integration can be done explicitly, the corresponding credible interval may be impossible to create, and numerical approaches may fail. In this case, we propose using the Markov chain Monte Carlo (MCMC) method to approximate (31).

3.1. The Metropolis–Hastings Algorithm within Gibbs Sampling. We use MCMC technique in this subsection, which approximates the generation of random variables from a posterior distribution $\phi(\delta, \beta)$, then computes the Bayesian estimation of δ, β, $s(t)$, and $h(t)$, as well as the corresponding credible intervals. There are many different MCMC schemes to choose from, and it might be difficult to decide which one to use. Within Gibbs sampling, we primarily focus on a special form of the MCMC known as Metropolis-Hastings, which was developed by Metropolis et al. [29] and then extended by Hastings [30].

From (30), the posterior conditional density function of δ given β can be obtained as follows:

$$\begin{aligned} \phi_\delta^*(\delta|\beta,\underline{x}) &\propto \delta^{m+a-1}e^{-b\delta}\left[\prod_{i=1}^m e^{-\beta\left(e^{\delta x_i}-1\right)+\delta x_i}\right] \\ &\left[\prod_{i=1}^J e^{-\beta R_i\left(e^{\delta x_i}-1\right)}\right]\left[e^{-\beta R_m^* e^{\delta x_m}}\right]. \end{aligned} \tag{32}$$

Similarly, the posterior conditional density function of β given δ can be obtained as follows:

$$\phi_\beta^*(\beta|\delta,\underline{x}) \propto \beta^{m+c-1} e^{-d\beta}\left[\prod_{i=1}^{m} e^{-\beta\left(e^{\delta x_i}-1\right)}\right]\left[\prod_{i=1}^{J} e^{-\beta R_i\left(e^{\delta x_i}-1\right)}\right]\left[e^{-\beta R_m^*\left(e^{\delta x_m}-1\right)}\right]. \quad (33)$$

Both δ and β posterior density functions in equations (32) and (33) cannot be reduced analytically to well-known distributions. As a result, standard methods cannot be used to sample directly, although the plots indicate that they are similar to the normal distribution. To generate random numbers from this distribution, we use Metropolis-Hastings sampling with a normal proposal distribution. The following approach is proposed for generating δ and β from posterior density functions and obtaining Bayesian estimates of δ and β, as well as the corresponding credible intervals.

(1) Begin with $(\delta_{(0)}, \beta_{(0)})$ as the initial value.
(2) Put $j = 1$.
(3) Using $N(\delta_{(j-1)}, \mathrm{Var}(\widehat{\delta}))$ as the proposal distribution, generate $\delta_{(j)}$ from $\phi_\delta^*(\delta|\beta,\underline{x})$.
(4) Using $N(\beta_{(j-1)}, \mathrm{Var}(\widehat{\beta}))$ as the proposal distribution, generate $\beta_{(i)}$ from $\phi_\beta^*(\beta|\delta,\underline{x})$.
(5) Calculate $\delta_{(j)}$ and $\beta_{(i)}$.
(6) Calculate $s^{(j)}(t)$ and $h^{(j)}(t)$ and substitute $\delta_{(j)}$ and $\beta_{(i)}$ into (5) and (6), respectively.
(7) Put $j = j + 1$
(8) Steps $3-7$ should be repeated N times.
(9) Using the squared error loss function, the Bayesian estimates of δ and β are calculated as follows:

$$\delta^* = \frac{\sum_{j=M+1}^{N}\delta_{(j)}}{N-M},\quad \beta^* = \frac{\sum_{j=M+1}^{N}\beta_{(j)}}{N-M}, \quad (34)$$

where M is burn-in.

(10) Using the squared error loss function, the approximate Bayesian estimates of $s(t)$ and $h(t)$ are obtained as follows:

$$s^*(t) = \frac{\sum_{j=M+1}^{N} s_{(j)}(t)}{N-M},\quad h^*(t) = \frac{\sum_{j=M+1}^{N} h_{(j)}(t)}{N-M}. \quad (35)$$

(11) Sort $\delta_{(M+1)}, \delta_{(M+2)}, \ldots, \delta_{(N)}$ and $\beta_{(M+1)}, \beta_{(M+2)}, \ldots, \beta_{(N)}$ as $\delta^{(1)} < \delta^{(2)} < \ldots < \delta^{(N-M)}$ and $\beta^{(1)} < \beta^{(2)} < \ldots < \beta^{(N-M)}$, respectively. The $100(1-\gamma)\%$ symmetric credible intervals of δ and β can be then calculated as follows:

$$\left(\delta^{(k_1)}, \delta^{(k_2)}\right) an\ d\left(\beta^{(k_1)}, \beta^{(k_2)}\right), \quad (36)$$

where $k_1 = (N-M)\gamma/2$ and $k_2 = (N-M)(1-\gamma/2)$.

(12) Sort $s_{(M+1)}(t), s_{(M+2)}(t), \ldots, s_{(N)}(t)$ and $h_{(M+1)}(t), h_{(M+2)}(t), \ldots, h_{(N)}(t)$ as $s^{(1)}(t) < s^{(2)}(t) < \ldots < s^{(N-M)}(t)$ and $h^{(1)}(t) < h^{(2)}(t) < \ldots < h^{(N-M)}(t)$, respectively. The $100(1-\gamma)\%$ symmetric credible intervals of $s(t)$ and $h(t)$ can be then calculated as follows:

$$\left(s^{(k_1)}(t), s^{(k_2)}(t)\right),\ \left(h^{(k_1)}(t), h^{(k_2)}(t)\right), \quad (37)$$

4. Bootstrap Confidence Interval

When the effective sample size m is large, the approximate and Bayesian confidence intervals are adequate. When m is small, the bootstrap resampling approach is preferable for constructing the confidence interval. The procedure for obtaining the confidence interval for δ, β, $s(t)$, and $h(t)$ using the bootstrap resampling approach is described as follows:

Step 1: based on n, m, R, and T, compute $\widehat{\delta}_{ML}$ and $\widehat{\beta}_{ML}$.
Step 2: use $\widehat{\delta}_{ML}$ and $\widehat{\beta}_{ML}$ and the same values of n, m, R, and T to generate a bootstrap resample.
Step 3: calculate the bootstrap estimates $\widehat{\delta}^B$, $\widehat{\beta}^B$, $\widehat{s}^B(t)$, and $\widehat{h}^B(t)$ based the generated bootstrap sample.
Step 4: repeat steps 2-3 up to N times to get $\widehat{\delta}_1^B, \ldots, \widehat{\delta}_N^B$ and $\widehat{\beta}_1^B, \ldots, \widehat{\beta}_N^B$ as well as $\widehat{s}_1^B(t), \ldots, \widehat{s}_N^B(t)$ and $\widehat{h}_1^B(t), \ldots, \widehat{h}_N^B(t)$.
Step 5: rearrange these bootstrap estimates obtained in step 4 in ascending order as $\widehat{\delta}_{[1]}^B, \ldots, \widehat{\delta}_{[N]}^B$; $\widehat{\beta}_{[1]}^B, \ldots, \widehat{\beta}_{[N]}^B$; $\widehat{s}_{[1]}^B(t), \ldots, \widehat{s}_{[N]}^B(t)$; $\widehat{h}_{[1]}^B(t), \ldots, \widehat{h}_{[N]}^B(t)$.
Step 6: The $100(1-\gamma)\%$ bootstrap confidence intervals for δ, β, and $h(t)$ are, respectively, given by the following equations:

$$\left(\widehat{\delta}_{[N\gamma/2]}^B, \widehat{\delta}_{[N(1-\gamma/2)]}^B\right),\ \left(\widehat{\beta}_{N\gamma/2}^B, \widehat{\beta}_{[N(1-\gamma/2)]}^B\right), \quad (38)$$

and

$$\left(\widehat{s}_{[N\gamma/2]}^B(t), \widehat{s}_{[N(1-\gamma/2)]}^B(t)\right),\ \left(\widehat{h}_{[N\gamma/2]}^B(t), \widehat{h}_{[N(1-\gamma/2)]}^B(t)\right). \quad (39)$$

5. Monte Carlo Simulation Study

In this section, Monte Carlo simulation is used to compare the different estimation methods presented in the previous sections. We employed a variety of sample sizes and different sets of R_i's, as shown in Table 1. We next simulated adaptive Type-II progressively censored samples from the Gompertz distribution with parameters $\delta = 0.8$ and $\beta = 1.1$ when $T = 0.5$ using the algorithm reported by Ng et al. [23]. We computed the ML and Bayesian estimates for the two unknown parameters as well as the corresponding reliability and hazard rate function (when $t = 0.3$) in each case. We also

Table 1: The different censoring schemes R with different choices of n and m.

n	m	CS	R
40	30	1	$\{2(0), 1, 5(0), 1, 6(0), 1, 1, 3(0), 1, 0, 2, 4(0), 2, 0, 1\}$
50	30	2	$\{0, 1, 2(0)1, 2(0), 2, 0, 2, 0, 1, 1, 2(0), 1, 1, 0, 2, 0, 1, 0, 2, 0, 2, 3(0), 3, 0\}$
50	40	3	$\{0, 1, 2(0), 1, 6(0), 1, 1, 2(0), 1, 1, 3(0), 1, 8(0), 1, 1, 8(0), 1\}$
60	30	4	$\{3(0), 5, 0, 8, 2(0), 2, 2(0), 7, 3(0), 1, 3, 3(0), 1, 2(0), 3(0), 1, 2(0), 2\}$
60	40	5	$\{2(0), 2, 0, 4, 4(0), 3, 3(0), 2, 0, 3, 3(0), 1, 3(0), 2, 2(0), 1, 2(0), 1, 9(0), 1\}$
60	50	6	$\{8(0), 1, 3(0), 1, 2(0), 1, 4(0), 1, 2(0), 1, 4(0), 1, 1, 4(0), 1, 4(0), 1, 8(0), 1\}$
70	50	7	$\{2, 3(0), 2, 3(0), 2, 6(0), 1, 2, 3(0), 1, 3(0), 2, 4(0), 2, 1, 4(0), 2, 4(0), 2, 8(0), 1\}$

$r(0)$ means that 0 repeated r times.

Table 2: The average, bias, and MSE of the ML and Bayesian estimates of δ and β.

CS		*ML*		*I − Bayesian*		*NI − Bayesian*	
		δ	β	δ	β	δ	β
1	*Average*	1.2939	1.1742	0.9738	1.0455	1.1702	1.7719
	Bias	0.1939	0.3742	0.1262	0.2455	0.0702	0.9719
	MSE	0.4302	2.8172	0.230	0.697	0.414	2.3288
2	*Average*	1.3574	1.213	0.9852	1.0479	1.2194	1.8378
	Bias	0.2574	0.413	0.1148	0.2479	0.1194	1.0378
	MSE	0.573	3.1107	0.205	0.728	0.537	2.5401
3	*Average*	1.226	0.9489	0.987	1.0417	1.0988	1.6165
	Bias	0.126	0.1489	0.113	0.2417	0.0012	0.8165
	MSE	0.2219	0.9991	0.206	0.711	0.24	1.8356
4	*Average*	1.2948	1.103	0.9832	1.0421	1.1594	1.7499
	Bias	0.1948	0.303	0.1168	0.2421	0.0594	0.9499
	MSE	0.413	2.1398	0.213	0.708	0.4098	2.2479
5	*Average*	1.229	1.0282	0.9876	1.0412	1.1054	1.6715
	Bias	0.129	0.2282	0.1124	0.2412	0.0054	0.8715
	MSE	0.263	1.3206	0.212	0.727	0.2836	2.0354
6	*Average*	1.1954	0.9553	0.9889	1.0396	1.0776	1.5937
	Bias	0.0954	0.1553	0.1111	0.2396	0.0224	0.7937
	MSE	0.1972	0.7391	0.213	0.725	0.2229	1.7763
7	*Average*	1.2083	0.9901	0.9869	1.0347	1.0816	1.6389
	Bias	0.1083	0.1901	0.1131	0.2347	0.0184	0.8389
	MSE	0.2258	1.1242	0.213	0.703	0.2516	1.94

Table 3: The average, bias, and MSE of the ML and Bayesian estimates of $s(0.3)$ and $h(0.3)$.

CS		*ML*		*I − Bayesian*		*NI − Bayesian*	
		$s(0.3)$	$h(0.3)$	$s(0.3)$	$h(0.3)$	$s(0.3)$	$h(0.3)$
1	*Average*	1.2939	1.1742	0.7022	1.3637	0.6126	1.9465
	Bias	0.1939	0.3742	0.0292	0.1396	0.1188	0.7224
	MSE	0.4302	2.8172	0.0415	0.2228	0.1456	0.8599
2	*Average*	0.7364	1.2362	0.6981	1.3903	0.5976	2.0674
	Bias	0.005	0.0121	0.0333	0.1662	0.1338	0.8433
	MSE	0.0566	0.2474	0.0483	0.2652	0.1644	1.0186
3	*Average*	0.7351	1.2246	0.6998	1.379	0.6365	1.7755
	Bias	0.0037	0.0005	0.0316	0.1549	0.0949	0.5514
	MSE	0.0482	0.2141	0.0425	0.2282	0.1241	0.6905
4	*Average*	0.7005	1.3757	0.7005	1.3757	0.6177	1.9097
	Bias	0.0309	0.1516	0.0309	0.1516	0.1137	0.6856
	MSE	0.0445	0.2415	0.0445	0.2415	0.1412	0.8237
5	*Average*	0.7332	1.2349	0.7	1.3782	0.633	1.7974
	Bias	0.0018	0.0108	0.0314	0.1541	0.0984	0.5733
	MSE	0.0484	0.2135	0.0428	0.2302	0.1267	0.7063
6	*Average*	0.7314	1.2398	0.7002	1.3766	0.6443	1.7211
	Bias	0.0016	0.0157	0.0312	0.1525	0.0871	0.497
	MSE	0.045	0.2013	0.0415	0.2218	0.1158	0.6306
7	*Average*	0.7326	1.235	0.7019	1.3673	0.6401	1.7471
	Bias	0.0012	0.0109	0.0295	0.1432	0.0913	0.523
	MSE	0.0462	0.2058	0.041	0.2196	0.1196	0.6562

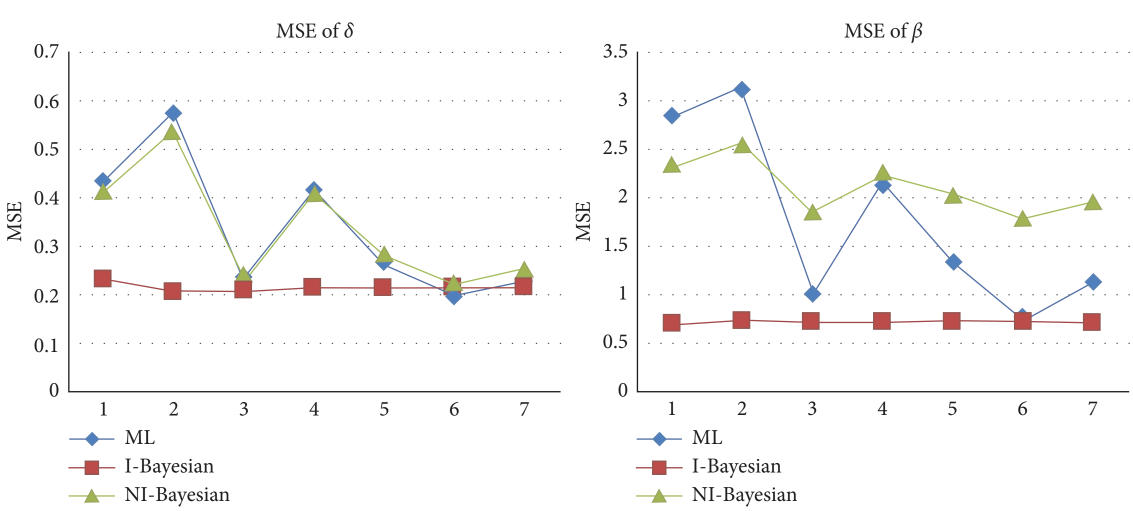

Figure 4: The mean square error of δ and β.

Table 4: The length and Cov of the approximate, Bayesian, and bootstrap estimation intervals of δ and β.

CS		*ML*		*I − Bayesian*		*NI − Bayesian*		*Bootstrap*	
		δ	β	δ	β	δ	β	δ	β
1	*Length*	2.5082	6.5906	0.7013	1.0275	1.9233	5.2458	2.3698	2.6144
	Cov	0.948	0.807	0.991	0.993	0.895	0.896	0.998	0.966
2	*Length*	2.7847	7.4145	0.7319	1.0214	2.0863	5.4309	1.9121	2.6017
	Cov	0.926	0.799	0.989	0.982	0.877	0.87	1.	0.973
3	*Length*	1.8456	3.3299	0.6555	1.0036	1.5283	4.6595	1.4977	2.6376
	Cov	0.966	0.846	0.985	0.976	0.907	0.908	1.	0.986
4	*Length*	2.456	5.7975	0.7132	1.0114	1.8377	5.1261	1.7292	2.995
	Cov	0.945	0.81	0.991	0.984	0.886	0.88	1.	0.977
5	*Length*	1.9902	4.0991	0.6687	0.999	1.5641	4.7739	2.085	3.2472
	Cov	0.938	0.856	0.972	0.963	0.86	0.863	1.	0.975
6	*Length*	1.7135	2.9553	0.6407	0.9868	1.404	4.4039	2.4193	2.8444
	Cov	0.954	0.869	0.976	0.967	0.872	0.876	0.996	0.971
7	*Length*	1.8267	3.4283	0.6569	0.9851	1.4752	4.5466	1.541	1.9798
	Cov	0.956	0.871	0.981	0.967	0.871	0.856	1.	0.985

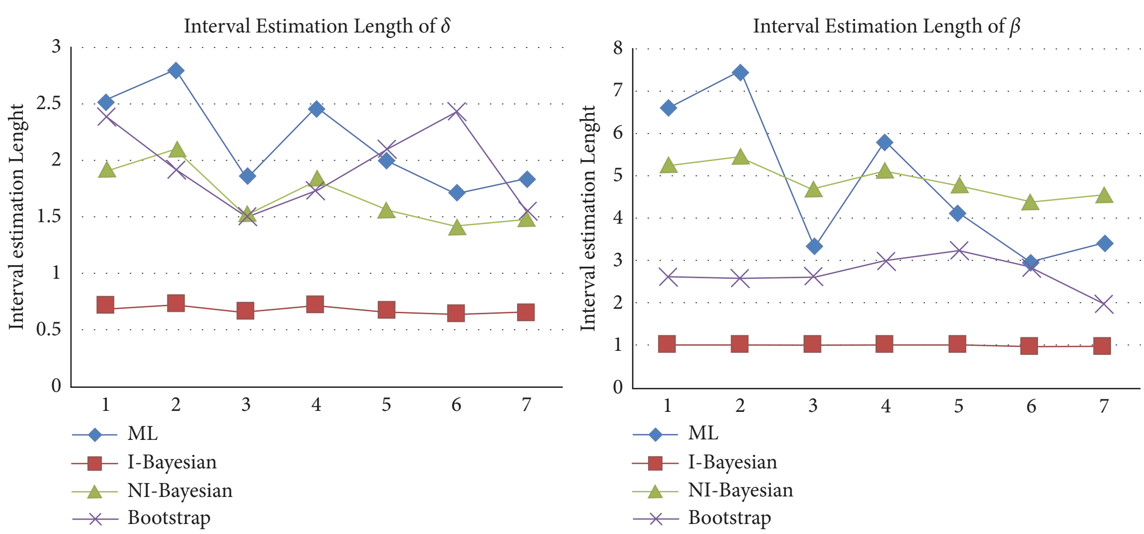

Figure 5: The interval estimation length of δ and β.

Table 5: The length and Cov of the approximate, Bayesian, and bootstrap estimation intervals of $s(0.3)$ and $h(0.3)$.

CS		*ML*		*I − Bayesian*		*NI − Bayesian*		*Bootstrap*	
		$s(0.3)$	$h(0.3)$	$s(0.3)$	$h(0.3)$	$s(0.3)$	$h(0.3)$	$s(0.3)$	$h(0.3)$
1	*Length*	2.5082	6.5906	0.0706	0.7841	0.1286	1.023	0.219	1.5105
	Cov	0.948	0.887	0.875	0.952	0.873	0.914	0.993	0.990
2	*Length*	0.218	0.956	0.1112	0.8895	0.14	1.1271	0.1969	1.3199
	Cov	0.925	0.947	0.874	0.959	0.877	0.882	0.987	0.979
3	*Length*	0.205	0.9109	0.0199	0.5995	0.0958	0.8068	0.0992	1.8401
	Cov	0.957	0.961	0.935	0.924	0.947	0.883	0.990	0.976
4	*Length*	0.0891	0.8568	0.0891	0.8568	0.1339	1.0571	0.1078	1.1518
	Cov	0.964	0.936	0.947	0.936	0.942	0.924	0.982	0.977
5	*Length*	0.2042	0.8995	0.0485	0.7047	0.1045	0.8649	0.0992	2.0694
	Cov	0.950	0.959	0.939	0.921	0.917	0.925	0.980	0.964
6	*Length*	0.1882	0.8295	0.0266	0.5567	0.0846	0.7308	0.1693	2.7344
	Cov	0.952	0.956	0.933	0.955	0.936	0.918	0.971	0.952
7	*Length*	0.1823	0.7959	0.019	0.5645	0.0936	0.7765	0.080	1.9612
	Cov	0.936	0.949	0.926	0.965	0.885	0.929	0.981	0.968

Table 6: Real data set: 86 times between failures.

4.79	7.45	10.22	15.76	26.10	28.59	35.52	41.49	42.66	44.36
45.53	58.27	62.96	74.70	81.63	100.71	102.06	104.83	110.79	118.36
122.73	145.03	149.40	152.80	156.85	162.20	164.97	168.60	173.82	179.95
182.72	195.72	203.93	206.06	222.26	238.27	241.25	249.99	256.17	282.57
282.62	284.11	294.45	318.86	323.46	329.11	340.30	344.67	353.94	398.56
405.70	407.51	422.36	429.93	461.47	482.62	491.46	511.83	526.64	532.23
537.13	543.06	560.75	561.60	589.96	592.09	610.75	615.65	630.52	673.74
687.92	698.15	753.05	768.25	801.06	828.22	849.97	885.02	892.27	911.90
951.69	962.59	965.04	976.98	986.92	1025.94				

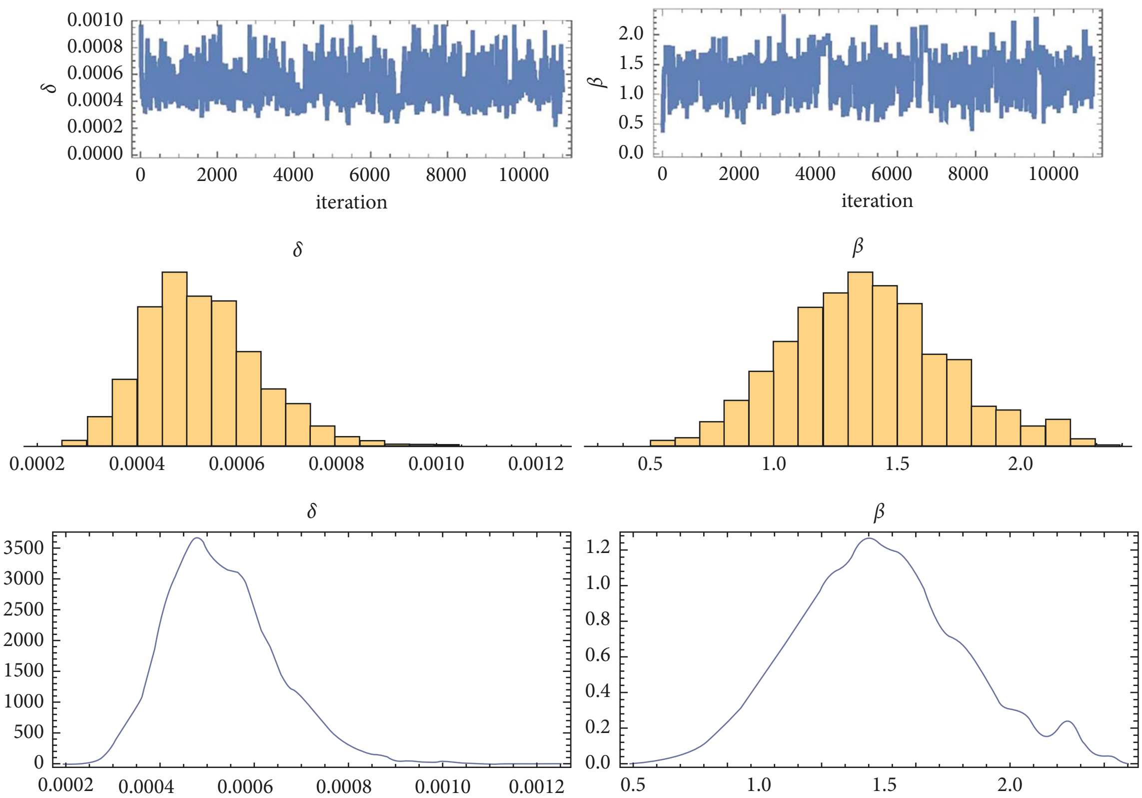

Figure 6: The MCMC trace plots and the marginal posterior density with histograms of δ and β.

calculated the 95% approximate and Bayesian and Bootstrap estimation intervals for δ, β, $s(0.3)$, and $h(0.3)$.

We used a noninformative gamma prior (NI-Bayesian: $a = b = c = d = 0$) and an informative gamma prior (I-Bayesian: $a = 10, b = 8, c = 9, d = 7$) to compute Bayesian estimates and credible intervals based on 11000 MCMC samples, discarding the first 1000 values as burn-in. We repeated the process 1000 times and computed the average, bias, and mean squared error (MSE) of the ML and Bayesian estimates for δ and β, as shown in Table 2 and Figure 4, and for $s(0.3)$ and $h(0.3)$, as shown in Table 3. Also, we calculated the length and the coverage probability (Cov) for the 95% approximate and Bayesian and Bootstrap estimation intervals for δ and β, as shown in Table 4 and Figure 5, and for $s(0.3)$ and $h(0.3)$, as shown in Table 5.

6. Illustrative Example

In this section, some numerical results based on real data sets are presented to demonstrate inferential procedures.

Real data set: Lyu [31] has represented 86 times between failures, as shown in Table 6.

Srivastava [32] tested the model's validity using the data supplied by Lyu [31]. He used the Kolmogorov–Smirnov test to plot a graph of the empirical distribution function and fitted distribution function. He also discussed Q-Q plots for model validation. He has stated that the Gompertz model may well fit the data presented above. Now, we simulate an adaptive Type-II progressively censored sample from this data with $n = 86$, $m = 40$, $T = 300$, and $R = \{1, 1, 0, 1, 2, 1, 0, 2, 1, 3, 1, 1, 0, 1, 2, 1, 1, 2, 1, 3, 1, 1, 0, 1, 2, 1, 1, 2, 1, 3, 1, 1, 1, 0, 1, 0, 1, 1, 2, 1\}$. This leads to $j = 13$. Thus, the resulting adaptive Type-II progressive censored sample is shown in Table 7.

We assume the noninformative prior, where $(a, b, c, d) = (0, 0, 0, 0)$, because we have no prior information about the unknown parameters. In Table 8, the ML and noninformative prior Bayesian estimates for the parameters δ, β, $s(0.3)$, and $h(0.3)$, as well as the lower and upper

Table 7: The resulting adaptive Type-II progressive censored sample.

4.79	7.45	10.22	15.76	26.10	28.59	35.52	41.49	42.66	45.53
58.27	62.96	81.63	100.71	104.83	110.79	118.36	122.73	145.03	149.40
152.80	156.85	162.20	164.97	182.72	195.72	203.93	238.27	249.99	282.57
340.30	344.67	407.51	429.93	673.74					

Table 8: The ML, Bayesian estimate, and the lower and upper bounds of β, δ, $s(0.3)$, and $h(0.3)$.

Methods of estimation	Parameter	Point estimate	Lower bound	Upper bound	Interval length
Maximum likelihood	δ	0.0006	0.0003	0.001	0.0007
	β	1.0689	0.3162	1.8216	1.5054
	$s(0.3)$	0.9998	0.9997	0.9999	0.0002
	$h(0.3)$	0.0007	0.0005	0.0009	0.0004
$MCMCnoninformativeprior$	δ	0.0005	0.0003	0.0008	0.0005
	β	1.4296	0.7727	2.1932	1.4205
	$s(0.3)$	0.9998	0.9997	0.9999	0.0002
	$h(0.3)$	0.0007	0.0005	0.001	0.0005

bounds of 95% confidence intervals for them, using the asymptotic distributions and noninformative prior Bayesian methods are presented. The MCMC trace plots and the marginal posterior density with histograms of δ and β are shown in Figure 6.

7. Concluding Remarks

In this work, we looked at several estimating approaches using an adaptive Type-II progressive censored sample from the Gompertz distribution. For the unknown parameters, we calculated the ML and Bayesian estimators, as well as the corresponding reliability and hazard rate functions. For the unknown parameters and their corresponding reliability and hazard rate functions, we constructed approximate, Bayesian (using MCMC approach), and Bootstrap confidence intervals. Furthermore, we conducted simulation studies using a variety of sample sizes and censoring schemes to compare and evaluate the suggested estimate methods' effectiveness. Finally, we used a numerical example based on real data to show the computations of the methods proposed in this study.

From Tables 2–5 and 8 and Figures 4–6, we can notice that

(1) The results obtained using the Bayesian method are better than those obtained using the ML method.

(2) The results of the Bayesian method with informative prior are better than those of the Bayesian technique with noninformative prior.

(3) The Bayesian method with noninformative prior produces results that are quite close to the results using the ML method. Because the Bayesian method is computationally more expensive, it is always preferable to use the ML method rather than the Bayesian method with noninformative prior.

(4) The average length of all confidence intervals and the mean squared error of all estimations decrease as the number of data increases.

(5) The marginal posterior density plots and histograms are approximately symmetric around their means, and the MCMC trace plots of δ and β converge. As a result, the unknown parameters may be estimated using the MCMC generated sample.

Data Availability

All data generated or analyzed during this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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