Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 1500577, 12 pages
https://doi.org/10.1155/2022/1500577

Research Article

A Coupled System of Caputo–Hadamard Fractional Hybrid Differential Equations with Three-Point Boundary Conditions

Meraa Arab and Muath Awadalla

Department of Mathematics and Statistics, College of Science, King Faisal University, Hafuf, Al Ahsa 31982, Saudi Arabia

Correspondence should be addressed to Muath Awadalla; mawadalla@kfu.edu.sa

Received 15 April 2022; Revised 22 May 2022; Accepted 27 May 2022; Published 11 July 2022

Academic Editor: Hamdi Gassara

This article presents a study of the existence and uniqueness of solutions for a system of hybrid fractional differential equations involving fractional derivatives of the Caputo-Hadamard type with three-point hybrid boundary conditions. In addition to this, the "Hyres–Ulam" stability of the solutions for this type of equation is verified, and finally a numerical example was presented to support our theoretical results.

1. Introduction

Fractional calculus has sprouted as a remarkable domain of realization given its extensive applications in the mathematical modeling of numerous complex and nonlocal nonlinear systems. Fractional differential equations were applied in several fields. In the recent past, fractional differentials have drawn the attention of researchers in various fields of research of engineering, bioengineering, mathematics, physics, viscosity, electrochemistry, and other physical processes (see [1–3]).

The difference between the ordinary differential equation and the fractional differential equation is that the latter is an equation that contains fractional derivatives and also comes in a relationship so that the definition of the fractional derivative is an integral equation on the other side of this equation. The Hadamard fractional derivative contains a logarithmic function of an arbitrary exponent in the kernel of the integral appearing in its definition.

For the details of Hadamard fractional calculus, we mention the reader to the articles [4–6]. Fractional differential equations involving Hadamard derivatives attracted remarkable interest in the latest years; for example, see [7–20] and [21, 22].

Freshly, some authors have studied different characteristics of Hybrid FDEs including the existence of solutions (see [23–37]), and some go further and studied Hyers–Ulam stability for FDEs by different mathematical theories; see [28–30] for some details.

In 2019, the authors [12] published a study investigating the existence results for the following boundary value problem:

$$\left\{ {}^{H}D^{p} z(\omega) = f(\omega, z(\omega)), \omega \in [1, T], 0 < p \leq 1, c_1 z(1) + c_2 z(T) = c_3^{H} I^{q} z(\eta) + \varphi, 0 < q \leq 1. \right. \tag{1}$$

where ${}^{H}D^{p}$ is the Caputo–Hadamard fractional derivative; let B denote Banach space. $f \in [1,T] \times B \longrightarrow B$ is a continuous function, $c_i \in \mathbb{R}, i = 1,2,3$, and $\eta \in (1,T), \varphi \in B$.

In 2022, the authors [38] published a study investigating the existence results for the following FDE:

$$
\begin{gathered}
{}^{H}D^{\alpha,\beta;\psi}\left(\frac{z(\varpi)}{w(\varpi, z(\varpi))}\right) = \chi(\varpi, y(\varpi)), \quad \varpi \in (0,T], 0 < \alpha < 1, 0 \leq \beta \leq 1, \\
\lim_{\varpi \longrightarrow 0+} (\psi(\varpi) - \psi(0))^{1-\xi} z(\varpi) = z_0 \in \mathbb{R},
\end{gathered} \tag{2}
$$

where ${}^{H}D^{\alpha,\beta;\psi}$ denoted Psi-Hilfer fractional derivative of order α and type β, $w \in C([0,T] \times \mathbb{R}, \mathbb{R}\backslash\{0\})$,, $\chi \in C([0,T] \times \mathbb{R}, \mathbb{R})$., $\xi = \mu + \nu(1-\mu)$.

Some researchers focus on having solutions to a system of equations only (see [8, 25, 26, 29, 31]). However, others went deeper in their research beyond the issue of verifying the issue of the existence of a solution to such equations and studied the issue of the stability of these solutions (see [21, 35–37]).

This paper aims to investigate the existence of solutions for the following nonlinear sequential hybrid fractional differential equation given by

$$
\begin{gathered}
{}^{CH}D^{p}\left(\frac{u(t)}{z_1(t,u(t),v(t))}\right) = w_1(t,u(t),v(t)), \quad t \in [1,e], p \in (1,2], \\
{}^{CH}D^{q}\left(\frac{v(t)}{z_2(t,u(t),v(t))}\right) = w_2(t,u(t),v(t)), \quad q \in (1,2].
\end{gathered} \tag{3}
$$

Subject to the following boundary conditions:

$$
\begin{gathered}
\left(\frac{u(t)}{z_1(t,u(t),v(t))}\right)_{t=1} = 0, \\
{}^{CH}D\left(\frac{u(t)}{z_1(t,u(t),v(t))}\right)_{t=e} = \lambda_1 {}^{CH}D\left(\frac{u(t)}{z_1(t,u(t),v(t))}\right)_{t=\eta_1}, \\
\left(\frac{v(t)}{z_2(t,u(t),v(t))}\right)_{t=1} = 0, \\
{}^{CH}D\left(\frac{v(t)}{z_2(t,u(t),v(t))}\right)_{t=e} = \lambda_2 {}^{CH}D\left(\frac{v(t)}{z_2(t,u(t),v(t))}\right)_{t=\eta_2},
\end{gathered} \tag{4}
$$

where ${}^{CH}D^{\gamma}, \gamma = \{p,q\}$ is the Caputo–Hadamard fractional derivative of order $1 < \gamma \leq 2, \lambda_1, \lambda_2 \in [0,1)$, and. $\eta_1, \eta_2 \in (1,e)$.

Recently, interest in fractional differential equations has become so considerable that we can say that this field is an independent science in its own right. In fact, the follower of this topic in the literature notices immediately that a good number of researchers interested in the field did not pay much attention to the hybrid type, even the stability is not always investigated.

The findings of this paper must be novel and generalize several earlier findings that are important to the research. To the best of our knowledge, there are no articles that discuss boundary value problems for systems of fractional differential equations with y-Caputo and no articles that investigate Ulam–Hyers stability for differential equations that contain y-Caputo derivatives.

This work is organized as follows. The second section is devoted to illuminating the fundamental principles of fractional calculus and the associated definitions and lemmas. Leray-Schauder, Krasnoselskii, and Banach fixed point theorems are applied in Section 3 to show their existence and uniqueness results. In Section 4, the stability of Hyers–Ulam solutions is examined, and a set of requirements are established that ensure the stability of these solutions. Section 5 provides some examples provided to sustain the theoretical results. In Section 6, a conclusion with future work is also introduced.

2. Preliminaries

This section is dedicated to presenting some definitions, Lemmas, and theorems related to the fixed point concept of solutions of differential equations, which will be used to verify the existence of a solution to the system of equations given by (3)

Definition 1 (see [12]). The Hadamard fractional integral of order ν for a continuous function φ is defined as

$$
{}^{H}I^{\nu}\varphi(\omega) = \frac{1}{\Gamma(\nu)} \int_a^{\omega} \left(\ln \frac{\omega}{\tau}\right)^{q-1} \frac{1}{\tau} \varphi(\tau) d\tau, \nu > 0. \tag{5}
$$

Definition 2 (see [12]). The Hadamard fractional derivative of order $\nu > 0$ for a continuous function φ: $[a,\infty) \longrightarrow \mathbb{R}$ is defined as

$$
{}^{H}D^{\nu}\varphi(\omega) = \delta^{n}\left({}^{H}I^{\nu}\varphi\right)(\omega), n-1 < \nu < n, n = [\nu] + 1, \tag{6}
$$

where $\delta = \omega(d/d\omega)$, $[\nu]$ denotes the integer part of the real number ν.

Definition 3 (see [12]). The Caputo–Hadamard fractional derivative of order ν for at least $n-$times differentiable function φ: $[a,\infty) \longrightarrow \mathbb{R}$ is defined as

$$
{}^{CH}D^{\nu}\varphi(\omega) = \frac{1}{\Gamma(n-\nu)} \int_a^{\omega} \left(\ln \frac{\omega}{\tau}\right)^{n-\nu-1} \delta^{n} \frac{g(\tau)}{\tau} d\tau. \tag{7}
$$

Remark 1. For properties and more notes on the Caputo-Hadamard derivative, we ask the reader to refer to reference [8].

Lemma 1 (see [12]). *Let* $u \in C_{\delta}^{n}([a,T],\mathbb{R})$, *where* $C_{\delta}^{n}[a,T] = \{u: [a,T] \longrightarrow \mathbb{R}: \delta^{(n-1)} \quad u \in C[a,T]\}$. *Then,* ${}^{H}I^{\nu}({}^{H}D^{\nu}u)(\omega) = u(\omega) - \sum_{k=1}^{n} c_k (\ln \omega/a)^{\nu-k}$, *and*

$$^{H}I^{\gamma}\left(^{CH}D^{\gamma}u\right)(\omega)=u(\omega)-\sum_{k=0}^{n-1}c_k\left(\ln\frac{\omega}{a}\right)^k. \tag{8}$$

Theorem 1 (Banach's contraction mapping principle [23]). *Let (S,d) be a complete metric space, $H: S\longrightarrow S$ is a contraction, then*

(i) H has a unique fixed point $s\in S$, that is, $H(s)=s$

(ii) $\forall s_0\in S$, we have $\lim_{n\longrightarrow\infty}H^n(u_0)=u$

Theorem 2 (nonlinear alternative of Leray–Schauder type [23]).

Assume that V is an open subset of a Banach space U, $0\in V$ and $F: \overline{V}\longrightarrow U$ be a contraction such that $F(\overline{V})$ is bounded, then either

(i) F has a fixed point in $\overline{V}$ or

(ii) $\exists\mu\in(0,1)$ and $v\in\partial V$ such that $v=\mu F(v)$ holds

Theorem 3 (Arzela–Ascoli theorem [23]). *$F\subset C(U,\mathbb{R})$ is compact if and only if it is closed, bounded, and equicontinuous.*

Theorem 4 (Krasnoselskii's theorem [23]). *Let $(E,\|\cdot\|)$ be Banach space, B be closed convex subset of E, A be an open subset of B, and $p\in A$. Assume that $G: \overline{A}\longrightarrow B$ can be written as $G=G_1+G_2$. In addition, $G(\overline{A})$ is bounded set in B satisfying the following:*

(i) It is continuous and completely continuous

(ii) It is a contraction, that is, there is a continuous nondecreasing function

$\phi: [0,\infty]\longrightarrow[0,\infty]$ with $\phi(x)>x, x>0$, such that $|G_2(x)-G_2(y)|\le\phi(\|x-y\|)$, for any $x,y\in\overline{A}$.

Then, one of the following holds:

(i) G has a fixed point in $\overline{A}$

(ii) There are $a\in\partial A$ and $\lambda\in(0,1)$ with $a=\lambda G(a)+(1-\lambda)p$

In the next section, we will present the most important results of this study, as we will review finding a solution to (3). Using this result, we obtain the operator and then we develop theories that study the existence and uniqueness of solutions for such a system of equations.

3. Existence Result

Lemma 2. *Given $h\in C([1,e],\mathbb{R})$, the integral solution to the problem*

$$\left\{^{CH}D^{p}\left(\frac{u(t)}{z_1(t,u(t),v(t))}\right)=h(t), t\in[1,e], p\in(1,2], \left(\frac{u(t)}{z_1(t,u(t),v(t))}\right)_{t=1}=0, {}^{CH}D\left(\frac{u(t)}{z_1(t,u(t),v(t))}\right)_{t=e}=\lambda_1{}^{CH}D\left(\frac{u(t)}{z_1(t,u(t),v(t))}\right)_{t=\eta_1},\right. \tag{9}$$

is given by

$$u(t)=z_1(t,u(t),v(t))\times\left(\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}h(r)\frac{dr}{r}+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}h(r)\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}h(r)\frac{dr}{r}\right]\right). \tag{10}$$

Proof. First, we apply $^{H}I_1^{p}$ to the equation $^{CH}D^{p}(u(t)/z_1(t,u(t),v(t)))=h(t)$; using the properties of Caputo–Hadamard fractional derivatives, we get

$$\frac{u(t)}{z_1(t,u(t),v(t))}=-{}^{H}I_1^{p}h(t)+a_0+a_1\log t. \tag{11}$$

Observe that $^{CH}Du(t)/z_1(t,u(t),v(t))=-{}^{H}I_1^{p-1}h(t)+a_1$.

The first boundary condition $(u(t)/z_1(t,u(t),v(t)))_{t=1}=0$, yields $a_0=0$; thus,

$$\frac{u(t)}{z_1(t,u(t),v(t))}=-{}^{H}I_1^{p}h(t)+a_1\log t. \tag{12}$$

Using the second boundary condition $^{CH}D(u(t)/z_1(t,u(t),v(t)))_{t=e}=\lambda_1{}^{CH}D(u(t)/z_1(t,u(t),v(t)))_{t=\eta_1}$ yields

$$a_1=\frac{1}{1-\lambda_1}\left[{}^{H}I_1^{p-1}h(e)-\lambda_1{}^{H}I_1^{p-1}h(\eta_1)\right], \tag{13}$$

which on substituting in (11) gives

$$\frac{u(t)}{z_1(t,u(t),v(t))}=-{}^{H}I_1^{p}h(t)+\frac{1}{1-\lambda_1}\left[{}^{H}I_1^{p-1}h(e)-\lambda_1{}^{H}I_1^{p-1}h(\eta_1)\right]\log t. \tag{14}$$

Alternatively, we have

$$u(t) = z_1(t, u(t), v(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} h(r)\frac{dr}{r} \right) + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} h(r)\frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} h(r)\frac{dr}{r} \right], \quad t \in [1, e]. \tag{15}$$

The proof is completed.

Let $H = C[1, e]$ denote the Banach space of all continuous functions with the norm defined as $\|h\| = \sup_{0\le t\le 1} |h(t)|$. The product space $(H \times H, \|(u, v)\|)$ with the norm $\|(u, v)\| = \|u\| + \|v\|,, \forall (u, v) \in H \times H$ is indeed a Banach space too. By lemma 2, we define an operator $\mathrm{\lambda}\!\!\!^-: H \times H \longrightarrow H \times H$ as

$$\bar{\lambda}(u, v)(t) = \begin{pmatrix} \bar{\lambda}_1(u, v)(t) \\ \bar{\lambda}_2(u, v)(t) \end{pmatrix}, \tag{16}$$

where

$$\begin{aligned} \bar{\lambda}_1(u, v)(t) &= z_1(t, u(t), v(t)) \\ &\times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} w_1(r, u(r), v(r))\frac{dr}{r} + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \right. \\ &\left. \cdot \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} w_1(r, u(r), v(r))\frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} w_1(r, u(r), v(r))\frac{dr}{r} \right] \right), \\ \bar{\lambda}_2(u, v)(t) &= z_2(t, u(t), v(t)) \\ &\times \left(\frac{-1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{r}\right)^{q-1} w_2(r, u(r), v(r))\frac{dr}{r} + \frac{\log t}{(1-\lambda_2)\Gamma(q-1)} \right. \\ &\left. \cdot \left[\int_1^e \left(\log\frac{e}{r}\right)^{q-2} w_2(r, u(r), v(r))\frac{dr}{r} - \lambda_2 \int_1^{\eta_2} \left(\log\frac{\eta_2}{r}\right)^{q-2} w_2(r, u(r), v(r))\frac{dr}{r} \right] \right). \end{aligned} \tag{17}$$

To construct the necessary conditions for the results of uniqueness and existence of a problem (3), let us consider the following hypotheses:

(1) ($\Xi 1$) Let the functions z_1, z_2 be assumed to be continuous and bounded, that is, $\exists \lambda_{z_1}, \lambda_{z_2} > 0$ such that

$$\begin{aligned} |z_1(t, u, v)| &\le \lambda_{z_1}, \\ |z_2(t, u, v)| &\le \lambda_{z_2}, \forall (t, u, v) \in [1, e] \times \mathbb{R}^2. \end{aligned} \tag{18}$$

(2) ($\Xi 2$) Let the functions w_1, w_2 are assumed to be continuous, and $\exists v_i, \tau_i > 0$, $(i = 1, 2)$ such that

$$\begin{aligned} |w_1(t, u_1, v_1) - w_1(t, u_2, v_2)| &\le v_1|u_1 - u_2| + v_2|v_1 - v_2|, \\ |w_2(t, u_1, v_1) - w_2(t, u_2, v_2)| &\le \tau_1|u_1 - u_2| + \tau_2|v_1 - v_2|, \quad \forall t \in [1, e], u_i, v_i \in \mathbb{R}, (i = 1, 2). \end{aligned} \tag{19}$$

(3) ($\Xi 3$) There are positive constants ω_0, θ_0 and $\omega_i, \theta_i \ge 0$ $(i = 1, 2)$ such that $|w_1(t, u, v)| \le \omega_0 + \omega_1|u| + \omega_2|v|$, and $|w_2(t, u, v)| \le \theta_0 + \theta_1|u| + \theta_2|v|$, $\forall t \in [1, e], u, v \in \mathbb{R}$, $(i = 1, 2)$.

(4) ($\Xi 4$) Let $R \subset H \times H$ be a bounded set, then $\exists \kappa_i > 0$, $(i = 1, 2)$ such that $|w_1(t, u(t), v(t))| \le \kappa_1$, and $|w_2(t, u(t), v(t))| \le \kappa_2$, $\forall (u, v) \in R$.

To facilitate the calculations as follows, let us say

$$\Lambda_1 = \sup_{1\le t\le e} \left\{ \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} \frac{dr}{r} + \right. \right.$$

$$+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}\frac{dr}{r}\right]\Bigg)\Bigg\},\leq\frac{1}{\Gamma(p+1)}+\frac{1}{(1-\lambda_1)\Gamma(p)}\left[1+\lambda_1(\log\eta_1)^{p-1}\right],$$
$$\Lambda_2=\sup_{1\leq t\leq e}\Bigg\{\Bigg(\frac{-1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{r}\right)^{q-1}\frac{dr}{r}+$$
$$+\frac{\log t}{(1-\lambda_2)\Gamma(q-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{q-2}\frac{dr}{r}-\lambda_2\int_1^{\eta_2}\left(\log\frac{\eta_2}{r}\right)^{q-2}\frac{dr}{r}\right]\Bigg)\Bigg\},\leq\frac{1}{\Gamma(q+1)}+\frac{1}{(1-\lambda_2)\Gamma(q)}\left[1+\lambda_2(\log\eta_2)^{q-1}\right].\tag{20}$$

□

Theorem 5. *If both* ($\Xi 1$) *and* ($\Xi 2$) *are satisfied and assume that* $[\lambda_{z_1}\Lambda_1(v_1+v_2)+\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)]<1$, *then the boundary value problem (3) has a unique solution.*

Proof. Let $N_{w_1}=\sup_{1\leq t\leq e}|w_1(t,0,0)|$, $N_{w_2}=\sup_{1\leq t\leq e}|w_2(t,0,0)|$, then we select $\varepsilon\geq\lambda_{z_1}\Lambda_1 N_{w_1}+\lambda_{z_2}\Lambda_2 N_{w_2}/1-(\lambda_{z_1}\Lambda_1(v_1+v_2)+\lambda_{z_2}\Lambda_2(\tau_1+\tau_2))$; next, we show $\overline{\lambda}\overline{Q_\varepsilon}\subset\overline{Q_\varepsilon}$, where

$$\overline{Q_\varepsilon}=\{(u,v)\in H\times H\colon\ \|(u,v)\|\leq\varepsilon\}.\tag{21}$$

Observe that $|w_1(t,u,v)|=|w_1(t,u,v)-w_1(t,0,0)+w_1(t,0,0)|\leq v_1\|u\|+v_2\|v\|+N_{w_1}\leq(v_1+v_2)\varepsilon+N_{w_1}$.

For any $(u,v)\in\overline{Q_\varepsilon}, t\in[1,e]$, we have

$$\begin{aligned}
\left|\lambda_1(u,v)(t)\right|&=|z_1(t,u(t),v(t))\times\Bigg(\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}w_1(r,u(r),v(r))\frac{dr}{r}\\
&\quad+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}w_1(r,u(r),v(r))\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}w_1(r,u(r),v(r))\frac{dr}{r}\right]\Bigg)|\\
&\leq\lambda_{z_1}\sup_{1\leq t\leq e}\Bigg\{\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}|w_1(r,u(r),v(r))|\frac{dr}{r}\\
&\quad+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}|w_1(r,u(r),v(r))|\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}|w_1(r,u(r),v(r))|\frac{dr}{r}\right]\\
&\leq\lambda_{z_1}\left[(v_1+v_2)\varepsilon+N_{w_1}\right]\sup_{1\leq t\leq e}\Bigg\{\Bigg(\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}\frac{dr}{r}\\
&\quad+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}\frac{dr}{r}\right]\Bigg)\Bigg\}\leq\lambda_{z_1}\left[(v_1+v_2)\varepsilon+N_{w_1}\right]\Lambda_1.
\end{aligned}\tag{22}$$

Similar to what was done above, we get

$$\left\|\lambda_2(u,v)\right\|\leq\lambda_{z_2}\left[(\tau_1+\tau_2)\varepsilon+N_{w_2}\right]\Lambda_2.\tag{23}$$

From (22) and (23), we deduce that $\|\lambda(u,v)\|\leq\varepsilon$.

Next, for any $(u_1,v_1),(u_2,v_2)\in H\times H, \forall t\in[1,e]$, we have

$$\begin{aligned}
\left|\lambda_1(u_1,v_1)(t)-\lambda_1(u_2,v_2)(t)\right|&\leq\lambda_{z_1}\sup_{1\leq t\leq e}\Bigg\{\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}\left|w_1(r,u_1(r),v_1(r))-w_1(r,u_2(r),v_2(r))\right|\frac{dr}{r}+\\
&\quad+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\Bigg[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}\left|w_1(r,u_1(r),v_1(r))-w_1(r,u_2(r),v_2(r))\right|\frac{dr}{r}\\
&\quad-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}\left|w_1(r,u_1(r),v_1(r))-w_1(r,u_2(r),v_2(r))\right|\frac{dr}{r}\Bigg]
\end{aligned}$$

$$\leq \lambda_{z_1}\left(v_1\|u_1-u_2\|+v_2\|v_1-v_2\|\right)\sup_{1\leq t\leq e}\left\{\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}\frac{dr}{r}\right.$$
$$\left.+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}\frac{dr}{r}\right]\right\}\leq \lambda_{z_1}\Lambda_1\left(v_1\|u_1-u_2\|+v_2\|v_1-v_2\|\right)$$
$$\leq \lambda_{z_1}\Lambda_1\left(v_1+v_2\right)\left(\|u_1-u_2\|+\|v_1-v_2\|\right). \tag{24}$$

Similarly, we can find

$$\|\lambda_2(u_1,v_1)-\lambda_2(u_2,v_2)\|\leq \lambda_{z_2}\Lambda_2\left(\tau_1+\tau_2\right)\left(\|u_1-u_2\|+\|v_1-v_2\|\right). \tag{25}$$

Combining (24) and (25) yields

$$\|\lambda(u_1,v_1)-\lambda(u_2,v_2)\|\leq\left[\lambda_{z_1}\Lambda_1\left(v_1+v_2\right)+\lambda_{z_2}\Lambda_2\left(\tau_1+\tau_2\right)\right]\times\left(\|u_1-u_2\|+\|v_1-v_2\|\right). \tag{26}$$

Since $[\lambda_{z_1}\Lambda_1(v_1+v_2)+\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)]<1$, then $\|\lambda(u_1,v_1)-\lambda(u_2,v_2)\|\leq(\|u_1-u_2\|+\|v_1-v_2\|)$, that is, the defined operator λ is a contraction; consequently, Banach fixed point theorem applies; thus, problem (3) has a unique solution on $[1,e]$. □

Theorem 6. *If* (Ξ1), (Ξ3), *and* (Ξ4) *are satisfied and if* $(\lambda_{z_1}\Lambda_1\omega_1+\lambda_{z_2}\Lambda_2\theta_1)<1$ *and* $(\lambda_{z_1}\Lambda_1\omega_2+\lambda_{z_2}\Lambda_2\theta_2)<1$, *then the boundary value problem (3) has at least one solution.*

Proof. First, we show that the operator $\lambda: H\times H\longrightarrow H\times H$ is completely continuous; obviously, the operator is continuous as a result that z_1, z_2, w_1, and w_2 are all assumed to be continuous.

By the aid of (Ξ4), $\forall(u,v)\in R$, we have

$$\left|\lambda_1(u,v)(t)\right|=\left|z_1(t,u(t),v(t))\times\left(\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}w_1(r,u(r),v(r))\frac{dr}{r}\right.\right.$$
$$\left.\left.+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}w_1(r,u(r),v(r))\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}w_1(r,u(r),v(r))\frac{dr}{r}\right]\right)\right|$$
$$\leq\lambda_{z_1}\sup_{1\leq t\leq e}\left\{\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}\left|w_1(r,u(r),v(r))\right|\frac{dr}{r}\right.$$
$$\left.+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}\left|w_1(r,u(r),v(r))\right|\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}\left|w_1(r,u(r),v(r))\right|\frac{dr}{r}\right]\right.$$
$$\leq\lambda_{z_1}\kappa_1\sup_{1\leq t\leq e}\left\{\left(\frac{-1}{\Gamma(p)}\int_1^t\left(\log\frac{t}{r}\right)^{p-1}\frac{dr}{r}+\frac{\log t}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}\frac{dr}{r}\right]\right)\right\}$$
$$\leq\lambda_{z_1}\Lambda_1\kappa_1. \tag{27}$$

Similarly,

$$\|\lambda_2(u,v)\|\leq\lambda_{z_2}\Lambda_2\kappa_2. \tag{28}$$

Combining inequalities (27) and (28) yields $\|\lambda(u,v)\|\leq\lambda_{z_1}\Lambda_1\kappa_1+\lambda_{z_2}\Lambda_2\kappa_2$, that is, the operator λ is uniformly bounded.

Next, to verify the equicontinuity for the operator λ, we let $t_1,t_2\in[1,e]$, $(t_1<t_2)$ and then

$$\left|\lambda_1(u,v)(t_2)-\lambda_1(u,v)(t_1)\right|\leq\lambda_{z_1}\kappa_1\sup_{1\leq t\leq e}\left\{\frac{-1}{\Gamma(p)}\int_1^{t_1}\left(\left(\log\frac{t_2}{r}\right)^{p-1}-\left(\log\frac{t_1}{r}\right)^{p-1}\right)\frac{dr}{r}+\frac{-1}{\Gamma(p)}\int_{t_1}^{t_2}\left(\left(\log\frac{t_2}{r}\right)^{p-1}\right)\frac{dr}{r}\right.$$
$$\left.+\frac{\log t_2-\log t_1}{(1-\lambda_1)\Gamma(p-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{p-2}\frac{dr}{r}-\lambda_1\int_1^{\eta_1}\left(\log\frac{\eta_1}{r}\right)^{p-2}\frac{dr}{r}\right]\right\}. \tag{29}$$

$$\left|\bar{\lambda}_2(u,v)(t_2)-\bar{\lambda}_2(u,v)(t_1)\right|\le\lambda_{z_2}\kappa_2\sup_{1\le t\le e}\left\{\frac{-1}{\Gamma(q)}\int_1^{t_1}\left(\left(\log\frac{t_2}{r}\right)^{q-1}-\left(\log\frac{t_1}{r}\right)^{q-1}\right)\frac{dr}{r}+\frac{-1}{\Gamma(q)}\int_{t_1}^{t_2}\left(\left(\log\frac{t_2}{r}\right)^{q-1}\right)\frac{dr}{r}\right.$$
$$\left.+\frac{\log t_2-\log t_1}{(1-\lambda_2)\Gamma(q-1)}\left[\int_1^e\left(\log\frac{e}{r}\right)^{q-2}\frac{dr}{r}-\lambda_2\int_1^{\eta_2}\left(\log\frac{\eta_2}{r}\right)^{q-2}\frac{dr}{r}\right]\right\}. \tag{30}$$

The R.H.S for both (29) and (30) tends to zero as $t_1\longrightarrow t_2$, and they are both independent on (u,v). This implies that the operator $\bar{\lambda}(u,v)$ is equicontinuous and yields that the operator $\bar{\lambda}(u,v)$ is completely continuous.

Finally, we establish the bounded set given by $S=\{(u,v)\in H\times H|(u,v)=\beta\bar{\lambda}(u,v),\beta\in[0,1]\}$, then $\forall t\in[1,e]$; the equation $(u,v)=\beta\bar{\lambda}(u,v)$ gives

$$\begin{aligned}u(t)&=\beta\bar{\lambda}_1(u,v)(t),\\ v(t)&=\beta\bar{\lambda}_2(u,v)(t).\end{aligned} \tag{31}$$

Using the hypothesis $(\Xi3)$, we get

$$\begin{aligned}\|u\|&\le\lambda_{z_1}\Lambda_1(\omega_0+\omega_1\|u\|+\omega_2\|v\|),\\ \|v\|&\le\lambda_{z_2}\Lambda_2(\theta_0+\theta_1\|u\|+\theta_2\|v\|).\end{aligned} \tag{32}$$

Consequently, we have

$$\begin{aligned}\|u\|+\|v\|\le{}&\left(\lambda_{z_1}\Lambda_1\omega_0+\lambda_{z_2}\Lambda_2\theta_0\right)+\left(\lambda_{z_1}\Lambda_1\omega_1+\lambda_{z_2}\Lambda_2\theta_1\right)\|u\|\\ &+\left(\lambda_{z_1}\Lambda_1\omega_2+\lambda_{z_2}\Lambda_2\theta_2\right)\|v\|.\end{aligned} \tag{33}$$

Inequality (33) can be written as follows:

$$\|(u,v)\|\le\frac{\left(\lambda_{z_1}\Lambda_1\omega_0+\lambda_{z_2}\Lambda_2\theta_0\right)}{\Lambda_0}. \tag{34}$$

where $\Lambda_0=\min\left\{1-(\lambda_{z_1}\Lambda_1\quad\omega_1+\lambda_{z_2}\Lambda_2\theta_1),1-(\lambda_{z_1}\Lambda_1\omega_2+\lambda_{z_2}\Lambda_2\theta_2)\right\}$.

Inequality (34) shows that S is bounded. Hence, Leray–Schauder alternative applies, which implies that problem (3) has at least one solution. □

4. Stability

In the current section, we are interested in studying the U–H of the proposed system of the coupled sequential fractional differential BVPs (1).

Definition 4. The system of the coupled sequential fractional differential BVPs (1) is stable with in U–H sense if a real number $c=\max(c_1,c_2)>0$ exists so that for any $\varepsilon=\max(\varepsilon_1,\varepsilon_2)>0$ and for any $(\bar{u},\bar{v})\in H\times H$ satisfying

$$\begin{cases}\left|{}^{CH}D^p\left(\dfrac{\bar{u}(t)}{z_1(t,\bar{u}(t),\bar{v}(t))}\right)-w_1(t,\bar{u}(t),\bar{v}(t))\right|<\varepsilon_1,t\in[1,e],\\[2ex] \left|{}^{CH}D^q\left(\dfrac{\bar{u}(t)}{z_2(t,\bar{u}(t),\bar{v}(t))}\right)-w_2(t,\bar{u}(t),\bar{v}(t))\right|<\varepsilon_2.\end{cases} \tag{35}$$

There exists a unique solution $(u,v)\in H\times H$ of (3) with

$$\|(u,v)-(\bar{u},\bar{v})\|<c\varepsilon. \tag{36}$$

It is clear that $(\bar{u},\bar{v})\in H\times H$ satisfies inequality (35) if there exists a function $(f_1,f_2)\in H\times H$ (which depends on $(\bar{u},\bar{v})$), such that

(i) $|f_1(t)|<\varepsilon_1$ and $|f_2(t)|<\varepsilon_2$, $t\in[1,e]$

(ii) for $t\in[1,e]$

And

$$\left\{{}^{CH}D^p\left(\frac{\bar{u}(t)}{z_1(t,\bar{u}(t),\bar{v}(t))}\right)=w_1(t,\bar{u}(t),\bar{v}(t))+f_1(t),{}^{CH}D^q\left(\frac{\bar{u}(t)}{z_2(t,\bar{u}(t),\bar{v}(t))}\right)=w_2(t,\bar{u}(t),\bar{v}(t))+f_2(t).\right. \tag{37}$$

Theorem 7. *Suppose that* $(\Xi2)$ *is fulfilled. Moreover,*

$$\begin{aligned}&\lambda_{z_1}\Lambda_1(v_1+v_2)<1,\\ &\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)<1,\\ &\Delta=\left(1-\lambda_{z_1}\Lambda_1(v_1+v_2)\right)\left(1-\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)\right)\\ &\qquad-\lambda_{z_1}\Lambda_1(v_1+v_2)\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)>0.\end{aligned} \tag{38}$$

Then, the system of coupled sequential fractional differential BVPs (1) is U–H stable.

Proof. Assume that for $\varepsilon_1,\varepsilon_2>0$ a couple $(\bar{u},\bar{v})\in H\times H$ satisfies the inequalities (35). Introduce the following operators.

Then,

$$\begin{aligned}
\overline{u}(t) &= z_1(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} w_1(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} \right. \\
&\quad \left. + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} w_1(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} w_1(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} \right] \right), \\
&\quad + z_1(t,u(t),v(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} f_1(r) \frac{dr}{r} \right. \\
&\quad \left. + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} f_1(r) \frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} f_1(r) \frac{dr}{r} \right] \right),
\end{aligned} \tag{39}$$

$$\begin{aligned}
\overline{v}(t) &= z_2(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{r}\right)^{q-1} w_2(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} \right. \\
&\quad \left. + \frac{\log t}{(1-\lambda_2)\Gamma(q-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{q-2} w_2(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} - \lambda_2 \int_1^{\eta_2} \left(\log\frac{\eta_2}{r}\right)^{q-2} w_2(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} \right] \right), \\
&\quad + z_2(t,u(t),v(t)) \times \left(\frac{-1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{r}\right)^{q-1} f_2(r) \frac{dr}{r} \right. \\
&\quad \left. + \frac{\log t}{(1-\lambda_2)\Gamma(q-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{q-2} f_2(r) \frac{dr}{r} - \lambda_2 \int_1^{\eta_2} \left(\log\frac{\eta_2}{r}\right)^{q-2} f_2(r) \frac{dr}{r} \right] \right).
\end{aligned} \tag{40}$$

From (39) and (40), we obtain

$$\begin{aligned}
&\left| \overline{u}(t) - z_1(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} w_1(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \right. \right. \\
&\quad \left. \left. \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} w_1(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} w_1(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} \right] \right) \right| \\
&\leq \left| z_1(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} f_1(r) \frac{dr}{r} + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \right. \right. \\
&\quad \left. \left. \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} f_1(r) \frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} f_1(r) \frac{dr}{r} \right] \right) \right|.
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\left| \overline{v}(t) - z_2(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{r}\right)^{q-1} w_2(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} + \frac{\log t}{(1-\lambda_2)\Gamma(q-1)} \right. \right. \\
&\quad \left. \left. \left[\int_1^e \left(\log\frac{e}{r}\right)^{q-2} w_2(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} - \lambda_2 \int_1^{\eta_2} \left(\log\frac{\eta_2}{r}\right)^{q-2} w_2(r,\overline{u}(r),\overline{v}(r)) \frac{dr}{r} \right] \right) \right| \\
&\leq \left| z_2(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{r}\right)^{q-1} f_2(r) \frac{dr}{r} + \frac{\log t}{(1-\lambda_2)\Gamma(q-1)} \right. \right. \\
&\quad \left. \left. \left[\int_1^e \left(\log\frac{e}{r}\right)^{q-2} f_2(r) \frac{dr}{r} - \lambda_2 \int_1^{\eta_2} \left(\log\frac{\eta_2}{r}\right)^{q-2} f_2(r) \frac{dr}{r} \right] \right) \right|.
\end{aligned} \tag{42}$$

From (41) and (42), we obtain

$$\left|\overline{u}(t) - z_1(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} w_1(r,\overline{u}(r),\overline{v}(r))\frac{dr}{r} + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} w_1(r,\overline{u}(r),\overline{v}(r))\frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} w_1(r,\overline{u}(r),\overline{v}(r))\frac{dr}{r}\right]\right)\right| \leq \lambda_{z_1}\Lambda_1\|f_1\| \leq \lambda_{z_1}\Lambda_1\varepsilon_1. \tag{43}$$

$$\left|\overline{v}(t) - z_2(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{r}\right)^{q-1} w_2(r,\overline{u}(r),\overline{v}(r))\frac{dr}{r} + \frac{\log t}{(1-\lambda_2)\Gamma(q-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{q-2} w_2(r,\overline{u}(r),\overline{v}(r))\frac{dr}{r} - \lambda_2 \int_1^{\eta_2} \left(\log\frac{\eta_2}{r}\right)^{q-2} w_2(r,\overline{u}(r),\overline{v}(r))\frac{dr}{r}\right]\right)\right| \leq \lambda_{z_2}\Lambda_2\|f_2\| \leq \lambda_{z_2}\Lambda_2\varepsilon_2. \tag{44}$$

Let $(u,v) \in H \times H$ be a solution of (3). Thanks to lemma 2, it is equivalent to the following integral equations:

$$\begin{aligned} u(t) &= z_1(t,u(t),v(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} w_1(r,u(r),v(r))\frac{dr}{r} \right. \\ &\quad \left. + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} w_1(r,u(r),v(r))\frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} w_1(r,u(r),v(r))\frac{dr}{r}\right]\right), \\ v(t) &= z_2(t,u(t),v(t)) \times \left(\frac{-1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{r}\right)^{q-1} w_2(r,u(r),v(r))\frac{dr}{r} \right. \\ &\quad \left. + \frac{\log t}{(1-\lambda_2)\Gamma(q-1)} \left[\int_1^e \left(\log\frac{e}{r}\right)^{q-2} w_2(r,u(r),v(r))\frac{dr}{r} - \lambda_2 \int_1^{\eta_2} \left(\log\frac{\eta_2}{r}\right)^{q-2} w_2(r,u(r),v(r))\frac{dr}{r}\right]\right). \end{aligned} \tag{45}$$

By the same arguments in theorem 2, we get

$$\begin{aligned} |u(t) - \overline{u}(t) = &\left|\bar{\lambda}_1(u,v)(t) - \bar{\lambda}_1(\overline{u},\overline{v})(t) - z_1(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} f_1(r)\frac{dr}{r} + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \right.\right. \\ &\left.\left. \cdot \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} f_1(r)\frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} f_1(r)\frac{dr}{r}\right]\right)\right| \\ &\leq \left|\bar{\lambda}_1(u,v)(t) - \bar{\lambda}_1(\overline{u},\overline{v})(t)\right| + \left|z_1(t,\overline{u}(t),\overline{v}(t)) \times \left(\frac{-1}{\Gamma(p)} \int_1^t \left(\log\frac{t}{r}\right)^{p-1} f_1(r)\frac{dr}{r} + \frac{\log t}{(1-\lambda_1)\Gamma(p-1)} \right.\right. \\ &\left.\left. \cdot \left[\int_1^e \left(\log\frac{e}{r}\right)^{p-2} f_1(r)\frac{dr}{r} - \lambda_1 \int_1^{\eta_1} \left(\log\frac{\eta_1}{r}\right)^{p-2} f_1(r)\frac{dr}{r}\right]\right)\right| \\ &\leq \lambda_{z_1}\Lambda_1(v_1 + v_2)(\|u - \overline{u}\| + \|v - \overline{v}\|) + \lambda_{z_1}\Lambda_1\varepsilon_1. \end{aligned} \tag{46}$$

$$
\begin{aligned}
|v(t)-\overline{v}(t)| &= \Bigg|\bar{\lambda}_2(u,v)(t)-\bar{\lambda}_2(\overline{u},\overline{v})(t)-z_2(t,\overline{u}(t),\overline{v}(t))\times\Bigg(\frac{-1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{r}\right)^{q-1}f_2(r)\frac{dr}{r}+\frac{\log t}{(1-\lambda_2)\Gamma(q-1)}\\
&\quad\cdot\left[\int_1^e\left(\log\frac{e}{r}\right)^{q-2}f_2(r)\frac{dr}{r}-\lambda_2\int_1^{\eta_2}\left(\log\frac{\eta_2}{r}\right)^{q-2}f_2(r)\frac{dr}{r}\right]\Bigg)\Bigg|\\
&\le\left|\bar{\lambda}_2(u,v)(t)-\bar{\lambda}_2(\overline{u},\overline{v})(t)\right|+\Bigg|z_2(t,\overline{u}(t),\overline{v}(t))\times\Bigg(\frac{-1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{r}\right)^{q-1}f_2(r)\frac{dr}{r}+\frac{\log t}{(1-\lambda_2)\Gamma(q-1)}\\
&\quad\cdot\left[\int_1^e\left(\log\frac{e}{r}\right)^{q-2}f_2(r)\frac{dr}{r}-\lambda_2\int_1^{\eta_2}\left(\log\frac{\eta_2}{r}\right)^{q-2}f_2(r)\frac{dr}{r}\right]\Bigg)\Bigg|\\
&\le\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)(\|u-\overline{u}\|+\|v-\overline{v}\|)+\lambda_{z_2}\Lambda_2\varepsilon_2.
\end{aligned}
\tag{47}
$$

It follows that

$$
\begin{aligned}
&\|u-\overline{u}\|-\lambda_{z_1}\Lambda_1(v_1+v_2)(\|u-\overline{u}\|+\|v-\overline{v}\|)\le\lambda_{z_1}\Lambda_1\varepsilon_1,\\
&\|v-\overline{v}\|-\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)(\|u-\overline{u}\|+\|v-\overline{v}\|)\le\lambda_{z_2}\Lambda_2\varepsilon_2.
\end{aligned}
\tag{48}
$$

Representing these inequalities as matrices, we get

$$
\begin{pmatrix} 1-\lambda_{z_1}\Lambda_1(v_1+v_2) & -\lambda_{z_1}\Lambda_1(v_1+v_2)\\ 1-\lambda_{z_2}\Lambda_2(\tau_1+\tau_2) & -\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)\end{pmatrix}\begin{pmatrix}\|u-\overline{u}\|\\ \|v-\overline{v}\|\end{pmatrix}\le\begin{pmatrix}\lambda_{z_1}\Lambda_1\varepsilon_1\\ \lambda_{z_2}\Lambda_2\varepsilon_2\end{pmatrix}.
\tag{49}
$$

Solving the above inequality, we get

$$
\begin{aligned}
\|u-\overline{u}\|&\le\frac{1-\lambda_{z_1}\Lambda_1(v_1+v_2)}{\Delta}\lambda_{z_1}\Lambda_1\varepsilon_1+\frac{\lambda_{z_1}\Lambda_1(v_1+v_2)}{\Delta}\lambda_{z_2}\Lambda_2\varepsilon_2,\\
\|v-\overline{v}\|&\le\frac{\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)}{\Delta}\lambda_{z_1}\Lambda_1\varepsilon_1+\frac{1-\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)}{\Delta}\lambda_{z_2}\Lambda_2\varepsilon_2,
\end{aligned}
\tag{50}
$$

where $\Delta=(1-\lambda_{z_1}\Lambda_1(v_1+v_2))(1-\lambda_{z_2}\Lambda_2(\tau_1+\tau_2))-\lambda_{z_1}\Lambda_1(v_1+v_2)\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)\neq 0$.

Thus,

$$
\begin{aligned}
\|u-\overline{u}\|+\|v-\overline{v}\|&\le\left(\frac{1-\lambda_{z_1}\Lambda_1(v_1+v_2)}{\Delta}+\frac{\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)}{\Delta}\right)\lambda_{z_1}\Lambda_1\varepsilon_1\\
&\quad+\left(\frac{1-\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)}{\Delta}+\frac{\lambda_{z_1}\Lambda_1(v_1+v_2)}{\Delta}\right)\lambda_{z_2}\Lambda_2\varepsilon_2.
\end{aligned}
\tag{51}
$$

For $\varepsilon=\max(\varepsilon_1,\varepsilon_2)$ and

$$
c=\frac{\left(1-\lambda_{z_1}\Lambda_1(v_1+v_2)+\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)\right)\lambda_{z_1}\Lambda_1+\left(1-\lambda_{z_2}\Lambda_2(\tau_1+\tau_2)+\lambda_{z_1}\Lambda_1(v_1+v_2)\right)\lambda_{z_2}\Lambda_2}{\Delta},
\tag{52}
$$

we get

$$
\|(u,\overline{u})-(v,\overline{v})\|\le\|u-\overline{u}\|+\|v-\overline{v}\|\le c\varepsilon.
\tag{53}
$$

Therefore, by means of Definition 4, the solution to problem (3) is U–H stable. □

5. Example

In this part, we present an applied example to support the theoretical results we reached in the previous part; consider the following system:

$$
\begin{cases}
{}^{CH}D^{\frac{4}{3}}\left(\dfrac{u(t)}{5/6|\sin u(t)|+7/6}\right)=7e^{5t}+\dfrac{1}{9\sqrt{t^3+24}}\dfrac{|v|}{1+|v|}+\dfrac{1}{45}\tan^{-1}u, t\in[1,e],\\
{}^{CH}D^{\frac{5}{4}}\left(\dfrac{v(t)}{1/6|\cos v(t)|}\right)=2e^{-3t}\sqrt{t}+\dfrac{1}{20}\left(\tan^{-1}u+\tan^{-1}v\right),\\
\left(\dfrac{u(t)}{1/2|\sin u(t)|+7/5}\right)_{t=1}=0, {}^{CH}D\left(\dfrac{u(t)}{1/2|\sin u(t)|+7/5}\right)_{t=e}=\dfrac{3}{4}{}^{CH}D\left(\dfrac{u(t)}{1/2|\sin u(t)|+7/5}\right)_{t=2},\\
\left(\dfrac{v(t)}{1/3|\cos v(t)|}\right)_{t=1}=0, {}^{CH}D\left(\dfrac{v(t)}{1/3|\cos v(t)|}\right)_{t=e}=\dfrac{3}{4}{}^{CH}D\left(\dfrac{v(t)}{1/3|\cos v(t)|}\right)_{t=\frac{5}{2}}.
\end{cases}
\tag{54}
$$

Here, $3/4 = \lambda_1 = \lambda_2$, $p = 4/3, q = 5/4, \eta_1 = 2, \eta_2 = 5/2, z_1(t, u(t), v(t)) = 5/6|\sin u(t)| + 7/6, z_2(t, u(t), v(t)) = 1/6|\cos v(t)|$.

$$w_1(t, u(t), v(t)) = 7e^{5t} + \frac{1}{9\sqrt{t^3 + 24}}\frac{|v|}{1 + |v|} + \frac{1}{45}\tan^{-1}u,$$

$$w_2(t, u(t), v(t)) = 2e^{-3t}\sqrt{t} + \frac{1}{17}\left(\tan^{-1} u + \tan^{-1} v\right). \tag{55}$$

Observe that

$$\begin{aligned}
&\left|w_1(t, u_1, v_1) - w_1(t, u_2, v_2)\right| \le \frac{1}{45}\left|u_2 - u_1\right| + \frac{1}{45}\left|v_2 - v_1\right|,\\
&\left|w_2(t, u_1, v_1) - w_2(t, u_2, v_2)\right| \le \frac{1}{17}\left|u_2 - u_1\right| + \frac{1}{17}\left|v_2 - v_1\right|,\\
&\left[\lambda_{z_1}\Lambda_1(v_1 + v_2) + \lambda_{z_2}\Lambda_2(\tau_1 + \tau_2)\right] \le 2 \times (8.294) \times \frac{1}{45} + \frac{1}{6}\\
&\qquad \times (8.534) \times \frac{1}{17} = 0.452288 < 1.
\end{aligned} \tag{56}$$

Thus, the boundary value problem (54) satisfies all the conditions of Theorem 5; accordingly, we conclude that the BVP has a unique solution on $[1, e]$.

6. Conclusion

In previous works, researchers investigated the existence and uniqueness of linear fractional differential equations involving Caputo–Hadamard. The legacy of this work lies in verifying the existence and uniqueness of solutions to a coupled system of Caputo–Hadamard hybrid fractional differential equations with Hybrid boundary conditions. Our major findings are demonstrated using the Banach fixed point theorem and the alternative of Leray–Schauder. The stability of the solutions involved in the Hyers–Ulam type was investigated. We provide an example to demonstrate the study results. Caputo–Hadamard calculus has its own prominence. For example, some researchers showed that by considering different different fractional derivative's order, a particular natural phenomenon can be remodeled with more accuracy when fractional derivative is replaced by the ordinary one.

In future studies, researchers can verify the existence, uniqueness, and stability of the solutions for the system of equations given by (3) using the ψ-Hilfer fractional derivative or any other derivatives such as the fractional Katugambula derivative. In addition, this system can be used in practical applications of the subject by taking our results as proven facts [39, 40].

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Project no. GRANT242).

References

[1] C. Ionescu, A. Lopes, D. Copot, J. T. Machado, and J. H. Bates, “The role of fractional calculus in modeling biological phenomena: a review,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 51, pp. 141–159, 2017.

[2] R. L. Magin, “Fractional calculus models of complex dynamics in biological tissues,” *Computers & Mathematics with Applications*, vol. 59, no. 5, pp. 1586–1593, 2010.

[3] R. Toledo-Hernandez, V. Rico-Ramirez, G. A. Iglesias-Silva, and U. M. Diwekar, “A fractional calculus approach to the dynamic optimization of biological reactive systems. Part I: fractional models for biological reactions,” *Chemical Engineering Science*, vol. 117, pp. 217–228, 2014.

[4] Y. Adjabi, F. Jarad, D. Baleanu, and T. Abdeljawad, “On Cauchy problems with Caputo Hadamard fractional derivatives,” *Journal of Computational Analysis and Applications*, vol. 21, no. 4, pp. 661–681, 2016.

[5] R. Almeida, “Caputo–Hadamard fractional derivatives of variable order,” *Numerical Functional Analysis and Optimization*, vol. 38, no. 1, pp. 1–19, 2017.

[6] Y. Y. Gambo, F. Jarad, D. Baleanu, and T. Abdeljawad, “On Caputo modification of the Hadamard fractional derivatives,” *Advances in Difference Equations*, vol. 2014, no. 1, 2014.

[7] Y. Arioua and N. Benhamidouche, *Boundary Value Problem for Caputo-Hadamard Fractional Differential Equations*, 2017.

[8] F. Jarad, T. Abdeljawad, and D. Baleanu, “Caputo-type modification of the Hadamard fractional derivatives,” *Advances in Difference Equations*, vol. 2012, no. 1, 2012.

[9] X. Zhang, “The general solution of differential equations with Caputo-Hadamard fractional derivatives and impulsive effect,” *Advances in Difference Equations*, vol. 2015, no. 1, 2015.

[10] X. Zhang, “On impulsive partial differential equations with Caputo-Hadamard fractional derivatives,” *Advances in Difference Equations*, vol. 2016, no. 1, 2016.

[11] X. Zhang, Z. Liu, H. Peng, X. Zhang, and S. Yang, “The general solution of differential equations with Caputo-Hadamard fractional derivatives and noninstantaneous impulses,” *Advances in Mathematical Physics*, vol. 2017, Article ID 3094173, 11 pages, 2017.

[12] A. Boutiara, K. Guerbati, and M. Benbachir, “Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces,” *Aims Mathematics*, vol. 5, no. 1, pp. 259–272, 2019.

[13] B. B. He and H. C. Zhou, “Caputo–hadamard fractional halanay inequality,” *Applied Mathematics Letters*, vol. 125, Article ID 107723, 2022.

[14] A. Ardjouni, A. Lachouri, and A. Djoudi, “Existence and uniqueness results for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations,” *Open J. Math. Anal*, vol. 3, no. 2, pp. 106–111, 2019.

[15] E. Fan, C. Li, and Z. Li, “Numerical approaches to Caputo–Hadamard fractional derivatives with applications to long-term integration of fractional differential systems,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 106, Article ID 106096, 2022.

[16] L. Ma, "Comparison theorems for Caputo–Hadamard fractional differential equations," *Fractals*, vol. 27, no. 3, Article ID 1950036, 2019.

[17] S. Abbas, M. Benchohra, N. Hamidi, and J. Henderson, "Caputo-Hadamard fractional differential equations in Banach spaces," *Fractional Calculus and Applied Analysis*, vol. 21, no. 4, pp. 1027–1045, 2018.

[18] S. Hamani, W. Benhamida, and J. Henderson, "Boundary value problems for Caputo-Hadamard fractional differential equations," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 2, no. 3, pp. 138–145, 2018.

[19] M. Gohar, C. Li, and C. Yin, "On Caputo–Hadamard fractional differential equations," *International Journal of Computer Mathematics*, vol. 97, no. 7, pp. 1459–1483, 2020.

[20] A. Ardjouni and A. Djoudi, "Existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations," *Results in Nonlinear Analysis*, vol. 2, no. 3, pp. 136–142, 2019.

[21] G. Nazir, K. Shah, T. Abdeljawad, H. Khalil, and R. A. Khan, "Using a prior estimate method to investigate sequential hybrid fractional differential equations," *Fractals*, vol. 28, no. 8, Article ID 2040004, 2020.

[22] E. Ameer, H. Aydi, M. Arshad, and M. De la Sen, "Hybrid Ćirić type graphic Υ, Λ-contraction mappings with applications to electric circuit and fractional differential equations," *Symmetry*, vol. 12, no. 3, 2020.

[23] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, "Existence results for a system of coupled hybrid fractional differential equations," *The Scientific World Journal*, vol. 2014, pp. 1–6, 2014.

[24] M. I. Abbas and M. A. Ragusa, "On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function," *Symmetry*, vol. 13, no. 2, 2021.

[25] S. T. Sutar and K. D. Kucche, "On nonlinear hybrid fractional differential equations with Atangana-Baleanu-Caputo derivative," *Chaos, Solitons & Fractals*, vol. 143, Article ID 110557, 2021.

[26] Z. Ullah, A. Ali, R. A. Khan, and M. Iqbal, "Existence results to a class of hybrid fractional differential equations," *Matrix Science Mathematic*, vol. 2, pp. 13–17, 2018.

[27] S. Sitho, S. K. Ntouyas, and J. Tariboon, "Existence results for hybrid fractional integro-differential equations," *Boundary Value Problems*, vol. 2015, no. 1, 2015.

[28] H. Khan, C. Tunc, W. Chen, and A. Khan, "Existence theorems and Hyers–Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator," *Journal of Applied Analysis & Computation*, vol. 8, no. 4, pp. 1211–1226, 2018.

[29] S. Ferraoun and Z. Dahmani, "Existence and stability of solutions of a class of hybrid fractional differential equations involving RL-operator," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 4, pp. 885–903, 2020.

[30] W. Al-Sadi, H. Zhenyou, and A. Alkhazzan, "Existence and stability of a positive solution for nonlinear hybrid fractional differential equations with singularity," *Journal of Taibah University for Science*, vol. 13, no. 1, pp. 951–960, 2019.

[31] B. Ahmad and S. K. Ntouyas, "Initial-value problems for hybrid Hadamard fractional differential equations," *The Electronic Journal of Differential Equations*, vol. 2014, no. 161, 2014.

[32] S. Mashayekhi and M. Razzaghi, "Numerical solution of distributed order fractional differential equations by hybrid functions," *Journal of Computational Physics*, vol. 315, pp. 169–181, 2016.

[33] A. Amara, "Existence results for hybrid fractional differential equations with tree-point boundary conditions," *AIMS Mathematics*, vol. 5, no. 2, pp. 1074–1088, 2020.

[34] A. Ali, K. Shah, and R. A. Khan, "Existence of solution to a coupled system of hybrid fractional differential equations," *Bulletin of Mathematical Analysis and Applications*, vol. 9, no. 1, pp. 9–18, 2017.

[35] A. D. Mali, K. D. Kucche, and J. Vanterler da Costa Sousa, "On coupled system of nonlinear ψ-Hilfer hybrid fractional differential equations," *International Journal of Nonlinear Sciences and Numerical Stimulation*, 2021.

[36] Z. Baitiche, K. Guerbati, M. Benchohra, and Y. Zhou, "Boundary value problems for hybrid Caputo fractional differential equations," *Mathematics*, vol. 7, no. 3, 2019.

[37] R. A. Khan, S. Gul, F. Jarad, and H. Khan, "Existence results for a general class of sequential hybrid fractional differential equations," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.

[38] K. D. Kucche and A. D. Mali, "On the nonlinear $$\Psi $$ Ψ-Hilfer hybrid fractional differential equations," *Computational and Applied Mathematics*, vol. 41, no. 3, 2022.

[39] D. S. Oliveira and E. C. de Oliveira, "Hilfer–Katugampola fractional derivatives," *Computational and Applied Mathematics*, vol. 37, no. 3, pp. 3672–3690, 2018.

[40] U. N. Katugampola, "A new approach to generalized fractional derivatives," 2011.