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Research Article

Computing Connection Distance Index of Derived Graphs

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Distance based topological indices (TIs) play a vital role in the study of various structural and chemical aspects for the molecular graphs. The first distance-based TI is used to find the boiling point of paraffin. The connection distance (CD) index is a latest developed TI that is defined as the sum of all the products of distances between pair of vertices with the sum of their respective connection numbers . In this paper, we computed CD indices of the different derived graphs (subdivision graph $S(G)$, vertex-semitotal graph $R(G)$, edge-semitotal graph $Q(G)$ and total graph $T(G)$ obtained from the graph G under various operations of subdivision in the form of degree distance (DD) and CD indices of the basic graphs including some other algebraic expressions.

1. Introduction

A topological index (TI) is a function from the set of simple graphs $\mathbb{F}$ to the set of real numbers that assigns a unique number to each graph G belonging to $\mathbb{F}$. More importantly, it remains constant for the isomorphic graphs that is if $G_1 \cong G_2$ then $TI(G_1) = TI(G_2)$. Many topological indices have been introduced for the molecular graphs in the chemical graph theory to predict the certain structural and chemical properties such as vaporization, freezing point, boiling point, volume, density, weight and physicochemical properties of chemical bound [1]. Moreover, most rapidly growing fields of science such as chemistry, mathematics and information sciences provide effective research achievements in the last decade and one of the combinations of these subjects is called by cheminformatics which studies two relationships techniques quantitative structures property relationships (QSPR) and quantitative structure activity relationships (QSAR) for the different molecular structures, see [2–4].

Wiener (1947) [5] was first scientist who laid the foundation of chemical graph theory. In chemical graph theory, atoms and bounds are taken as vertices and edges. He discovered through his research work that there is a close correlation between the sums of the distances among pairs of vertices and the boiling points of paraffine. Almost, after passage of three quarters of the 20th century (1972) [6] discovered degree-based indices, (First and Second Zagreb indices), which were utilized to calculate the total π-electron energy of molecules. After these developments, many TIs were introduced which are found to be highly useful for the study of different physicochemical properties of chemical compounds. Degree distance index [7] and Gutman index [8] are the most important distance-degree based TIs. Ali and Trinajstic [9] (2018) restudied the first Zagreb connection index (ZC_1), second Zagreb connection index (ZC_2) and the modified first Zagreb connection index (ZC_1^*). Recently, Javaid et al. [10] defined a new distance-based TI called by connection distance index as the sum of all the products of distances between pair of vertices with the sum of their respective connection numbers.

Yan et al. [11] defined the new graphs called as subdivided graph $S(G)$, vertex-total graph $R(G)$, edge-total graph $Q(G)$ and total graph $T(G)$ with the help of the subdivision-related operations S, R, Q and T respectively. Xu et al. [12] have determined the degree distance index of the derived graphs. Bahadur et al. [13] have determined Gutman index of derived graphs. In particular, on different families of graphs for the result about degree based TIs, see [14, 15] (line graph), [16, 17] (sum-graphs), [18, 19] (derived graphs), [20, 21] (extremal graphs), [22, 23] (generalized sum graphs), [24] (nanotube) and distance based TIs, see [25–27].

In this paper, firstly, we have determined the exact values of connection numbers of old (black) vertices and new (white) for derived graph $S(G)$ and bounded values of connection numbers for $\{C_3, C_4\}-$ graph of $R(G), Q(G)$ and $T(G)$ were determined. Then, finally, bounded and exact values of connection distance (CD) of derived graphs $\{S(G), R(G), Q(G), T(G)\}$ are determined by Theorems and Corollaries. Section 2 consists of applications, Section 3 comprises some related definitions, Section 4 covers the main results of connection distance index (CD) for the derived graphs and Section 5 sum up our findings.

2. Applications

From the last two decades, the worth investing problem of finding out the physicochemical properties of the molecular structures is attracting the attention of chemists and mathematicians continuously. The TIs have been used to: predict the solubility of fullerene C_{60} [28], find the ionic liquids densities [1], optimize QSPR models (Simplex optimisation of generalized TIs) [29], manufacture the anticancer drugs [30, 31], calculate molecular van der Waals areas or volumes [32] and measure the vaporisation, sublimation, formation & combustion for the monocarboxylic acids $(C_2H_4O_2 - C_{20}H_{40}O_2)$ [4]. Recently, International Academy of Mathematical Chemistry checked thirteen physicochemical properties of octane isomers (heat capacity at T constant, heat capacity at P constant, density, boiling point, entropy, acentric factor, enthalpy of formation, octanol-water partition coefficient, enthalpy of vaporisation, standard enthalpy of formation, molar volume and total surface tension) with the help of the connection number-based TIs and declared that the chemical capability of the Zagreb connection indices is better than the ordinary Zagreb indices for the entropy and acentric factor of the octane isomers, for detail see [9]. In addition, Javaid et al. [33] presents comparison of correlation coefficients between different TIs and eleven physicochemical properties of octane isomers (boiling point, heat capacity, entropy, density,mean radius, change in heat of vaporization, standard heat of formation, acentric factor, enthalpy of vaporization and standard enthalpy of vaporization) which shows that Gutman connection index is a very useful TI for the prediction of entropy, acentric factor, enthalpy of vaporization and standard enthalpy of vaporization.

3. Preliminaries

Throughout in this research paper, we take a simple and connected graphs G with $V(G) = \{u_i\colon 1 \le i \le n\}$ and $E(G) = \{e_i\colon 1 \le i \le m\}$ such that $|V(G)| = n$ and $|E(G)| = m$. For more basic notations of graphs, see [34, 35]. Now, we define some most frequent used definition as follows

- The number of vertices at distance one from a vertex u_i is called its degree and it is denoted by $\delta(u_i)$.
- Degree of an edge $e_i = u_j u_k$ is given by $\delta(e_i) = \delta(u_j) + \delta(u_k) - 2$, where $1 \le i \le m$ for some $1 \le j, k \le n$.
- The distance between two vertices $u_i, u_j \in V(G)$ is denoted by $d(u_i, u_j)$ and defined as the length of the shortest path between both the vertices u_i and u_j for $1 \le i, j \le n$.
- The distance between two edges $e_i = u_p u_q$ and $e_j = u_r u_s$ is defined as $d_G(e_i, e_j) = \min\{d_G(u_p, u_r), d_G(u_p, u_s), d_G(u_q, u_r), d_G(u_q, u_s)\}$, where $1 \le i, j \le m$ and $1 \le p, q, r, s \le n$.
- The distance between one vertex u_i and one edge $e_j = u_p u_q$ is defined as $d_G(u_i, e_j) = \min\{d_G(u_i, u_p), d_G(u_i, u_q)\}$, where $1 \le j \le m$ and $1 \le i, p, q \le n$.
- The number of vertices at distance two from a vertex u_i is called its connection number of u_i and it is denoted by $\tau(u_i)$ for $1 \le i \le n$.

More details about aforesaid definitions can be obtained from [9, 36, 37].

Some important and related TIs are the following:

Definition 1 (see [5]). Let G be a connected graph of order n, then its Wiener index is

$$W(G) = \frac{1}{2} \sum_{u_i, u_j \in V(G)} d_G(u_i, u_j), \tag{1}$$

where $1 \le i, j \le n$.

Definition 2 (see [6]). Let G be a connected graph of order n, then first and second Zagreb index are defined as

$$M_1(G) = \sum_{u_i u_j \in E(G)} \left[\delta_G(u_i) + \delta_G(u_j)\right] = \sum_{u_i \in V(G)} \left[\delta_G(u_i)^2.\right. \tag{2}$$

$$M_2(G) = \sum_{u_i u_j \in E(G)} \left[\delta_G(u_i)\delta_G(u_j)\right], \tag{3}$$

where $1 \le i, j \le n$.

Definition 3 (see [38]). Let G be a connected graph of order n, then first and second Zagreb coindex are defined as

$$\overline{M_1(G)} = \sum_{u_i u_j \notin E(G)} \left[\delta_G(u_i) + \delta_G(u_j)\right]. \tag{4}$$

$$\overline{M_2(G)} = \sum_{u_i u_j \notin E(G)} \left[\delta_G(u_i)\delta_G(u_j)\right], \tag{5}$$

where $1 \le i, j \le n$.

Definition 4 (see [6]). Let G be a connected graph of order n and size m, then edge version of Wiener index is defined as

$$W_e(G) = \sum_{\{e_i,e_j\}\subseteq E(G)} \left[d_G(e_i,e_j)+1\right], \qquad (6)$$

where $1 \le i, j \le m$.

Definition 5 (see [35]). Let G be a connected graph of order n, then the degree distance index of G is

$$D\,D(G) = \frac{1}{2}\sum_{u_i,u_j\in V(G)} \left\{d(u_i,u_j)\left(\delta(u_i)+\delta(u_j)\right)\right\}, \qquad (7)$$

where $1 \le i, j \le n$.

Definition 6 (see [12]). Let G be a connected graph of order n and size m, then the edge version of degree distance index of G is

$$DD_e(G) = \sum_{\{e_i,e_j\}\subseteq E(G)} \left[d_e(e_i,e_j)+1\right]\left[\delta(e_i)+\delta(e_j)\right)], \qquad (8)$$

where $1 \le i, j \le m$.

Definition 7 (see [8]). Let G be a connected graph, then the Gutman index of G is

$$\text{Gut}(G) = \frac{1}{2}\sum_{u_i,u_j\in V(G)} \left\{d(u_i,u_j)\left(\delta(u_i)\delta(u_j)\right)\right\}, \qquad (9)$$

where $1 \le i, j \le n$.

Definition 8 (see [10]). Connection Distance index of a graph G (denoted by $C\,D(G)$) is defined as

$$C\,D(G) = \sum_{\{u_i,u_j\}\subseteq V(G)} d_G(u_i,u_j)\left[\tau_G(u_i)+\tau_G(u_j)\right], \qquad (10)$$

where $1 \le i, j \le n$ or

$$C\,D(G) = \frac{1}{2}\sum_{u,v\in V(G)} \{d(u,v)(\tau(u)+\tau(v))\}, \qquad (11)$$

Definition 9 (see [10]). Gutman Connection index of a graph G (denoted by $GC(G)$) is defines as

$$GC(G) = \sum_{\{u_i,u_j\}\subseteq V(G)} d_G(u_i,u_j)\left[\tau_G(u_i)\tau_G(u_j)\right], \qquad (12)$$

where $1 \le i, j \le n$ or

$$GC(G) = \frac{1}{2}\sum_{u,v\in V(G)} \{d(u,v)(\tau(u)\tau(v))\}. \qquad (13)$$

Yan et al. [11] defined the four operations S, R, Q and T on the graph G and obtained the four new graphs from G as follows:

- $S(G)$ is obtained from G if a new vertex w_i is inserted in every edge e_i of G. The already existing vertices u_i of G are named as old or black vertices while new vertices w_i are also named new or white vertices.
- $R(G)$ is formed by assigning a new vertex w_i corresponding to each edge e_i of G and this new vertex w_i is connected with the end vertices of the respective edge e_i.
- $Q(G)$ is formed from $S(G)$ if two white vertices w_i and w_j are further joined together when corresponding edges e_i and e_j have one common end vertex.
- $T(G)$ is formed from $R(G)$ if two white vertices w_i and w_j are further joined together when corresponding edges e_i and e_j have one common end vertex.

4. Relation Between Connection Numbers and Degrees

In order to determine connection numbers or bounded values of connection numbers of derived graphs $S(G)$, $R(G)$, $Q(G)$ and $T(G)$, we will develop Lemma 1 to Lemma 4. Before to the proofs of these results, in particular we consider Figure 1 and Figure 2 to present some trees, cycles and graphs consisting of cycles of order 3 and 4.

Lemma 1. *For a simple and connected graph G (i)* $\tau_{S(G)}(u_i) = \delta_{(G)}(u_i)$ *and (ii)* $\tau_{S(G)}(w_i) = \delta_{(G)}(u_j) + \delta_{(G)}(u_k) - 2 = \delta(e_i)$ *where* w_i *is a new vertex corresponding to edge* $e_i = u_ju_k$.

Proof 1. On applying S-operation, subdivision graph $S(G)$, each vertex u_i is connected with $\delta_{(G)}(u_i) = d_i$ new (white) vertices. So, $\delta_{(G)}(u_i) = d_i$ new (white) vertices w_i are at distance one from old vertices u_i. Further d_i old vertices u_j are at distance two from u_i in $S(G)$ (which are at distance one in G)

Also, each new vertex (white) w_i corresponds to each edge $e_i = u_ju_k$ is at distance one from u_j and u_k. So, number of vertices at distance 2 from w_i are $\delta_{(G)}(u_j) - 1 + \delta_{(G)}(u_k) - 1$ (see Figure 3 and Figure 4). Therefore $\tau_{S(G)}(w_i) = \delta_{(G)}(u_j) + \delta_{(G)}(u_k) - 2 = \delta_{(G)}(e_i)$. □

Lemma 2. *Let G be a simple and connected graph*

(a) If G is a $\{C_3, C_4\}-$ *free graph, then*

- *(i)* $\tau_{R(G)}(u_i) = 2\tau_{(G)}(u_i)$ *and*
- *(ii)* $\tau_{R(G)}(w_i) = 2[\delta_{(G)}(u_j) + \delta_{(G)}(u_k)] - 4 = 2[\delta_{(G)}(e_i)]$.

(b) If G is a $\{C_3, C_4\}-$ *graph, then*

- *(i)* $\tau_{R(G)}(u_i) \leqslant 2\tau_{(G)}(u_i) + r$, *where* $r = \max\{r_i\}$ *and* r_i *are number of* C_3 *and* C_4 *cycles connected with* u_i *in G.*
- *(ii)* $\tau_{R(G)}(w_i) \leqslant 2[\delta_{(G)}(u_j) + \delta_{(G)}(u_k)] - 4 - s = 2[\delta_{(G)}(e_i)] - s$, *where* $s = \max\{s_i\}$ *and* s_i *are number of* C_3 *cycles connected with* u_i *in G.*

Proof 2. On applying $R-operation$, a new vertex w_i is joined with the corresponding vertices of edge $e_i = u_ju_k$.

Case (a)

If G is a $\{C_3, C_4\}-$ free graph, then the degree of each vertex u_i in $R(G)$ is twice the degree u_i in G i.e. $\delta_{R(G)}(u_i) = 2\delta_G(u_i)$ (see Figure 5) . Then their connection number of

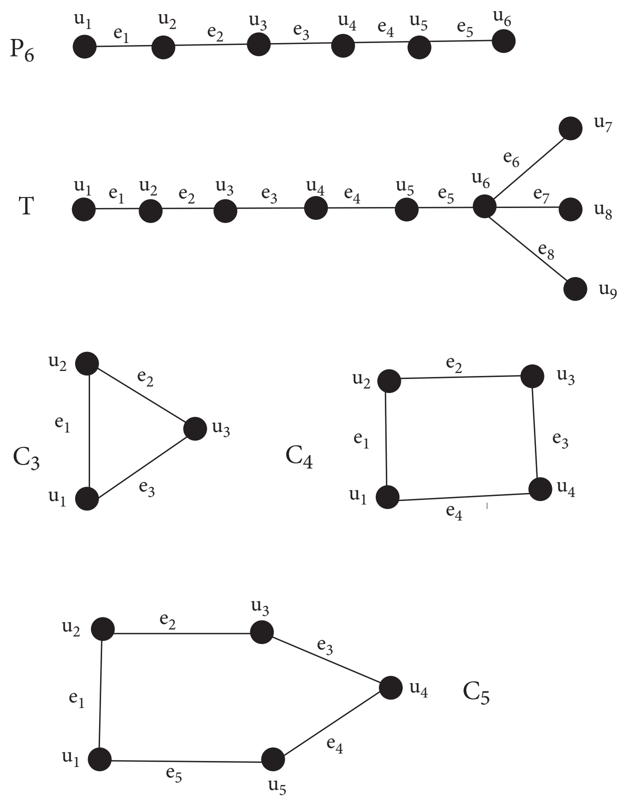

FIGURE 1: Some trees and cycles.

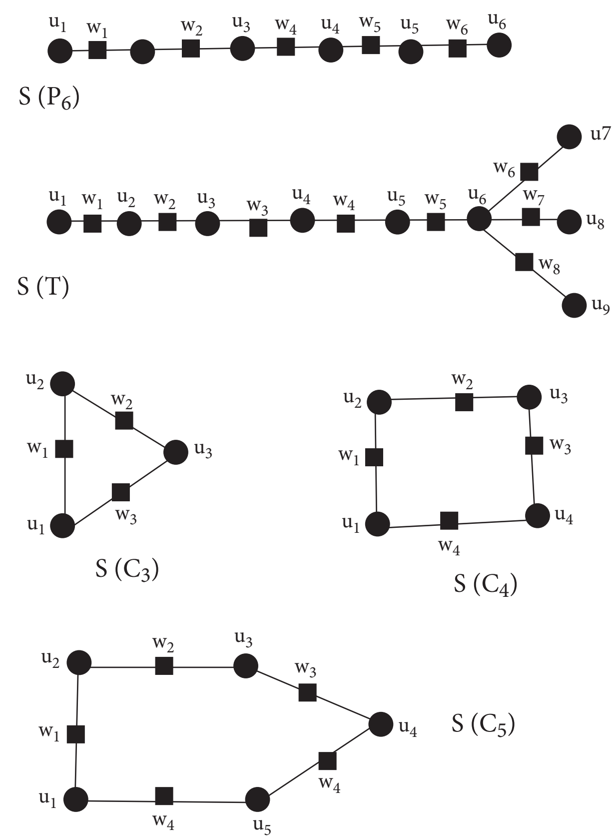

FIGURE 3: S-operation of some trees and cycles.

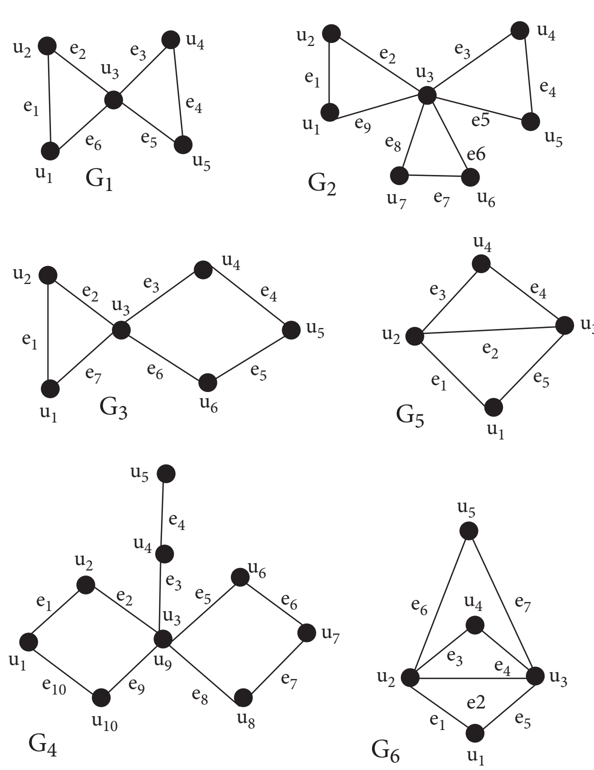

FIGURE 2: Some simple and connected $\{C_3, C_4\}$– graphs.

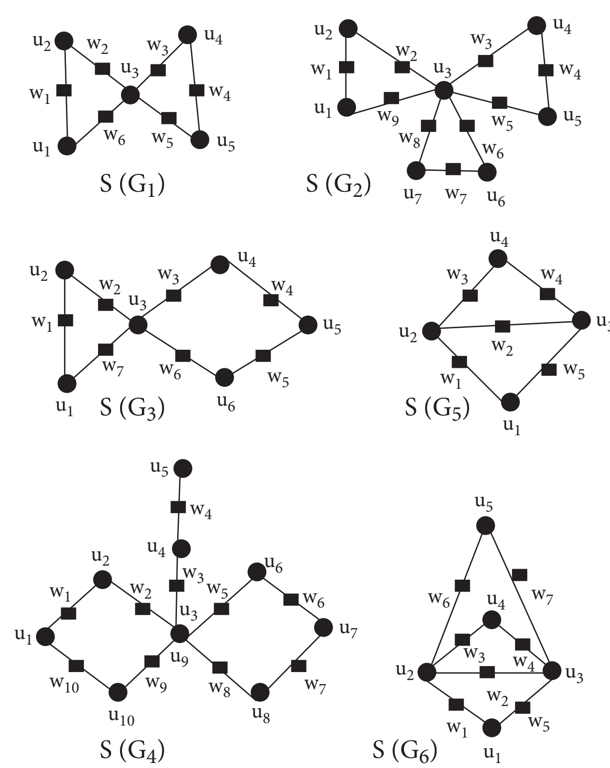

FIGURE 4: S-operation of simple and connected $\{C_3, C_4\}$– graphs.

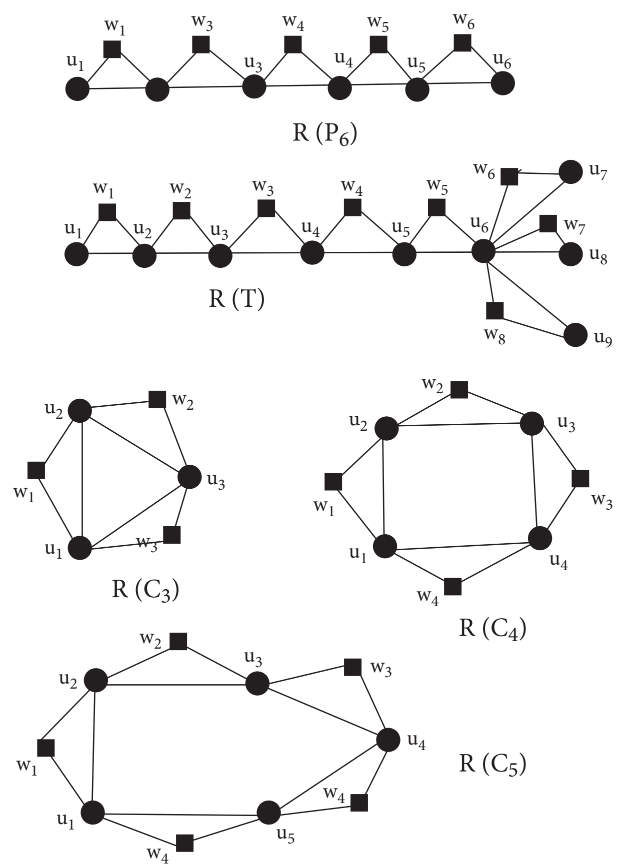

Figure 5: R-operation of trees and cycles.

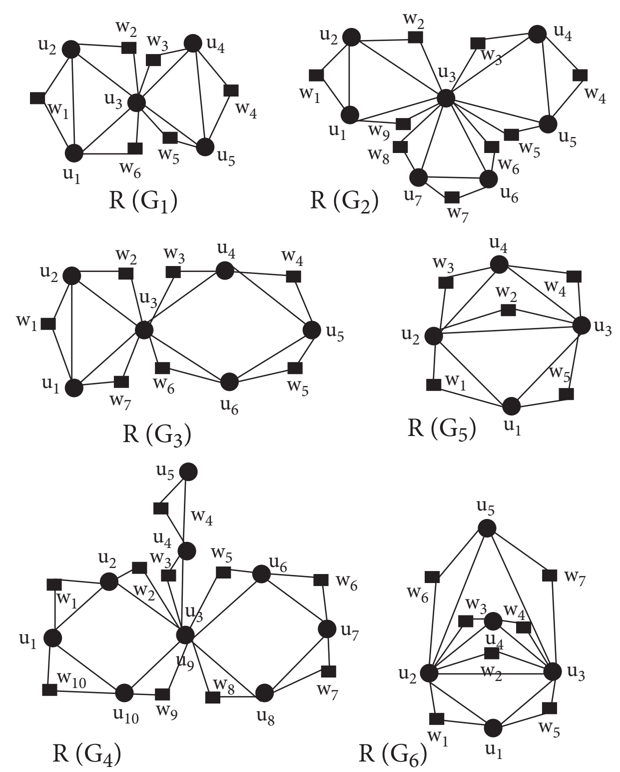

Figure 6: R-operation of simple and connected $\{C_3, C_4\}-$ graphs.

each vertex u_i in $R(G)$ is twice the connection number u_i in G i.e. $\tau_{R(G)}(u_i) = 2\tau_{(G)}(u_i)$. Also u_j and u_k are at distance one from w_i, but in R(G), degree of u_j and u_k are also doubled, then number of vertices at distance 2 of w_i are $2[\delta_{(G)}(u_j) - 1 + \delta_{(G)}(u_k) - 1] = 2[\delta_{(G)}(u_j) + \delta_{(G)}(u_k)] - 42\delta_G(e_i) = 2\delta_G(e_i)$. Also for $n \geq 5$, $\delta_{C_n}(u_i) = 2$, $\tau_{C_n}(u_i) = 2$, $\tau_{R(C_n)}(u_i) = 4$, $\tau_{R(C_n)}(w_i) = 4$, $\Rightarrow \tau_{R(C_n)}(u_i) = 2\tau_{(C_n)}(u_i)$ and $\tau_{R(C_n)}(w_i) = 2[\delta_{(C_n)}(u_j) + \delta_{(C_n)}(u_k)] - 4 = 2\delta_{C_n}(e_i)$, $i = 1, 2 \ldots n$.

Case (b)

If G is a $\{C_3, C_4\}-$ graph, then consider $\delta_{C_3}(u_i) = 2$, $\delta_{C_3}(u_i) = 2, \tau_{C_3}(u_i) = 0$, $\tau_{R(C_3)}(u_i) = 1$, $\tau_{R(C_3)}(w_i) = 3 \Rightarrow \tau_{R(C_3)}(u_i) = 2\tau_{(C_3)}(u_i) + 1$ and $\tau_{R(C_3)}(w_i) = 2[\delta_{(C_3)}(u_j) + \delta_{(C_3)}(u_k)] - 5,= 2\delta_G(e_i) - 1\ i = 1, 2, 3$.

$\delta_{C_4}(u_i) = 2, \tau_{C_4}(u_i) = 1$, $\tau_{R(C_4)}(u_i) = 3$, $\tau_{R(C_4)}(w_i) = 4$, $\Rightarrow \tau_{R(C_4)}(u_i) = 2\tau_{(C_4)}(u_i) + 1$ and $\tau_{R(C_4)}(w_i) = 2[\delta_{(C_4)}(u_j) + \delta_{(C_4)}(u_k)] - 4 = 2\delta_{C_4}(e_i)$, $i = 1, 2, 3, 4$.

Consider $R(G_1)$ graph in which two C_3 are connected by a common vertex u_3 as shown in Figure 6, then $\delta_{(G_1)}(u_i) = 2$, $\tau_{G_1}(u_i) = 2, \tau_{R(G_1)}(u_i) = 5 \Rightarrow \tau_{R(G_1)}(u_i) = 2\tau_{(G_1)}(u_i) + 1$ for $i = 1, 2, 4, 5$ and $\delta_{(G_1)}(u_3) = 4$, $\tau_{G_1}(u_3) = 0, \tau_{R(G_1)}(u_3) = 2 \Rightarrow \tau_{R(G_1)}(u_3) = 2\tau_{(G_1)}(u_3) + 2$ as u_3 lies in two C_3 graphs. Also $\tau_{R(G_1)}(w_i) = 3 \Rightarrow \tau_{R(G_1)}(w_i) = 2[\delta_{(G_1)}(u_j) + \delta_{(G_1)}(u_k)] - 5 = 2\delta_G(e_i) - 1$ for $i = 1, 4$ and $\tau_{R(G_1)}(w_i) = 7 \Rightarrow \tau_{R(G_1)}(w_i) = 2[\delta_{(G_1)}(u_j) + \delta_{(G_1)}(u_k)] - 5 = 2\delta_G(e_i) - 1$, for $i = 2, 3, 5, 6$.

Consider $R(G_2)$ graph in which three C_3 are connected by a common vertex u_3 as shown in Figure 6, then $\delta_{(G_2)}(u_i) = 2$, $\tau_{G_2}(u_i) = 4, \tau_{R(G_2)}(u_i) = 9 \Rightarrow \tau_{R(G_2)}(u_i) = 2\tau_{(G_2)}(u_i) + 1$ for $i = 1, 2, 4, 5, 6, 7$ and $\delta_{(G_2)}(u_3) = 6$, $\tau_{G_2}(u_3) = 0, \tau_{R(G_2)}(u_3) = 3 \Rightarrow \tau_{R(G_2)}(u_3) = 2\tau_{(G_2)}(u_3) + 3$ as u_3 lies in three C_3 graphs. Also $\tau_{R(G_2)}(w_i) = 3 \Rightarrow \tau_{R(G_1)}(w_i) = 2[\delta_{(G_2)}(u_j) + \delta_{(G_2)}(u_k)] - 5 = 2\delta_G(e_i) - 1$ for $i = 1, 4, 7$ and $\tau_{R(G_1)}(w_i) = 11 \Rightarrow \tau_{R(G_1)}(w_i) = 2[\delta_{(G_2)}(x_i) + \delta_{(G_2)}(y_i)] - 5 = 2\delta_G(e_i) - 1$, for $i = 2, 3, 5, 6, 8, 9$.

Consider $R(G_3)$ in which one C_3 and one C_4 are connected by a common vertex u_3 as shown in Figure 6, then $\delta_{(G_3)}(u_i) = 2$ for $i = 1, 2, 4, 5, 6$, $\tau_{G_3}(u_i) = 2$ for $i = 1, 2$, $\tau_{G_3}(u_i) = 3$ for $i = 4, 6, \tau_{G_3}(u_i) = 1$ for $i = 3, 5, \tau_{R(G_3)}(u_i) = 5 = 2\tau_{(G_3)}(u_i) + 1$, for $i = 1, 2\ \tau_{R(G_3)}(u_3) = 4 = 2\tau_{(G_3)}(u_3) + 2$, as u_3 lies in one C_3 and one C_4 graphs, $\tau_{R(G_3)}(u_i) = 7 = 2\tau_{(G_3)}(u_i) + 1$, for $i = 4, 6\ \tau_{R(G_3)}(u_5) = 3 = 2\tau_{(G_3)}(u_5) + 1$, Also $\tau_{R(G_3)}(w_1) = 3 = 2[\delta_{(G_3)}(u_1) + \delta_{(G_3)}(u_2)] - 5 = 2\delta_G(e_i) - 1$, $\tau_{R(G_3)}(w_2) = 7 = 2[\delta_{(G_3)}(u_2) + \delta_{(G_3)}(u_3)] - 5 = 2\delta_G(e_i) - 1$, $\tau_{R(G_3)}(w_3) = 8 = 2[\delta_{(G_3)}(u_3) + \delta_{(G_3)}(u_4)] - 4 = 2\delta_G(e_3)$, $\tau_{R(G_3)}(w_4) = 4 = 2[\delta_{(G_3)}(u_4) + \delta_{(G_3)}(u_5)] - 4 = 2\delta_G(e_4)$.

Consider $R(G_4)$ graph in which two C_4 graphs and one path are connected by vertex u_3 as shown in Figure 6. As $\delta_{(G_4)}(u_i) = 2$ for $i = 1, 2, 4, 6, 8$, $\delta_{(G_1)}(u_3) = 5$ and $\delta_{(G_4)}(u_5) = 1$, $\tau_{G_4}(u_1) = 1$, $\tau_{G_4}(u_2) = 4$, $\tau_{G_4}(u_3) = 3$, $\tau_{G_4}(u_4) = 4, \tau_{G_4}(u_5) = 1$, $\tau_{R(G_4)}(u_1) = 3 = 2\tau_{(G_4)}(u_1) + 1$, $\tau_{R(G_4)}(u_2) = 9 = 2\tau_{(G_4)}(u_2) + 1$, $\tau_{R(G_4)}(u_3) = 8 = 2\tau_{(G_4)}(u_3) + 2, \tau_{R(G_4)}(u_4) = 8 = 2\tau_{(G_4)}(u_1)$, $\tau_{R(G_4)}(u_5) = 2 = 2\tau_{(G_4)}(u_5)$, $\tau_{R(G_4)}(w_1) = 4 = 2[\delta_{(G_4)}(u_1) + \delta_{(G_4)}(u_2)] - 4 = 2\delta_G(e_1)$, $\tau_{R(G_4)}(w_2) = 10 = 2[\delta_{(G_4)}(u_2) + \delta_{(G_4)}(u_3)] - 4 = 2\delta_G(e_2)$, $\tau_{R(G_4)}(w_3) = 10 = 2[\delta_{(G_4)}(u_3) + \delta_{(G_4)}(u_4)] - 4 = 2\delta_G(e_3)$ $\tau_{R(G_4)}(w_4) = 2 = 2[\delta_{(G_4)}(u_4) + \delta_{(G_4)}(u_5)] - 4 = 2\delta_{G_4}(e_4)$.

Consider a graph $R(G_5)$, in which one edge u_2u_3 is common in two $C_3 - graphs$ as shown in Figure 6.

$\delta_{(G_5)}(u_i) = 2$ and $\tau_{G_5}(u_i) = 1$, $\tau_{R(G_5)}(u_1) = 4 = 2\tau_{(G_5)}(u_1) + 2$ as $N(u_1) \in$ two C_3 graphs, for $i = 1, 4$, $\delta_{(G_5)}(u_i) = 3$ and $\tau_{G_5}(u_i) = 0$, $\tau_{R(G_5)}(u_1) = 2 = 2\tau_{(G_5)}(u_1) + 2$ as $N(u_i) \in C_3$ graphs, for $i = 2, 3$, $\tau_{R(G_4)}(w_1) = 5 = 2[\delta_{(G_4)}(u_1) + \delta_{(G_4)}(u_2)] - 5 = 2\delta_{G_5}(e_1)$, similarly we can get w_2, w_3 and w_4, $\tau_{R(G_4)}(w_5) = 5 = 2[\delta_{(G_4)}(u_2) + \delta_{(G_4)}(u_4)] - 6 = 2\delta_{G_5}(e_5) - 2$, as $u_2u_3 \in$ two C_3 graph.

Now consider a graph $R(G_6)$ in which one edge u_2u_3 is common in three $C_3 - graphs$ as shown in Figure 6, $\delta_{(G_6)}(u_i) = 2$ and $\tau_{G_6}(u_i) = 2$, $\tau_{R(G_6)}(u_1) = 7 = 2\tau_{(G_6)}(u_1) + 3$ as $N(u_1) \in$ three C_3 graphs, for $i = 1, 4, 5$, $\delta_{(G_6)}(u_i) = 4$ and $\tau_{G_6}(u_i) = 0$, $\tau_{R(G_6)}(u_1) = 3 = 2\tau_{(G_6)}(u_1) + 3$ as $N(u_1) \in$ three C_3 graphs, for $i = 2, 3$, $\tau_{R(G_4)}(w_1) = 7 = 2[\delta_{(G_6)}(u_1) + \delta_{(G_6)}(u_2)] - 5 = 2\delta_{G_6}(e_1)$. Similarly, we can get w_3, w_4, w_5, w_6, and w_7, $\tau_{R(G_4)}(w_2) = 9 = 2[\delta_{(G_4)}(u_2) + \delta_{(G_4)}(u_4)] - 7 = 2\delta_{G_4}(e_2) - 3$, as $u_2u_3 \in$ three C_3 graph.

Now, we close our discussion by taking $r = \max\{r_i\}$ and $s = \max\{s_i\}$, then upper bound for connection number is taken is taken as $\tau_{R(G)}(u_i) = 2\tau_{(G)}(u_i) + r$ and $\tau_{R(G)}(w_i) = 2[\delta_{(G)}(x_i) + \delta_{(G)}(y_i)] - 4 = 2[\delta_{(G)}(e_i)]$. And lower bound for connection number is taken as $\tau_{R(G)}(u_i) = 2\tau_{(G)}(u_i)$ and $\tau_{R(G)}(w_i) = 2[\delta_{(G)}(x_i) + \delta_{(G)}(y_i)] - 4 - s = 2[\delta_{(G)}(e_i)] - s$. □

Lemma 3. *For a simple and connected graph G,*

(a) If G is a $\{C_3, C_4\}$– free graph, then

(i) $\tau_{Q(G)}(u_i) = \delta_{(G)}(u_i) + \tau_{(G)}(u_i)$ and
(ii) $\tau_{Q(G)}(w_i) = \tau_{(G)}(x_i) + \tau_{(G)}(y_i)$.

(b) If G is a $\{C_3, C_4\}$– graph, then

(i) $\tau_{Q(G)}(u_i) \leq \delta_{(G)}(u_i) + \tau_{(G)}(u_i) + r$ where $r = \max\{r_i\}$ and r_i is the number of C_3 and C_4 cycles connected with vertex u_i.
(ii) $\tau_{Q(G)}(w_i) \leq \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + s$ where $s = \max\{s_i\}$ and s_i is the number of C_3 cycles in graph G connected with edge e_i.

Proof 3

Case (a)

If G is a $\{C_3, C_4\}$– free graph, On applying first S-operation, the connection number of u_i becomes equal to δ_{u_i}. Then for Q-operation, further two new (white) vertices of $S(G)$ are also joined if their corresponding edges have a common vertex between them. So connection number of u_i in $Q(G)$ becomes equal to the sum of δ_{u_i} and $\tau_G(u_i)$.

(For explanation from Figure 7, first consider a tree T with nine vertices in which $\delta_{(T)}(u_i) = 1$ for $i = 1, 7, 8, 9$, $\delta_{(T)}(u_i) = 2$ for $i = 2, 3, 4, 5$. $\delta_{(T)}(u_6) = 4$, $\tau_{(G)}(u_i) = 1$ $i = 1, 2, 6$, $\tau_{(T)}(u_i) = 3$ $i = 3, 4$, $\tau_{(T)}(u_5) = 4$, $\tau_{(G)}(u_i) = 3$ $i = 7, 8, 9$. Then $\tau_{Q(T)}(u_1) = 2 = \delta_{(T)}(u_1) + \tau_{(T)}(u_1)$, $\tau_{Q(T)}(u_2) = 3 = \delta_{(T)}(u_2) + \tau_{(T)}(u_2)$, $\tau_{Q(T)}(u_3) = 4 = \delta_{(T)}(u_3) + \tau_{(T)}(u_3) = \tau_{Q(T)}(u_4)$, $\tau_{Q(T)}(u_5) = 6 = \delta_{(T)}(u_5) + \tau_{(T)}(u_5)$, $\tau_{Q(T)}(u_6) = 5 = \delta_{(T)}(u_6) + \tau_{(G)}(u_6)$, $\tau_{Q(T)}(u_7) = 4 = \delta_{(T)}(u_7) + \tau_{(T)}(u_7) = \tau_{Q(T)}(u_8) = \tau_{Q(T)}(u_9)$, $\tau_{Q(T)}(w_1) = 2 = \tau_{(T)}(u_1) + \tau_{(T)}(u_2)$, $\tau_{Q(T)}(w_2) = 3 = \tau_{(T)}(u_2) + \tau_{(T)}(u_3)$, $\tau_{Q(T)}(w_3) = \tau_{(T)}(u_3) + \tau_{(T)}(u_4)$, $\tau_{Q(T)}(w_4) = 6 = \tau_{(T)}(u_4) + \tau_{(T)}(u_5)$, $\tau_{Q(T)}(w_5) = 5 = \tau_{(T)}(u_5) + \tau_{(T)}(u_6)$, $\tau_{Q(T)}(w_6) = 4 = \tau_{(T)}(u_6) + \tau_{(T)}(u_7) = \tau_{Q(T)}(w_7) = \tau_{Q(T)}(w_8)$, here we conclude $\tau_{Q(T)}(u_i) = \delta_{(T)}(u_i) + \tau_{(T)}(u_i)$ and $\tau_{Q(G)}(T_i) = \tau_{(G)}(u_j) + \tau_{(T)}(u_k)$).

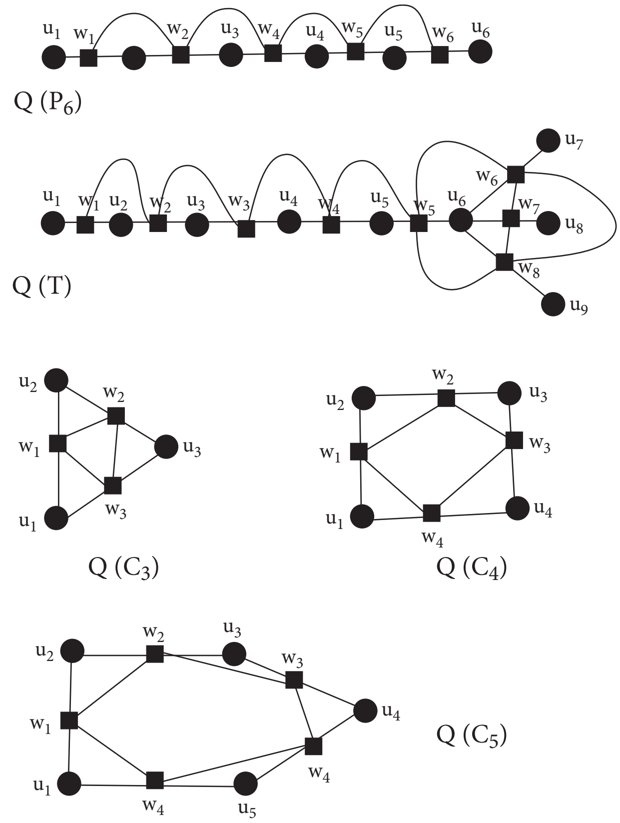

Figure 7: Q-operation of trees and cycles.

Also for $n \geq 5$, $\delta_{C_n}(u_i) = 2$, $\tau_{C_n}(u_i) = 2$, $\tau_{Q(C_n)}(u_i) = 4$, $\tau_{q(C_n)}(w_i) = 4$, $\Rightarrow \tau_{Q(C_n)}(u_i) = \delta_{(C_n)}(u_i) + \tau_{(C_n)}(u_i)$ and $\tau_{Q(G)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$.

Case(b)

Consider $\delta_{C_3}(u_i) = 2, \tau_{C_3}(u_i) = 0$, $\tau_{Q(C_3)}(u_i) = 3$, $\tau_{Q(C_3)}(w_i) = 1 \Rightarrow \tau_{Q(C_3)}(u_i) = \delta_{(C_3)}(u_i) + \tau_{(C_3)}(u_i) + 1$ and $\tau_{Q(C_3)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + 1$, $i = 1, 2, 3$.

$\delta_{C_4}(u_i) = 2, \tau_{C_4}(u_i) = 1$, $\tau_{Q(C_4)}(u_i) = 4$, $\tau_{Q(C_4)}(w_i) = 3$, $\Rightarrow \tau_{Q(C_4)}(u_i) = \delta_{(C_4)}(u_i) + \tau_{(C_4)}(u_i) + 1$ and $\tau_{Q(G)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$, $i = 1, 2, 3, 4$.

Consider $Q(G_1)$ graph in which two C_3 are connected by a common vertex u_3 as shown in Figure 8, then $\delta_{(G_1)}(u_i) = 2$, $\tau_{G_1}(u_i) = 2, \tau_{Q(G_1)}(u_i) = 5 \Rightarrow \tau_{Q(G_1)}(u_i) = \delta_{(G_1)}(u_i) + \tau_{(G_1)}(u_i) + 1$ for $i = 1, 2, 4, 5$ and $\delta_{(G_1)}(u_3) = 4$, $\tau_{G_1}(u_3) = 0, \tau_{Q(G_1)}(u_3) = 6 \Rightarrow \tau_{Q(G_1)}(u_3) = \delta_{(G_1)}(u_i) + \tau_{(G_1)}(u_3) + 2$ as u_3 lies in two C_3 graphs. Also $\tau_{Q(G_1)}(w_i) = 3 \Rightarrow \tau_{Q(G_1)}(w_i) \neq \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$, but $\tau_{Q(G_1)}(w_i) = \delta_{(G_1)}(u_3) - 1$ when u_1 and u_2 is of same degree and u_3 is their third vertex in C_3, Also $\tau_{Q(G_1)}(w_i) < \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + 1$ for $i = 1, 4$ and $\tau_{Q(G_1)}(w_i) = 4 \Rightarrow \tau_{Q(G_1)}(w_i) = \tau_{(G)}(x_i) + \tau_{(G)}(y_i) + 2$, for $i = 2, 3, 5, 6$.

Consider $Q(G_2)$ graph in which three C_3 are connected by a common vertex u_3 as shown in Figure 8, then $\delta_{(G_2)}(u_i) = 2$, $\tau_{G_2}(u_i) = 4, \tau_{Q(G_2)}(u_i) = 7 \Rightarrow \tau_{Q(G_2)}(u_i) = \delta_{(G_1)}(u_i) + \tau_{(G_2)}(u_i) + 1$ for $i = 1, 2, 4, 5, 6, 7$ and $\delta_{(G_2)}(u_3) = 6$, $\tau_{G_2}(u_3) = 0, \tau_{Q(G_2)}(u_3) = 9 \Rightarrow \tau_{Q(G_2)}(u_3) = \delta_{(G_1)}(u_3) + \tau_{(G_2)}(u_3) + 3$ as u_3 lies in three C_3 graphs. Also

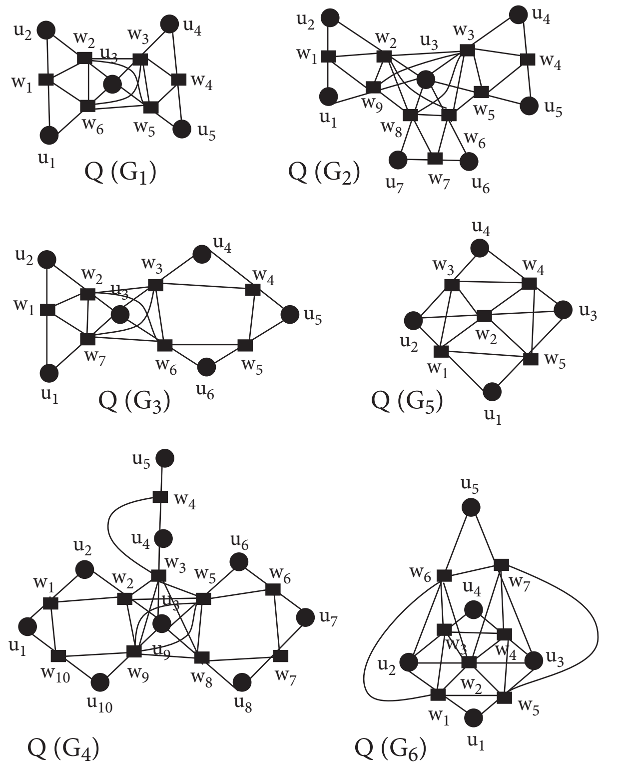

Figure 8: Q-operation of simple and connected $\{C_3, C_4\}-$ graphs.

$\tau_{Q(G_2)}(w_i) = 5 \Rightarrow \tau_{Q(G_1)}(w_i) \neq \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$ for $i = 1, 4, 7$ but $\tau_{Q(G_1)}(w_i) = \delta_{(G_1)}(u_3) - 1$ when u_1 and u_2 is of same degree and u_3 is their third vertex in C_3, Also $\tau_{Q(G_1)}(w_i) < \tau_{(G)}(x_i) + \tau_{(G)}(y_i) + 1$ for $i = 1, 4, 7$ and $\tau_{Q(G_1)}(w_i) = 7 \Rightarrow \tau_{Q(G_1)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + 3$, for $i = 2, 3, 5, 6, 8, 9$.

Consider $Q(G_3)$ graph in which one C_3 and one C_4 are connected by a common vertex u_3 as shown in Figure 8, then $\delta_{(G_3)}(u_i) = 2$ for $i = 1, 2, 4, 5, 6$, $\tau_{G_3}(u_i) = 2$ for $i = 1, 2$, $\tau_{G_3}(u_i) = 3$ for $i = 4, 6, \tau_{G_3}(u_i) = 1$ for $i = 3, \tau_{Q(G_3)}(u_1) = 5 = \delta_{(G_1)}(u_i) + \tau_{(G_3)}(u_1) + 1$, $\tau_{Q(G_3)}(u_3) = 7 = \delta_{(G_1)}(u_i) + \tau_{(G_3)}(u_3) + 2$, as u_3 lies in one C_3 and one C_4 graphs, $\tau_{Q(G_3)}(u_4) = 6 = \delta_{(G_1)}(u_4) + \tau_{(G_3)}(u_4) + 1$, $\tau_{Q(G_3)}(u_5) = 4 = \delta_{(G_1)}(u_5) + \tau_{(G_3)}(u_5) + 1, .$ Also $\tau_{Q(G_3)}(w_1) = 3 = \delta_{(G_1)}(u_3) - 1$ when u_1 and u_2 is of same degree and u_3 is their third vertex in C_3, Also $\tau_{Q(G_1)}(w_1) < \tau_{(G)}(u_1) + \tau_{(G)}(u_2) + 1$, $\tau_{Q(G_3)}(w_2) = 5 = \tau_{(G)}(u_2) + \tau_{(G)}(u_3) + 2$, $\tau_{Q(G_3)}(w_3) = 6 = \tau_{(G)}(u_4) + \tau_{(G)}(u_3) + 2, \tau_{Q(G_3)}(w_4) = 5 = \tau_{(G)}(u_4) + \tau_{(G)}(u_5) + 1$.

Consider $Q(G_4)$ graph in which two C_4 graphs and one path are connected by vertex u_3 as shown in Figure 8. As $\delta_{(G_4)}(u_i) = 2$ for $i = 1, 2, 4$, $\delta_{(G_1)}(u_3) = 5$ and $\delta_{(G_4)}(u_5) = 1$, $\tau_{G_4}(u_1) = 1$, $\tau_{G_4}(u_2) = 4$, $\tau_{G_4}(u_3) = 3$, $\tau_{G_4}(u_4) = 4, \tau_{G_4}(u_5) = 1$, $\tau_{Q(G_4)}(u_1) = 4 = \delta_{(G_4)}(u_1) + \tau_{(G_4)}(u_1) + 1$, $\tau_{Q(G_4)}(u_2) = 7 = \delta_{(G_4)}(u_2) + \tau_{(G_4)}(u_2) + 1, \tau_{Q(G_4)}(u_3) = 10 = \delta_{(G_4)}(u_3) + \tau_{(G_4)}(u_3) + 2, \tau_{Q(G_4)}(u_4) = 6 = \delta_{(G_4)}(u_4) + \tau_{(G_4)}(u_4)$, $\tau_{Q(G_4)}(u_5) = 4 = \delta_{(G_4)}(u_5) + \tau_{(G_4)}(u_5)$, $\tau_{Q(G_4)}(w_1) = 6 = \tau_{(G)}(u_1) + \tau_{(G)}(u_2) + 1$, $\tau_{Q(G_4)}(w_2) = 9 = \tau_{(G)}(u_2) + \tau_{(G)}(u_3) + 2, \tau_{Q(G_4)}(w_3) = 9 = \tau_{(G)}(u_3) + \tau_{(G)}(u_4) + 2$, $\tau_{Q(G_4)}(w_4) = 6 = \tau_{(G)}(u_4) + \tau_{(G)}(u_5)$.

Now we consider a graph $Q(G_5)$, in which one edge u_2u_4 is common in two $C_3 - graphs$ as shown in Figure 8 . $\delta_{(G_5)}(u_i) = 2$ and $\tau_{G_5}(u_i) = 1$, $\tau_{Q(G_5)}(u_1) = 5 = \delta_{(G_4)}(u_1) + \tau_{(G_5)}(u_1) + 2$ as $N(u_1) \in$ two C_3 graphs, for $i = 1, 3$, $\delta_{(G_5)}(u_i) = 3$ and $\tau_{G_5}(u_i) = 0, \tau_{Q(G_5)}(u_i) = 5 = \delta_{(G_4)}(u_i) + \tau_{(G_5)}(u_i) + 2$ as $N(u_i) \in C_3$ graphs, for $i = 2, 4$, $\tau_{Q(G_4)}(w_1) = 3 = \tau_{(G)}(u_1) + \tau_{(G)}(u_2) + 2$. Similarly, we can get w_2, w_3 and w_4, $\tau_{Q(G_4)}(w_5) = \tau_{(G)}(u_2) + \tau_{(G)}(u_4) + 2$, as $u_2u_4 \in$ two C_3 graph.

Now we consider a graph $Q(G_6)$ in which one edge u_2u_4 is common in three $C_3 - graphs$ as shown in Figure 8, $\delta_{(G_6)}(u_i) = 2$ and $\tau_{G_6}(u_i) = 2$, $\tau_{Q(G_6)}(u_1) = 7 = \delta_{(G_4)}(u_1) + \tau_{(G_6)}(u_1) + 3$ as $N(u_1) \in$ three C_3 graphs, for $i = 1, 3, 5$, $\delta_{(G_6)}(u_i) = 4$ and $\tau_{G_6}(u_i) = 0$, $\tau_{Q(G_6)}(u_i) = 3 = \delta_{(G_4)}(u_i) + \tau_{(G_5)}(u_i) + 3$ as $N(u_i) \in$ three C_3 graphs, for $i = 2, 4$, $\tau_{Q(G_4)}(w_1) = \tau_{(G)}(u_1) + \tau_{(G)}(u_2) + 2$ =6. Similarly, we can get w_2, w_3, w_4, w_6, and w_7, $\tau_{R(G_4)}(w_5) = 3 = \tau_{(G)}(u_2) + \tau_{(G)}(u_4) + 3$, as $u_2u_4 \in$ three C_3 graph.

Now, we close our discussion by taking, $r = \max\{r_i\}$ and $s = \max\{s_i\}$, $\tau_{Q(G)}(u_i) \leqslant \delta_{(G)}(u_i) + \tau_{(G)}(u_i) + r$ and $\tau_{Q(G)}(w_i) \leqslant \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + s$. □

Lemma 4. *For a simple, connected graph G,*

(a) If G is a $\{C_3, C_4\}-$ free graph, then

(i) $\tau_{T(G)}(u_i) = 2\tau_{(G)}(u_i)$ and
(ii) $\tau_{T(G)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$.

(b) If G is a $\{C_3, C_4\}-$ graph, then

(i) $\tau_{T(G)}(u_i) \leqslant 2\tau_{(G)}(u_i) + r$ where $r = \max\{r_i\}$ and vertex u_i is connected with r_i number of C_3 and C_4 cycles and
(ii) $\tau_{T(G)}(w_i) \leqslant \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + s$ where $s = \max\{s_i\}$ and edge e_i is connected with the s_i number of C_3 cycles in graph G.

Proof 4. After applying T-operation on G, we get m new vertices w_i corresponding to each edge $e_i = u_ju_k = x_iy_i$ for $u_j = x_i$, $u_k = y_i$.

Case (a)

If G is a $\{C_3, C_4\}-$ free graph, on applying first R-operation, the connection number of u_i becomes equal to τ_{u_i}. Then for T-operation, further two new (white) vertices of $R(G)$ are also joined if their corresponding edges have a common vertex between them. So connection number of u_i in $T(G)$ becomes equal to the twice of τ_{u_i}.

(For explanation from Figure 9, first consider $\delta_{(P_6)}(u_i) = 1$ for $i = 1, 6$, $\delta_{(P_6)}(u_i) = 2$ for $i = 2, 3, 4, 5, \tau_{(P_6)}(u_i) = 1$ $i = 1, 2, 5, 6, \tau_{(P_6)}(u_3) = 2 = \tau_{(P_6)}(u_4), \tau_{T(P_6)}(u_i) = 2$ for $i = 1, 2, 5, 6, \tau_{T(P_6)}(u_3) = 4 = \tau_{T(P_6)}(u_4)$ $\tau_{T(P_6)}(w_1) = 2 = \tau_{P_6}(u_1) + \tau_{((P_6))}(u_2) = \tau_{T(G)}(w_5)$, $\tau_{T((P_6))}(w_2) = 3 = \tau_{((P_6))}(u_2) + \tau_{((P_6))}(u_3) = \tau_{T(G)}(w_4)$, $\tau_{T((P_6))}(w_3) = 4 = \tau_{((P_6))}(u_3) + \tau_{((P_6))}(u_4)$ here we conclude $\tau_{T(P_n)}(u_i) = 2\tau_{(P_n)}(u_i)$ and $\tau(P_n)(w_i) = \tau_{(P_n)}(u_j) + \tau_{(P_n)}(u_k)$

Also for $n \geq 5, \delta_{C_n}(u_i) = 2$, $\tau_{C_n}(u_i) = 2$, $\tau_{T(C_n)}(u_i) = 4$, $\tau_{T(C_n)}(w_i) = 4$, $\Rightarrow \tau_{T(C_n)}(u_i) = 2\tau_{(C_n)}(u_i)$ and $\tau_{T(G)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$).

Case (b)

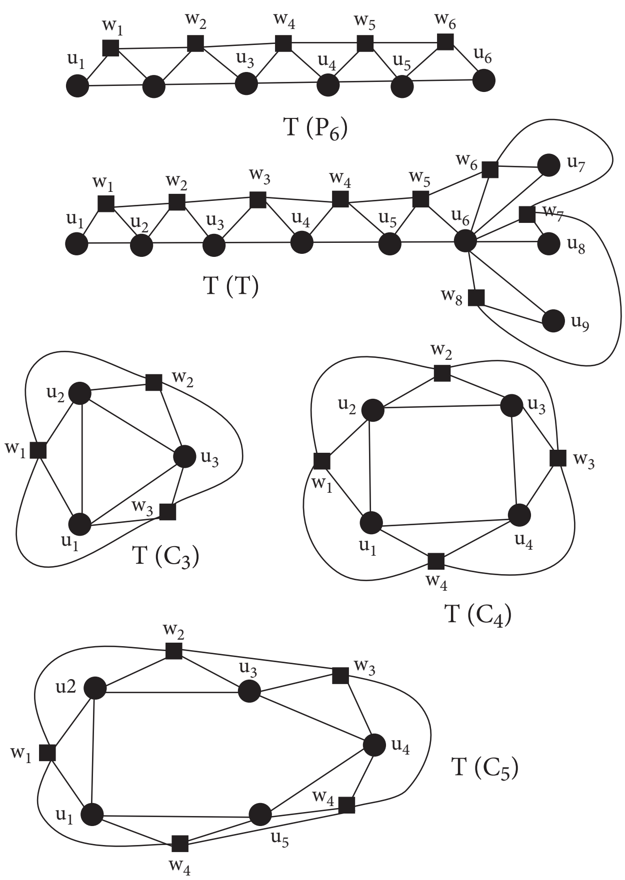

Figure 9: T-operation of trees and cycles.

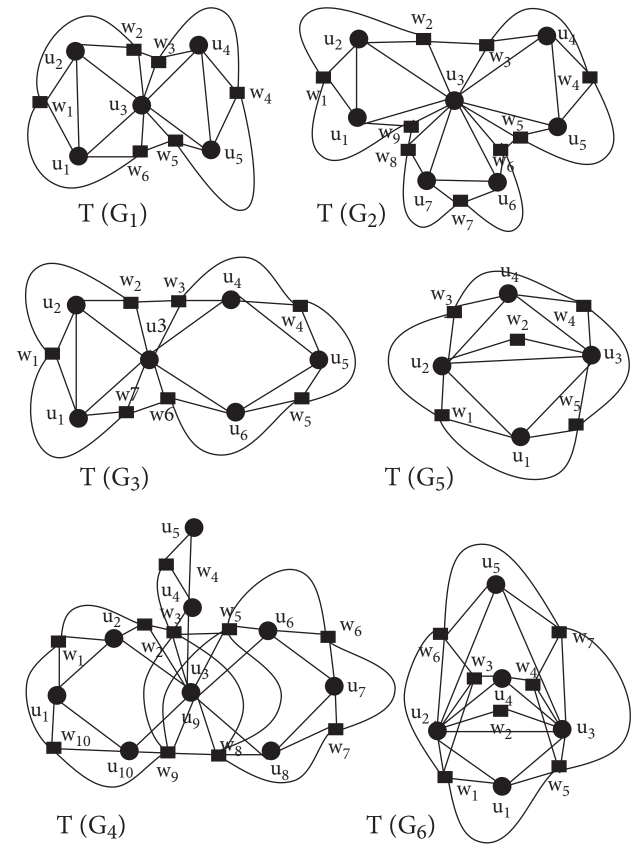

Figure 10: T-operation of simple and connected $\{C_3, C_4\}-$ graphs.

Now consider $\delta_{C_3}(u_i) = 2, \tau_{C_3}(u_i) = 0,\ \tau_{T(C_3)}(u_i) = 1,\ \tau_{T(C_3)}(w_i) = 1 \Rightarrow \tau_{T(C_3)}(u_i) = 2\tau_{(C_3)}(u_i) + 1$ and $\tau_{Q(C_3)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + 1,\ i=1,2,3$.

$\delta_{C_4}(u_i) = 2, \tau_{C_4}(u_i) = 1,\ \tau_{T(C_4)}(u_i) = 3,\ \tau_{Q(C_4)}(w_i) = 3, \Rightarrow \tau_{Q(C_4)}(u_i) = 2\tau_{(C_4)}(u_i) + 1$ and $\tau_{T(C_4)}(w_i) = \tau_{(G)}(u_j) + \tau_{(C_4)}(u_k) + 1,\ i = 1, 2, 3, 4$.

Consider $T(G_1)$ in which two C_3 are connected by a common vertex u_3 as shown in figure 10, then $\delta_{(G_1)}(u_i) = 2$, $\tau_{G_1}(u_i) = 2, \tau_{Q(G_1)}(u_i) = 5 \Rightarrow \tau_{T(G_1)}(u_i) = 2\tau_{(G_1)}(u_i) + 1$ for $i = 1, 2, 4, 5$ and $\delta_{(G_1)}(u_3) = 4,\ \tau_{G_1}(u_3) = 0, \tau_{T(G_1)}(u_3) = 2 \Rightarrow \tau_{T(G_1)}(u_3) = 2\tau_{(G_1)}(u_3) + 2$ as u_3 lies in two C_3 cycles. Also $\tau_{T(G_1)}(w_i) = 3 \Rightarrow \tau_{T(G_1)}(w_i) \neq \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$, but $\tau_{T(G_1)}(w_i) = \delta_{(G_1)}(u_3) - 1$ when u_1 and u_2 is of same degree 2 and u_3 is their third vertex in C_3, Also $\tau_{Q(G_1)}(w_i) < \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + 1$ for $i = 1, 4$ and $\tau_{T(G_1)}(w_i) = 4 \Rightarrow \tau_{T(G_1)}(w_i) = \tau_{(G)}(x_i) + \tau_{(G)}(y_i) + 2$, for $i = 2, 3, 5, 6$.

Consider $T(G_2)$ in which three C_3 are connected by a common vertex u_3 as shown in Figure 10, then $\delta_{(G_2)}(u_i) = 2$, $\tau_{G_2}(u_i) = 4, \tau_{T(G_2)}(u_i) = 9 \longrightarrow \tau_{T(G_2)}(u_i) = 2\tau_{(G_2)}(u_i) + 1$ for $i = 1, 2, 4, 5, 6, 7$ and $\delta_{(G_2)}(u_3) = 6,\ \tau_{G_2}(u_3) = 0$, $\tau_{T(G_2)}(u_3) = 3 \Rightarrow \tau_{T(G_2)}(u_3) = 2\tau_{(G_2)}(u_3) + 3$ as u_3 lies in three C_3 graphs. Also $\tau_{T(G_2)}(w_i) = 5 \Rightarrow \tau_{T(G_1)}(w_i) \neq \tau_{(G)}(u_j) + \tau_{(G)}(u_k)$ for $i = 1, 4, 7$ but $\tau_{T(G_1)}(w_i) = \delta_{(G_1)}(u_3) - 1$ when u_1 and u_2 is of same degree 2 and u_3 is their third vertex in C_3,moreover $\tau_{T(G_1)}(w_i) < \tau_{(G_2)}(x_i) + \tau_{(G_2)}(y_i) + 1$ for $i = 1, 4, 7$ and $\tau_{T(G_1)}(w_i) = 7 \Rightarrow \tau_{T(G_1)}(w_i) = \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + 3$, for $i = 2, 3, 5, 6, 8, 9$.

Consider $T(G_3)$ in which one C_3 and one C_4 are connected by a common vertex u_3 as shown in Figure 10, then $\delta_{(G_3)}(u_i) = 2$ for $i = 1, 2, 4, 5, 6$, $\tau_{G_3}(u_i) = 2$ for $i = 1, 2$, $\tau_{G_3}(u_i) = 3$ for $i = 4, 6, \tau_{G_3}(u_3) = 1, \tau_{T(G_3)}(u_1) = 5 = 2\tau_{(G_3)}(u_1) + 1$, $\tau_{T(G_3)}(u_3) = 4 = 2\tau_{(G_3)}(u_3) + 2$, as u_3 lies in one C_3 and one C_4 graphs, $\tau_{T(G_3)}(u_4) = 7 = 2\tau_{(G_3)}(u_4) + 1$, $\tau_{T(G_3)}(u_5) = 3 = 2\tau_{(G_3)}(u_5) + 1$. Also $\tau_{T(G_3)}(w_1) = 3 = \delta_{(G_1)}(u_3) - 1$ when u_1 and u_2 are of same degree and u_3 is their third vertex in C_3, Also $\tau_{T(G_1)}(w_1) \langle \tau_{(G_3)}(u_1) + \tau_{(G_3)}(u_2) + 1$, $\tau_{T(G_3)}(w_2) = 5 = \tau_{(G_3)}(u_2) + \tau_{(G_3)}(u_3) + 2$, $\tau_{T(G_3)}(w_3) = 6 = \tau_{(G)}(u_4) + \tau_{(G)}(u_3) + 2, \tau_{Q(G_3)}(w_4) = 5 = \tau_{(G_3)}(u_4) + \tau_{(G_3)}(u_5) + 1$.

Consider G_4 graph in which two C_4 graphs and one path are connected by vertex u_3 as shown in Figure 10. As $\delta_{(G_4)}(u_i) = 2$ for $i = 1, 2, 4$, $\delta_{(G_4)}(u_3) = 5$ and $\delta_{(G_4)}(u_5) = 1$, $\tau_{G_4}(u_1) = 1,\ \tau_{G_4}(u_2) = 4,\ \tau_{G_4}(u_3) = 3,\ \tau_{G_4}(u_4) = 4$, $\tau_{G_4}(u_5) = 1,\ \tau_{T(G_4)}(u_1) = 3 = 2\tau_{(G_4)}(u_1) + 1, \tau_{T(G_4)}(u_2) = 9 = 2\tau_{(G_4)}(u_2) + 1,\ \tau_{T(G_4)}(u_3) = 8 = 2\tau_{(G_4)}(u_3) + 2,\ \tau_{T(G_4)}(u_4) = 8 = 2\tau_{(G_4)}(u_4),\ \tau_{T(G_4)}(u_5) = 2 = 2\tau_{(G_4)}(u_5),\ \tau_{T(G_4)}(w_1) = 6 = \tau_{(G_4)}(u_1) + \tau_{(G_4)}(u_2) + 1,\ \tau_{T(G_4)}(w_2) = 9 = \tau_{(G_4)}(u_2) + \tau_{(G_4)}(u_3) + 2,\ \tau_{T(G_4)}(w_3) = 9 = \tau_{(G)}(u_3) + \tau_{(G)}(u_4) + 2$, $\tau_{T(G_4)}(w_4) = 5 = \tau_{(G_4)}(u_4) + \tau_{(G_4)}(u_5)$.

Now we consider a graph G_5, in which one edge u_2u_4 is common in two C_3- graphs . $\delta_{(G_5)}(u_i) = 2$ and $\tau_{G_5}(u_i) = 1$, $\tau_{T(G_5)}(u_1) = 3 = 2\tau_{(G_5)}(u_1) + 1$, for $i = 1, 3$, $\delta_{(G_5)}(u_i) = 3$ and $\tau_{G_5}(u_i) = 0, \tau_{T(G_5)}(u_i) = 2 = 2\tau_{(G_5)}(u_i) + 2$ as $u_i \in$ two C_3 cycles, for $i = 2, 4$, $\tau_{T(G_5)}(w_1) = 3 = \tau_{(G_5)}(u_1) + \tau_{(G_5)}(u_2) + 2$. Similarly, we can get w_2, w_3 and w_4,

$\tau_{T(G_4)}(w_5) = 2 = \tau_{(G_5)}(u_2) + \tau_{(G_5)}(u_4) + 2$, as $u_2u_4 \in$ two C_3 graph.

Now we consider a graph G_6 in which one edge u_2u_3 is common in three C_3-graphs as shown in Figure 10, $\delta_{(G_6)}(u_i) = 2$ and $\tau_{G_6}(u_i) = 2$, $\tau_{T(G_6)}(u_1) = 7 = 2\tau_{(G_6)}(u_1) + 3$ as $N(u_1) \in$ three C_3 graphs, for $i = 1, 4, 5$, $\delta_{(G_6)}(u_i) = 4$ and $\tau_{G_6}(u_i) = 0$, $\tau_{T(G_6)}(u_i) = 3 = 2\tau_{(G_5)}(u_i) + 3$ as $N(u_i) \in$ three C_3 graphs, for $i = 2, 4$, $\tau_{T(G_4)}(w_1) = 5 = \tau_{(G_6)}(u_2) + \tau_{(G_6)}(u_4) + 1$. Similarly, we can get w_2, w_3, w_4, w_6, and w_7, $\tau_{T(G_4)}(w_5) = 3 = \tau_{(G_6)}(u_2) + \tau_{(G_6)}(u_4) + 3$, as $u_2u_4 \in$ three C_3 graph.

Now, we close our discussion by taking $r = \max\{r_i\}$ and $s = \max\{s_i\}$, $\tau_{T(G)}(u_i) \leqslant \delta_{(G)}(u_i) + \tau_{(G)}(u_i) + r$ and $\tau_{T(G)}(w_i) \leqslant \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + s$.

5. Connection Distance Indices of derived graphs

Now, we will determine degree connection sum of derived graphs $S(G)$, $R(G)$, $Q(G)$ and $S(G)$ by the following theorems.

Theorem 1. *If $S(G)$ is subdivision graph of G, then*

$$C\ D(S(G)) = 2\ DD(G) + 2\ DD_e(G) + \frac{1}{2}nM_1 + m(m-n) + \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \delta_G(e_j)\right]d_G(u_i, e_j). \quad (14)$$

Proof 5. By □

lemma 5. $\tau_{S(G)}(u_i) = \delta_{(G)}(u_i)$ *and* $\tau_{S(G)}(w_i) = \delta_{(G)}(u_j) + \delta_{(G)}(u_k) - 2 = \tau_{S(G)}(e_i)$

Also

$$\begin{aligned}
d_S(u_i, u_j) &= 2d_G(u_i, u_j),\\
d_S(w_i, w_j) &= 2\left[d_G(e_i, e_j) + 1\right],\\
d_S(u_i, w_j) &= 2d_G(u_i, e_i) + 1,\\
C\ D(S(G)) &= \sum_{\{u,v\}\subseteq V(G)} d_{S(G)}(u_i, u_j)\left[\tau_{S(G)}(u_i) + \tau_{S(G)}(u_j)\right],\\
&= \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\tau_S(u_i) + \tau_S(u_j)\right]d_S(u_i, u_j) + \frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_S(w_i) + \tau_S(w_j)\right]d_S(w_i, w_j),\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_S(u_i) + \tau_S(w_j)\right]d_S(u_i, w_j),\\
&= \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\delta_G(u_i) + \delta_G(u_j)\right]2d_G(u_i, u_j) + \frac{1}{2}\sum_{i=1,j=1}^{m}\left[\delta_G(e_i) + \delta_G(e_j)\right]2\left[d_G(e_i, e_j) + 1\right],\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \delta_G(e_j)\right]\left[2d_G(u_i, e_j) + 1\right],\\
&= 2\ DD(G) + 2\ DD_e(G) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \delta_G(e_j)\right] + \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \delta_G(e_j)\right]d_G(u_i, e_j),\\
&= 2\ DD(G) + 2\ DD_e(G) + \frac{1}{2}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\delta_G(u_i)\right) + \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\delta_G(e_j)\right),\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \delta_G(e_j)\right]d_G(u_i, e_j),\\
&= 2\ DD(G) + 2\ DD_e(G) + m^2 + \frac{1}{2}n(M_1 - 2m) + \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \delta_G(e_j)\right]d_G(u_i, e_j),\\
&= 2\ DD(G) + 2\ DD_e(G) + \frac{1}{2}nM_1 + m(m-n) + \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \delta_G(e_j)\right]d_G(u_i, e_j).
\end{aligned} \quad (15)$$

Theorem 2. *If G be a $\{C_3, C_4\}$− graph and $R(G)$ is an vertex-semitotal graph of G, then*

$$(a) CD(R(G)) \leq 2CD(G) + 2rW(G) + 2DD_e(G) + (2m+n)(M_1 - 2m) + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) + \frac{r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + m\sum_{i=1}^{n}\tau_G(u_i) + \frac{rmn}{2},$$
$$(b) CD(R(G)) \geq 2CD(G) + 2DD_e(G) + (2m+n)(M_1 - 2m) + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) - s\sum_{i,j=1}^{m}d_G(e_i,e_i) + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) - \frac{s}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + m\sum_{i=1}^{n}\tau_G(u_i) - sm\left(2m + \frac{n}{2}\right). \quad (16)$$

Proof 6. (a) For upper bounds, we take by lemma 2, $\tau_{R(G)}(u_i) = 2\tau_{(G)}(u_i) + r$ and $\tau_{R(G)}(w_i) = \delta_{(G)}(u_j) + \delta_{(G)}(u_k) - 2 = \delta_G(e_i)$.

Also $d_R(u_i,u_j) = d_G(u_i,u_j)$, for $u_i,u_j \in V(G)$. . $d_R(w_i,w_j) = d_G(e_i,e_j) + 2$, for $e_i,e_j \in E(G)$. $d_R(u_i,w_j) = d_G(u_i,e_j) + 1$, for $u_i \in V(G)$ and $e_j \in E(G)$.

$$\begin{aligned}
CD(R(G)) &= \sum_{\{u_i,u_j\}\subseteq V(G)} d_{R(G)}(u_i,u_j)\left[\tau_{R(G)}(u_i) + \tau_{R(G)}(u_j)\right],\\
&= \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\tau_R(u_i) + \tau_R(u_j)\right]d_R(u_i,u_j) + \frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_R(w_i) + \tau_R(w_j)\right]d_R(w_i,w_j)\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_R(u_i) + \tau_R(w_j)\right]d_R(u_i,w_j)'\\
&\leq \sum\frac{1}{2}\sum_{i,j=1}^{n}\left[2\tau_G(u_i) + r + 2\tau_G(u_j) + r\right]d_G(u_i,u_j) + \frac{1}{2}\sum_{i,j=1}^{m}\left[2\delta_G(e_i) + 2\delta_G(e_j)\right]\left[d_G(e_i,e_j) + 2\right]\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[2\tau_G(u_i) + r + 2\delta_G(e_j)\right]\left[d_G(u_i,e_j) + 1\right],\\
&= \sum_{i,j=1}^{n}\left[\tau_G(u_i) + \tau_G(u_j)\right]d_G(u_i,u_j) + r\sum_{i,j=1}^{n}d_G(u_i,u_j) + \sum_{i,j=1}^{m}\left[\delta_G(e_i) + \delta_G(e_j)\right]\left[d_G(e_i,e_j) + 1\right]\\
&\quad + \sum_{i,j=1}^{m}\left[\delta_G(e_i) + \delta_G(e_j)\right] + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j)\\
&\quad + \frac{r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i) + \frac{r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(e_j).\\
&= 2CD(G) + 2rW(G) + 2DD_e(G) + 2m(M_1 - 2m) + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) +\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) + \frac{r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + m\sum_{i=1}^{n}\tau_G(u_i) + \frac{rmn}{2} + n(M_1 - 2m).\\
&= 2CD(G) + 2rW(G) + 2DD_e(G) + (2m+n)(M_1 - 2m) + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) +\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) + \frac{r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + m\sum_{i=1}^{n}\tau_G(u_i) + \frac{rmn}{2}.
\end{aligned} \quad (17)$$

For lower bounds, by

lemma 6. *$\tau_{R(G)}(u_i) = 2\tau_{(G)}(u_i)$ and $\tau_{R(G)}(w_i) = [\delta_{(G)}(u_j) + \delta_{(G)}(u_k) - 2] - s = \delta_G(e_i) - s$.*

$$
\begin{aligned}
C\,D(R(G)) &= \sum_{\{u_i,u_j\}\subseteq V(G)} d_{R(G)}(u_i,u_j)\left[\tau_{R(G)}(u_i)+\tau_{R(G)}(u_j)\right] \\
&= \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\tau_R(u_i)+\tau_R(u_j)\right]d_R(u_i,u_j) + \frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_R(w_i)+\tau_R(w_j)\right]d_R(w_i,w_j) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_R(u_i)+\tau_R(w_j)\right]d_R(u_i,w_j) \\
&\geq \frac{1}{2}\sum_{i,j=1}^{n}\left[2\tau_G(u_i)+2\tau_G(u_j)\right]d_G(u_i,u_j) + \frac{1}{2}\sum_{i,j=1}^{m}\left[2\delta_G(e_i)-s+2\delta_G(e_j)-s\right]\left[d_G(e_i,e_j)+2\right] \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[2\tau_G(u_i)+2\delta_G(e_j)-s\right]\left[d_G(u_i,e_j)+1\right]. \\
&= \sum_{i,j=1}^{n}\left[\tau_G(u_i)+\tau_G(u_j)\right]d_G(u_i,u_j) + \sum_{i,j=1}^{m}\left[\delta_G(e_i)+\delta_G(e_j)\right]\left[d_G(e_i,e_j)+1\right] \\
&\quad + \sum_{i,j=1}^{m}\left[\delta_G(e_i)+\delta_G(e_j)\right] - s\sum_{i,j=1}^{m} d_G(e_i,e_j) - 2s\sum_{i,j=1}^{m} + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j). \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) - \frac{s}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i) - \frac{s}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(e_j) \\
&= 2C\,D(G) + 2\,DD_e(G) + 2m(M_1-2m) - s\sum_{i,j=1}^{m} d_G(e_i,e_i) - 2sm^2 \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) - \frac{s}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + m\sum_{i=1}^{n}\tau_G(u_i) - \frac{\mathrm{smn}}{2} + n(M_1-2m). \\
&= 2C\,D(G) + 2\,DD_e(G) + (2m+n)(M_1-2m) + \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) - sm\left(2m+\frac{n}{2}\right) \\
&\quad - s\sum_{i,j=1}^{m} d_G(e_i,e_i) + \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) - \frac{s}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}d_G(u_i,e_j) + m\sum_{i=1}^{n}\tau_G(u_i).
\end{aligned}
\tag{18}
$$

If G is $\{C_3, C_4\}-$ free graph, then connection distance index (CD)of $R(G)$ can also be determined by taking $r = 0$ and $s = 0$ in Theorem 2.

Corollary 1. *If G be a $\{C_3, C_4\}-$ free graph,then*

$$
\begin{aligned}
C\,D(R(G)) = 2C\,D(G) + 2\,DD_e(G) + (2m+n)(M_1-2m) \\
+ \sum_{i=1}^{n}\sum_{j=1}^{m}\tau_G(u_i)d_G(u_i,e_j) \\
+ \sum_{i=1}^{n}\sum_{j=1}^{m}\delta_G(u_i)d_G(u_i,e_j) + m\sum_{i=1}^{n}\tau_G(u_i).
\end{aligned}
\tag{19}
$$

Theorem 3. *If $Q(G)$ is an edge-semitotal graph of G, then*

$$\begin{aligned} CD(Q(G)) =& DD(G) + CD(G) + 2rW(G) + 2mn + n\sum_{i=1}^{n}\tau_G(u_i) + rn^2 + 2sW_e(G) \\ &+\frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_G(x_i)+\tau_G(y_i)+\tau_G(x_j)+\tau_G(y_j)\right]\left[d_G(e_i,e_j)+1\right] \\ &+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i)+\tau_G(u_i)\right]\left[d_G(u_i,e_j)+1\right]+\frac{(r+s)}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[d_G(u_i,e_j)+1\right] \\ &+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_G(x_j)+\tau_G(y_j)\right]\left[d_G(e_i,e_j)+1\right]. \end{aligned} \tag{20}$$

Proof 7. By **lemma 7.** $\tau_Q(G)(u_i) \leqslant \delta_G(u_i) + \tau_G(u_i) + r$ and $\tau_{Q(G)}(w_i) \leqslant \tau_{(G)}(u_j) + \tau_{(G)}(u_k) + s$.

To avoid confusion, we take edge $e_i = x_i y_i = u_j u_k$ for some j and k, then $\tau_{Q(G)}(w_i) \leqslant \tau_{(G)}(x_i) + \tau_{(G)}(y_i) + s$. Also

$$\begin{aligned} d_Q(u_i,u_j) &= d_G(u_i,u_j)+1, \\ d_Q(w_i,w_j) &= d_G(e_i,e_j)+1, \\ d_Q(u_i,w_j) &= d_G(u_i,e_i)+1, \\ CD(Q(G)) &= \sum_{\{u_i,u_j\}\subseteq V(Q)} d_{Q(G)}(u_i,u_j)\left[\tau_{Q(G)}(u_i)+\tau_{Q(G)}(u_j)\right], \\ &= \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\tau_Q(u_i)+\tau_Q(u_j)\right]d_Q(u_i,u_j)+\frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_Q(w_i)+\tau_Q(w_j)\right]d_Q(w_i,w_j),+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_Q(u_i)+\tau_Q(w_j)\right]d_Q(u_i,w_j). \\ &\leqslant \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\delta_G(u_i)+\tau_G(u_i)+r+\delta_G(u_j)+\tau_G(u_j)+r\right]\left[d_G(u_i,u_j)+1\right] \\ &\quad+\frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_G(x_i)+\tau_G(y_i)+s+\tau_G(x_j)+\tau_G(y_j)+s\right]\left[d_G(e_i,e_j)+1\right] \\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i)+\tau_G(u_i)+r+\tau_G(x_j)+\tau_G(y_j)+s\right]\left[d_G(u_i,e_j)+1\right]. \\ &= \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\delta_G(u_i)+\delta_G(u_j)\right]\left[d_G(u_i,u_j)\right]+\frac{1}{2}\sum_{i=1,j=1}^{n}\left[\tau_G(u_i)+\tau_G(u_j)\right]\left[d_G(u_i,u_j)\right] \\ &\quad+r\sum_{i=1,j=1}^{n}\left[d_G(u_i,u_j)\right]+\frac{1}{2}\sum_{i=1,j=1}^{n}\left[\delta_G(u_i)+\delta_G(u_j)\right] \\ &\quad+\frac{1}{2}\sum_{i=1,j=1}^{n}\left[\tau_G(u_i)+\tau_G(u_j)\right]+r\sum_{i=1,j=1}^{n}+\frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_G(x_i)+\tau_G(y_i)+\tau_G(x_j)+\tau_G(y_j)\right]\left[d_G(e_i,e_j)+1\right]+s\sum_{i,j=1}^{m}\left[d_G(e_i,e_j)+1\right] \\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i)+\tau_G(u_j)\right]\left[d_G(u_i,e_j)+1\right]+\frac{(r+s)}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[d_G(u_i,e_j)+1\right]+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_G(x_j)+\tau_G(y_j)\right]\left[d_G(u_i,e_j)+1\right]. \\ &= DD(G)+CD(G)+2rW(G)+2mn+n\sum_{i=1}^{n}\tau_G(u_i)+rn^2+2sW_e(G) \\ &\quad+\frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_G(x_i)+\tau_G(y_i)+\tau_G(x_j)+\tau_G(y_j)\right]\left[d_G(e_i,e_j)+1\right] \\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i)+\tau_G(u_i)\right]\left[d_G(u_i,e_j)+1\right]+\frac{(r+s)}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[d_G(u_i,e_j)+1\right]+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_G(x_j)+\tau_G(y_j)\right]\left[d_G(e_i,e_j)+1\right]. \end{aligned} \tag{21}$$

If G is $\{C_3, C_4\}-$ free graph, then connection distance index (CD) of $Q(G)$ can also be determined by taking $r = 0$ and $s = 0$ in Theorem 3.

$$CD(Q(G)) = DD(G) + CD(G) + 2mn + n\sum_{i=1}^{n}\tau_G(u_i) + \frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_G(x_i) + \tau_G(y_i) + \tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i, e_j) + 1\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\delta_G(u_i) + \tau_G(u_i)\right]\left[d_G(u_i, e_j) + 1\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i, e_j) + 1\right]. \quad (22)$$

Theorem 4. *If $T = T(G)$ is an total graph of G, then*

$$CD(T(G)) = 2CD(G) + 2rW(G) + 2sW_e(G) + \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_G(u_i)\right]\left[d_G(u_i, e_j) + 1\right] + \frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_G(x_i) + \tau_G(y_i) + \tau_G(x_j) + \tau_G(y_j)\right] \cdot \left[d_G(e_i, e_j) + 1\right] + \frac{(r+s)}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[d_G(u_i, e_j) + 1\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i, e_j) + 1\right]. \quad (23)$$

Proof 8. To avoid confusion, we take edge $e_i = x_i y_i = u_j u_k$ for some j and k. By lemma 4, $\tau_{T(G)}(w_i) \leqslant \tau_{(G)}(x_i) + \tau_{(G)}(y_i) + s$ and $\tau_{T(G)}(u_i) = 2\tau_G(u_i) + r$

Also

$$\begin{aligned}
d_T(u, v) &= d_G(u, v),\\
d_T(w_i, w_j) &= d_G(e_i, e_j) + 1,\\
d_T(u_i, w_j) &= d_G(u_i, e_i) + 1,\\
CD(T(G)) &= \sum_{\{u,v\}\subseteq V(Q)} d_{T(G)}(u, v)\left[\tau_{T(G)}(u) + \tau_{T(G)}(v)\right],\\
&= \frac{1}{2}\sum_{i=1,j=1}^{n}\left[\tau_T(u_i) + \tau_Q(u_j)\right]d_T(u_i, u_j) + \frac{1}{2}\sum_{i=1,j=1}^{m}\left[\tau_T(w_i) + \tau_T(w_j)\right]d_T(w_i, w_j)\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\tau_T(u_i) + \tau_T(w_j)\right]d_T(u_i, w_j)\\
&\leqslant \frac{1}{2}\sum_{i=1,j=1}^{n}\left[2\tau_G(u_i) + r + 2\tau_G(u_j) + r\right]\left[d_G(u_i, u_j)\right]\\
&\quad + \frac{1}{2}\sum_{i,j=1}^{m}\left[\tau_G(x_i) + \tau_G(y_i) + s + \tau_G(x_j) + \tau_G(y_j) + s\right]\left[d_G(e_i, e_j) + 1\right]\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[2\tau_G(u_i) + r + \tau_G(x_j) + \tau_G(y_j) + s\right]\left[d_G(u_i, e_j) + 1\right]
\end{aligned}$$

Corollary 2. *If G is a $\{C_3, C_4\}-$ free graph, then*

$$
\begin{aligned}
&= \sum_{i=1,j=1}^{n} \left[\tau_G(u_i) + \tau_G(u_j)\right]\left[d_G(u_i,u_j)\right] + r \sum_{i=1,j=1}^{n} \left[d_G(u_i,u_j)\right] \\
&\quad + \frac{1}{2} \sum_{i=1,j=1}^{n} \left[\tau_G(x_j) + \tau_G(y_j) + \tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i,e_j) + 1\right] \\
&\quad + s \sum_{i,j=1}^{m} \left[d_G(e_i,e_j) + 1\right] + \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\tau_G(u_j)\right]\left[d_G(u_i,e_j) + 1\right] \\
&\quad + \frac{(r+s)}{2} \sum_{i=1}^{n}\sum_{j=1}^{m} \left[d_G(u_i,e_j) + 1\right] + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(u_i,e_j) + 1\right] \\
&= 2C\ D(G) + 2rW(G) + 2sW_e(G) + \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\tau_G(u_i)\right]\left[d_G(u_i,e_j) + 1\right] \\
&\quad + \frac{1}{2} \sum_{i=1,j=1}^{m} \left[\tau_G(x_i) + \tau_G(y_i) + \tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i,e_j) + 1\right] \\
&\quad + \frac{(r+s)}{2} \sum_{i=1}^{n}\sum_{j=1}^{m} \left[d_G(u_i,e_j) + 1\right] + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i,e_j) + 1\right].
\end{aligned}
\tag{24}
$$

If G is $\{C_3, C_4\}-$ free graph, then connection distance index (CD) of $T(G)$ can also be determined by taking $r = 0$ and $s = 0$ in Theorem 4.

Corollary 3. *If G is a $\{C_3, C_4\}-$ free graph, then*

$$
\begin{aligned}
C\ D(T(G)) &= 2CD(G) + \frac{1}{2} \sum_{i=1,j=1}^{m} \left[\tau_G(x_i) + \tau_G(y_i) + \tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i,e_j) + 1\right] \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\tau_G(u_i)\right]\left[d_G(u_i,e_j) + 1\right] + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\tau_G(x_j) + \tau_G(y_j)\right]\left[d_G(e_i,e_j) + 1\right].
\end{aligned}
\tag{25}
$$

6. Conclusion

In previous section, first of all, connection numbers of old (black) vertices and new (white) were determined for derived graph $F(G)$ where $F \in \{S, R, Q, T\}$ (by Lemma 1, Lemma 2,Lemma 3 and Lemma 4) . It was found that exact values of connection numbers for $\{C_3, C_4\}-$ free graph and $S(G)$ were determined. Bounded values of $R(G)$, $Q(G)$ and $T(G)$ for $\{C_3, C_4\}-$ graphs were determined. Then, finally, exact and bounded values of connection distance (CD) of derived graphs $\{S(G), R(G), Q(G), T(G)\}$ were determined by Theorems and Corollaries. International Academy of Mathematical Chemistry declared better reports for the chemical capability of the Zagreb connection indices than the ordinary Zagreb indices for the entropy and acentric factor of the octane isomers.

Data Availability

The whole data are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest

The authors declare no conflicts of interest.

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