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Research Article

Bi-Finite Difference Method to Solve Second-Order Nonlinear Hyperbolic Telegraph Equation in Two Dimensions

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This study introduces a computational scheme by the bi-finite difference method (Bi-FDM) to solve the hyperbolic telegraph equation in two dimensions. The proposed numerical method converts nonlinear two-dimensional hyperbolic telegraph equation of second order to difference equations that can be solved by the Mathematica program. Consistency and stability of the proposed scheme are discussed and found to be accurate of $O((h_x)^2 + (h_y)^2 + (\Delta\tau)^2)$ and conditionally stable, respectively. The efficiency and accuracy of Bi-FDM have been shown by comparing the numerical results of the presented problems with the exact solutions and other numerical techniques.

1. Introduction

For $\delta > 0$ and $\gamma > 0$, we consider the following second-order nonlinear hyperbolic telegraph equation in two dimensions on region $R = \{(x, y)\colon x \in [0,1], y \in [0,1]\}$ f$V_{\tau\tau} = V_{xx} + V_{yy} - 2\delta V_\tau - \gamma^2 V + G(x, y, \tau, V)$or temporal coordinate with seconds $\tau > 0$ [20]:

$$V_{\tau\tau} = V_{xx} + V_{yy} - 2\delta V_\tau - \gamma^2 V + G(x, y, \tau, V), \tag{1}$$

with the initial conditions,

$$V(x, y, 0) = f_1(x, y), V_\tau(x, y, 0) = f_2(x, y), x \in [0,1], y \in [0,1], \tag{2}$$

and the boundary conditions,

$$\begin{aligned} &V(0, y, \tau) = f_3(y, \tau), V(1, y, \tau) = f_4(y, \tau), y \in [0,1], \tau > 0,\\ &V(x, 0, \tau) = f_5(x, \tau), V(x, 1, \tau) = f_6(x, \tau), x \in [0,1], \tau > 0. \end{aligned} \tag{3}$$

Equation (1) is in spatial-temporal spaces and represents damped wave equation at $\delta > 0$ and $\gamma = 0$ and telegraph equation at $\delta > 0$ and $\gamma > 0$. The two-dimensional nonlinear hyperbolic telegraph (1) of second-order under conditions (2) and (3) arises in electric voltage and current in a double conductor, signal analysis, wave propagation, random walk theory, and atomic physics [1–4]. Also, the hyperbolic partial differential equations are useful in explaining and understanding structures vibrations such as buildings, beams, and machines [5]. So, most previous branches of science can be studied by telegraph equation which is more appropriate in modeling reaction-diffusion than ordinary diffusion equation. In most of the literature, the numerical schemes for hyperbolic telegraph equations of second order in one dimension and multidimensions recently developed it as follows. Mohanty et al. studied the stability of numerical schemes of telegraph equations [6–8]. Hyperbolic telegraph equations of one dimension are solved by cubic and quartic B-spline collocation methods [9–11]. In [12, 13], Mohanty et al. used high-order approximation to solve two-dimensional quasilinear hyperbolic equations. Many authors solved two-dimensional linear hyperbolic equations [14–18]. Two-dimensional nonlinear hyperbolic equations of second order are studied using modified B-spline method [19], bicubic B-spline method [20], meshless local and stronge forms [21], and high-order difference schemes [22]. Smith introduced finite difference methods [23]. Raslan and Ali used the finite difference method to solve MHD-duct flow model [24]. Baleanu et al. applied a nonstandard finite difference scheme for fractional chaotic system [25]. Erturk et al. found some electrical engineering applications as capacitor microphone system [26] and description motion of beam on nanowire [27]. Hasan et al. [28] presented formulations of fractional optimal control of two spherical structures and one hollow cylindrical structure which represented fractional partial differential equations in two dimensions and can be solved via our presented scheme and will discuss it in the future work.

The present study is organized as follows. In Section 2, we present the bi-finite difference method of the proposed problem. We discuss the consistency and stability of the proposed scheme as given in Sections 3 and Section 4, respectively. Section 5 investigates some numerical problems and explains them in some tables and figures. Finally, Section 6 provides the conclusion for this paper.

2. Bi-Finite Difference Method

We divide the region R into knots (x_i, y_j), for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$, in spatial directions with horizontal space in $x-$ direction $h_x = x_i - x_{i-1} = 1/n$, $i = 1, 2, \ldots, n$, and vertical space in $y-$ direction $h_y = y_j - y_{j-1} = 1/m$, $j = 1, 2, \ldots, m$, where n and m are $+$ve integers. Suppose that temporal direction with spacing $\Delta\tau > 0$ is $\tau_k = k * \Delta\tau$, for $k = 0, 1, \ldots, l$, where l is +ve integer. The finite difference method in two dimensions to find the approximate solution $v(x, y, \tau)$ corresponding to the exact solution $V(x, y, \tau)$ can be written as

$$v\left(x_i, y_j, \tau_k\right) = c_{i,j}^k, \tag{4}$$

and its partial derivative up to second order in spatial and temporal directions at various grid points (x_i, y_j, τ_k) is as follows:

$$\begin{aligned}
v_x\left(x_i, y_j, \tau_k\right) &\simeq \frac{c_{i+1,j}^k - c_{i-1,j}^k}{2h_x},\\
v_{xx}\left(x_i, y_j, \tau_k\right) &\simeq \frac{c_{i+1,j}^k + c_{i-1,j}^k - 2c_{i,j}^k}{\left(h_x\right)^2},\\
v_y\left(x_i, y_j, \tau_k\right) &\simeq \frac{c_{i,j+1}^k - c_{i,j-1}^k}{2h_y},\\
v_{yy}\left(x_i, y_j, \tau_k\right) &\simeq \frac{c_{i,j-1}^k + c_{i,j+1}^k - 2c_{i,j}^k}{\left(h_y\right)^2},\\
v_\tau\left(x_i, y_j, \tau_k\right) &\simeq \frac{c_{i,j}^{k+1} - c_{i,j}^{k-1}}{2\Delta\tau},\\
v_{\tau\tau}\left(x_i, y_j, \tau_k\right) &\simeq \frac{c_{i,j}^{k+1} + c_{i,j}^{k-1} - 2c_{i,j}^k}{(\Delta\tau)^2}.
\end{aligned} \tag{5}$$

Equation (1), at points (x_i, y_j, τ_k), for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$, and $k = 0, 1, \ldots, l$, where $V(x_i, y_j, \tau_k)$, is written as $V_{i,j}^k$:

$$\left(V_{\tau\tau}\right)_{i,j}^k = \left(V_{xx}\right)_{i,j}^k + \left(V_{yy}\right)_{i,j}^k - 2\delta\left(V_\tau\right)_{i,j}^k - \gamma^2 V_{i,j}^k + G_{i,j}^k. \tag{6}$$

Conditions (2) and (3) become

$$\begin{aligned}
V_{i,j}^0 &= f_1\left(x_i, y_j\right), & \left(V_\tau\right)_{i,j}^0 &= f_2\left(x_i, y_j\right),\\
V_{0,j}^k &= f_3\left(y_j, \tau_k\right), & V_{n,j}^k &= f_4\left(y_j, \tau_k\right),\\
V_{i,0}^k &= f_5\left(x_i, \tau_k\right), & V_{i,m}^k &= f_6\left(x_i, \tau_k\right).
\end{aligned} \tag{7}$$

Substituting (5) into (6), we obtain as follows:

$$\begin{aligned}
\frac{c_{i,j}^{k+1} + c_{i,j}^{k-1} - 2c_{i,j}^k}{(\Delta\tau)^2} &= \frac{c_{i+1,j}^k + c_{i-1,j}^k - 2c_{i,j}^k}{\left(h_x\right)^2} + \frac{c_{i,j-1}^k + c_{i,j+1}^k - 2c_{i,j}^k}{\left(h_y\right)^2}\\
&\quad - 2\delta\frac{c_{i,j}^{k+1} - c_{i,j}^{k-1}}{2\Delta\tau} - \gamma^2 c_{i,j}^k + G_{i,j}^k,
\end{aligned} \tag{8}$$

where (8) with conditions (7) can be solved.

3. Local Truncation Error

In this section, we introduce the local truncation error of our scheme.

Theorem 1. *The local truncation error of finite difference scheme (8) is* $O\left((h_x)^2 + (h_y)^2 + (\Delta\tau)^2\right)$.

Proof. By using Taylor's expansion in (8), we investigate the local truncation error in two-dimensional space and time as follows:

$$\begin{aligned}
T\left(c_{i,j}^k\right) &= \frac{\partial^2 c_{i,j}^k}{\partial t^2} - \frac{\partial^2 c_{i,j}^k}{\partial x^2} - \frac{\partial^2 c_{i,j}^k}{\partial y^2} + 2\delta\frac{\partial c_{i,j}^k}{\partial t} + \gamma^2 c_{i,j}^k - G_{i,j}^k\\
&\quad + \frac{\left(h_x\right)^2}{12}\frac{\partial^4 c_{i,j}^k}{\partial x^4} + \frac{\left(h_y\right)^2}{12}\frac{\partial^4 c_{i,j}^k}{\partial y^3} + \frac{\delta(\Delta\tau)^2}{3}\frac{\partial^3 c_{i,j}^k}{\partial t^3}\\
&\quad + \frac{(\Delta\tau)^2}{12}\frac{\partial^4 c_{i,j}^k}{\partial t^4} + \ldots,
\end{aligned} \tag{9}$$

$$\begin{aligned}
T\left(c_{i,j}^k\right) &= \frac{\left(h_x\right)^2}{12}\frac{\partial^4 c_{i,j}^k}{\partial x^4} + \frac{\left(h_y\right)^2}{12}\frac{\partial^4 c_{i,j}^k}{\partial y^3} + \frac{\delta(\Delta\tau)^2}{3}\frac{\partial^3 c_{i,j}^k}{\partial t^3}\\
&\quad + \frac{(\Delta\tau)^2}{12}\frac{\partial^4 c_{i,j}^k}{\partial t^4} + \cdots.
\end{aligned} \tag{10}$$

Hence, $T(c_{i,j}^k) \longrightarrow 0$ as $(h_x)^2, (h_y)^2, (\Delta\tau)^2 \longrightarrow 0$.

Then, the local truncation error of finite difference scheme (8) is

$$O\left(\left(h_x\right)^2 + \left(h_y\right)^2 + (\Delta\tau)^2\right). \tag{11}$$

□

4. Stability of Bi-Finite Difference Scheme

In this section, we test the stability of the difference scheme.

Theorem 2. *The difference scheme (8) is conditionally stable.*

Proof. Firstly, suppose $c_{i,j}^k = \zeta^k e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}$, and substituting with it in (8), we use the following linearization of the nonlinear term $G_{i,j}^k = MV_{i,j}^k$, where M is locally constant; we obtain

$$\begin{aligned}
&\left(\zeta^{k+1} e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right) + \left(\zeta^{k-1} e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right) - 2\left(\zeta^{k} e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right) \\
&= \left(\frac{\Delta\tau}{h_x}\right)^2 \left(\left(\zeta^k e^{\mathbb{I}\vartheta(i+1)h_x} e^{\mathbb{I}\varphi(jh_y)}\right) + \left(\zeta^k e^{\mathbb{I}\vartheta(i-1)h_x} e^{\mathbb{I}\varphi(jh_y)}\right)\right. \\
&\quad \left. - 2\left(\zeta^k e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right)\right) + \left(\frac{\Delta\tau}{h_y}\right)^2 \left(\left(\zeta^k e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(j+1)h_y}\right)\right. \\
&\quad \left. + \left(\zeta^k e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(j-1)h_y}\right) - 2\left(\zeta^k e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right)\right) \\
&\quad - \delta * \Delta\tau\left(\left(\zeta^{k+1} e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right) - \left(\zeta^{k-1} e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right)\right) \\
&\quad - \left(\gamma^2 - M\right)(\Delta\tau)^2 \left(\zeta^k e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}\right),
\end{aligned} \tag{12}$$

where $\mathbb{I} = \sqrt{-1}$, and we divided equation (12) by $\zeta^{k-1} e^{\mathbb{I}\vartheta(ih_x)} e^{\mathbb{I}\varphi(jh_y)}$ as

$$(1 + \delta * \Delta\tau)\zeta^2 + B\zeta^2 + (1 - \delta * \Delta\tau) = 0, \tag{13}$$

where

$$\begin{aligned}
B = &-2 + \left(\gamma^2 - M\right)(\Delta\tau)^2 + 2\left(\frac{\Delta\tau}{h_x}\right)^2 (1 - \cos\vartheta h_x) \\
&+ 2\left(\frac{\Delta\tau}{h_y}\right)^2 \left(1 - \cos\varphi h_y\right).
\end{aligned} \tag{14}$$

Then,

$$|\zeta| = \left|\frac{-B \pm \sqrt{B^2 + 4\delta^2\Delta\tau^2 - 4}}{2(1 + \delta\Delta\tau)}\right| \le 1. \tag{15}$$

If $|B| \leqslant 1 + \delta * \Delta\tau$ and $\delta * \Delta\tau \leqslant 1$, so, the bi-finite difference scheme is conditionally stable under the condition $|B| \leqslant 2$. □

5. Numerical Problems

In this section, we test some numerical experiments for several two-dimensional hyperbolic telegraph problems using the bi-finite difference scheme (Bi-FDM). We calculate L_∞-errors, L_2-errors, root mean square errors (RMSE), and relative errors (RE) to show the efficiency of the proposed numerical scheme by comparing the exact solutions and the numerical solutions in our presented method and other different methods. The formulas of the used errors in two-dimensional space at different levels τ are given by [16]

$$\begin{aligned}
L_\infty - \text{error} &= \max_{i,j=0}^{n,m} \left|V_{i,j}^k - v_{i,j}^k\right|, \\
L_2 - \text{error} &= \sqrt{h_x * h_y \sum_{i=0}^{n}\sum_{j=0}^{m} \left(V_{i,j}^k - v_{i,j}^k\right)^2}, \\
\text{Root mean square error (RMSE)} &= \sqrt{\frac{\sum_{i=0}^{n}\sum_{j=0}^{m} \left(V_{i,j}^k - v_{i,j}^k\right)^2}{n * m}}, \\
\text{Relative error (RE)} &= \sqrt{\frac{\sum_{i=0}^{n}\sum_{j=0}^{m} \left(V_{i,j}^k - v_{i,j}^k\right)^2}{\sum_{i=0}^{n}\sum_{j=0}^{m} \left(V_{i,j}^k\right)^2}}.
\end{aligned} \tag{16}$$

Problem 1. We consider two-dimensional second-order nonlinear hyperbolic telegraph equation [20]:

$$V_{\tau\tau} - V_{xx} - V_{yy} + 2\delta V_\tau + \gamma^2 V - 2(\cos\tau - \sin\tau)\sin x \sin y = 0. \tag{17}$$

Subject to the initial and boundary conditions, for $x, y \in [0, 1]$ and $\tau \geqslant 0$,

$$\begin{aligned}
&V(x, y, 0) = \sin x \cos y, V_\tau(x, y, 0) = 0, \\
&V(0, y, \tau) = 0, V(1, y, \tau) = \sin 1 \ \cos y \cos\tau, \\
&V(x, 0, \tau) = \sin x \cos\tau, V(x, 1, \tau) = \sin x \cos 1 \ \cos\tau.
\end{aligned} \tag{18}$$

and the exact solution is $V(x, y, \tau) = \sin x \cos y \cos\tau$, where $\delta = \gamma = 1$.

Table 1: L_2, L_∞, and relative errors at different time levels for Problem 1.

τ	L_2 error (Bi-FDM)	L_∞ (E)error (Bi-FDM)	R (E)E (Bi-FDM)	L_2 (E)[20]	L_∞ (E)[20]	R (E)E [19]
1	1.197–5	2.1390–5	7.1789–5	2.2277–4	5.6153–4	5.976E-3
2	4.129–6	8.9229–6	3.2132–5	1.0859–4	3.9413–4	8.501E-3
4	1.091–5	2.1178–5	5.4081–5	2.7579–4	6.7468–4	- - - - -
7	1.229–5	2.3968–5	5.2785–5	3.1130–4	7.7897–4	3.157E-3
10	1.346–5	2.6265–5	5.1976–5	3.4167–4	8.6373–4	2.287E-3

Keeping $\Delta\tau = 0.01$, Table 1 presents L_2, L_∞ errors and relative error for equal grid sizes $h_x = h_y = 0.1$, while Table 2 shows the same errors for different grid sizes $h_x = 0.1$ and $h_y = 0.2$ at different time levels. Our results are compared with the obtained results in [19, 20]. Obviously, our numerical results are preferable. Figure 1 and 2 show the surface plots of numerical and exact solution at $\tau = 1, 3$ for $h_x = h_y = 0.05$ and $\Delta\tau = 0.001$, respectively. The maximum absolute error plots are observed that, with increase in time, the errors are decreasing, as shown in Figure 3.

Problem 2. We consider two-dimensional second-order nonlinear hyperbolic telegraph equation [20]:

$$V_{\tau\tau} - V_{xx} - V_{yy} + 2\delta V_\tau + \gamma^2 V - \left(-2\delta + \gamma^2 - 1\right)e^{-\tau}\sinh x \sinh y = 0, \tag{19}$$

with the initial and boundary conditions, for $x, y \in [0,1]$ and $\tau \geqslant 0$,

$$\begin{aligned} &V(x,y,0) = \sinh x \sinh y, V_\tau(x,y,0) = -\sinh x \sinh y, \\ &V(0,y,\tau) = 0, V(1,y,\tau) = e^{-\tau}\sinh 1 \ \cosh y, \\ &V(x,0,\tau) = 0, V(x,1,\tau) = e^{-\tau}\sinh x \sinh 1. \end{aligned} \tag{20}$$

The exact solution is $V(x,y,\tau) = e^{-\tau}\sinh x \sinh y$, where $\delta = 10$ and $\gamma = 5$.

Root mean square error and relative error for equal grid sizes $h_x = h_y = 0.1$ and $\Delta\tau = 0.01$ are calculated at different time levels, as shown in Table 3. Comparison of obtained results with results of other methods in [19, 20] display that our numerical results are effective. Figure 4 appears the surface plots of numerical and exact solution at $\tau = 2$ for $h_x = h_y = 0.05$ and $\Delta\tau = 0.001$. Figure 5 presents error plots at various time levels. We find the maximum absolute errors are increasing with time increasing.

Problem 3. We consider two-dimensional second-order nonlinear hyperbolic telegraph equation [20]:

$$\begin{aligned} &V_{\tau\tau} - V_{xx} - V_{yy} + 2\delta V_\tau + \gamma^2 V \\ &\quad + \left(3\cos\tau + 2\delta\sin\tau - \gamma^2\cos\tau\right)\sinh x \sinh y = 0, \end{aligned} \tag{21}$$

with the initial and boundary conditions, for $x, y \in [0,1]$ and $\tau \geqslant 0$,

$$\begin{aligned} &V(x,y,0) = \sinh x \sinh y, V_\tau(x,y,0) = 0, \\ &V(0,y,\tau) = 0, V(1,y,\tau) = \sinh 1 \ \sinh y \cos\tau, \\ &V(x,0,\tau) = 0, V(x,1,\tau) = \sinh x \sinh 1 \ \cos\tau. \end{aligned} \tag{22}$$

The exact solution is $V(x,y,\tau) = \sinh x \sinh y \cos\tau$, where $\delta = 10$ and $\gamma = 5$.

We tabulate root mean square errors and relative errors for $h_x = h_y = 0.05$ and $\Delta\tau = 0.001$ at different time levels in Table 4. The acquired results are better than acquired results in [19, 20]. Surface plots of and exact and numerical solution at $\tau = 1$ for $h_x = h_y = 0.05$ and $\Delta\tau = 0.001$ are as observed in Figure 6.

Problem 4. We consider two-dimensional second-order nonlinear hyperbolic telegraph equation [20]:

$$V_{\tau\tau} - V_{xx} - V_{yy} + 2\delta V_\tau + \gamma^2 V + 2e^{x+y-\tau} = 0. \tag{23}$$

Subject to the initial and boundary conditions, for $x, y \in [0,1]$ and $\tau \geqslant 0$,

$$\begin{aligned} &V(x,y,0) = e^{x+y}, V_\tau(x,y,0) = -e^{x+y}, \\ &V(0,y,\tau) = e^{y-\tau}, V(1,y,\tau) = e^{1+y-\tau}, \\ &V(x,0,\tau) = e^{x-\tau}, V(x,1,\tau) = e^{x+1-\tau}. \end{aligned} \tag{24}$$

The exact solution is $V(x,y,\tau) = e^{x+y-\tau}$, where $\delta = \gamma = 1$.

We compute L_2, L_∞ norms and relative errors for equal grid sizes $h_x = h_y = 0.05$ and $\Delta\tau = 0.001$ at different time levels, as shown in Table 5. Table 5 clarifies our obtained results are in good agreement with results in [19, 20]. Figure 7 illustrates surface plots of numerical solution with exact solution at $\tau = 1$, for $h_x = h_y = 0.1$ and $\Delta\tau = 0.01$. Figure 8 shows error plots are decreasing versus time increasing at various time levels for $h_x = h_y = 0.1, \Delta\tau = 0.01$ and $h_x = h_y = 0.05, \Delta\tau = 0.001$, respectively.

Problem 5. We consider two-dimensional second-order nonlinear hyperbolic telegraph equation [20]:

$$\begin{aligned} &V_{\tau\tau} - V_{xx} - V_{yy} + 2\delta V_\tau + \gamma^2 V - \left(1+\gamma^2\right)x^2y^2\sinh\tau - 2\delta x^2y^2\cosh\tau \\ &\quad + 2\left(x^2+y^2\right)\sinh\tau + \sin\left(x^2y^2\sinh\tau\right) - \sin V = 0, \end{aligned} \tag{25}$$

with the initial and boundary conditions, for $x, y \in [0,1]$ and $\tau \geqslant 0$,

Table 2: L_2, L_∞, and relative errors for Problem 1 at different time levels with $h_x = 0.1, h_y = 0.2$, and $\Delta\tau = 0.01$.

τ	L_2 error (Bi-FDM)	L_∞ error (Bi-FDM)	RE (Bi-FDM)	L_2 [20]	L_∞ [20]
1	3.0401E-05	5.3037E-05	1.7200E-04	1.2561E-04	4.0150E-04
3	3.8642E-05	7.5546E-05	1.1931E-04	2.4419E-04	7.1428E-04
5	7.6946E-06	1.5049E-05	8.2919E-05	5.7786E-05	1.9111E-04

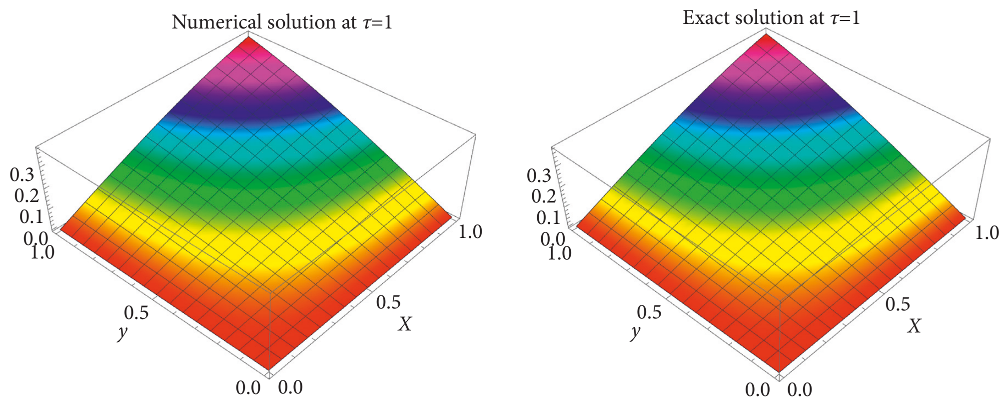

Figure 1: Numerical and exact surface plots at $\tau = 1$ for Problem 1.

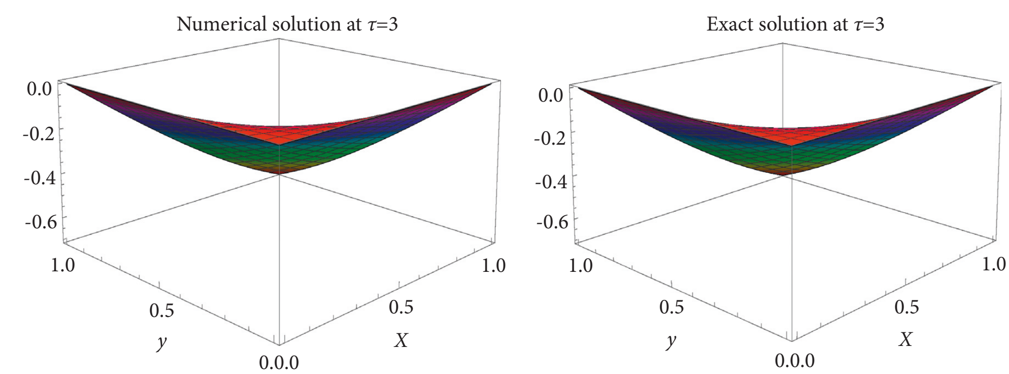

Figure 2: Numerical and exact surface plots at $\tau = 3$ for Problem 1.

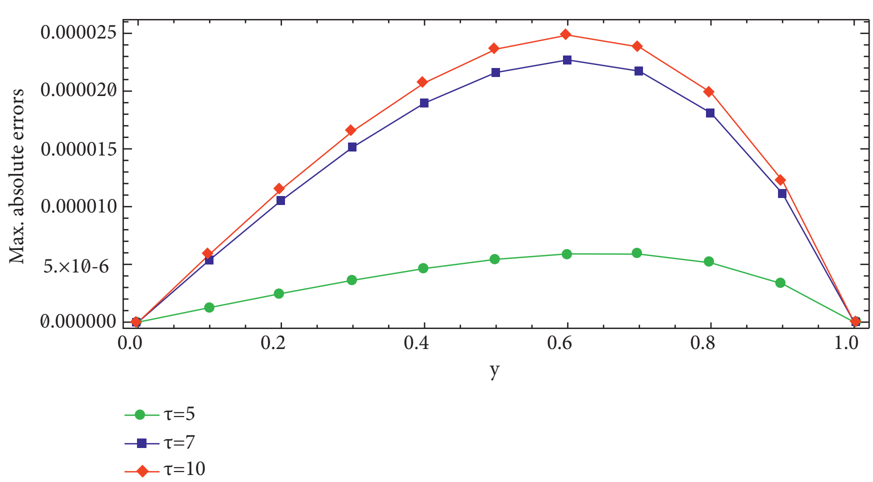

Figure 3: Maximum error plots at $\tau = 5, 7, 10$ and $x = 0.5$ with $h_x = h_y = 0.1$ and $\Delta\tau = 0.01$ for Problem 1.

Table 3: Root mean square and relative errors at different time levels for Problem 2.

τ	RMSE (Bi-FDM)	RE (Bi-FDM)	RMSE [20]	RE [19]
0 (E).5	6.2971E-06	2.1684E-05	2.6594E-04	1.3787–5
1	5.5545E-06	3.1535E-05	1.9039E-04	3.5957–5
3	9.3544E-07	3.9242E-05	2.8681E-05	1.0266–6
5	1.2770E-07	3.9585E-05	3.8988E-06	7.0735–5
10	8.6081E-10	3.9601E-05	2.6275E-08	1.6514–5

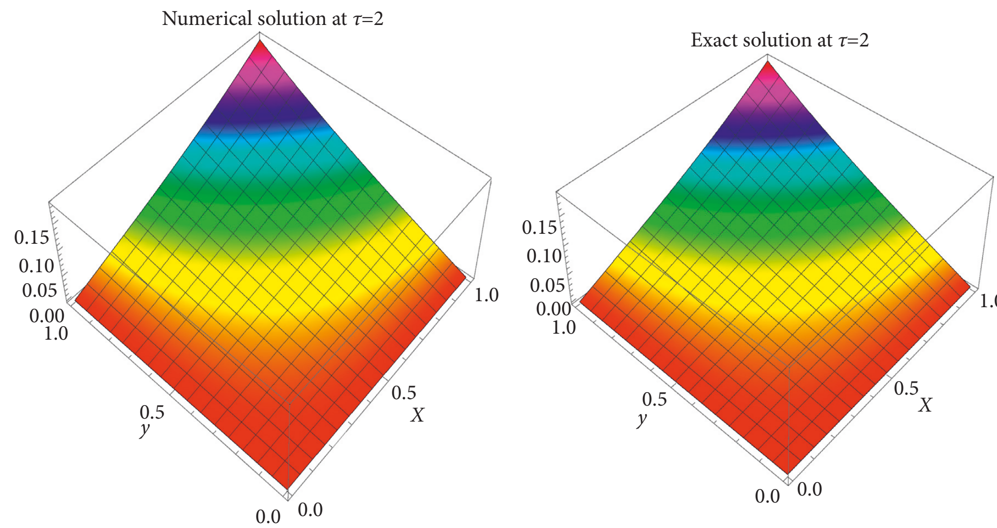

Figure 4: Numerical and exact surface plots at $\tau = 2$ for Problem 2.

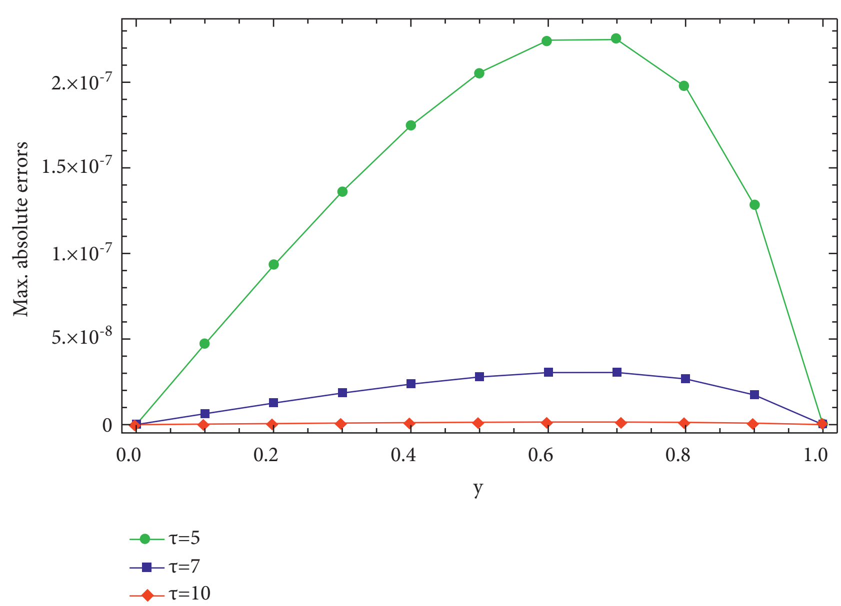

Figure 5: Maximum error plots at $\tau = 5, 7, 10$ and $x = 0.5$ with $h_x = h_y = 0.1$ and $\Delta\tau = 0.01$ for Problem 2.

Table 4: Root mean square and relative errors at different time levels for Problem 3.

τ	RMSE (Bi-FDM)	RE (Bi-FDM)	RMSE [20]	RE [19]
0 (E).5	1.6828E-06	4.3384E-06	8.9921E-05	1.3787–5
1	1.6932E-06	7.0904E-06	6.8547E-05	3.5957–5
2	2.3164E-07	1.2593E-06	2.9487E-05	4.4722–5
5	2.8018E-07	2.2346E-06	1.8225E-05	7.0735–5

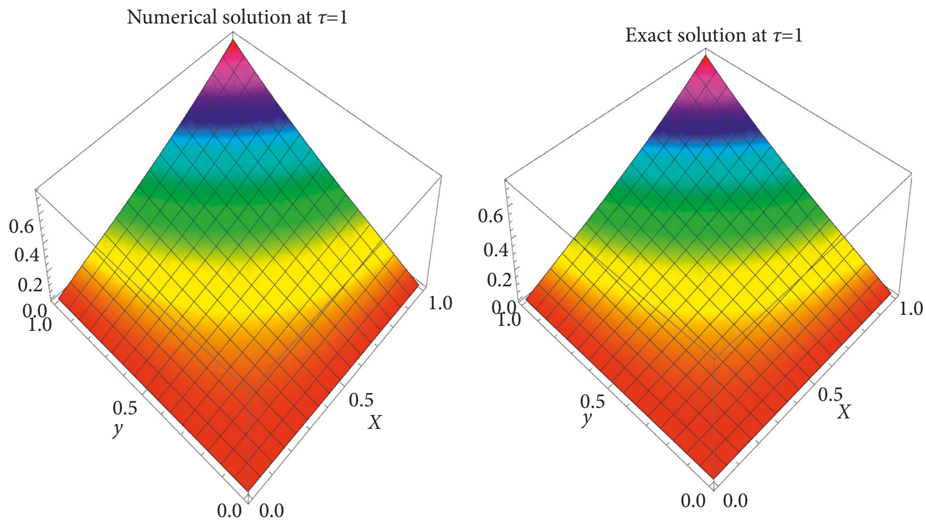

Figure 6: Numerical and exact surface plots at $\tau = 1$ for Problem 3.

Table 5: L_2, L_∞, and relative errors at different time levels for Problem 4.

τ	L_2 error (Bi-FDM)	L_∞ error (Bi-FDM)	RE (Bi-FDM)	L_2 (E)[19]	L_∞ (E)[20]	R (E)E [19]
1 (E)	2.309E-5	3.786E-5	1.8424–5	3.2351–3	1.0000–3	1.2906–4
3	9.048E-7	2.458E-6	5.3347–6	3.1028–4	1.3654–4	9.1555–5
5	6.557E-7	1.347E-6	2.8567–5	2.4495–5	2.8590–5	5.3354–5
10	3.940E-10	8.052E-10	2.5475–6	3.6505–6	9.2105–8	1.1812–5

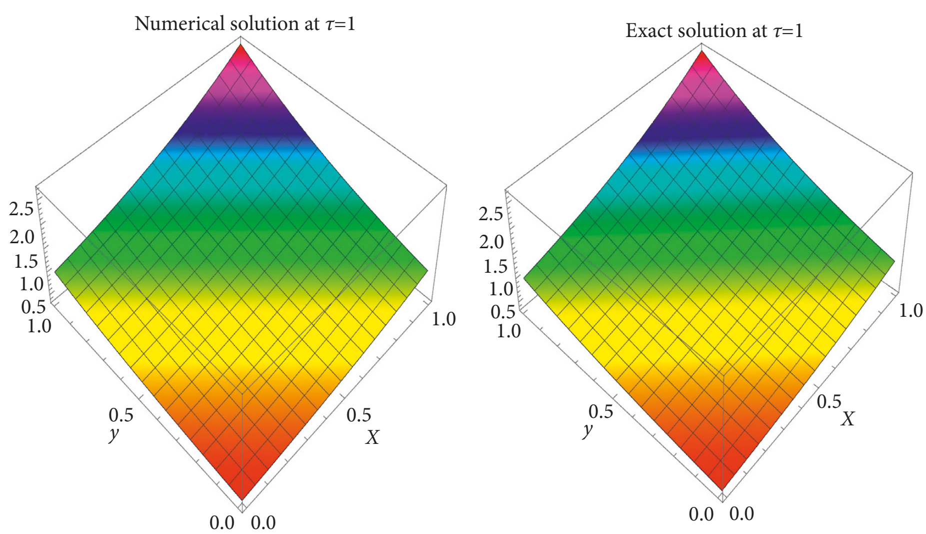

Figure 7: Numerical and exact surface plots at $\tau = 1$ for Problem 4.

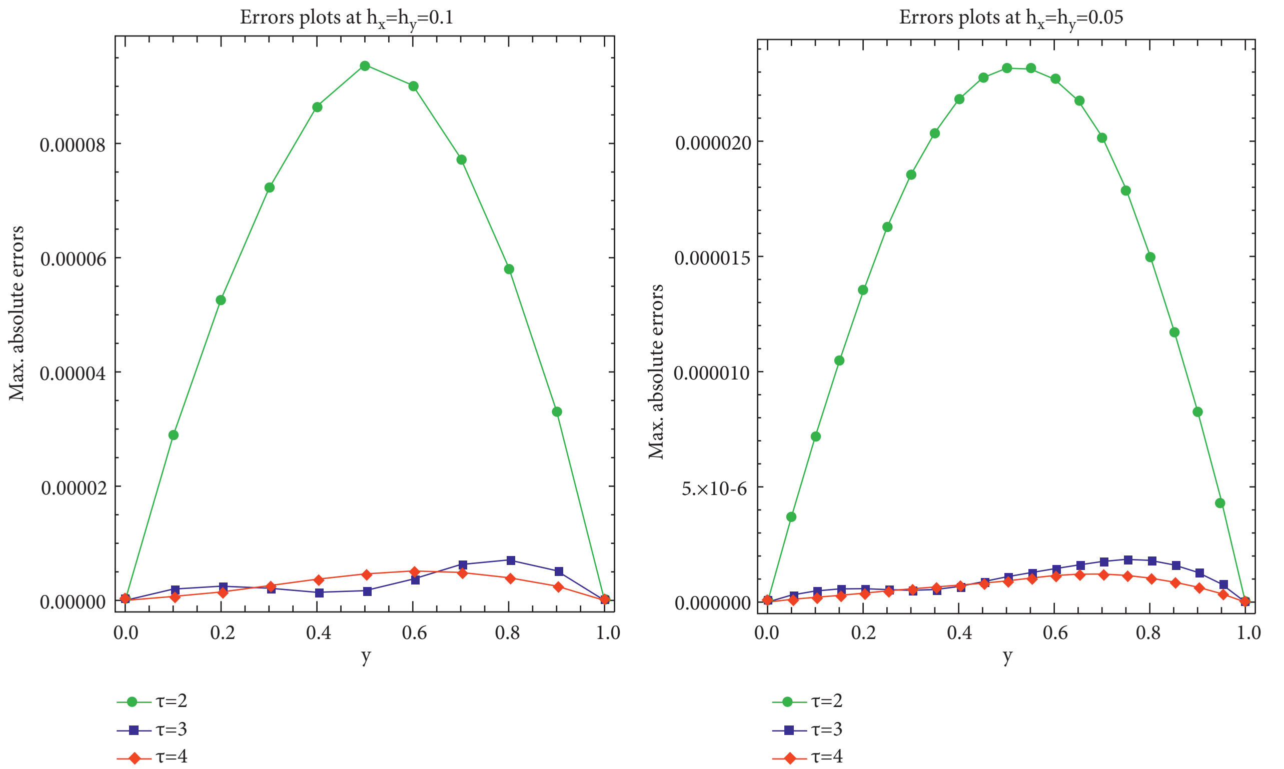

FIGURE 8: Maximum error plots at $\tau = 2, 3, 4$ and $x = 0.6$ for Problem 4.

TABLE 6: L_2 and L_∞ errors at different time levels for Problem 5.

τ	L_2 error (Bi-FDM)	L_∞ (E)error (Bi-FDM)	L_2 (E)[20]	L_∞ [20]
1	3.0796−8	1.2502−7	6.2420E-05	2.3975E-04
2	7.5325−8	3.0573−7	1.8389E-04	7.3788E-04
3	2.0166−7	8.1908−7	5.1608E-04	2.7000E-03
5	1.4865−6	6.0327−6	3.6000E-03	2.0900E-02

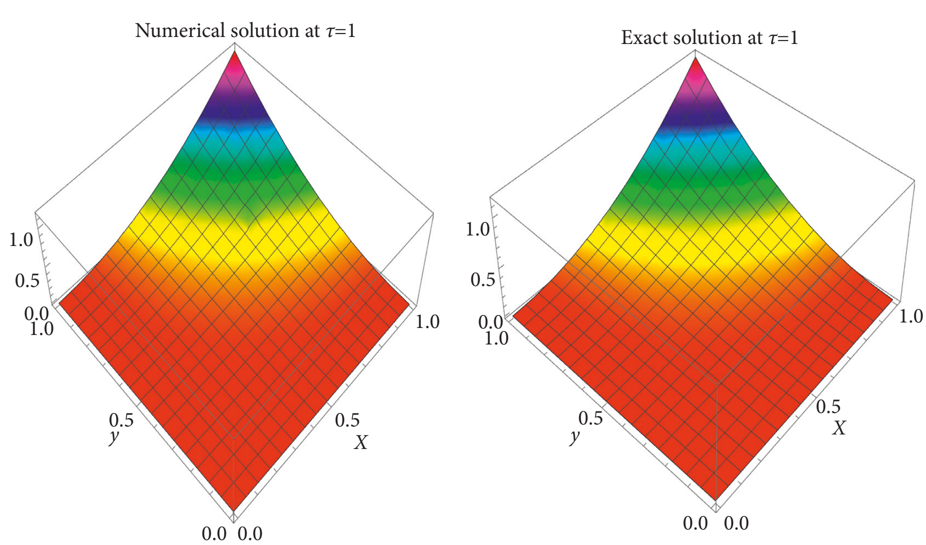

FIGURE 9: Numerical and exact surface plots for Problem 5 at $\tau = 1$.

$$\begin{aligned} &V(x,y,0)=0, V_\tau(x,y,0)=x^2y^2,\\ &V(0,y,\tau)=0, V(1,y,\tau)=y^2\sinh\tau,\\ &V(x,0,\tau)=0, V(x,1,\tau)=x^2\sinh\tau, \end{aligned} \tag{26}$$

and the exact solution is $V(x,y,\tau)=x^2y^2\sinh\tau$, where $\delta=10$ and $\gamma=50$.

We evaluate L_2, L_∞ errors for $h_x=h_y=0.1$ and $\Delta\tau=0.01$ at different time levels. We get precise results by comparing with results in [20] as in Table 6, and Figure 9 clears surface plots of numerical and exact solution at $\tau=1$ for $h_x=h_y=0.1$ and $\Delta\tau=0.01$.

6. Conclusion

This study employed the bi-finite difference method (Bi-FDM) as a collocation method to solve a nonlinear hyperbolic telegraph equation of second order in two dimensions. The local truncation error of the proposed scheme is derived. The numerical scheme has conditionally stable. Five different forms of the equation under study were studied to verify the accuracy of the proposed method, and their L_2, L_∞ norms were calculated. According to the obtained results, the proposed numerical scheme is efficient and in good agreement with other works in [19, 20]. In addition, the Bi-FDM allows us to produce an extremely better estimation of the hidden aspects for a system of the different applications under study.

Data Availability

All data included in this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors have read and approved the final manuscript.

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