Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 1426724, 8 pages
https://doi.org/10.1155/2022/1426724

Research Article

Algebraic Properties of (ω, θ)-Complex Fuzzy Subgroups

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Received 28 May 2022; Revised 21 July 2022; Accepted 17 August 2022; Published 29 September 2022

Academic Editor: Musavarah Sarwar

This paper defined the notion of (ω, θ)-complex fuzzy sets, (ω, θ)-complex fuzzy subgroupoid, and (ω, θ)-complex fuzzy subgroups and described important examples under (ω, θ)-complex fuzzy sets. Additionally, we discussed the conjugacy class of group with respect to (ω, θ)-complex fuzzy normal subgroups. We define (ω, θ)-complex fuzzy cosets and elaborate the certain operation of this analog to group theoretic operation. We prove that factor group with regard to (ω, θ)-complex fuzzy normal subgroup forms a group and establishes an ordinary homomorphism from group to its factor group with regard to (ω, θ)-complex fuzzy normal subgroup. Moreover, we create the (ω, θ)-complex fuzzy subgroup of factor group.

1. Introduction

The study of group theory has a crucial orbit in mathematics owing to its utilitarian applications in different scientific domains including cryptography, mathematical biology, and chaos theory. The extensive understanding of the group allows us to evaluate the dynamic behavior of the real-world physical problems that have been effectively employed in mathematical problem modeling to tackle many complicated problems in the field of science and technology. On the other hand, the theory of fuzzy set plays a significant role for dealing uncertainty and vagueness. Zadeh [1] first proposed the concept of fuzzy sets and their operations. Subsequently, many researchers have analyzed various aspects of this theory and its applications. This theory also plays a vital role for studying differential equations. Many differential equations exhibit uncertainty and vagueness in their solutions [2]. Fuzzy set theory becomes more important to deal this type of information. Imai et al. [3] proposed the study of BCK/BCI-algebras in 1966 as a generalized conviction of set-theoretic difference. Rosenfeld [4] built up the structure of fuzzy subgroups on fuzzy sets in 1971. In 1979, fuzzy groups were redefined by Anthony and Sherwood [5]. The notions of inclusion, intersection, union, convexity, relation, etc., are extended to such sets, and different properties of these convictions in the context of fuzzy sets are established in [6]. Liu [7] introduced the concept of fuzzy invariant subgroups. In 1988, Choudhury [8] introduced the notion of fuzzy homomorphism between two groups and studied its effect on fuzzy subgroups. Mashour et. al. depicted many different key properties of fuzzy subgroups in [9]. Filep [10] extended the structure and construction of fuzzy subgroups of groups in 1992. Many researchers actively engaged the development of fuzzy set in group theory [11]. Gupta and Qi [12] presented the some typical T-operators and reviewed T-norm, T-conorm, and negation function under T-operators. Malik and Mordeson [13] defined the concept fuzzy subgroups of Abelian groups and discussed their various algebraic properties. Mishref [14] depicted the fuzzy normal series of finite groups. The idea of complex fuzzy set was introduced by Ramot et al. [15, 16] in 2002. In 2009, Zhang [17] et al. developed the various algebraic setups of complex fuzzy sets. In 2009, a new structure and construction of Q-fuzzy groups were described by Solairaju [18]. Al-Husban and Salleh [19]elaborated the notion of complex fuzzy hypergroups based on complex fuzzy spaces. The new applications of complex fuzzy sets in metric spaces, graph theory, group theory, and ring theory were introduced in [20–23]. Ma et al. [24] presented some new mathematical operations and laws of complex fuzzy set such as absorption law, involution law, symmetrical difference formula, simple difference, and disjunctive sum. Moreover, they developed a new algorithm using complex fuzzy sets for applications signals. Trevijano and Elorza [25] initiated the new concept of annihilator of fuzzy subgroups and discussed their various algebraic properties. The recent applications of complex fuzzy sets in ring theory and BCk/BCI algebras may be viewed in [26, 27]. Imtiaz [28] et al explored the new structure of ξ-complex fuzzy sets and ξ-complex fuzzy subgroups. Many authors [29–34] proposed several interesting techniques and approaches to solve complicated systems in fuzzy group theory, fuzzy ring theory, and fuzzy fractional calculus whereas the computational effects are very vague and straightforward. Gulzar [35] et al. presented the novel concept of complex fuzzy subfields. Verma [36] et al. discussed a systematic review on the advancement in study of fuzzy variational problems. Alolaiyan et al. [38] developed the structure of (α, β)-complex fuzzy subgroups.

In this article, we present a new concept (ω, θ)-complex fuzzy set (ω, θ)-CFS. First, we introduce some basic notions and some important properties which are needed in our paper. Second, in section three, we define (ω, θ)-complex fuzzy subgroups $((\omega, \theta))$-CFSGs and (ω, θ)-complex fuzzy normal subgroups and study their properties. Finally, we define the (ω, θ)-complex fuzzy cosets, and we obtain the quotient group regarding to (ω, θ)-complex normal fuzzy group (ω, θ)-CFNSG.

2. Preliminaries

In this section, we recall important concepts of complex fuzzy set and complex fuzzy subgroup.

Definition 1 (see [1]). A fuzzy subset is just like a function from a universal set to a unit interval [0, 1].

Definition 2 (see[15]). A CFS M of the universe of discourse X is mapping from nonempty set to unit disk and is described by the rule α_M: $X \longrightarrow \{x \in Ç\colon |z| \leq 1\}$, where Ç is set of complex numbers. The $\alpha_M(x) = \zeta_M(x)e^{i\varphi_M(x)}$ is complex membership function of CFS M, where $i = \sqrt{-1}$.

Definition 3 (see [22]). Let R be a underlying universe and $\delta = \{(x, \delta(x))\colon x \in R\}$ be fuzzy subset. Then, π-fuzzy subset of R is described as

$$\delta_\pi = \{(x, \delta_\pi(x))\colon \delta_\pi(x) = 2\pi\delta(x), \quad x \in R\}. \tag{1}$$

Definition 4 (see [22]). Let G be a group. A π-fuzzy set δ_π of G is a π-fuzzy subgroup of G if

(1) $\delta_\pi(xy) \geq \min\{\delta_\pi(x), \delta_\pi(y)\}$ for all $x, \quad y \in G$
(2) $\delta_\pi(x^{-1}) \geq \delta_\pi(x^{-1})$ for all $x \in G$

Definition 5 (see [22]). Let G be a underlying universe where $\delta = \{(x, \zeta_\delta(x)e^{i\varphi_\delta(x)})\colon x \in R\}$ and $\gamma = \{(x, \zeta_\gamma(x)e^{i\varphi_\gamma(x)})\colon x \in R\}$ be CFSs of G. Then,

(1) A CFS δ is homogeneous CFS if $\forall x, \quad y \in R$, $\zeta_\delta(x) \leq \zeta_\delta(y)$ if and only if $\varphi_\delta(x) \leq \varphi_\delta(y)$
(2) A CFS δ is homogeneous CFS with γ if, $\forall x, \quad y \in R$ such that $\zeta_\delta(x) \leq \zeta_\gamma(y)$ if and only if $\varphi_\delta(x) \leq \varphi_\gamma(y)$

Definition 6 (see [17]). Let $\delta = \{(x, \zeta_\delta(x)e^{i\varphi_\delta(x)})\colon x \in G\}$ and $\gamma = \{(x, \zeta_\gamma(x)e^{i\varphi_\gamma(x)})\colon x \in G\}$ be a CFSs of set X. Then, the following set-theoretic operation is defined as

(1) $(\delta \cap \gamma)(x) = \zeta_{(\delta\cap\gamma)}(x)e^{i\varphi_{(\delta\cap\gamma)}(x)} = \min\{\zeta_\delta(x)e^{i\varphi_\delta(x)}, \zeta_\gamma(x)e^{i\varphi_\gamma(x)}\}, \quad \forall x \in G$
(2) $(\delta \cup \gamma)(x) = \zeta_{(\delta\cup\gamma)}(x)e^{i\varphi_{(\delta\cup\gamma)}(x)} = \max\{\zeta_\delta(x)e^{i\varphi_\delta(x)}, \zeta_\gamma(x)e^{i\varphi_\gamma(x)}\}, \quad \forall x \in G$

Definition 7 (see [22]). Let G be a group and $A = \{(x, \mu_A(x) e^{i\varphi_A(x)})\colon x \in G\}$ be a homogeneous CFS of G. Then, A is said to be CFSG of G if the given below axioms are satisfied.

(1) $\zeta_\delta(xy)e^{i\varphi_\delta(xy)} \geq \min\{\zeta_\delta(x)e^{i\varphi_\delta(x)}, \zeta_\delta(y)e^{i\varphi_\delta(y)}\}$
(2) $\zeta_\delta(x^{-1})e^{i\varphi_\delta(x^{-1})} \geq \zeta_\delta x)e^{i\varphi_\delta(x)}$, for all $x, \quad y \in G$

Definition 8 (see [22]). Let G be group. A CFS $\delta = \{(x, \zeta_\delta(x)e^{i\varphi_\delta(x)})\colon x \in G\}$ of G is called a CFNSG of G if: $\zeta_\delta(xy)e^{i\varphi_\delta(xy)} = \zeta_\delta(yx)e^{i\varphi_\delta(yx)}$ for all $x, \quad y \in G$.

3. (ω, θ)-Complex Fuzzy Subgroups

In this section, we define (ω, θ)-CFSG and study their properties. We investigate the properties of (ω, θ)-CFNSGs and erect the quotient group with respect to (ω, θ)-CFNSGs.

Definition 9. Let $\delta = \{((x, \zeta_\delta x)e^{i\varphi_\delta(x)}); x \in G\}$ be complex fuzzy set of group G. For any $\omega \in [0,1]$ and $\theta \in [0, 2\pi]$, then the complex fuzzy set $\delta_{(\omega,\theta)}$ is an (ω, θ)-complex fuzzy set of G with regard to complex fuzzy set δ and is defined as $\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)} = \zeta_\delta(x)e^{i\varphi_\delta(x)} * \omega e^{i\theta} = (\zeta_\delta(x).\omega)e^{i(\varphi_{\delta_\theta}(x).\theta/2\pi)}$.

Example 1. Consider a universal set $Z = \{a, b, c, d, e\}$. The complex fuzzy set δ of Z is described as $\delta = \{(a, 0.2\ e^{(\pi/3i)}), (b, 0.1e^{(\pi/4i)}), (c, 0.4e^{(\pi/6i)}), (d, 0.7e^{(\pi/3i)}), (e, 0.3e^{(\pi/5i)})\}$. In the view Definition 9, we get (ω, θ)-complex fuzzy set of Z with respect to complex fuzzy set δ, for the value of $\omega = 0.2$, and $\theta = e^{\pi i}$, as follows: $\delta_{(\omega,\theta)} = \{(a, 0.04e^{(\pi/6i)}), (b, 0.02e^{(\pi/8i)}), (c, 0.08e^{(\pi/12i)}), (d, 0.14e\ (\pi/6i)), (e, 0.06e^{(\pi/10i)})\}$.

Definition 10. Let $\delta_{(\omega,\theta)}$ and $\gamma_{(\omega,\theta)}$ be two (ω, θ)-CFSs of G. Then, A (ω, θ)−CFS $\delta_{(\omega,\theta)}$ is homogeneous (ω, θ)-CFS if for all $x, y \in G$, we have $\zeta_{\delta_\omega}(x) \leq \zeta_{\delta_\omega}(y)$ if and only if $\varphi_{\delta_\theta}(x) \leq \varphi_{\delta_\theta}(y)$. A (ω, θ)-CFS $\delta_{(\omega,\theta)}$ is homogeneous (ω, θ)-CFS with $\gamma_{(\omega,\theta)}$ if for all $x, y \in G$, we have $\zeta_{\delta_\omega}(x) \leq \zeta_{\delta_\omega}(y)$ if and only if $\varphi_{\delta_\theta}(x) \leq \varphi_{\delta_\theta}(y)$.

Remark 1. If $\omega = 1$ and $\theta = 2\pi$ in the above definition, we obtain a classical complex fuzzy set.

Remark 2. Let $\delta_{(\omega,\theta)}$ and $\gamma_{(\omega,\theta)}$ be two (ω, θ)-CFSs of G. Then, $(\delta \cap \gamma)_{(\omega,\theta)} = \delta_{(\omega,\theta)} \cap \gamma_{(\omega,\theta)}$.

Definition 11. Let $\delta_{(\omega,\theta)}$ be an (ω, θ)-CFS of group G, for $\omega \in [0,1]$ and $\theta \in (0, 2\pi)$. A (ω, θ)-complex fuzzy subgroupoid of G is a fuzzy algebraic structure which satisfies the following condition:

$$\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} \geq \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}\right\}. \quad (2)$$

Example 2. Consider the group G of fourth roots of unity under usual multiplication. Define

$$\delta(x) = \begin{cases} 0.4e^{\pi/3i}, & \text{if } x \in \{1, i\} \\ 0.2e^{\pi/6i}, & \text{if } x \in \{-1, -i\} \end{cases}. \quad (3)$$

Take $\delta_{(\omega,\theta)}$ with $\omega = 0.1$ and $\theta = \pi$. Then, $\zeta_{\delta_{0.1}}(xy)$ $e^{i\varphi_{\delta_\pi}(xy)} \geq \min\left\{\zeta_{\delta_{0.1}}(x)e^{i\varphi_{\delta_\pi}(x)}, \zeta_{\delta_{0.1}}(y)e^{i\varphi_{\delta_\pi}(y)}\right\}$ for each $x, y \in \{1, -1, i, -i\}$. Thus, $\delta_{(\omega,\theta)}$ is $(\omega.\theta)$-complex fuzzy subgroupoid of G.

Definition 12. Let $\delta_{(\omega,\theta)}$ be an $(\omega, \theta)-$ CFS of group G for $\omega \in [0, 1]$ and $\theta \in [0, 2\pi]$. A (ω, θ)-CFSG of the group G is a fuzzy algebraic structure that satisfies the following axioms:

(1) $\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} \geq \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}\right\}$
(2) $\zeta_{\delta_\omega}(x^{-1})e^{i\varphi_{\delta_\theta}(x^{-1})} \geq \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}$ for all $x, y \in G$

Example 3. Consider the dihedral group $D_3 = \langle j, s\colon j^3 = s^2 = e, sj = j^2 s\rangle$. Let δ be CFS of G such that

$$\delta = \begin{cases} 0.8e^{\pi/3i} \text{ if } x \in \langle s\rangle \\ 0.7e^{\pi/6i} \text{ otherwise} \end{cases}. \quad (4)$$

Define $\delta_{(\omega,\theta)}$ for the value $\omega = 0.5$ and $\theta = \pi/2$ as follows:

$$\delta_{(0.5,\pi/2)}(x) = \begin{cases} 0.4e^{\pi/12i} \text{ if } x \in \langle s\rangle \\ 0.35e^{\pi/24i} \text{ otherwise} \end{cases}. \quad (5)$$

Take $x = sj$, $y = s$, and $xy = j^2$, we have

$$\zeta_{\delta_{0.5}}(xy)e^{i\varphi_{\delta_{\pi/2}}(xy)} = 0.35e^{\pi/24i}. \quad (6)$$

$$\begin{aligned} &\min\left\{\zeta_{\delta_{0.5}}(x)e^{i\varphi_{\delta_{\pi/2}}(x)}, \zeta_{\delta_{0.5}}(y)e^{i\varphi_{\delta_{\pi/2}}(y)}\right\}, \\ &= \min\left\{0.35e^{\pi/24i}, 0.4e^{\pi/12i}\right\} = 0.35e^{\pi/24i}. \end{aligned} \quad (7)$$

Moreover,

$$\delta_{(0.5,\pi/2)}\left(x^{-1}\right) \geq \delta_{(0.5,\pi/2)}(x). \quad (8)$$

Similarly, both conditions of Definition 12 are satisfied for all elements of D_3. Hence, $\delta_{(0.5,\pi/2)}(x)$ is CFSG of D_3.

Remark 3. If $\delta_{(\omega,\theta)}$ be an $(\omega, \theta)-$CFS of the group G for $\omega \in [0, 1]$. Then,

$$\zeta_{\delta_\omega}\left(x^{-1}y\right)e^{i\varphi_{\delta_\theta}(x^{-1}y)} \geq \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}\right\}. \quad (9)$$

Theorem 1. *Let $\delta_{(\omega,\theta)}$ be an $(\omega, \theta)-$CFSG of the group $G, \forall x, y \in G$. Then,*

(1) $\zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)} \leq \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$

(2) $\zeta_{\delta_\omega}(xy^{-1})e^{i\varphi_{\delta_\theta}(xy^{-1})} = \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$ which implies that $\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)} = \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}$

Proof. The proof of this theorem is obvious. □

Theorem 2. *Let G be finite group. If $\delta_{(\omega,\theta)}$ is an $(\omega, \theta)-$complex fuzzy subgroupoid of G, then $\delta_{(\omega,\theta)}$ is (ω, θ)-CFSG of G.*

Proof. Assume that x is a member of G, thus, the order of x is a finite number n such that $x^n = e$, where e is the identity element of the group G. We know that $x^{-1} = x^{n-1}$ by the application of Definition 11. Then, we have

$$\begin{aligned} &\zeta_{\delta_\omega}\left(x^{-1}\right)e^{i\varphi_{\delta_\theta}(x^{-1})} \\ &= \zeta_{\delta_\omega}\left(x^{n-1}\right)e^{i\varphi_{\delta_\theta}(x^{n-1})} \\ &= \zeta_{\delta_\omega}\left(x^{n-2}x\right)e^{i\varphi_{\delta_\theta}(x^{n-2}x)} \geq \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}. \end{aligned} \quad (10)$$

□

Theorem 3. *Let G be a group. If $\delta_{(\omega,\theta)}$ be an (ω, θ)-CFSG of G and for some $x \in G$, $\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)} = \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$, then $\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} = \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}$ for all $y \in G$.*

Proof. Given that $\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)} = \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$. Then, from Theorem 2, we have

$$\zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)} \leq \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \forall y \in G. \quad (11)$$

Consider

$$\begin{aligned} &\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} \geq \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}\right\}, \\ &\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} \geq \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}.\text{From Theorem 3.9.} \end{aligned} \quad (12)$$

Now, assume that

$$\begin{aligned} \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)} &= \zeta_\delta\left(x^{-1}xy\right)^{i\varphi_\delta(x^{-1}xy)} \\ &\geq \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)}\right\}. \end{aligned} \quad (13)$$

Again, from Theorem 2, we have

$$\min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)}\right\} = \zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)}. \quad (14)$$

Therefore, we obtain

$$\zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)} \geq \zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)}, \forall y \in G. \quad (15)$$

From (12) and (15), we have

$$\zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)} = \zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)}, \forall y \in G. \quad (16)$$

□

Theorem 4. *.If δ is CFSG of the group G, then δ is an (ω, θ)-CFSG of G.*

Proof. Assume that δ is a CFSG of G, for all $x, y \in G$. Consider

$$\begin{aligned}\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} &= \zeta_\delta(xy)e^{i\varphi_\delta(xy)} * \omega e^{i\theta}\\ &\geq \min\left\{\zeta_\delta(x)e^{i\varphi_\delta(x)}, \zeta_\delta(y)e^{i\varphi_\delta(y)}\right\} * \omega e^{i\theta}\\ &= \min\left\{\zeta_\delta(x)e^{i\varphi_\delta(x)} * \omega e^{i\theta}, \zeta_\delta(y)e^{i\varphi_\delta(y)} * \omega e^{i\theta}\right\}\\ &= \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}\right\}.\end{aligned} \tag{17}$$

Further, we assume that

$$\begin{aligned}\zeta_{\delta_\omega}\left(x^{-1}\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}\right)} &= \zeta_\delta\left(x^{-1}\right)^{i\varphi_\delta\left(x^{-1}\right)} * \omega e^{i\theta}\\ &\geq \zeta_\delta(x)e^{i\varphi_\delta(x)} * \omega e^{i\theta}\\ &= \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}.\end{aligned} \tag{18}$$

□

Theorem 5. *Let $P_{(\omega,\theta)}$ and $Q_{(\omega,\theta)}$ be two (ω,θ)-CFSGs of G, then $P_{(\omega,\theta)} \cap Q_{(\omega,\theta)}$ is also (ω,θ)-CFSGs of G.*

Proof. Given that $P_{(\omega,\theta)}$ and $Q_{(\omega,\theta)}$ be two (ω,θ)-CFSGs of G, for any $x, y \in G$. Consider

$$\begin{aligned}\zeta_{(P\cap Q)_\omega}(xy)e^{\varphi(P\cap Q)_\theta(xy)} &= \zeta_{P_\omega\cap Q_\omega}(xy)e^{i\varphi_{P_\theta\cap Q_\theta}(xy)}\\ &= \min\left\{\zeta_{P_\omega}(xy)e^{i\varphi_{P_{x\theta}}(xy)}, \zeta_{Q_\omega}(xy)e^{i\varphi_{Q_{x\theta}}(xy)}\right.\\ &\geq \min\left\{\begin{array}{l}\min\left\{\zeta_{P_\omega}(x)e^{i\varphi_{P_\theta}(x)}, \zeta_{P_\omega}(y)e^{i\varphi_{P_\theta}(y)}\right\},\\ \min\left\{\zeta_{Q_\omega}(x)e^{i\varphi_{Q_\theta}(x)}, \zeta_{Q_\omega}(y)e^{i\varphi_{Q_\theta}(y)}\right\}.\end{array}\right.\\ &= \min\left\{\begin{array}{l}\min\left\{\zeta_{P_\omega}(x)e^{i\varphi_{P_\theta}(x)}, \zeta_{Q_\omega}(x)e^{i\varphi_{Q_\theta}(x)}\right\},\\ \min\left\{\zeta_{P_\omega}(y)e^{i\varphi_{P_\theta}(y)}, \zeta_{Q_\omega}(y)e^{i\varphi_{Q_\theta}(y)}\right\}.\end{array}\right.\\ &= \min\left\{\zeta_{P_\omega\cap Q_\omega}(x)e^{i\varphi_{Q_\theta\cap P_\theta}(x)}, \mu_{Q_\omega\cap P_\omega}(y)e^{i\varphi_{P_\theta\cap Q_\theta}(y)}\right\},\\ &= \min\left\{\zeta_{(P\cap Q)_\omega}(x)e^{\varphi_{(P\cap Q)_\theta}(x)}, \zeta_{(P\cap Q)_\omega}(y)e^{\varphi_{(P\cap Q)_\theta}(y)}\right\}.\end{aligned} \tag{19}$$

Furthermore,

$$\begin{aligned}&\zeta_{(P\cap Q)_\omega}\left(x^{-1}\right)e^{\varphi_{(P\cap Q)_\theta}\left(x^{-1}\right)}\\ &= \zeta_{P_\omega\cap Q_\omega}\left(x^{-1}\right)e^{i\varphi_{Q_\theta\cap P_\theta}\left(x^{-1}\right)}\\ &= \min\left\{\zeta_{P_\omega}\left(x^{-1}\right)e^{i\varphi_{P_\theta}\left(x^{-1}\right)}, \zeta_{Q_\omega}\left(x^{-1}\right)e^{i\varphi_{Q_\theta}\left(x^{-1}\right)}\right\}\\ &\geq \min\left\{\zeta_{P_\omega}(x)e^{i\varphi_{P_\theta}(x)}, \zeta_{Q_\omega}(x)e^{i\varphi_{Q_\theta}(x)}\right\}\\ &= \zeta_{(P\cap Q)_\omega}(x)e^{\varphi_{(P\cap Q)_\theta}(x)}.\end{aligned} \tag{20}$$

Consequently, $P_{(\omega,\theta)} \cap Q_{(\omega,\theta)}$ is (ω,θ) of G. □

Remark 4. The converse of Theorem 10 may not be true.

Example 4. Consider a permutation group S_4. Consider two (ω,θ)-CFSGs $\delta_{(0.7,2\pi)}$ and $\gamma_{(0.7,2\pi)}$ of S_4, we take $\omega e^{i\theta} = 0.7e^{2\pi}$ are described as

$$\delta_{(0.7,2\pi)}(x) = \begin{cases}0.6e^{\pi/3} & \text{if } x \in \langle(13)\rangle\\ 0.5e^{\pi/5} & \text{otherwise}\end{cases}. \tag{21}$$

$$\gamma_{(0.7,2\pi)}(x) = \begin{cases}0.7e^{2\pi} & \text{if } x \in \langle(1324)\rangle\\ 0.4e^{\pi/6} & \text{otherwise}\end{cases}. \tag{22}$$

This implies that

$$\left(\delta_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)}\right)(x) = \begin{cases}0.7e^{2\pi} & \text{if } x \in \langle(1324)\rangle\\ 0.6e^{\pi/3} & \text{if } x \in \langle(13)\rangle - \langle(1324)\rangle\\ 0.5e^{\pi/5} & \text{otherwise}\end{cases}. \tag{23}$$

Take $x = (12)(34), y = (13)$, and $xy = (1432)$. Therefore,

$(\delta_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})\ (x) = 0.7e^{2\pi}$, $(\delta_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})\ (y) = 0.6e^{\pi/3}$ and $(\delta_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(xy) = 0.5e^{\pi/5}$ $(\delta_{(0.7,2\pi)} \cup \gamma_{(0.7,\ 2\pi)})(xy) \ngeq \min\{(\delta_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})\ (x), (\delta_{(0.7,2\pi)} \cup \gamma_{(0.7,2\pi)})(y)\}$.

Hence, this proves the claim.

Definition 13. Let $\delta_{(\omega,\theta)}$ be an (ω,θ)-CFSG of the group G. Then, (ω,θ)-CFS $x\delta_{(\omega,\theta)}(p) = (p, \zeta_{x\delta_\omega}(p)e^{i\varphi_{x\delta_\theta}(p)})$ of G is said to be a (ω,θ)-complex fuzzy left coset of G established by (ω,θ) and x. It is defined as

$$\begin{aligned}\zeta_{x\delta_\omega}(p)e^{i\varphi_{x\delta_\theta}(p)} &= \zeta_{\delta_\omega}\left(x^{-1}p\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}p\right)}\\ &= \zeta_\delta\left(x^{-1}p\right)e^{i\varphi_\delta\left(x^{-1}(p)\right)} * \omega e^{i\theta}, \forall p, x \in G.\end{aligned} \tag{24}$$

On the same way, we describe the (ω,θ)-complex fuzzy right coset $\delta_{(\omega,\theta)}x(p) = (p, \zeta_{\delta_\omega x}(p)e^{i\varphi_{\delta_\theta x}(p)}), x \in G$ is described as

$$\zeta_{\delta_\omega x}(p)e^{i\varphi_{\delta_\theta x}(p)} = \zeta_{\delta_\omega}\left(px^{-1}\right)e^{i\varphi_{\delta_\theta}\left(px^{-1}\right)} = \zeta_\delta\left(px^{-1}\right)e^{i\varphi_\delta\left(px^{-1}\right)} * \omega e^{i\theta}\forall p, x \in G. \tag{25}$$

Example 5. Take $G = \{(1), (1234), (1432), (13)(24), (14)(23), (12)(34), (24), (13)\}$ a group of order 8. Describe (ω, θ)-CFSG of G, $\omega = 1$ and $\theta = \pi/2$, such that

$$\delta_{(1,\pi/2)}(p) = \begin{cases} 0.6e^{\pi/2} & \text{if } x \in \{(1), (13)(24)\} \\ 0.5e^{\pi/4} & \text{if } x \in \{(14)(23), (12)(34)\}. \\ 0.4e^{\pi/6} & \text{otherwise} \end{cases} \tag{26}$$

From Definition 13, we have $\zeta_{x\delta_{(1,\pi/2)}}(p)e^{i\varphi_{x\delta_{(1,\pi/2)}}(p)} = \zeta_{\delta_{(1,\pi/2)}}(x^{-1}(p))e^{i\varphi_{\delta_{(1,\pi/2)}}}(x^{-1}(p))$. Hence, the $(1, \pi/2)$-complex fuzzy left coset of $\delta_{(1,\pi/2)}(p)$ in G for $x = (24)$ is defined as

$$x\delta_{(1,\pi/2)}(p) = \begin{cases} 0.6e^{\pi/2} & \text{if } x \in \{(13), (24)\} \\ 0.5e^{\pi/4} & \text{if } x \in \{(1432), (1234)\}. \\ 0.4e^{\pi/6} & \text{otherwise} \end{cases} \tag{27}$$

In the same way, we can define $(1, \pi/2)$-complex fuzzy right coset of $\delta_{(1,\pi/2)}(p)$, for all $x \in G$.

Definition 14. Let G be group and $\delta_{(\omega,\theta)}$ be an (ω, θ)-CFSG of G. Then, $\delta_{(\omega,\theta)}$ is said to be a complex fuzzy normal subgroup (invariant) (ω, θ)-CFNSG if $\delta_{(\omega,\theta)}(xy) = \delta_{(\omega,\theta)}(yx)$. Equivalently, (ω, θ)-CFSG $\delta_{(\omega,\theta)}$ is (ω, θ)-CFNSG of the group G if $\delta_{(\omega,\theta)}x(y) = x\delta_{(\omega,\theta)}(y)$, for all $x, y \in G$.

Remark 5. Let $\delta_{(\omega,\theta)}$ be an (ω, θ)-CFNSG of the group G. Then, $\delta_{(\omega,\theta)}(y^{-1}xy) = \delta_{(\omega,\theta)}(x)$ for all $x, y \in G$.

Theorem 6. *Every CFNSG of the group G is an (ω, θ)-CFNSG of G.*

Proof. Suppose that z, x are elements of G. Then,

$$\zeta_\delta\left(x^{-1}z\right)e^{i\varphi_\delta\left(x^{-1}z\right)} = \zeta_\delta\left(z^{-1}x\right)e^{i\varphi_\delta\left(z^{-1}x\right)}. \tag{28}$$

This implies that

$$\zeta_\delta\left(x^{-1}z\right)e^{i\varphi_\delta\left(x^{-1}z\right)} * \omega e^{i\theta} = \min\zeta_\delta\left(z^{-1}x\right)e^{i\varphi_\delta\left(z^{-1}x\right)} * \omega e^{i\theta}, \tag{29}$$

which implies that

$$\zeta_{x\delta_\omega}(z)e^{i\varphi_{x\delta_\theta}(z)} = \zeta_{\delta_\omega x}(z)e^{i\varphi_{x\delta_\theta}(z)}. \tag{30}$$

This implies that $x\delta_{(\omega,\theta)}(z) = \delta_{(\omega,\theta)}x(z)$. Consequently, $\delta_{(\omega,\theta)}$ is (ω, θ)-CFNSG of G. □

Theorem 7. *Let $\delta_{(\omega,\theta)}$ be an (ω, θ)-CFSG of the group G. Then, $\delta_{(\omega,\theta)}$ remains the same in the conjugacy class of G if and only if $\delta_{(\omega,\theta)}$ is an (ω, θ)-CFNSG.*

Proof. Suppose that $\delta_{(\omega,\theta)}$ is an (ω, θ)-CFNSG of the group G. Then,

$$\zeta_{\delta_\omega}\left(y^{-1}xy\right)e^{i\varphi_{\delta_\theta}\left(y^{-1}xy\right)} = \zeta_{\delta_\omega}\left(xyy^{-1}\right)e^{i\varphi_{\delta_\theta}\left(xyy^{-1}\right)} = \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \forall x, y \in G. \tag{31}$$

Conversely, assume that $\delta_{(\omega,\theta)}$ remains the same in every conjugate class of G. Then,

$$\begin{aligned}\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} &= \zeta_{\delta_\omega}\left(xyxx^{-1}\right)e^{i\varphi_{\delta_\theta}\left(xyxx^{-1}\right)} \\ &= \zeta_{\delta_\omega}\left(x(yx)x^{-1}\right)e^{i\varphi_{\delta_\theta}\left(x(yx)x^{-1}\right)} \\ &= \zeta_{\delta_\omega}(yx)e^{i\varphi_{\delta_\theta}(yx)}, \forall x, y \in G.\end{aligned} \tag{32}$$

□

Theorem 8. *Let G be a group and $\delta_{(\omega,\theta)}$ be an (ω, θ)-CFSG of G. Then, $\delta_{(\omega,\theta)}$ is an (ω, θ)-CFNSG if and only if $\zeta_{\delta_\omega}([x, y])e^{i\varphi_{\delta_\theta}([x,y])} \geq \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}}(x), \forall x, y \in G$.*

Proof. Take $\delta_{(\omega,\theta)}$ is an (ω, θ)-CFNSG of G. Let x, y be two members of G. Assume that

$$\begin{aligned}\zeta_{\delta_\omega}\left(x^{-1}y^{-1}xy\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}y^{-1}xy\right)} &\geq \min\left\{\zeta_{\delta_\omega}\left(y^{-1}xy\right)e^{i\varphi_{\delta_\theta}\left(y^{-1}xy\right)}, \zeta_{\delta_\omega}\left(x^{-1}\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}\right)}\right\} \\ &= \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}\right\}\zeta_{\delta_\omega}([x, y])e^{i\varphi_{\delta_\theta}([x,y])} \geq \zeta_\delta(x)^{i\varphi_{\delta_\theta}(x)}.\end{aligned} \tag{33}$$

Conversely, suppose that $\zeta_{\delta_\omega}([x,y])e^{i\varphi_{\delta_\theta}([x,y])} \geq \zeta_\delta(x) e^{i\varphi_{\delta_\theta}(x)}$. Let $x, r \in G$

$$\begin{aligned}\text{Consider}\zeta_{\delta_\omega}\left(x^{-1}rx\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}rx\right)} &= \zeta_{\delta_\omega}\left(rr^{-1}x^{-1}rx\right)e^{i\varphi_{\delta_\theta}\left(rr^{-1}x^{-1}rx\right)} \\ &\geq \min\left\{\zeta_{\delta_\omega}(r)e^{i\varphi_{\delta_\theta}(r)}, \zeta_{\delta_\omega}([r,x])e^{i\varphi_{\delta_\theta}([r,x])}\right\} \\ &= \zeta_{\delta_\omega}(r)e^{i\varphi_{\delta_\theta}(r)},\end{aligned} \tag{34}$$

$$\text{Thus}, \zeta_{\delta_\omega}\left(x^{-1}rx\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}rx\right)} \geq \zeta_{\delta_\omega}(r)e^{i\varphi_{\delta_\theta}(r)}\ \forall r, x \in G. \tag{35}$$

$$\begin{aligned}\text{Now}, \zeta_{\delta_\omega}(r)e^{i\varphi_{\delta_\theta}(r)} &= \zeta_{\delta_\omega}\left(xx^{-1}rxx^{-1}\right)e^{i\varphi_{\delta_\theta}\left(xx^{-1}rxx^{-1}\right)} \\ &\geq \min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}\left(x^{-1}rx\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}rx\right)}\right\}.\end{aligned} \tag{36}$$

Now, we present two potential cases: □

Case 1. If $\min\left\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(x^{-1}rx)e^{i\varphi_{\delta_\theta}(x^{-1}rx)}\right\} = \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}$

Then, we obtain $\zeta_{\delta_\omega}(r)e^{i\varphi_{\delta_\theta}(r)} \geq \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \forall r, x \in G$, which means that $\delta_{(\omega,\theta)}$ remains a constant, and the result is valid trivially.

Case 2. If $\min\{\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)}, \zeta_{\delta_\omega}(x^{-1}rx)e^{i\varphi_{\delta_\theta}(x^{-1}rx)}\} = \zeta_{\delta_\omega}(x^{-1}rx)e^{i\varphi_{\delta_\theta}(x^{-1}rx)}$.

Then, from (36), we have

$$\zeta_{\delta_\omega}(r)e^{i\varphi_{\delta_\theta}(r)} \geq \zeta_{\delta_\omega}\left(x^{-1}rx\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}rx\right)}. \tag{37}$$

From Equations (35) and (37), we have

$$\zeta_{\delta_\omega}(r)e^{i\varphi_{\delta_\theta}(r)} = \zeta_{\delta_\omega}\left(x^{-1}rx\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}rx\right)}. \tag{38}$$

Hence, $\delta_{(\omega,\theta)}$ is a constant

Theorem 9. *If $\delta_{(\omega,\theta)}$ is an (ω,θ)-complex fuzzy normal subgroup of G, Then, the set $\delta^e_{(\omega,\theta)} = \left\{p \in G\colon \delta_{(\omega,\theta)}(p^{-1}) = \delta_{(\omega,\theta)}(e)\right\}$ is an invariant subgroup of G.*

Proof. We have $\delta^e_{(\omega,\theta)} \neq \phi$ because $e \in \delta^e_{(\omega,\theta)}$. Let $p, q \in \delta^e_{(\omega,\theta)}$ be any members. Consider $\zeta_{\delta_\omega}(pq)e^{i\varphi_{\delta_\theta}(pq)} \geq \min\left\{\zeta_{\delta_\omega}(p) e^{i\varphi_{\delta_\theta}(p)}, \zeta_{\delta_\omega}(q)e^{i\varphi_{\delta_\theta}(q)}\right\} = \min\left\{\zeta\delta_\omega(e)e^{i\varphi_{\delta_\theta}(e)}, \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}\right\}$ which means that $\zeta_{\delta_\omega(pq)}e^{i\varphi_{\delta_\theta}(pq)} \geq \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$. However, $\zeta_{\delta_\omega(pq)}e^{i\varphi_{\delta_\theta}(pq)} \geq \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$. Therefore, $\zeta_{\delta_\omega(pq)}e^{i\varphi_{\delta_\theta}(pq)} \geq \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$. It means that $\delta_{(\omega,\theta)}(p^{-1}) = \delta_{(\omega,\theta)}(e)$; hence, $pq \in \delta^e_{(\omega,\theta)}$. Furthermore, $\zeta_{\delta_\omega}(q^{-1})e^{i\varphi_{\delta_\theta}(q^{-1})} \geq \zeta_{\delta_\omega}(q)e^{i\varphi_{\delta_\theta}(q)} = \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$. However, $\zeta_{\delta_\omega}(q)e^{i\varphi_{\delta_\theta}(q)} \leq \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$. Thus, $\delta^e_{(\omega,\theta)}$ is subgroup of group G. Let $p \in \delta^e_{(\omega,\theta)}$ and $q \in G$. We know $\zeta_{\delta_\omega}(q^{-1}pq)e^{i\varphi_{\delta_\theta}(q^{-1}pq)} = \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$. This implies that $q^{-1}pq \in \delta^e_{(\omega,\theta)}$. Therefore, $\delta^e_{(\omega,\theta)}$ is invariant subgroup. □

Theorem 10. *Let $\delta_{(\omega,\theta)}$ is an (ω,θ)-CFNSG of G. Then,*

(i) $x\delta_{(\omega,\theta)} = y\delta_{(\omega,\theta)}$ if and only if $x^{-1}y \in \delta^e_{(\omega,\theta)}$

(ii) $\delta_{(\omega,\theta)}x = \delta_{(\omega,\theta)}y$if and only if$xy^{-1} \in \delta^e_{(\omega,\theta)}$

Proof

(i) For any $x, y \in G$, we have $x\delta_{(\omega,\theta)} = y\delta_{(\omega,\theta)}$. Consider

$$\begin{aligned}&\zeta_{\delta_\omega}\left(x^{-1}y\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}y\right)} \\ &= \zeta_\delta\left(x^{-1}y\right)e^{i\varphi_\delta\left(x^{-1}y\right)} * \omega e^{i\theta} \\ &= \zeta_{x\delta_\omega}(y)e^{i\varphi_{x\delta_\omega}(y)} \\ &= \zeta_{y\delta_\omega}(y)e^{i\varphi_{y\delta_\omega}(y)} \\ &= \zeta_{\delta_\omega}\left(y^{-1}y\right)e^{i\varphi_{\delta_\theta}\left(y^{-1}y\right)} \\ &= \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}.\end{aligned} \tag{39}$$

Therefore, $x^{-1}y \in \delta^e_{(\omega,\theta)}$. Conversely, let $x^{-1}y \in \delta^e_{(\omega,\theta)}$ implies that $\zeta_{\delta_\omega}(x^{-1}y)e^{i\varphi_{\delta_\theta}(x^{-1}y)} = \zeta_\delta(e) e^{i\varphi_\delta(e)} * \omega e^{i\theta}$. Consider

$$\begin{aligned}&\zeta_{x\delta_\omega}(a)e^{i\varphi_{x\delta_\omega}(a)} \\ &= \zeta_\delta\left(x^{-1}a\right)e^{i\varphi_\delta\left(x^{-1}a\right)} * \omega e^{i\theta} \\ &= \zeta_{\delta_\omega}\left(x^{-1}a\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}a\right)} = \zeta_{\delta_\omega}\left(x^{-1}y\right)\left(y^{-1}a\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}a\right)} \\ &\geq \min\left\{\zeta_{\delta_\omega}\left(x^{-1}y\right)e^{i\varphi_{\delta_\theta}\left(x^{-1}y\right)}, \zeta_{\delta_\omega}\left(y^{-1}a\right)e^{i\varphi_{\delta_\theta}\left(y^{-1}a\right)}\right\} \\ &= \min\left\{\zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}, \zeta_{\delta_\omega}\left(y^{-1}a\right)e^{i\varphi_{\delta_\theta}\left(y^{-1}a\right)}\right\} \\ &= \zeta_{\delta_\omega}\left(y^{-1}a\right)e^{i\varphi_{\delta_\theta}\left(y^{-1}a\right)} \\ &= \zeta_{y\delta_\omega}(a)e^{i\varphi_{y\delta_\omega}(a)}.\end{aligned} \tag{40}$$

Exchange the function of x and y, and we get $\zeta_{y\delta_\omega}(a)e^{i\varphi_{y\delta_\theta}(a)} \geq \zeta_{x\delta_\omega}(a)e^{i\varphi_{x\delta_\theta}(a)}$. Therefore, $\zeta_{x\delta_\omega}(a) e^{i\varphi_{x\delta_\theta}(a)} = \zeta_{y\delta_\omega}(a)e^{i\varphi_{y\delta_\theta}(a)}$.

(ii) In the same way, we can show that as part (i). □

Theorem 11. *Let H be a group and $\delta_{(\omega,\theta)}$ be an (ω,θ)-CFNSG of H and let x, y, r and t be any members in H.*

If $x\delta_{(\omega,\theta)} = r\delta_{(\omega,\theta)}$ *and* $y\delta_{(\omega,\theta)} = t\delta_{(\omega,\theta)}$. *Then,* $xy\delta_{(\omega,\theta)} = rt\delta_{(\omega,\theta)}$.

Proof. We have $x\delta_{(\omega,\theta)} = r\delta_{(\omega,\theta)}$ and $y\delta_{(\omega,\theta)} = t\delta_{(\omega,\theta)}$. This means that $x^{-1}r, y^{-1}t \in \delta^{e}_{(\omega,\theta)}$. Consider $(xy)^{-1}(rt) = y^{-1}(x^{-1}r)t = y^{-1}(x^{-1}r)(yy^{-1})t = [y^{-1} \quad (x^{-1}r)(y)](y^{-1}t)$. As $\delta^{e}_{(\omega,\theta)}$ is normal subgroup of H. Thus, $(xy)^{-1}(rt) \in \delta^{e}_{(\omega,\theta)}$. As result, $xy\delta_{(\omega,\theta)} = rt\delta_{(\omega,\theta)}$. □

Theorem 12. *Assume that* $H/\delta_{(\omega,\theta)} = \{x\delta_{(\omega,\theta)}; x \in H\}$ *is the assemblage of all* (ω,θ)*-complex fuzzy cosets of* $\delta_{(\omega,\theta)}$ *is an* (ω,θ)*-CFNSG of the group* H. *Therefore, we can define a well-defined binary operation* $\star$ *of* $H/\delta_{(\omega,\theta)}$ *and is described as* $x\delta_{(\omega,\theta)}\star y\delta_{(\omega,\theta)} = xy\delta_{(\omega,\theta)} \forall x, y \in H$.

Proof. We have $x\delta_{(\omega,\theta)} = y\delta_{(\omega,\theta)}$ and $f\delta_{(\omega,\theta)} = g\delta_{(\omega,\theta)}$ for any $f, g, x, y \in H$. Let $c \in H$ by any element. Then, $[x\delta_{(\omega,\theta)}\star f\delta_{(\omega,\theta)}](c) = (xf\delta_{(\omega,\theta)}(c)) = (c, \zeta_{xf\delta_\omega}(c)e^{i\varphi_{x\delta_\omega}(c)})$. Consider

$$
\begin{aligned}
&\zeta_{xf\delta_\omega}(c)e^{i\varphi_{x\delta_\omega}(c)} \\
&= \zeta_{xf\delta_\omega}(c)e^{i\varphi_{x\delta_\omega}(c)}\star\omega e^{i\theta} \\
&= \zeta_{\delta_\omega}\left((xf)^{-1}c\right)e^{i\varphi_{\delta_\theta}\left((xf)^{-1}c\right)} \\
&= \zeta_{\delta_\omega}\left(\left(f^{-1}x^{-1}\right)c\right)e^{i\varphi_{\delta_\theta}\left((f^{-1}x^{-1})c\right)} \\
&= \zeta_{\delta_\omega}\left(f^{-1}\left(x^{-1}c\right)\right)e^{i\varphi_{\delta_\theta}\left(f^{-1}(x^{-1}c)\right)} \\
&= \zeta_{f\delta_\omega}\left(x^{-1}c\right)e^{i\varphi_{x\delta_\theta}\left(x^{-1}c\right)} \\
&= \zeta_{g\delta_\omega}\left(x^{-1}c\right)e^{i\varphi_{x\delta_\theta}\left(x^{-1}c\right)} \\
&= \zeta_{\delta}\left(g^{-1}\left(x^{-1}c\right)\right)e^{i\varphi_{\delta}\left(g^{-1}(x^{-1}c)\right)} \\
&= \zeta_{\delta}\left(x^{-1}\left(cg^{-1}\right)\right)e^{i\varphi_{\delta}\left(x^{-1}(cg^{-1})\right)} \\
&= \zeta_{x\delta_\omega}\left(cg^{-1}\right)e^{i\varphi_{x\delta_\theta}\left(cg^{-1}\right)} \\
&= \zeta_{y\delta_\omega}\left(cg^{-1}\right)e^{i\varphi_{y\delta_\theta}\left(cg^{-1}\right)} \\
&= \zeta_{\delta_\omega}\left(y^{-1}\left(cg^{-1}\right)\right)e^{i\varphi_{\delta_\theta}\left(y^{-1}(cg^{-1})\right)} \\
&= \zeta_{\delta_\omega}\left(\left(y^{-1}c\right)g^{-1}\right)e^{i\varphi_{\delta_\theta}\left((y^{-1}c)g^{-1}\right)} \\
&= \zeta_{\delta_\omega}\left(\left(y^{-1}g^{-1}\right)c\right)e^{i\varphi_{\delta_\theta}\left((y^{-1}g^{-1})c\right)} \\
&= \zeta_{\delta_\omega}\left((yg)^{-1}c\right)e^{i\varphi_{\delta_\theta}\left((yg)^{-1}c\right)} \\
&= \zeta_{yg\delta_\omega}(c)e^{i\varphi_{yg\delta_\theta}(c)}.
\end{aligned}
\tag{41}
$$

Therefore, $\star$ is well-defined operation and satisfied the associative property on $H/\delta_{(\omega,\theta)}$. Furthermore, $\zeta_{\delta_\omega}e^{i\varphi_{\delta_\theta}}\star\zeta_{x\delta_\omega}e^{i\varphi_{x\delta_\theta}} = \zeta_{e\delta_\omega}e^{i\varphi_{e\delta_\theta}}\star\zeta_{x\delta_\omega}e^{i\varphi_{x\delta_\theta}} = \zeta_{x\delta_\omega}e^{i\varphi_{x\delta_\theta}}$, and this means that $\zeta_{\delta_\omega}e^{i\varphi_{\delta}}$ is an identity element of $H/\delta_{(\omega,\theta)}$. Obviously, the inverse of each member of $H/\delta_{(\omega,\theta)}$ exists if $\zeta_{x\delta_\omega}e^{i\varphi_{x\delta_\theta}} \in H/\delta_{(\omega,\theta)}$, and then, there exists an element $\zeta_{x^{-1}\delta_\omega}e^{i\varphi_{x^{-1}\delta_\theta}} \in H/\delta_{(\omega,\theta)}$ such that $\zeta_{x^{-1}x\delta_\omega}e^{i\varphi_{x^{-1}x\delta_\theta}} = \zeta_{\delta_\omega}e^{i\varphi_{\delta_\theta}}$. Consequently, $H/\delta_{(\omega,\theta)}$ is a group. □

Lemma 1. *Let* G *be group then we can define* $L\colon G \longrightarrow G/\delta_{(\omega,\theta)}$ *to be a classical homomorphism from* G *onto* $G/\delta_{(\omega,\theta)}$ *and is described by* $L(x) = x\delta_{(\omega,\theta)}$ *with the kernel* $L = \delta^{e}_{(\omega,\theta)}$.

Proof. Let x, y be an arbitrary member of G, and then,

$$
\begin{aligned}
L(xy) = xy\delta_{(\omega,\theta)} &= \mu_{xy\delta_\omega}e^{i\varphi_{xy\delta_\theta}} = \zeta_{x\delta_\omega}e^{i\varphi_{x\delta_\theta}}\star\zeta_{y\delta_\omega}e^{i\varphi_{y\delta_\theta}} \\
&= x\delta_{(\omega,\theta)}\star y\delta_{(\omega,\theta)} = L(x)\star L(y).
\end{aligned}
\tag{42}
$$

Therefore, L is a homomorphism. In the same way, f is also a homomorphism

$$
\begin{aligned}
\text{Now, Ker}L &= \left\{x \in G\colon L(x) = e\delta_{(\omega,\theta)}\right\} \\
&= \left\{x \in G\colon x\delta_{(\omega,\theta)} = e\delta_{(\omega,\theta)}\right\} \\
&= \left\{x \in G\colon xe^{-1} \in \delta^{e}_{(\omega,\theta)}\right\} \\
&= \left\{x \in G\colon x \in \delta^{e}_{(\omega,\theta)}\right\} \\
&= \delta^{e}_{(\omega,\theta)}.
\end{aligned}
\tag{43}
$$

□

Theorem 13. *Let* $\delta^{e}_{(\omega,\theta)}$ *be a normal subgroup of* G. *If* $\delta_{(\omega,\theta)} = \{(x, \zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)})\colon x \in G\}$ *is* (ω,θ)*-CFSG, then the* (ω,θ)*-CFS* $\overline{\delta}_{(\omega,\theta)} = \{(x\delta^{e}_{(\omega,\theta)}, \overline{\zeta}_{\delta_\omega}(x\delta^{e}_{(\omega,\theta)}e^{i\overline{\varphi}_{\delta_\theta}(x\delta^{e}_{(\omega,\theta)})}))\colon x \in G\}$ *of* $G/\delta^{e}_{(\omega,\theta)}$ *is also* (ω,θ)*-CFSG of* $G/\delta^{e}_{(\omega,\theta)}$, *where* $\overline{\zeta}_{\delta_\omega}(x\delta^{e}_{(\omega,\theta)}e^{i\overline{\varphi}_{\delta_\theta}(x\delta^{e}_{(\omega,\theta)})}) = \max\{\zeta_{\delta_\omega}(xa)e^{i\varphi_{\delta_\theta}(xa)}\colon a \in \delta^{e}_{(\omega,\theta)}\}$.

Remark 6. If $\delta_{(\omega,\theta)}$ is an (ω,θ)-CFSG of G, $x \in G$, and $\zeta_{\delta_\omega}(xy)e^{i\varphi_{\delta_\theta}(xy)} = \zeta_{\delta_\omega}(y)e^{i\varphi_{\delta_\theta}(y)}$, for all $y \in G$, then $\zeta_{\delta_\omega}(x)e^{i\varphi_{\delta_\theta}(x)} = \zeta_{\delta_\omega}(e)e^{i\varphi_{\delta_\theta}(e)}$.

4. Conclusion

We have presented a new theory of (ω,θ)-CFSs and (ω,θ)-CFSGs as a powerful extension of complex fuzzy sets and complex fuzzy subgroups. Moreover, we have created the (ω,θ)-CFSG of the factor group. Each concept introduced in this paper is explained with clear examples. This algebraic structure will play an important role to handle many other group theory problems under a new complex fuzzy environment [37, 38].

Data Availability

No real data were used to support this study. The data used in this study are hypothetical and anyone can use them by citing this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

The authors would like to thank the Deanship of Scientic Research of King Abdulaziz University, Jeddah, Saudi Arabia, for technical and financial support.

References

[1] L. A. Zadeh, "Fuzzy Sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] M. K. Alaoui, F. M. Alharbi, and S. Zaland, "Novel analysis of fuzzy physical models by generalized fractional fuzzy operators," *Journal of Function spaces*, vol. 202212 pages, Article ID 2504031, 2022.

[3] Y. Imai and K. Iseki, "On axiom system of propositional calculi. XIV," *Proceedings of the Japan Academy*, vol. 42, no. 1, pp. 19–22, 1966.

[4] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.

[5] J. M. Anthony and H. Sherwood, "Fuzzy groups redefined," *Journal of Mathematical Analysis and Applications*, vol. 69, no. 1, pp. 124–130, 1979.

[6] M. Mizumoto and K. Tanaka, "Fuzzy sets and their operations," *Information and Control*, vol. 48, no. 1, pp. 30–48, 1981.

[7] W. J. Liu, "Fuzzy invariant subgroups and fuzzy ideals," *Fuzzy Sets and Systems*, vol. 8, no. 2, pp. 133–139, 1982.

[8] F. P. Choudhury, A. B. Chakraborty, and S. S. Khare, "A note on fuzzy subgroups and fuzzy homomorphism," *Journal of Mathematical Analysis and Applications*, vol. 131, no. 2, pp. 537–553, 1988.

[9] A. S. Mashour, H. Ghanim, and F. I. Sidky, "Normal fuzzy subgroups," *Series.mathematics*, vol. 20, no. 2, pp. 53–59, 1990.

[10] L. Filep, "Structure and construction of fuzzy subgroups of a group," *Fuzzy Sets and Systems*, vol. 51, no. 1, pp. 105–109, 1992.

[11] I. J. Kumar, P. K. Saxena, and P. Yadav, "Fuzzy normal subgroups and fuzzy quotients," *Fuzzy Sets and Systems*, vol. 46, no. 1, pp. 121–132, 1992.

[12] M. M. Gupta and J. Qi, "Theory of T-norm and fuzzy inference methods," *Fuzzy Sets and Systems*, vol. 40, pp. 431–450, 1991.

[13] D. S. Malik, J. N. Mordeson, and P. S. Nair, "Fuzzy normal subgroups in fuzzy subgroups," *Journal of the Korean Mathematical Society*, vol. 29, no. 1, pp. 1–8, 1992.

[14] M. Atif Mishref, "Normal fuzzy subgroups and fuzzy normal series of finite groups," *Fuzzy Sets and Systems*, vol. 72, no. 3, pp. 379–383, 1995.

[15] D. Ramot, R. Milo, and M. Friedman, "Kandel A Complex fuzzy sets," *IEEE Transactions on Fuzzy Systems*, vol. 10, pp. 450–461, 2002.

[16] D. Ramot, M. Friedman, G. Langholz, and A. Kandel, "Complex fuzzy logic," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 4, pp. 450–461, 2003.

[17] G. Q. Zhang, T. S. Dillon, K. Y. Cai, J. Ma, and J. Lu, "Operation properties and δ-equalities of complex fuzzy sets," *International Journal of Approximate Reasoning*, vol. 50, pp. 1227–1249, 2009.

[18] A. Solairaju and R. Nagarajan, "A new structure and construction of Q-fuzzy subgroup," *Advances in Fuzzy Mathematics*, vol. 4, pp. 23–29, 2009.

[19] A. Al-Husban and A. R. Salleh, "Complex fuzzy group based on complex fuzzy space," *Global Journal of Pure and Applied Mathematics*, vol. 12, pp. 1433–1450, 2016.

[20] D. Singh, V. Joshi, M. Imdad, and P. Kumam, "A novel framework of complex valued fuzzy metric spaces and fixed point theorems1," *Journal of Intelligent and Fuzzy Systems*, vol. 30, no. 6, pp. 3227–3238, 2016.

[21] P. Thirunavukarasu, R. Suresh, and K. K. Viswanathan, "Energy of complex fuzzy graph," *Int. J. Math. Sci. Eng. Appls*, vol. 10, pp. 243–248, 2016.

[22] M. O. Alsarahead and A. G. Ahmad, "Complex fuzzy subgroups," *Applied Mathematical Sciences*, vol. 11, pp. 2011–2021, 2017.

[23] M. O. Alsarahead and A. G. Ahmad, "Complex fuzzy subrings," *International Journal of Pure and Applied Mathematics*, vol. 117, pp. 563–577, 2017.

[24] X. Ma, J. Zhan, M. Khan, M. Zeeshan, S. Anis, and A. S. Awan, "Complex fuzzy sets with Applications in signals," *Computational and Applied Mathematics*, vol. 38, no. 4, pp. 150–234, 2019.

[25] S. Ardanza-Trevijano, M. J. Chasco, and J. Elorza, "The annihilator of fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 369, pp. 122–131, 2019.

[26] M. Gulistan, N. Yaqoob, S. Nawaz, and M. Azhar, "A study of (α,β)-complex fuzzy hyperideals in non-associative hyperrings," *Journal of Intelligent and Fuzzy Systems*, vol. 36, pp. 1–17, 2019.

[27] Y. B. Yun and X. L. Xin, "Complex fuzzy sets with application in BCK/BCI-Algebras," *Bull. Sect. Log.*vol. 48, pp. 173–185, 2019.

[28] A. Imtiaz, U. Shuaib, H. Alolaiyan, A. Razaq, and M. Gulistan, "On structural Properties of ξ-complex fuzzy sets and their application," *Complexity International*, vol. 2020, Article ID 2038724, 13 pages, 2020.

[29] M. Gulzar, G. Abbas, and F. Dilawar, "Algebraic properties of ω-Q-fuzzy subgroup," *International Journal of Mathematics and Computer Science*, vol. 15, pp. 265–274, 2020.

[30] M. Gulzar, D. Alghazzawi, M. H. Mateen, and N. Kausar, "A certain class of t-intuitionistic fuzzy subgroups," *IEEE Access*, vol. 8, pp. 163260–163268, 2020.

[31] M. Gulzar, M. H. Mateen, D. Alghazzawi, and N. Kausar, "A novel applications of complex intuitionistic fuzzy sets in group theory," *IEEE Access*, vol. 8, pp. 196075–196085, 2020.

[32] M. Gulzar, D. Alghazzawi, M. H. Mateen, and M. Premkumar, "On some characterization of Q complex fuzzy subrings," *The Journal of Mathematics and Computer Science*, vol. 22, no. 3, pp. 295–305, 2021.

[33] U. Ali, H. A. Alyousef, K. Ahmed, and S. Ali, "Solving Nonlinear Fractional differential equations for Contractive and weakly compatible Mapping in neutrosophic metric spaces," *Journal of Function Spaces*, vol. 202219 pages, Article ID 1491683, 2022.

[34] M. Aresh, El-Tantawy, B. M. Alotaibi, and S. Zaland, "Study of fuzzy fractional third-order dispersive KdV equation in a plasma under Atangana-Bleanu Derivative," *Journal of Function Spaces*, vol. 2022, Article ID 7922001, 13 pages, 2022.

[35] M. Gulzar, F. Dilawar, D. Alghazzawi, and M. H. Mateen, "A note on complex fuzzy subfield," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 21, no. 2, pp. 1048–1056, 2021.

[36] M. Verma, C. Y. Chen, A. Kilicman, and R. Mat Hasim, "A systematic Review on the advancement in the study of fuzzy variational problems," *Journal of Function spaces*, vol. 202214 pages, Article ID 8037562, 2022.

[37] N. P. Mukherjee and P. Bhattacharya, "Fuzzy normal subgroups and fuzzy cosets," *Information Sciences*, vol. 34, no. 3, pp. 225–239, 1984.

[38] H. Alolaiyan, H. A. Alshehri, M. H. Mateen, D. Pamucar, and M. Gulzar, "A novel algebraic structure of (α,β)-complex fuzzy subgroups," *Entropy*, vol. 23, no. 8, p. 992, 2021.