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Research Article

Some Integrals Involving Meijer's and Hypergeometric Functions Using the Coherent State Technique

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In this paper, we have examined a particular case of coherent states, defined so that their structure constants depend only on the products of the energy eigenvalues of the examined systems. In this manner, we have built all three kinds of coherent states (Barut–Girardello, Klauder–Perelomov, and Gazeau–Klauder). From the equation of the unitary operator decomposition, we have highlighted and used a so-called fundamental integral, to obtain some new integrals involving Meijer's and hypergeometric functions. All calculations are made using the properties of the diagonal operator ordering technique. This is implicitly proved that the coherent state technique can be useful not only in different branches of physics (quantum mechanics, quantum optics, quantum information theory, and so on) but also in the deduction of new integrals involving generalized Meijer's and hypergeometric functions. This approach can be considered as suitable "feedback" from physics to mathematics.

1. Introduction

The special functions have been introduced and investigated extensively due mainly to their applications in diverse areas in mathematics, applied mathematics, physics, and engineering. Most of the special functions, which have various physical and technical applications, can be expressed in terms of generalized hypergeometric functions. Srivastava and Karlsson [1] introduced and analyzed the extension of the generalized hypergeometric functions utilizing the extended Pochhammer symbol. Other new extensions of Pochhammer symbol can be found in [2, 3]. Several interesting integral representations of the Euler type and Laplace type for some Gauss hypergeometric functions of three variables are given in [4]. Another extension of the generalized hypergeometric functions is defined in [5] where some of their properties are presented such as integral representations, derivative formulas, recurrence relations, and others [5, 6].

The hypergeometric functions are used in a wide range of applications, such as integral representations, generating functions, recurrence relations, finite and infinite sums, analytic continuation, asymptotic behaviour, and in perturbation theory or in quantum theory [5]. Today, it is a fact that there exists a strong relation between hypergeometric functions and some special integrals which appear in different branches of physics (quantum mechanics, quantum optics, quantum information theory, and so on).

In the last decades, the coherent state (CS) formalism, introduced almost a century ago by Schrödinger [7], as quasi-classical state, has proven to be a useful approach for several applications in quantum mechanics, quantum optics, and quantum information theory and practice. Over time, these CSs, originally formulated only for the one-dimensional quantum harmonic oscillator (HO-1D), have been extended to other quantum systems. Thus, CSs were defined in the following manner: Barut–Girardello CSs (BG-CSs) [8], Klauder–Perelomov CSs (KP-CSs) [9], and Gazeau–Klauder CSs (GK-CSs) [10].

For the HO-1D, all three manners of defining the CSs lead to the same results, i.e., to the same expression of CSs. For this reason, these CSs are also named *canonical* or *linear* coherent states. The situation is different if we consider other quantum systems (named nonlinear): the three manners of building CSs lead to different expressions. Consequently, all these CSs are named *nonlinear*.

Appl and Schiller [11] observed that all CSs (linear and nonlinear) are in fact particular cases of generalized coherent states, defined so that their normalizing function is a generalized hypergeometric function (G-HGF) $pF_q(\{a_i\}_1^p; \{b_j\}_1^q; |z|^2)$, where p and q are positive numbers. The sets $\{a_1, a_2, \ldots, a_p,\} \equiv \{a_i\}_1^p$ and $\{b_1, b_2, \ldots, b_q,\} \equiv \{b_j\}_1^q$ are real (or complex) parameters.

Decades after their introduction, CSs seemed to be neglected in terms of scientific interest. However, in the second half of the twentieth century, there was a tendency to "link" the definition of CSs to the theory of quantum groups associated with the system under examination [8, 9, 12–15]. Unfortunately, there are just a few quantum systems for which we are able to construct the associated group generators. The impact situation can be easily avoided if we consider a dimensionless Hamiltonian $\widehat{H}$, having only the discrete energy spectrum with eigenvalues $e(n)$:

$$\widehat{H}|n\rangle = e(n)|n\rangle. \tag{1}$$

We suppose that these energy eigenvalues are ascendant and nondegenerate: $e(0) = 1\langle e(1)\langle e(2)\langle \ldots \langle e(n)$.

On the other hand, let us choose a pair of two Hermitian operators, the creation $\widehat{E}_+$ and the annihilation $\widehat{E}_-$, whose actions on the Fock vectors $|n\rangle$ are

$$\begin{aligned} \widehat{E}_-|n\rangle &= \sqrt{e(n)}|n-1\rangle, \\ \widehat{E}_+|n\rangle &= \sqrt{e(n+1)}|n+1\rangle, \\ \widehat{E}_+\widehat{E}_-|n\rangle &= e(n)|n\rangle. \end{aligned} \tag{2}$$

In other words, we have $\widehat{H} = \widehat{E}_+\widehat{E}_-$, and we point out here that the above pair of operators *is not mandatory as the group generators* of the quantum group attached to the examined quantum systems *are not supersymmetric partners*.

On the other hand, Hongyi [16] introduced a new calculation technique, called the *integration with ordered products* (IWOP), applicable only to HO-1D, which greatly facilitates calculations and leads to a range of results, including new mathematical results. In a series of previous works, we have generalized the IWOP technique and applied it to the pair of creation $\widehat{E}_+$, and annihilation $\widehat{E}_-$, operators associated with any quantum system, linear or nonlinear. Thus, it was born the diagonal operator ordering technique (DOOT) was born, with which we obtained a series of useful results [17–19]. An additional result of using DOOT appears in this paper: we obtain a series of new integration relations involving Meijer's $G_{p,q}^{m,n}(|z|^2|\ldots)$ and generalized hypergeometric functions $pF_q(\ldots;\ldots;|z|^2)$. This is, in fact, the main purpose of this paper.

The paper is organized as follows. In Section 2, we present some primary elements, necessary to define coherent states, using elements of the diagonal operator ordering technique (DOOT). Section 3 is dedicated to defining the different types of coherent states (CSs) and revealing their main properties. Special attention is paid to the relationship of solving the unit operator, i.e., to its development (decomposition) with respect to the projectors of the coherent states. In Section 4, we use a DOOT rule, according to which, inside the sign # #, the normally ordered operators can be seen as simple numbers and, consequently, we can perform all algebraic operations according to the usual rules. Consequently, several general integrals are obtained in which the generalized hypergeometric functions participate. Their verification is done by customizing the indices of these functions, which leads to known results obtained by other methods. In Section 5, some observations are made in connection with the results obtained previously. Implicitly, the results show that the formalism of coherent states corroborated with the DOOT technique can be useful also for deducing new integrals from complex or real space.

2. Preliminaries

Generally, the energy eigenvalues $e(n)$ are dependent of a set of real (or complex) parameters $\{a_i\}_1^p$ and $\{b_j\}_1^q$. Examining the expression of energy eigenvalues, we can say that for each quantum system, there exist a *specific* and *unique* set of integers and positive numbers p and q, respectively, and a set of parameters $\{a_i\}_1^p$ and $\{b_j\}_1^q$. Let us suppose that the dimensionless energy eigenvalues are nondegenerate and have the general expression (this assertion will be implicitly motivated by the following):

$$e(l) = Cl\frac{\prod_{j=1}^{q}\left(b_j - 1 + l\right)}{\prod_{i=1}^{p}\left(a_i - 1 + l\right)}, l = 1, 2, 3, \ldots, \tag{3}$$

where C is a dimensionless real constant.

We use for their product the following notation:

$$\rho_{p,q}\left(\frac{b}{a}|n\right) \equiv \prod_{l=1}^{n} e(l) = C^n n! \frac{\prod_{j=1}^{q}\left(b_j\right)_n}{\prod_{i=1}^{p}\left(a_i\right)_n}, \tag{4}$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol and $\Gamma(x)$ is Euler's gamma function.

It is useful to write the positive constants $\rho_{p,q}(b/a|n)$ (called "structure constants," when referring to coherent states) also in the same manner:

$$\rho_{p,q}\left(\frac{b}{a}|n\right) = \Gamma_{p,q}\left(\frac{a}{b}\right)C^n\Gamma(n+1)\frac{\prod_{j=1}^{q}\Gamma\left(b_j+n\right)}{\prod_{i=1}^{p}\Gamma(a_i+n)},$$
$$\Gamma_{p,q}\left(\frac{a}{b}\right) \equiv \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma\left(b_j\right)}. \quad (5)$$

The repeated action of the creation operator $\widehat{E}_+$ on the vacuum (or ground) state $|0\rangle$ leads to the following result:

$$\left(\widehat{E}_+\right)^n|0\rangle = \sqrt{\prod_{l=0}^{n} e(n)|n\rangle} = \sqrt{\rho_{p,q}\left(\frac{b}{a}|n\right)}|n\rangle, \quad (6)$$

$$\begin{pmatrix} |n\rangle \\ \langle n| \end{pmatrix} = \frac{1}{\sqrt{\rho_{p,q}(b/a|n)}}\begin{pmatrix} \left(\widehat{E}_+\right)^n|0\rangle \\ \langle 0|\left(\widehat{E}_+\right)^n \end{pmatrix}. \quad (7)$$

The completion relation (or the identity operator decomposition) for Fock vectors, if we assume that the Fock vector space can be finite or infinite dimensional, i.e., with dimension $n_{\max} \equiv M \le \infty$, is

$$\sum_{n=0}^{M} |n\rangle\langle n| = 1, \quad (8)$$

and then it becomes

$$\sum_{n=0}^{M}\frac{1}{\rho_{p,q}(b/a|n)}\left(\widehat{E}_+\right)^n|0\rangle\langle 0|\left(\widehat{E}_-\right)^n = 1. \quad (9)$$

Due to a fruitful operator ordering technique—the diagonal operator ordering technique (DOOT) (see [17] (which is a generalization of the integration with ordered products (IWOP)) and [16]), the above relation can be rearranged as follows:

$$|0\rangle\langle 0|\sum_{n=0}^{M}\frac{1}{\rho_{p,q}(b/a|n)}\#\left(\widehat{E}_+\widehat{E}_-\right)^n\# = 1. \quad (10)$$

The above sum is just a generalized hypergeometric function (G-HGF), if $M \longrightarrow \infty$, or a generalized hypergeometric polynomial of the degree M (G-HGP), if $M < \infty$, with the ordered operator product $\#\widehat{E}_+\widehat{E}_-\#$ as argument, generally defined as

$${}_pF_q^M\left(\{a_i\}_1^p; \left\{b_j\right\}_1^q; \frac{1}{C}x\right) = \sum_{n=0}^{M}\frac{1}{\rho_{p,q}(b/a|n)}x^n = \sum_{n=0}^{M}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}\left(b_j\right)_n}\frac{(x/C)^n}{n!}. \quad (11)$$

Consequently, the vacuum projector is

$$|0\rangle\langle 0| = \frac{1}{\#_{\mathrm{p}}F_q^M\left(\{a_i\}_1^p; \left\{b_j\right\}_1^q; \widehat{E}_+\widehat{E}_-/C\right)\#}. \quad (12)$$

Evidently, as we will see later, the vacuum projector is the same for all three kinds of CSs, i.e., it is independent of the definition of CSs.

This G-HGF is characteristic for every quantum system. At the same time, the definition of G-HGF is the motivation for choosing the general expression of energy eigenvalues (3). Consequently, the function $\#_pF_q^M(\{a_i\}_1^p; \left\{b_j\right\}_1^q; x/C)\#$ can be called "characteristic generalized hypergeometric functions or polynomials" or "associated hypergeometric functions or polynomials" of the quantum system.

Generally, an arbitrary set of CSs $|z\rangle$ is labelled by a complex number $z = |z|\exp(i\phi)$, $|z| \le R \le \infty$, $\phi \in (0, 2\pi)$ and can be developed in the Fock vector basis $|n\rangle$ as

$$|z\rangle = \frac{1}{\sqrt{N\left(|z|^2\right)}}\sum_{n=0}^{M}\frac{z^n}{\sqrt{\rho_{p,q}(b/a|n)}}|n\rangle. \quad (13)$$

All CSs $|z\rangle$ must satisfy some conditions, sometimes named Klauder's minimal prescriptions[10]: the continuity of labelling, i.e., if $z' \longrightarrow z$, then compulsory $\| |z'\rangle - |z\rangle \| \longrightarrow 0$; as well as the condition that CSs must be normalized to unity and nonorthogonal:

$$\langle z|z'\rangle = \begin{cases} 1, & \text{for } z' = z \\ \ne 0, & \text{for } z' \ne z \end{cases}. \quad (14)$$

Moreover, the most important condition is the so-called *completion relation* or *the resolution of the unity operator* of the CS projectors:

$$\int d\mu(z)|z\rangle\langle z| = 1, \quad (15)$$

with such an integration measure

$$d\mu(z) = \frac{d^2z}{\pi}h(|z|) = \frac{d\phi}{2\pi}d\left(|z|^2\right)h(|z|), \quad (16)$$

to ensure a positive weight function $h(|z|)$ which must be determined separately, for each examined quantum system and each kind of CSs.

To find the weight function $h(|z|)$, we must have

$$\sum_{n,n\prime=0}^{M}\frac{|n\rangle\langle n'|}{\sqrt{\rho_{p,q}(b/a|n)}\sqrt{\rho_{p,q}(b/a|n\prime)}}\int_0^R d\left(|z|^2\right)\frac{h(|z|)}{N\left(|z|^2\right)}\int_0^{2\pi}\frac{d\phi}{2\pi}\left(z^*\right)^n\left(z'\right)^{n'} = 1. \quad (17)$$

The normalization function $N(|z|^2)$ is obtained from the normalization condition of CSs $\langle z|z\rangle = 1$, so that

$$N\left(|z|^2\right) = {}_pF_q^M\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C}|z|^2\right). \tag{18}$$

The angular integral is $(|z|^2)^n\delta_{nn\prime}$, and, if we put $\tilde{h}(|z|) \equiv h(|z|)/N(|z|^2)$, we have to solve an integral moment problem (the so-called Stieltjes or Hausdorff moment, depending on whether $R \longrightarrow \infty$ or $R < \infty$). Consequently, the normalization function $N(|z|^2)$ is either a generalized hypergeometric function (if $R = \infty$) or a generalized hypergeometric polynomial of the degree M, if $R < \infty$.

Inserting the expression of the integration measure in the completion relation, to leave at the completion relation (or the identity operator decomposition) for Fock vectors, after angular integration, we obtain

$$\int_0^R d\left(|z|^2\right)\frac{h(|z|)}{{}_pF_q^M\left(\{a_i\}_1^p; \{b_j\}_1^q; 1/C|z|^2\right)}\left(|z|^2\right)^n = \rho_{p,q}\left(\frac{b}{a}|n\right). \tag{19}$$

The convergence radii for any kind of CSs are determined by calculating the limits (see, e.g., [20]):

$$R = \frac{1}{\lim\limits_{n\longrightarrow\infty}}\sqrt[n]{\rho_{p,q}\left(\frac{b}{a}|n\right)} > 0, \tag{20}$$

$$R = \lim_{n\longrightarrow\infty}\frac{\rho_{p,q}(b/a|n)}{\rho_{p,q}(b/a|n+1)} = \frac{1}{C}\lim_{n\longrightarrow\infty} n^{p-q-1}\frac{1}{1+1/n} = \begin{cases} \infty, & \text{if } p-q-1>0, \\ \dfrac{1}{C}<\infty, & \text{if } p-q-1=0, \\ 0, & \text{if } p-q-1<0, \end{cases} \tag{21}$$

and the CSs exist (that is, they make physical sense) only if $R \neq 0$.

3. Different Kinds of Coherent States

Generally, for a certain quantum system, there exist three kinds or manners to define the CSs: (a) the Barut–Girardello CSs (BG-CSs); (b) the Klauder–Perelomov CSs (KP-CSs); and (c) the Gazeau–Klauder CSs (GK-CSs). The corresponding expressions for these three kinds of CSs are different or divergent, except the case of HO-1D where these three definitions are convergent, i.e., their results are identical.

To highlight, in turn, the most important characteristics of the three types of CSs, we will retain the following:

(a) The Barut–Girardello coherent states (BG-CSs) are defined as the eigenfunctions of the lowering operator $\hat{E}_-$ (see [8]):

$$\hat{E}_-|z\rangle_{BG} = z|z\rangle_{BG}, \tag{22}$$

and their expansion in the Fock vector basis is

$$|z\rangle_{BG} = \frac{1}{\sqrt{N_{BG}\left(|z|^2\right)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_{p,q}(b/a|n)}}|n\rangle. \tag{23}$$

The normalization function $N_{BG}(|z|^2)$ is the following generalized hypergeometric function:

$$N_{BG}\left(|z|^2\right) = \sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^n}{\rho_{p,q}(b/a|n)} = pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C}|z|^2\right). \tag{24}$$

To find the weight function $h_{BG}(|z|)$ of the integration measure, we appeal to the completion relations for the CSs, as well as for the Fock vectors. Then, we have

$$\sum_{n,n\prime=0}^{\infty}\frac{|n\rangle\langle n'|}{\sqrt{\rho_{p,q}(b/a|n)\rho_{p,q}(b/a|n')}}\int_0^R d\left(|z|^2\right)\left(|z|^2\right)^n\frac{h_{BG}(|z|)}{N_{BG}\left(|z|^2\right)}\int_0^{2\pi}\frac{d\phi}{2\pi}\left(z^*\right)^{n'}z^n = 1. \tag{25}$$

The angular integral is $(|z|^2)^n\delta_{nn\prime}$ and if we put $\tilde{h}_{BG}(|z|) \equiv h_{BG}(|z|)/N_{BG}(|z|^2)$, we have to solve an integral moment problem (Stieltjes or Hausdorff, depending on whether $R = \infty$ or $R < \infty$) [9]:

$$\int_0^R d\left(|z|^2\right)\left(|z|^2\right)^n \widetilde{h}_{BG}(|z|) = \rho_{p,q}\left(\frac{b}{a}|n\right) = \Gamma_{p,q}\left(\frac{a}{b}\right)C^n\Gamma(n+1)\frac{\sum_{j=1}^{q}\Gamma\left(b_j+n\right)}{\sum_{i=1}^{p}\Gamma\left(a_i+n\right)}. \quad (26)$$

By changing the exponent $n = s - 1$, we get

$$\int_0^R d\left(|z|^2\right)\left(|z|^2\right)^{s-1}\widetilde{h}_{BG}(|z|) = \frac{1}{C}\Gamma_{p,q}\left(\frac{a}{b}\right)C^s\Gamma(s)\frac{\sum_{j=1}^{q}\Gamma\left(b_j-1+s\right)}{\sum_{i=1}^{p}\Gamma\left(a_i-1+s\right)}. \quad (27)$$

Using the general relation for the classical integral to one Meijer's G-function [21],

$$\int_0^\infty dx\, x^{s-1}G_{p,q}^{m,n}\left(\omega x \middle| \begin{matrix}\{a_i\}_1^n; & \{a_i\}_{n+1}^p\\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q\end{matrix}\right) = \frac{1}{\omega^s}\frac{\prod_{j=1}^{m}\Gamma\left(b_j+s\right)\prod_{i=1}^{n}\Gamma\left(1-a_i-s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-s\right)\prod_{i=n+1}^{p}\Gamma\left(a_i+s\right)}, \quad (28)$$

and identifying the constants, we obtain

$$\widetilde{h}_{BG}(|z|) = \frac{1}{C}\Gamma_{p,q}\left(\frac{a}{d}\right)d\left(|z|^2\right)G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_1^p\\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right), \quad (29)$$

so that finally the integration measure is

$$d\mu_{BG}(z) = \frac{1}{C}\Gamma_{p,q}\left(\frac{a}{b}\right)\frac{d\phi}{2\pi}d\left(|z|^2\right)N_{BG}\left(|z|^2\right)G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_1^p\\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right). \quad (30)$$

Replacing the expression of $\widetilde{h}_{BG}(|z|)$ in (27), we obtain an important integral of the real variable $|z|^2$ (see [11]):

$$\int_0^R d\left(|z|^2\right)\left(|z|^2\right)^{s-1}G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_1^p\\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right) = \frac{C}{\Gamma_{p,q}(a/b)}\rho_{p,q}\left(\frac{b}{a}|s-1\right). \quad (31)$$

Due to their importance for the rest of this paper, we will call this equation the *fundamental Barut–Girardello integral* (f-BG-int).

If we use (7) and (12) as well as the DOOT rules [17], we can write the BG-CSs as

$$|z\rangle_{BG} = \frac{1}{\sqrt{N_{BG}\left(|z|^2\right)}}\sum_{n=0}^{\infty}\frac{\left(z\widehat{E}_+\right)^n}{\rho_{p,q}(b/a|n)}|0\rangle = \frac{1}{\sqrt{N_{BG}\left(|z|^2\right)}}pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C}z\widehat{E}_+\right)|0\rangle, \quad (32)$$

and the resolution of the unity operator (15) becomes

$$\int \frac{d^2z}{\pi} G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right) \#_p F_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C} z\widehat{E}_+\right) \times$$
$$\times pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C} z^* \widehat{E}_-\right)\# = \frac{C}{\Gamma_{p,q}(a/b)} \#_p F_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C} \widehat{E}_+\widehat{E}_-\right)\#. \quad (33)$$

This integral will be useful to calculate some real integrals involving the hypergeometric functions.

If we perform the angular integral, we obtain

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \#_p F_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C} z\widehat{E}_+\right)_p F_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C} z^* \widehat{E}_-\right)\# = \sum_{n=0}^{\infty} \frac{\#\left(\widehat{E}_+\widehat{E}_-\right)^n\#}{\left[\rho_{p,q}(b/a|n)\right]^2}\left(\frac{1}{C^2}|z|^2\right)^n$$
$$= \#_{2p}F_{2q+1}\left(\{a_i\}_1^p, \{a_i\}_1^p; 1, \{b_j\}_1^q, \{b_j\}_1^q; \frac{1}{C^2}|z|^2 \widehat{E}_+\widehat{E}_-\right)\#. \quad (34)$$

Then, it remains for us to perform the integral in the real space, with variable $|z|^2$. Since in the DOOT formalism the operators (under the sign of integration) are treated as numbers, we can also replace, using DOOT rules, operators $\widehat{E}_+$ and $\widehat{E}_-$ with numbers A and B, so that in the end, we get

$$\int_0^{\infty} d\left(|z|^2\right) G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right) 2pF_{2q+1}\left(\{a_i\}_1^p, \{a_i\}_1^p; 1, \{b_j\}_1^q, \{b_j\}_1^q; \frac{AB}{C^2}|z|^2\right)$$
$$= \frac{C}{\Gamma_{p,q}(a/b)} pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{AB}{C}\right). \quad (35)$$

(b) The Klauder–Perelomov coherent states (KP-CSs) are defined as the result of the action of the generalized displacement unitary operator $\#\exp(z\widehat{E}_+ - z^*\widehat{E}_-)\#$ on the ground (or vacuum) state $|0\rangle$ (see [9]):

$$|z\rangle_{KP} = \frac{1}{\sqrt{N_{KP}\left(|z|^2\right)}} exp\left(z\widehat{E}_+\right)|0\rangle$$
$$= \frac{1}{\sqrt{N_{KP}\left(|z|^2\right)}} \sum_{n=0}^{M} \frac{z^n}{n!}\left(\widehat{E}_+\right)^n|0\rangle = \frac{1}{\sqrt{N_{KP}\left(|z|^2\right)}} \sum_{n=0}^{M} \frac{\sqrt{\rho_{p,q}(b/a|n)}}{n!} z^n|n\rangle. \quad (36)$$

The normalization function is then

$$N_{KP}\left(|z|^2\right) = \sum_{n=0}^{M} \frac{\rho_{p,q}(b/a|n)}{(n!)^2}(|z|)^n = \sum_{n=0}^{M} \frac{\prod_{j=1}^q (b_j)_n}{\prod_{i=1}^p (a_i)_n} \frac{\left(C|z|^2\right)^n}{n!} = {}_qF_p^M\left(\{b_j\}_1^q; \{a_i\}_1^p; C|z|^2\right). \quad (37)$$

Comparing with the normalization function $N_{BG}(|z|^2)$ of the BG-CSs, it is to observe the interchanging of the sets of parameters $\{a_i\}_1^p$ and $\{b_j\}_1^q$ as well as the inversion of the constant in front of the real variable $|z|_2$ of the generalized hypergeometric function.

We must determine the expression of the weight function $h_{KP}(|z|)$ of the integration measure $d\mu_{KP}(z) = (d^2z/\pi)h_{KP}(|z|)$ by the help of the resolution of the unity operator. Using the DOOT, this relation becomes

$$|0\rangle\langle 0| \int_0^R d\left(|z|^2\right) \frac{h_{KP}(|z|)}{N_{KP}\left(|z|^2\right)} \int_0^{2\pi} \frac{d\phi}{2\pi} \#\exp\left(z\widehat{E}_+\right)\exp\left(z^*\widehat{E}_-\right)\# = 1. \tag{38}$$

After separate expansion of exponentials into power series, the angular integral becomes ([22])

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \#\exp\left(z\,\widehat{E}_+\right) exp\left(z^*\widehat{E}_-\right)\# = \sum_{n=0}^{\infty} \frac{\#\left(\widehat{E}_+\widehat{E}_-\right)^n\#}{(n!)^2}\left(|z|^2\right)^n$$
$$= \#_0F_1\left(;1;\widehat{E}_+\widehat{E}_-|z|^2\right)\# = \#I_0\left(2|z|\sqrt{\widehat{E}_+\widehat{E}_-}\right)\#. \tag{39}$$

Following a successively similar path as in the case of BG-CSs, we will have

$$\frac{1}{\#_pF_q^M\left(\{a_i\}_1^p;\{b_j\}_1^q;(1/C)\widehat{E}_+\widehat{E}_-\right)\#} \sum_{n=0}^{\infty} \frac{\#\left(\widehat{E}_+\widehat{E}_-\right)^n\#}{(n!)^2} \int_0^R d\left(|z|^2\right)\left(|z|^2\right)^n \widetilde{h}_{KP}(|z|) = 1. \tag{40}$$

Hence, it follows that the integral must be of the moment problem type, with $n = s - 1$

:

$$\int_0^R d\left(|z|^2\right)\left(|z|^2\right)^{s-1}\widetilde{h}_{KP}(|z|) = \frac{[\Gamma(s)]^2}{\rho_{p,q}(b/a|s-1)} = \frac{C}{\Gamma_{p,q}(a/b)} \frac{1}{C^s}\Gamma(s) \frac{\prod_{i=1}^p \Gamma(a_i - 1 + s)}{\prod_{j=1}^q \Gamma\left(b_j - 1 + s\right)}. \tag{41}$$

Identifying the constants, after obtaining the solution of the above integral equation [21], the integration measure is finally written as

$$d\mu_{KP}(z) = \frac{C}{\Gamma_{p,q}(a/b)} \frac{d\phi}{2\pi} d\left(|z|^2\right) N_{KP}\left(|z|^2\right) G_{q,p+1}^{p+1,0}\left(\frac{1}{C}|z|^2 \middle| \begin{matrix} /; & \{b_j-1\}_1^q \\ 0,\{a_i-1\}_1^p; & / \end{matrix}\right). \tag{42}$$

Consequently, the decomposition relation of the unity operator turns into the following integral equation:

$$\int \frac{d^2z}{\pi} G_{q,p+1}^{p+1,0}\left(C|z|^2 \middle| \begin{matrix} /; & \{b_j-1\}_1^q \\ 0,\{a_i-1\}_1^p; & / \end{matrix}\right) \#\exp\left(z\widehat{E}_+\right) exp\left(z^*\widehat{E}_-\right)\#$$
$$= \frac{1}{C}\Gamma_{p,q}\left(\frac{a}{b}\right) \#_pF_q\left(\{a_i\}_1^p;\{b_j\}_1^q;\frac{1}{C}\widehat{E}_+\widehat{E}_-\right)\#. \tag{43}$$

After performing the angular integral, we obtain the integral in the real space, with variable $|z|^2$ (similarly, as for BG-CSs, we have used DOOT rules and replaced the operators $\widehat{E}_+$ and $\widehat{E}_-$ with scalars A and B), so that in the end, we get

$$\int_0^\infty d\left(|z|^2\right) G_{q,p+1}^{p+1,0}\left(C|z|^2 \middle| \begin{matrix} /; & \{b_j-1\}_1^q \\ 0, \{a_i-1\}_1^p; & / \end{matrix} \right) I_0\left(2|z|\sqrt{AB}\right) = \frac{\Gamma_{p,q}(a/b)}{C}\, pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C}AB\right). \tag{44}$$

Finally, let us point out that for these CSs, the *fundamental Klauder–Perelomov integral* (f-KP-int) is then

$$\int_0^R d\left(|z|^2\right)\left(\pi |z|^2\right)^{s-1} G_{q,p+1}^{p+1,0}\left(C|z|^2 \middle| \begin{matrix} /; & \{b_j-1\}_1^q \\ 0, \{a_i-1\}_1^p; & / \end{matrix} \right) = \frac{\Gamma_{p,q}(a/b)}{C} \frac{[\Gamma(s)]^2}{\rho_{p,q}(b/a|s-1)}. \tag{45}$$

(c) The Gazeau–Klauder coherent states (GK-CSs) were introduced by Gazeau and Klauder [10] and have the following expression:

$$|J,\gamma\rangle = \frac{1}{\sqrt{N_{GK}(J)}} \sum_{n=0}^{\infty} \frac{(\sqrt{J})^n}{\sqrt{\rho_{p,q}(b/a|n)}} e^{-i\gamma e(n)} |n\rangle, \tag{46}$$

where $0 \le J \le \infty$ is a real number labelling the GK-CSs and $-\infty \le \gamma \le +\infty$ is a real characteristic parameter. The normalization function $N_{GK}(J)$ is obtained, as usual, from the normalization condition $\langle J|J\rangle = 1$:

$$N_{GK}(J) = \sum_{n=0}^{\infty} \frac{J^n}{\rho_{p,q}(b/a|n)} = {}_pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{1}{C}J\right). \tag{47}$$

We can see that they have the same mathematical structure as $N_{BG}(|z|^2)$.

The integration measures

$$d\mu_{GK}(J,\gamma) = \frac{d\gamma}{2R} dJ\, h_{GK}(J) \tag{48}$$

must be understood to satisfy the limit [10]

$$\int d\mu_{GK}(J,\gamma) \ldots = \lim_{R\longrightarrow\infty} \int_{-R}^{+R} \frac{d\gamma}{2R} \ldots \int_0^\infty dJ\, h_{GK}(J) \ldots. \tag{49}$$

The GK-CSs can be obtained if three steps are taken [19]:

(i) Firstly, let us define BG-CSs, but for the *real* variable J, denoted by $|J\rangle$:

$$\widehat{E}_-|J\rangle = J|J\rangle. \tag{50}$$

(ii) Secondly, let us develop $|J\rangle$ into the Fock vector base, using the standard procedure:

$$|J\rangle = \frac{1}{\sqrt{N_{GK}(J)}} \sum_{n=0}^{\infty} \frac{(\sqrt{J})^n}{\sqrt{\rho_{p,q}(b/a|n)}} |n\rangle. \tag{51}$$

(iii) The third step is to act with the exponential operator $\exp(-i\gamma \widehat{H})$ on the state $|J\rangle$ (the parameter γ and the Hamiltonian $\widehat{H}$ are considered less dimensional):

$$|J,\gamma\rangle = \exp(-i\gamma\widehat{H})|J\rangle = \frac{1}{\sqrt{N_{GK}(J)}} \sum_{n=0}^{M} \frac{(\sqrt{J})^n}{\sqrt{\rho_{p,q}(b/a|n)}} e^{-i\gamma e(n)} |n\rangle. \tag{52}$$

The resolution of the unity operator:

$$\int d\mu_{GK}(J,\gamma) |J,\gamma\rangle\langle J,\gamma| = 1. \tag{53}$$

After some similar calculations as before, we arrive at the expression

$$\sum_{n,n\prime=0}^{\infty} \frac{|n\rangle\langle n'|}{\sqrt{\rho_{p,q}(b/a|n)}\sqrt{\rho_{p,q}(b/a|n')}} \lim_{R\longrightarrow\infty} \int_0^R dJ \frac{h_{GK}(J)}{N_{GK}(J)} (\sqrt{J})^n (\sqrt{J})^{n'} \int_{-R}^{+R} \frac{d\gamma}{2R} e^{i\gamma[e(n)-e(n')]} = 1. \tag{54}$$

The integral with respect to the variable γ is $\delta_{n,n'}$ and so we must solve the following moment problem:

$$\int_0^\infty dJ\ \widetilde{h}_{GK}(J)J^n = \rho_{p,q}\left(\frac{b}{a}|n\right), \tag{55}$$

which is identical to that of BG-CSs (26). Consequently, the integration measure is

$$d\mu_{KP}(J,\gamma) = \frac{1}{C}\Gamma_{p,q}\left(\frac{a}{b}\right)\frac{d\gamma}{2R}dJ\ N_{GK}(J)G_{p,q+1}^{q+1,0}\left(\frac{1}{C}J \middle| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0, \{b_j-1\}_1^m; & / \end{matrix}\right). \tag{56}$$

Let us replace this expression by the decomposition relation of the unity operator:

$$\begin{aligned} &\frac{1}{C}\Gamma_{p,q}\left(\frac{a}{b}\right)\sum_{n,n_l=0}^{M}\frac{|n><n'|}{\sqrt{\rho_{p,q}(b/a|n)}\sqrt{\rho_{p,q}(b/a|n')}} \\ &\times \lim_{R\longrightarrow\infty}\int_0^R dJG_{p,q+1}^{q+1,0}\left(\frac{1}{C}J \middle| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0, \{b_j-1\}_1^m; & / \end{matrix}\right)(\sqrt{J})^{n+n'}\lim_{R\longrightarrow\infty}\frac{1}{2R}\int_{-R}^{+R}d\gamma exp(-\mathrm{i}\gamma[e(n)-e(n')]) = 1. \end{aligned} \tag{57}$$

Performing the integral with respect to γ and using (17), finally we obtain

$$\int_0^\infty dJJ^{s-1}G_{p,q+1}^{q+1,0}\left(\frac{1}{C}J \middle| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right) = \frac{C}{\Gamma_{p,q}(a/b)}\rho_{p,q}\left(\frac{b}{a}|s-1\right), \tag{58}$$

which is identical, from the mathematical point of view, to (31).

An interesting situation arises in the case of *quantum systems with a linear energy spectrum*, for which $e(l) = Cl$, so $p = q$, $\{a_i\}_1^p = \{b_j\}_1^q$, and $\rho_{0,0}(/|n) = C^n n!$.

The GK-CSs for these systems are

$$\begin{aligned} |J,\gamma\rangle &= \frac{1}{\sqrt{N_{GK}(J)}}\sum_{n=0}^{M}\frac{(\sqrt{J})^n}{\sqrt{\rho_{p,q}(b/a|n)}}e^{-\mathrm{i}\gamma Cn}|n\rangle = \frac{1}{\sqrt{N_{GK}(J)}}\sum_{n=0}^{M}\frac{\left(\widehat{E}_+e^{-\mathrm{i}\gamma C}\sqrt{J}\right)^n}{\sqrt{\rho_{0,0}(/|n)}}|0\rangle \\ &= \frac{1}{\sqrt{N_{GK}(J)}}0F_0\left(;;\frac{1}{\sqrt{C}}\widehat{E}_+e^{-\mathrm{i}\gamma C}\sqrt{J}\right)|0\rangle = \frac{1}{\sqrt{N_{GK}(J)}}exp\left(\frac{1}{\sqrt{C}}\widehat{E}_+e^{-\mathrm{i}\gamma C}\sqrt{J}\right)|0\rangle, \end{aligned} \tag{59}$$

where the normalization function is

$$N_{GK}(J) = \exp\left(\frac{1}{C}J\right)|0\rangle\langle 0| = \frac{1}{\#_0F_0\left(;\,;(1/C)\hat{E}_+\hat{E}_-\right)\#} = \# e^{-\frac{1}{C}\hat{E}_+\hat{E}_-}\#. \quad (60)$$

Let us check if the integral resulting from the decomposition of the unit operator is true:

$$\lim_{R\longrightarrow\infty}\int_0^R dJ G_{0,1}^{1,0}\left(\frac{1}{C}J\,|0\,;\,/\right)\int_{-R}^{+R}\frac{d\gamma}{2R}\#\exp\left(\frac{1}{\sqrt{C}}\hat{E}_+e^{-i\gamma C}\sqrt{J}\right)|0\rangle\langle 0|\exp\left(\frac{1}{\sqrt{C}}\hat{E}_-e^{+i\gamma C}\sqrt{J}\right)\# = 1. \quad (61)$$

Assuming that [23]

$$G_{0,1}^{1,0}\left(\frac{1}{C}J\,|0\,;\,/\right) = \exp\left(\frac{1}{C}J\right) \quad (62)$$

and developing in series the complex exponentials, after integration with respect to γ, it is easy to show that the above integral is true.

By the way, linear energy spectrum systems also satisfy the conditions of temporal stability and action identity [10].

4. Results and Discussion

Let us turn our attention to the integrals which are called the *fundamental CS integrals*.i.e., equations (31), (45), and (58) are identical from a mathematical point of view. What differ are only the indices of Meijer's G functions. Because f-BG-int and f-GK-int are the same, consequently, we will pay attention only to f-BG-int and f-KP-int. We will use these integrals to obtain some new generalized integrals involving Meijer's and hypergeometric functions.

First, let us consider a new hypergeometric function

$${}_rF_s\left(\{c_i\}_1^r;\{d_j\}_1^s;A|z|^2\right) = \sum_{m=0}^{\infty}\frac{1}{\rho_{r,s}(d/c|m)}\left(A|z|^2\right)^m, \quad (63)$$

with the structure constants

$$\rho_{r,s}\left(\frac{d}{c}|m\right) = \frac{1}{A^m}m!\frac{\prod_{j=1}^s(d_j)_m}{\prod_{i=1}^r(c_i)_m}. \quad (64)$$

Let us calculate the integral

$$I(G_{BG}*F) \equiv \int_0^\infty d\left(|z|^2\right)G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2\,\middle|\,\begin{matrix}/\,; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q\,; & /\end{matrix}\right){}_rF_s\left(\{c_i\}_1^r;\{d_j\}_1^s;A|z|^2\right). \quad (65)$$

If we write the function $rF_s(\{c_i\}_1^r;\{d_j\}_1^s;A|z|^2)$ as a power series, we obtain successively

$$\begin{aligned}I(G_{BG}*F) &= \sum_{m=0}^{\infty}\frac{A^m}{\rho_{r,s}(d/c|m)}\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^m G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2\,\middle|\,\begin{matrix}/; & \{a_i-1\}_1^p\\ 0,\{b_j-1\}_1^q; & /\end{matrix}\right)\\ &= \frac{C}{\Gamma_{p,q}(a/b)}\sum_{m=0}^{\infty}\frac{\rho_{p,q}(b/a|m)}{\rho_{r,s}(d/c|m)}A^m = \frac{C}{\Gamma_{p,q}(a/b)}\sum_{m=0}^{\infty}\frac{\prod_{j=1}^q(b_j)_m\prod_{i=1}^r(c_i)_m(1)_m}{\prod_{i=1}^p(a_i)_m\prod_{j=1}^s(d_j)_m}\frac{(CA)^m}{m!}.\end{aligned} \quad (66)$$

Finally, we obtain a new integral, which is one of the goals of the present article:

$$\begin{aligned}&\int_0^\infty d\left(|z|^2\right)G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2\,\middle|\,\begin{matrix}/\,; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q\,; & /\end{matrix}\right){}_rF_s\left(\{c_i\}_1^r;\{d_j\}_1^s;A|z|^2\right)\\ &= \frac{C}{\Gamma_{p,q}(a/b)}q+r+1\,F_{p+s}\left(\{b_j\}_1^q,\{c_i\}_1^r,1;\{a_i\}_1^p,\{d_j\}_1^s;CA\right).\end{aligned} \quad (67)$$

To verify the correctness of this result, firstly we express the hypergeometric function through Meijer's function (see [21]):

$$ {}_rF_s\left(\{c_i\}_1^r;\{d_j\}_1^s;A|z|^2\right)=\frac{1}{\Gamma_{r,s}(c/d)}G_{r,s+1}^{1,r}\left(-A|z|^2\middle|\begin{matrix}\{1-c_i\}_1^r; & /\\ 0; & \{1-d_j\}_1^s\end{matrix}\right), \tag{68}$$

and then we use classical Meijer's integral for two G functions [15]:

$$\begin{aligned}&\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^{\alpha-1}G_{p,q}^{m,n}\left(\frac{1}{C}|z|^2\middle|\begin{matrix}\{a_i\}_1^n; & \{a_i\}_{n+1}^p\\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q\end{matrix}\right)G_{u,v}^{s,t}\left(-A|z|^2\middle|\begin{matrix}\{c_i\}_1^t; & \{c_i\}_{t+1}^u\\ \{d_j\}_1^s; & \{d_j\}_{s+1}^v\end{matrix}\right)\\ &=\frac{1}{(-A)^\alpha}G_{p+v,q+u}^{m+t,n+s}\left(-\frac{1}{CA}\middle|\begin{matrix}\{a_i\}_1^n, & \{1-\alpha-d_j\}_1^s; & \{1-\alpha-d_j\}_{s+1}^v, & \{a_i\}_{n+1}^p\\ \{b_j\}_1^m, & \{1-\alpha-c_i\}_1^t; & \{1-\alpha-c_i\}_{t+1}^u, & \{b_j\}_{m+1}^q\end{matrix}\right),\end{aligned} \tag{69}$$

as well as the following transformations and argument simplifications of Meijer's functions:

$$G_{p,q}^{m,n}\left(\frac{1}{|z|^2}\middle|\begin{matrix}\{a_i\}_1^p\\ \{b_j\}_1^q\end{matrix}\right)=G_{q,p}^{n,m}\left(|z|^2\middle|\begin{matrix}\{1-b_j\}_1^q\\ \{1-a_i\}_1^p\end{matrix}\right), \tag{70}$$

$$G_{p,q}^{m,n}\left(|z|^2\middle|\begin{matrix}\{\alpha+a_i\}_1^p\\ \{\alpha+b_j\}_1^q\end{matrix}\right)=\left(|z|^2\right)^\alpha G_{p,q}^{m,n}\left(|z|^2\middle|\begin{matrix}\{a_i\}_1^p\\ \{b_j\}_1^q\end{matrix}\right). \tag{71}$$

Let us we calculate the same kind of integrals, but using f-KP-int, i.e.,

$$I\left(G_{KP}*F\right)\equiv\int_0^\infty d\left(|z|^2\right)G_{q,p+1}^{p+1,0}\left(\frac{1}{C}|z|^2\middle|\begin{matrix}/; & \{b_j-1\}_1^q\\ 0,\{a_i-1\}_1^p; & /\end{matrix}\right){}_rF_s\left(\{c_i\}_1^r;\ \{d_j\}_1^s;\ A|z|^2\right). \tag{72}$$

Following the same procedure and steps after straightforward calculations, we obtain

$$\begin{aligned}&\int_0^\infty d\left(|z|^2\right)G_{q,p+1}^{p+1,0}\left(\frac{1}{C}|z|^2\middle|\begin{matrix}/; & \{b_j-1\}_1^q\\ 0,\{a_i-1\}_1^p; & /\end{matrix}\right){}_rF_s\left(\{c_i\}_1^r;\{d_j\}_1^s;A|z|^2\right)\\ &=\frac{\Gamma_{p,q}(a/b)}{C}\,p+r+1F_{q+s}\left(\{a_i\}_1^p,\{c_i\}_1^r,1;\{b_j\}_1^q,\{d_j\}_1^s;CA\right).\end{aligned} \tag{73}$$

To these new kinds of integrals, we can also add the integrals previously obtained, which can be considered as a particular case of the previous:

$$\int_0^\infty d\left(|z|^2\right) G_{p,q+1}^{q+1,0}\left(\frac{1}{C}|z|^2 \,\middle|\, \begin{matrix} / \,; & \{a_i-1\}_1^p \\ 0\,, \ \{b_j-1\}_1^q \,; & / \end{matrix}\right) 2pF_{2q+1}\left(\{a_i\}_1^p, \{a_i\}_1^p; 1, \{b_j\}_1^q, \{b_j\}_1^q; \frac{AB}{C^2}|z|^2\right) = \frac{C}{\Gamma_{p,q}(a/b)}\, pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{AB}{C}\right). \tag{74}$$

Let us examine some examples, to confirm the correctness of the obtained integrals.

Example 1. Let us consider the coefficients $p=1$ and $q=1$, as well as $a_1=1$, so we have

$$G_{1,2}^{2,0}\left(\frac{1}{C}|z|^2 \,\middle|\, \begin{matrix} /; & a_1-1 \\ 0, b_1-1; & / \end{matrix}\right) = G_{0,1}^{1,0}\left(\frac{1}{C}|z|^2 b_1 - 1\right) = \frac{1}{C^{b_1-1}}\left(|z|^2\right)^{b_1-1} e^{-(1/C)|z|^2}. \tag{75}$$

Integral (67) is then

$$\int_0^\infty d\left(|z|^2\right) G_{1,2}^{2,0}\left(\frac{1}{C}|z|^2 \,\middle|\, \begin{matrix} /; & a_1-1 \\ 0, b_1-1; & / \end{matrix}\right) {}_rF_s\left(\{c_i\}_1^r; \{d_j\}_1^s; A|z|^2\right) = C\,\Gamma(b_1)\, {}_{r+1}F_s\left(b_1, \{c_i\}_1^r; \{d_j\}_1^s; CA\right). \tag{76}$$

This result is in accordance with the integral 7.525.5, pp. 814 of Gradshteyn and Ryshik's book [24]:

$$\int_0^\infty dx e^{-x} x^{s-1}\, {}_pF_q\left(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; a|z|^2\right) = \Gamma(s)\, {}_{p+1}F_q\left(s, a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; a\right). \tag{77}$$

On the other hand, this equation can be regarded as the Laplace transform of the generalized hypergeometric function [25].

Example 2. Let us choose $p=1$, $q=1$, $m=2$, $n=0$, $C=1$, and $a_1, b_1 \neq 0$. Then,

$$G_{1,2}^{2,0}\left(|z|^2 \,\middle|\, \begin{matrix} /; & a_1-1 \\ 0, b_1-1; & / \end{matrix}\right) = e^{-|z|^2} U\left(a_1-b_1, 2-b_1; |z|^2\right), \tag{78}$$

where $U(a_1-b_1, 2-b_1; |z|^2)$ is the Tricomi confluent hypergeometric function.

According to equation (4), we have

$$\int_0^\infty d\left(|z|^2\right) G_{1,2}^{2,0}\left(|z|^2 \,\middle|\, \begin{matrix} /; & a_1-1 \\ 0, b_1-1; & / \end{matrix}\right) {}_rF_s\left(\{c_i\}_1^r; \{d_j\}_1^s; A|z|^2\right) = \Gamma(b_1)\, {}_{r+2}F_{s+1}\left(b_1, \{c_i\}_1^r, 1; a_1, \{d_j\}_1^s; A\right), \tag{79}$$

which can be verified using the integral calculus [26]:

$$\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^{\alpha-1} e^{-|z|^2} U\left(a, b, |z|^2\right) = \frac{\Gamma(1-b+\alpha)\Gamma(\alpha)}{\Gamma(a-b+\alpha+1)}, \max(0, \mathrm{Re}(b)-1) < \mathrm{Re}(\alpha). \tag{80}$$

Generally, by particularizing all coefficients p, q, r, and s, as well as the set of parameters $\{a_i\}_1^p$, $\{b_j\}_1^q$, $\{c_i\}_1^r$, and $\{d_j\}_1^s$, it is possible to obtain a lot of new integrals (some of them unknown yet).

5. Conclusions

In the present paper, we try to show that, apart from the role that the formalism of coherent states plays in different branches of physics (quantum mechanics, quantum optics, and quantum information theory), the formalism of coherent states can also be useful in mathematics, in the field of special functions. Namely, to solve the equation of the decomposition of the unity operator (which constitutes the fundamental property of coherent states), we arrive at a new type of integrals. Beginning from these integrals, we have deduced a set of new integrals involving Meijer's and generalized hypergeometric functions. After the particularization of their coefficients p, q, r, and s, as well as the set of parameters $\{a_i\}_1^p$, $\{b_j\}_1^q$, $\{c_i\}_1^r$, and $\{d_j\}_1^s$, it is possible, on the one hand, to recover a lot of known integrals and, on the other hand, to obtain some new unknown integrals. The above calculation was possible because we used the *diagonal operator ordering technique* (DOOT). From the attached examples, it is to observe that, from the integration point of view, the creation $\widehat{E}_+$ and the annihilation $\widehat{E}_-$ operators that appear under the integral's sign can be treated as numbers, and consequently they can be replaced by simple numerical constants. This is another consequence of using the DOOT formalism.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] H. M. Srivastava and P. W. Karlsson, "Multiple Gaussian hypergeometric series," *Halsted Press (Ellis Horwood Limited, Chichester)*, John Wiley & Sons, New York, Chichester, Brisbane and Toronto, 1985.

[2] V. Sahai and A. Verma, "On an extension of the generalized Pochhammer symboland its applications to hypergeometric functions," *Asian-European Journal of Mathematics*, vol. 9, no. No. 2, Article ID 1650064, 2016.

[3] V. Sahai and A. Verma, "Finite summation formulas of Srivastava's general triple hypergeometric function," *Asian-European Journal of Mathematics*, vol. 12, no. 02, Article ID 1950020, 2019.

[4] M. G. Bin-Saad, M. J. S. Shahwan, J. A. Younis, H. Aydi, and M. A. Abd El Salam, "On Gaussian hypergeometric functions of three variables: some new integral representations," *Journal of Mathematics*, vol. 2022, Article ID 1914498, 16 pages, 2022.

[5] A. Verma, J. A. Younis, H. Aydi, and M. A. Abd El Salam, "Certain recursion formulas for new hypergeometric functions of four variables," *Palestine Journal of Mathematics*, vol. 11, no. 2, p. þ, 2022.

[6] J. Younis, A. Verma, H. Aydi, K. S. Nisar, and H. Alsami, "Recursion formulas for certain quadruple hypergeometric functions," *Advances in Difference Equations*, vol. 12, 2021.

[7] E. Schrödinger, "Der stetige Übergang von der Mikro- zur Makromechanik," *Naturwissenschaften*, vol. 14, no. 28, pp. 664–666, 1926.

[8] A. O. Barut and L. Girardello, "New "Coherent" States associated with non-compact groups," *Communications in Mathematical Physics*, vol. 21, no. 1, pp. 41–55, 1971.

[9] J. R. Klauder, K. A. Penson, and J.-M. Sixdeniers, "Constructing coherent states through solutions of Stieltjes and Hausdorff moment problems," *Physical Review A*, vol. A64, no. 1, Article ID 013817, 2001.

[10] J. P. Gazeau and J. R. Klauder, "Coherent states for systems with discrete and continuous spectrum," *Journal of Physics A: Mathematical and General*, vol. 32, no. 1, pp. 123–132, 1999.

[11] T. Appl and D. H. Schiller, "Generalized hypergeometric coherent states," *Journal of Physics A: Mathematical and General*, vol. 37, no. 7, pp. 2731–2750, 2004.

[12] A. M. Perelomov, "Coherent states for arbitrary Lie group," *Communications in Mathematical Physics*, vol. 26, no. 3, pp. 222–236, 1972.

[13] R. Gilmore, "Geometry of symmetrized states," *Annals of Physics*, vol. 74, no. 2, pp. 391–463, 1972.

[14] R. Gilmore, "On properties of coherent states," *Revista Mexicana de Física*, vol. 143–18, 1974.

[15] E. Onofri, "A note on coherent state representations of Lie groups," *Journal of Mathematical Physics (Melville, NY, United States)*, vol. 16, no. 5, pp. 1087–1089, 1975.

[16] F. Hongyi, "The development of Dirac's symbolic method by virtue the IWOP technique," *Communications in Theoretical Physics*, vol. 31, no. 2, pp. 285–290, 1999.

[17] D. Popov and M. Popov, "Some operatorial properties of the generalized hypergeometric coherent states," *Physica Scripta*, vol. 90, no. 3, Article ID 035101, 2015.

[18] D. Popov, S. H. Dong, and M. Popov, "Diagonal ordering operation technique applied to Morse oscillator," *Annals of Physics (San Diego, CA, United States)*, vol. 362, pp. 449–472, 2015.

[19] D. Popov, R. Negrea, and M. Popov, "Gazeau–Klauder coherent states examined from the viewpoint of diagonal ordering operation technique," *Chinese Physics B*, vol. 25, no. 7, Article ID 070301, 2016.

[20] 2014 https://socratic.org/questions/how-do-you-test-a-power-series-for-convergence.

[21] A. M. Mathai and R. K. Saxena, "Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences," *Lect. Notes Math*, Vol. 348, Springer-Verlag, , Berlin, 1973.

[22] https://functions.wolfram.com/Bessel-TypeFunctions/BesselI/26/01/02/.

[23] https://functions.wolfram.com/HypergeometricFunctions/MeijerG/03/01/03/01/.

[24] I. S. Gradshteyn and I. M. Ryshik, *Table of Integrals, Series and Products*, Academic Press, Amsterdam, 2007.

[25] Y. L. Luke, "Inequalities for generalized hypergeometric functions," *Journal of Approximation Theory*, vol. 5, no. 1, pp. 41–65, 1972.

[26] https://functions.wolfram.com/HypergeometricFunctions/HypergeometricU/21/02/02/.