Research Article

Chaotic Threshold of a Nonlinear Zener Systems Based on the Melnikov Method

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1. Introduction

With the development of the society, viscoelastic materials have been developed rapidly in the field of vibration control and have shown superior benefits of reducing noise and vibration. As a result, it is widely applied in aviation [1, 2], machinery [3], civil engineering [4, 5], and many other fields, such as magnetorheological damper in automobile suspension [6], metal rubber in spacecraft vibration reduction [7], and air spring in high-speed train bogie [8]. To better analyze the characteristics of viscoelastic materials, a variety of simplified mechanical models were put forward. Among them, there are the two most classical models which are the Kelvin model and the Maxwell model, but each model has its drawbacks. The Kelvin model cannot reflect stress relaxation, and the Maxwell model cannot present the creep [9]. Therefore, scholars proposed the Zener model composed of a spring and a Maxwell element in parallel, which can not only be degenerate into the Kelvin model or the Maxwell model but also reflect the stress relaxation and creep characteristics of viscoelastic materials.

Specifically, in viscoelastic Zener system, there has been a lot of research studies on the vibration isolation performance and periodic solution of the model. Ruzicka [10] made a comprehensive analysis of its forced vibration response. Brennan et al. [11] studied the effects of stiffness and damping on the performance of the viscoelastic Zener system under the free or forced vibration. Wang et al. [12] presented the design method for the optimal damping of a three-parameter vibration isolator. Lou et al. [13] found that the Zener system has better impact resistance. After that, Brennan et al. [14–18] have conducted extensive research on
nonlinear isolation systems. In [14], the vibration transmission characteristics of single degree-of-freedom passive vibration isolation systems with different nonlinear dampers were considered. In [15], the dynamic behaviors of a nonlinear two degree-of-freedom system are discussed and the interaction between the linear dynamics of the shaker and the nonlinear dynamics of the isolator is captured. In addition, in [16, 17], the method of nonlinear generation and propagation is found and the dynamic behaviors of the nonlinear system by using the method of dynamic stiffness are explained. To further study the performance of the Zener system, Shi et al. [18] established a four-parameter Zener model and found that the vibration isolation performance of the isolator has been improved in the high-frequency band. Wang et al. [19] and Lucas et al. [20] used nonlinear cubic stiffness elements to replace the spring in Maxwell element and linear mainspring to focus on the vibration isolation behavior, respectively, which improves the transmissibility of the Zener model damping isolator. Liu et al. [21] introduced geometric nonlinearity into the viscoelastic Zener model, which effectively suppresses the dynamic response in a resonant frequency band. Wei et al. [22] and Wang et al. [23] applied nonlinear damping to improve the viscoelastic Zener model, and they found that damped viscoelasticity to simulate the velocity correlation of the support is conducive to reflecting the structural response more realistically. Fan et al. [24] extended the multiscale method to approximately solve the nonlinear Zener model and discussed the influence of system parameters on the dynamic behavior and stability of the system. The studies of viscoelastic Zener system in the above literature only focus on the analytical solutions of linear and simple nonlinear resonance.

In recent years, many scholars still attach great importance to the research on complicated dynamic behaviors such as bifurcation and chaos of nonlinear systems. As one of the analytical methods for judging the chaotic threshold of nonlinear systems, the Melnikov method has been widely used since it was proposed. The Melnikov method [25, 26] can determine whether the system has a transverse zero point by measuring the distance between the stable and unstable manifolds of the hyperbolic fixed point in the nonlinear system and obtains the chaotic threshold in the sense of Smale horseshoe. Cao et al. [27] studied multistable and multiwell dynamic characteristics of a class of coupled typical geometric nonlinear systems and gave the analytical chaotic parameter conditions of homoclinic orbit and heteroclinic orbit based on the Melnikov method. Luo et al. [28] investigated the bifurcation and chaos of transverse vibration of viscoelastic radial transmission structure under nonlinear parametric excitation by the numerical method. For a Duffing oscillator with delayed displacement and velocity feedbacks under harmonic excitation, the necessary conditions for generating homoclinic orbit and heteroclinic orbit were, respectively, investigated by Shen et al. [29] and Wen et al. [30]. Li et al. [31] applied the Melnikov method to globally analyze the Duffing oscillator of the simultaneous primary and super-harmonic resonance. Moreover, Stanton et al. [32] and Li et al. [33, 34] studied the bistable and tristable piezoelectric energy harvesting system via the Melnikov method and concluded that electromechanical coupling factors affect its dynamic behavior. Their works obtained a qualitative research method for homoclinic bifurcation of energy harvesting system under harmonic excitation. The aforementioned literature mainly focuses on the homoclinic or heteroclinic bifurcation thresholds of the two-dimensional plane system.

At present, many researchers are interested in high-dimensional nonlinear systems to explore interesting dynamic behaviors. Yao et al. [35] discussed the multipulse global bifurcation and chaotic dynamics of nonlinear and nonplanar oscillation in the viscoelastic motion band of parametric excited state under resonance via the extended Melnikov method. Their works showed that the extended Melnikov method can be applied to the research of Shilnikov-type multipulse homoclinic bifurcation and chaotic dynamics of high-dimensional nonlinear systems. Davood et al. [36] considered the chaotic behavior of viscoelastic plates under the simultaneous action of subsonic flow and external excitation based on the extended Melnikov method. A detailed study on the dynamics of bistable impact oscillators with double rigid constraints by using the extended Melnikov method and dynamic experiments were conducted by Li et al. [37]. Zhang et al. [38] used the extended Melnikov method to investigate the multipulse jump, two-parameter co-oblique orbit, and chaotic dynamics of eccentric rotating ring truss antenna under parametric and external excitation. Furthermore, Wei et al. [39–44] discussed the hidden attractors contained in 3D, 4D, and 5D autonomous systems in detail, and their work further analyzed these phenomena produced by the system and put forward practical applications.

With regard to the nonlinear Zener model, the analysis of chaotic threshold is one of the focal points. The nonlinear Zener dynamical system with a Maxwell viscoelastic element adds a half degree of freedom, which in turn increases the complexity of the system. Most studies use the harmonic balance method to calculate and analyze the approximate analytical solution of the system, and few studies introduce nonlinear factors into the viscoelastic Zener model to analyze its dynamic behavior. Moreover, the Melnikov method used in most kinds of literature was to study the homoclinic or heteroclinic bifurcation thresholds of two-dimensional plane systems, and the application of the Melnikov method in this kind of nonlinear system proposed in this study is rarely reported. Based on the excellent characteristics of viscoelastic materials, the nonlinear viscoelastic Zener model is seen as the research object in the study. Note that the factors such as damping, viscoelastic Maxwell element, and external excitation are regarded as small disturbances in the plane integrable system to study the chaotic motion at resonance. In the analysis of chaotic threshold, it is found that the system enters the chaotic state in two different ways within a certain excitation amplitude range. The effects of system parameters on the necessary condition for chaos are further discussed by numerical simulation and analytical solution. It shows that the proper selection of system parameters can suppress the bifurcation and chaos of the system.
The study is arranged as follows. In Section 2, the Melnikov necessary condition for homoclinic bifurcation on the harmonic excitation condition is obtained. Section 3 presents the bifurcation curves, Poincare maps, and the largest Lyapunov exponents by the numerical method. Section 4 analyses the effects of nonlinear coefficients and stiffness and damping coefficients in Maxwell element under the necessary condition for chaos using analytical and numerical solutions, and the correctness of the analytical solution is proved. Conclusions are drawn in Section 5.

2. Melnikov Analysis of Nonlinear Viscoelastic System

This section investigates the nonlinear Zener system with linear damping, linear stiffness, and nonlinear terms, as shown in Figure 1, where $m$ and $k$ are the mass and linear stiffness coefficient of the system, respectively, $a$ is the corresponding nonlinear stiffness coefficient, $c$ and $k_i$ are the damping coefficient and stiffness coefficient of Maxwell viscoelastic element, and $F$ and $\omega$ are the amplitude and frequency of external excitation, respectively.

Obviously, this nonlinear Zener model is a 1.5 DOF system. According to Newton’s second law, the dynamic equation is written as [10, 24]

$$m\ddot{x} + kx + ax^3 + c(\dot{x} - \dot{y}) = F\cos\omega t, k_i y + c(\dot{y} - \dot{x}) = 0.$$  

(1)

Here, considering the situation of $k < 0$ in system (1) and introducing the transformation,

$$\frac{c}{m} = 2\xi, \frac{k}{m} = -\omega_0^2, \frac{a}{m} = y, \frac{k_i}{k} = -\xi, \frac{F}{m} = \xi\delta,$$  

(2)

where $\xi$ is a small real parameter, and equation (1) can be simplified into the following form:

$$\begin{cases} x - \omega_0^2 x + \gamma x^3 + 2\xi\delta(\dot{x} - \dot{y}) = \xi\delta\cos\omega t, \\ \xi\omega_0^2 \dot{y} + 2\xi\delta(\dot{y} - \dot{x}) = 0. \end{cases}$$  

(3)

If $x_1 = x$, $x_2 = \dot{x}$, and $x_3 = y$, the state equation of equation (2) is

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \omega_0^2 x_1 - \gamma x_1^3 + \xi\delta\cos\omega t - 2\xi\delta(x_2 - \dot{x}_1), \\
\dot{x}_3 &= x_2 - \frac{\chi\omega_0^2}{2\eta}\dot{x}_3.
\end{align*}$$  

(4)

In this study, assuming that the factors such as damping, viscoelastic Maxwell element, and external excitation are the small disturbances of equation (3) corresponding to Hamilton system, the undisturbed system is

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \omega_0^2 x_1 - \gamma x_1^3. \\
\dot{x}_3 &= x_2 - \frac{\chi\omega_0^2}{2\eta}\dot{x}_3.
\end{align*}$$  

(5)

Then, the homoclinic orbit through integral equation (4) is

$$\begin{align*}
x_{1w} &= \pm\sqrt{\frac{2}{y}}\omega_0\text{sech}(\omega_0 t) \\
x_{2w} &= \mp\sqrt{\frac{2}{y}}\omega_0\text{sech}(\omega_0 t)\tanh(\omega_0 t).
\end{align*}$$  

(6)

Substituting equation (5) into the third equation of equation (3), we can obtain

$$\dot{x}_3 = x_2 - \frac{\chi\omega_0^2}{2\eta}x_{3w}.$$  

(7)

Then, the calculation result of equation (6) is

$$x_{3w} = \mp\sqrt{\frac{2}{y}}\omega_0\text{sech}^2(\omega_0 t)\tanh(\omega_0 t)\text{sech}(\omega_0 t)\text{tanh}(\omega_0 t)\text{sech}(\omega_0 t)\text{sech}(\omega_0 t)\text{sech}(\omega_0 t).$$  

(8)

where $F_1(a, b, c, d)$ is hypergeometric function and $G(a, b)$ is polygamma function. To validate the analytic solution, (7) is numerically simulated by MATLAB, and Figure 2 demonstrates the consistency between the analytical and the simulation results. From the sophisticated form of the analytical solution of $x_{3w}$, the numerical solution is adopted in the following analysis.

Define the undisturbed system that does not contain $x_3$:

$$\dot{x}_w = h(x_w) + \xi g(x_w, t).$$  

(9)
Among them,

$$
x_w = \begin{bmatrix} x_{1w} \\ x_{2w} \end{bmatrix}, \quad h(x_w) = \begin{bmatrix} x_{2w} \\ \frac{1}{2} \omega_0^2 x_{1w} - \gamma x_{1w}^3 \end{bmatrix}, \quad g(x_w) = \begin{bmatrix} 0 \\ \delta \cos \omega t - 2\eta (x_{2w} - \dot{x}_{3w}) \end{bmatrix}.
$$

According to the Melnikov method, the Melnikov function is established as

$$
M(\tau) = \int_{-\infty}^{+\infty} h(x_{1w}, \dot{x}_{1w}, \dot{x}_{2w}, t + \tau) \, dt
= \int_{-\infty}^{+\infty} x_{2w} \left[ \delta \cos \omega t - 2\eta (x_{2w} - \dot{x}_{3w}) \right] \, dt,
$$

where $\cdot$ denotes a wedge product. The definite integral of equation (9) can be divided into the following three parts:

$$
M_1(\tau) = \int_{-\infty}^{+\infty} \left[ \frac{1}{2} \omega_0^2 x_{1w} - \gamma x_{1w}^3 \right] \, dt
= \pm \frac{2\pi \delta \omega \text{sech}(\pi \omega/2\omega_0) \sin(\omega t)}{\sqrt{\gamma}}
$$

$$
M_2(\tau) = \int_{-\infty}^{+\infty} 2\eta x_{2w} \, dt = \frac{8\eta \omega_0^3}{3\sqrt{\gamma}}
$$

$$
M_3(\tau) = \int_{-\infty}^{+\infty} 2\eta x_{2w} \dot{x}_{3w} \, dt = \frac{2\sqrt{2} \eta \omega_0^3}{\sqrt{\gamma}} I_g.
$$

where $I_g = \int_{-\infty}^{+\infty} \text{sech}(\omega_0 t) \tanh(\omega_0 t) \dot{x}_{3w} \, dt$. Since the analytical solution of $I_g$ is hard to be solved, the form of Simpson numerical integration is expressed approximately. Through the analysis of the above results, the necessary condition for homoclinic orbit cross-section intersection in equation (1) is

$$
\frac{2\pi \delta \omega \text{sech}(\pi \omega/2\omega_0) \sin(\omega t)}{\sqrt{\gamma}} > \frac{8\eta \omega_0^3}{3\sqrt{\gamma}} + \frac{2\sqrt{2} \eta \omega_0^3}{\sqrt{\gamma}} I_g > 0.
$$

Substituting the original system parameters into equation (13), it could yield

$$
3\sqrt{2\pi \sqrt{m \xi \omega F \text{sech} \left( \frac{\pi \omega}{2\sqrt{-k/m}} \right)}} > \left| \frac{k}{m} - 3\sqrt{2} c k \sqrt{\frac{\alpha}{m} I_g} \right|,
$$

that is,

$$
|F| > \frac{\cosh(\pi \omega/2\sqrt{-k/m}) (4ck\omega_0 + 3\sqrt{2} c k \sqrt{\alpha/m} I_g)}{3\sqrt{2}\pi \sqrt{\alpha \omega}}.
$$

When inequality (15) holds, the homoclinic orbit of system (1) can cross intersect so that system (1) may enter chaos in the sense of Smale horseshoe. Notably, if the model in this study is simplified to a single degree-of-freedom system, the analytical result of the threshold of chaos is consistent with those in [25, 29], which indirectly proves the correctness of the analytical results in this study.

### 3. Numerical Simulation

By taking the system parameters $m = 1$, $k = -1$, $k_1 = 0.5$, $\alpha = 1$, $c = 0.2$, $\omega = 1.1$, and $\epsilon = 0.1$ and substituting them into equation (13), the critical excitation amplitude $F_{\text{min}}$ for chaos is 0.133. In other words, when $F > 0.133$, the homoclinic orbit cross section of equation (1) may occur, and the greater excitation amplitude is than the threshold, the more likely chaos is to occur in the system.

In order to analyze the critical value of generating chaos more accurately, the trapezoidal method of free interpolation [45, 46] and the Lyapunov exponent are introduced to simulate equation (1) at different excitation amplitudes. The Lyapunov exponent, which is a reliable quantitative analysis method to judge whether the system, enters chaos by measuring the average change rate of separation or convergence of adjacent phase orbit traces and is calculated by the Jacobian method. Here, the initial value of $F$ is 0 and the initial value $(x_{01}, \dot{x}_{01}, y_{01}, \dot{y}_{01})$ is $(0.1, 0, 0, 0)$, and the simulation time $t$ is set as $2\pi t/\omega$. When $F$ changes from 0 to 5 with the step size as 0.02, one can get the bifurcation diagram (Figure 3) and the largest Lyapunov exponent chart (Figure 4), respectively. From Figures 3 and 4, it can be found that there are two critical values, where the first critical value about 0.3 is the numerical threshold and the second critical excitation amplitude around 3 denotes the second occurrence from periodicity to chaos. Thus, the detailed bifurcation diagram and the variation of the largest Lyapunov...
The step sizes of Figures 5 and 7 are selected as 0.001 and 0.0005 individually. Around the first critical value, i.e., when \( F > 0.203 \), the system enters the chaotic state in a crisis way as shown in Figure 5, which is consistent with the sharp increase of the largest Lyapunov exponent of \( F \) at 0.3 in Figure 6. Figure 8 portrays the periodic orbit via taking \( F = 0.203 \) and the initial value \((x_{02}, \dot{x}_{02}, y_{02}, \dot{y}_{02}) = (1.25715268823769, 0.11307241041528, 0.158363049145063, -0.119905053458755)\). Then, when \( F = 0.204 \) and the initial value \((x_{02}, \dot{x}_{02}, y_{02}, \dot{y}_{02}) = (1.26119984719097, 0.115510861798501, 0.167496720727088, -0.121280433769164)\), the displacement time history, phase trajectory, and Poincare map are obtained as plotted in Figure 9. From Figures 8 and 9, it can be concluded that the homoclinic orbit breaks, and the system goes to chaos in a crisis way when the excitation amplitude increases from 0.203 to 0.204. Otherwise, at the second critical value, Figure 7 presents that, with the continuous increase of excitation, the system enters another chaos in the form of period-doubling bifurcation, where \( F = 3.2 \). At this point, the displacement time history, phase trajectory, and Poincare map are acquired by taking the initial value \((x_{03}, \dot{x}_{03}, y_{03}, \dot{y}_{03})\) of the system as \((2.2320514013694006, 0.23099234373530025, -1.856588531805132, -2.434406911433444)\), as shown in Figure 10.

### 4. Influence of System Parameters on Chaotic Motion

In this section, we are interested in the effects of nonlinear coefficient and stiffness and damping coefficients in Maxwell element under the necessary condition for chaos through analytical solution and numerical simulation. The numerical simulation uses ode23t in MATLAB to calculate the nonlinear Zener system, that is, (1), and the unstable response in the calculation result is rounded off to ensure the stability of the solution.

The system parameters, \( m = 1, k = -1, k_1 = 0.5, c = 0.2, \omega = 1.1, \) and \( \epsilon = 0.1 \), are selected to analyze the influence of nonlinear term \( \alpha \) with the value range of 0 to 2. The relationship curves between chaotic threshold \( F \) and nonlinear coefficient \( \alpha \) are obtained by MATLAB numerical simulation and Melnikov analytical method, respectively, in this study, as depicted in Figure 11. The numerical solution of the
The critical excitation amplitude is composed of small circles, and the analytical solution is consisted of asterisks. From the analysis of Figure 11, it can be found the chaotic threshold gradually decreases as the nonlinear coefficient increases. Through the comparison of analytical and numerical results, it can be observed that although there are quantitative

![Figure 6: The trend of the largest Lyapunov exponent of equation (1).](image1)

![Figure 7: Bifurcation diagram of equation (1) changing with $F$.](image2)

![Figure 8: Periodic orbits for $F=0.203$.](image3)
differences between the two results, the overall trend is the same. This difference is reasonable because the Melnikov method is an approximate method to find chaotic motion. In short, the increase of nonlinear term can promote the bifurcation and chaos of the system.

Other two sets of system parameters are selected as $m = 1$, $k = 1$, $\alpha = 1$, $c = 0.2$, $\omega = 1.1$, $\epsilon = 0.1$ and $m = 1$, $k = 1$, $k_1 = 0.5$, $\alpha = 1$, $\omega = 1.1$, and $\epsilon = 0.1$, respectively, and the effects of stiffness coefficient and damping coefficient in viscoelastic Maxwell device are considered. At this time, $k_1$ takes 0 to 3 and $c$ takes 0 to 2, and the relationship curves between $F$ and $k_1$ and the change trend of $F$ and with $c$ are shown in Figures 12 and 13, respectively. Figure 12 indicates that the tendencies of the numerical and analytical solutions are all increasing first and then remaining are unchanged with the increase of stiffness coefficient. It can be concluded from Figure 13 that the threshold $F$ first increases and then decreases, and the error between the numerical solution and analytical solution increases with the increase of damping. Therefore, within a certain damping range, the
Figure 11: Influence of nonlinear coefficient.

Figure 12: Effect of stiffness coefficient.

Figure 13: Influence of damping coefficient.
proper selection of damping coefficient can inhibit the bifurcation and chaos of the nonlinear Zener system.

5. Conclusion

In this study, the chaotic dynamics of the nonlinear Zener model with a viscoelastic Maxwell device is investigated based on the Melnikov method, and the analytically necessary condition for chaos in the sense of Smale horseshoes is obtained. The system parameters satisfying the main resonance condition are selected and simulated by numerical iteration method. From the analysis of numerical simulation, it can be found that there are two different paths to chaos in a certain range of excitation amplitude, and these phenomena are confirmed by analyzing its dynamic response. At the first critical value, the system enters chaos in a crisis way. While at the second critical value, the system generates chaos in the way of period-doubling bifurcation. The effects of system parameters on the necessary condition for chaos are analyzed, and by comparing the threshold of chaos generated by the numerical and analytical solutions under different parameters, it is found that the trend of the analytical solution is consistent with the numerical iterative simulation, and the qualitative results are the same. It can be concluded that the chaotic threshold changes differently with the change of system parameters, which provides a basis for the research of this kind of the viscoelastic system. Above all, the chaotic motion of the system at resonance is studied in this study. The method of considering the factors such as damping, viscoelastic Maxwell element, and external excitation as the small disturbance of the system plane integrable system can be extended and applied to other similar nonlinear viscoelastic systems, which provides a new research idea for the dynamics of nonlinear viscoelastic systems.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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