Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 1879152, 12 pages
https://doi.org/10.1155/2022/1879152

Research Article

Qualitative Analysis for Multiterm Langevin Systems with Generalized Caputo Fractional Operators of Different Orders

Saeed M. Ali [1] **and Mohammed S. Abdo** [2]

[1] *Department of Basic Engineering Sciences, College of Engineering, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 34151, Saudi Arabia*
[2] *Department of Mathematics, Hodeidah University, Al-Hudaydah, Yemen*

Correspondence should be addressed to Saeed M. Ali; smsali@iau.edu.sa and Mohammed S. Abdo; msabdo@hoduniv.net.ye

Received 20 January 2022; Accepted 9 February 2022; Published 30 March 2022

Academic Editor: Abdellatif Ben Makhlouf

In this research work, we study two types of fractional boundary value problems for multi-term Langevin systems with generalized Caputo fractional operators of different orders. The existence and uniqueness results are acquired by applying Sadovskii's and Banach's fixed point theorems, whereas the guarantee of the existence of solutions is shown by Ulam–Hyer's stability. Our reported results cover many outcomes as special cases. An example is provided to illustrate and validate our results.

1. Introduction

Fractional order models are more convenient than integer-order ones as fractional derivatives give superb tools for the portrayal of memory and inherited processes. Fractional differential equations (FDEs) have been applied in numerous fields, such as engineering, physical science, chemistry, financial matters, electrodynamics, aerodynamics, and dynamical systems (see [1–8]).

Some new contributions to Langevin FDEs have been investigated (see [9–13]) and the references referred to in that). There are many definitions of fractional integrals and derivatives, e.g., Riemann–Liouville type, Caputo type, Hadamard type, Hilfer type, and Erdelyi–Kober type, etc. With regard to the exciting development of local fractional calculus, it has been applied to deal with numerous different nondifferentiable problems in many applied fields (see [14–16]).

The generalized Riemann–Liouville definition with respect to another function was first presented by Osler [17]. Then, Kilbas et al. [2] presented important characteristics of this operator. The Caputo version called φ-Caputo fractional derivative has been introduced by Almeida [18]. Some amazing properties and generalized Laplace transform for the same operator were introduced by Jarad and Abdeljawad [19]. This recently defined fractional operator could model more precisely the process utilizing differential kernel. In order to evolve these definitions, special kernels and some kinds of operators are selected to apply on FDEs; for more details, we refer to some recent results associated with this development (see [20–29]).

In 2018, Almeida et al. [30] considered the following φ-Caputo type FDE with initial conditions:

$$\begin{cases} {}^{C}\mathfrak{D}_{a^+}^{\alpha;\varphi} \upsilon(\rho) = g(\rho, \upsilon(\rho)), & \rho \in [\mathrm{a}, \mathrm{b}], \\ \upsilon(a) = \upsilon_a, \quad \upsilon_{\varphi}^{(k)}(a) = \upsilon_a^k, & k = 1, \ldots, n-1. \end{cases} \tag{1}$$

Ahmad and Nieto [31] considered the following Caputo-type Langevin FDE with boundary conditions:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}\left({}^{C}\mathfrak{D}^{\kappa} + \lambda\right)\upsilon(\rho) = g(\rho, \upsilon(\rho)), \\ \upsilon(0) = \gamma_1, \quad \upsilon(1) = \gamma_2. \end{cases} \tag{2}$$

In 2020, Laadjal et al. [32] studied a Caputo-type multiterm Langevin FDE with boundary conditions of the form:

$$\left\{ {}^{C}\mathfrak{D}^{\alpha}\left({}^{C}\mathfrak{D}^{\kappa}+\lambda\right)v(\rho)-\sum_{i=1}^{m}\xi_i{}^{C}\mathfrak{D}^{\alpha}\left({}^{C}\mathfrak{D}^{\kappa_i}+\lambda_i\right)v(\rho)=g(\rho,v(\rho)), v(0)=v_{\varphi}'(0)=v(1)=0. \right. \quad (3)$$

In this work, we study two classes of generalized Caputo Langevin equations with two various fractional orders. This work is inspired by the recent works of Laadjal et al. [32] and Almeida et al. [30]. Precisely, we consider the following Langevin-type fractional differential problems (FDPs):

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa;\varphi}+\mu\right)v(\rho)=g(\rho,v(\rho)), \\ v(0)=v_{\varphi}'(0)=v(1)=0, \end{cases} \quad (4)$$

$$\left\{ {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa;\varphi}+\mu\right)v(\rho)-\sum_{i=1}^{m}\xi_i{}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa_i;\varphi}+\mu_i\right)v(\rho)=g(\rho,v(\rho)), v(0)=v_{\varphi}'(0)=v(1)=0, \right. \quad (5)$$

where $\rho\in\mho=[0,1]$, $0<\kappa_1<\ldots<\kappa_m\le 1$, $1<\kappa\le 2$, $0<\alpha\le 1$, $1\le\kappa-\kappa_i<2$, $(i=1,2,\ldots,m)$, $m\in\mathbb{N}$, $\xi_i,\mu_i\in\mathbb{R}$, the symbol ${}^{C}\mathfrak{D}^{\theta;\varphi}$ denotes the generalized fractional derivative in the Caputo sense of order $\theta\in\{\alpha,\kappa,\kappa_i\}$, and $g\colon\mho\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous nonlinear function.

Here, we investigate the existence and uniqueness of solutions for FDPs (1.4) and (1.5) involving generalized Caputo fractional derivatives of various orders. Moreover, the guarantee of the existence of solutions is proved by UH stability. The studied results expand and generalize many results by selecting special cases of the φ function.

The paper is organized as follows: In Section 2, we give some definitions and lemmas that are used in the research paper. Section 3 derives equivalent fractional integral equations to the linear variants of Langevin FDPs (1.4) and (1.5). Section 4 deals with the qualitative analysis of proposed problems. In Section 5, we give an example to substantiate the main outcomes.

2. Preliminaries and Lemmas

We are beginning this portion by endowment with some essential definitions and results required for forthcoming analysis.

We consider the Banach space $\mathscr{C}(\mho,\mathbb{R})$ with the norm $\|v\|_{\infty}=\max\{|v(\rho)|,\rho\in\mho\}$.

Let $v\colon\mho\longrightarrow\mathbb{R}$ be an integrable function and $\varphi\in\mathscr{C}^n(\mho,\mathbb{R})$ an increasing function such that $\varphi'(\rho)\neq 0$, for any $\rho\in\mho$.

Definition 1 (see [2]). The φ-Riemann–Liouville fractional integral of a function v of order θ is described by

$$\mathfrak{J}_{a+}^{\theta;\varphi}v(\rho)=\frac{1}{\Gamma(\theta)}\int_a^{\rho}\varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^{\theta-1}v(\varsigma)d\varsigma. \quad (6)$$

Definition 2 (see [2]). The φ-Riemann–Liouville fractional derivative of a function v of order θ is described by

$$\mathfrak{D}_{a+}^{\theta;\varphi}v(\rho)=\left(\frac{1}{\varphi'(\rho)}\frac{d}{d\rho}\right)^{n}\mathfrak{J}_{a+}^{n-\theta;\varphi}v(\rho), \quad (7)$$

where $n=[\alpha]+1$, $n\in\mathbb{N}$.

Definition 3 (see [18]). The φ-Caputo fractional derivative of a function $v\in\mathscr{C}^n(\mho,\mathbb{R})$ of order θ is described by

$${}^{C}\mathfrak{D}_{a+}^{\theta;\varphi}v(\rho)=\mathfrak{J}_{a+}^{(n-\theta);\varphi}v_{\varphi}^{[n]}(\rho), \quad (8)$$

where $v_{\varphi}^{[n]}(\rho)=(1/\varphi'(\rho)d/d\rho)^n v(\rho)$ and $n=[\alpha]+1$, $n\in\mathbb{N}$.

Lemma 1 (see [2, 18]). *Let $r_1,r_2>0$. Then,*

(1) $\mathfrak{J}_{a+}^{r_1;\varphi}(\varphi(\rho)-\varphi(a))^{r_2-1}(\rho)=\Gamma(r_2)/\Gamma(r_1+r_2)(\varphi(\rho)-\varphi(a))^{r_1+r_2-1}$

(2) ${}^{C}\mathfrak{D}_{a+}^{r_1;\varphi}(\varphi(\rho)-\varphi(a))^{r_2-1}(\rho)=\Gamma(r_2)/\Gamma(r_2-r_1)(\varphi(\rho)-\varphi(a))^{r_1+r_2-1}$

(3) ${}^{C}\mathfrak{D}_{a+}^{r_1;\varphi}(\varphi(\rho)-\varphi(a))^{k}(\rho)=0$, *for* $r_1>k\in\mathbb{N}$

Lemma 2 (see [18]). *If $v\in\mathscr{C}^n(\mho,\mathbb{R})$ and $r_1\in(n-1,n)$, then*

(1) $\mathfrak{J}_{a+}^{r_1;\varphi}{}^{C}\mathfrak{D}_{a+}^{r_1;\varphi}v(\rho)=v(\rho)-\sum_{k=0}^{n-1}v_{\varphi}^{[n]}(a^+)/\ k!(\varphi(\rho)-\varphi(a))^k$.

In particular,
if $r_1\in(0,1)$, we have

$$\mathfrak{J}_{a+}^{r_1;\varphi}{}^{C}\mathfrak{D}_{a+}^{r_1;\varphi}v(\rho)=v(\rho)-v(a). \quad (9)$$

If $r_1\in(1,2)$, we have

$$\mathfrak{J}_{a+}^{r_1;\varphi}{}^{C}\mathfrak{D}_{a+}^{r_1;\varphi}v(\rho)=v(\rho)-v(a)-v_{\varphi}'(a)(\varphi(\rho)-\varphi(a)). \quad (10)$$

Moreover, if $v\in\mathscr{C}(\mho,\mathbb{R})$, then

$${}^{C}\mathfrak{D}_{a+}^{r_1;\varphi}\mathfrak{J}_{a+}^{r_1;\varphi}v(\rho)=v(\rho). \quad (11)$$

Lemma 3 (see [18]). *Let $0<\theta<1$. Then, the unique solution of the following linear FDP is as follows:*

$$\begin{cases} {}^{C}\mathfrak{D}_{0+}^{\theta;\varphi}v(\rho)=g(\rho), \quad \rho\in[0,\rho], \\ v(0)=v_0, \end{cases} \quad (12)$$

is obtained as

$$v(\rho)=v_0+\frac{1}{\Gamma(\theta)}\int_0^{\rho}\varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^{\theta-1}g(\varsigma)d\varsigma. \quad (13)$$

Lemma 4. *Let $1 < \theta < 2$. Then, the unique solution of the following linear FDP is as follows:*

$$\begin{cases} {}^{C}\mathfrak{D}_{0+}^{\theta;\varphi} v(\rho) = g(\rho), \quad \rho \in [0,\rho], \\ v(0) = v_0, \quad v_{\varphi}'(0) = v_1, \end{cases} \tag{14}$$

is obtained as

$$v(\rho) = v_0 + (\varphi(\rho) - \varphi(0))\left[v_1 - \mathfrak{I}_{0+}^{\theta-1;\varphi} g(\rho)\right] + \mathfrak{I}_{0+}^{\theta;\varphi} g(\rho). \tag{15}$$

Proof. In view of Lemma 2, we have

$$v(\rho) = c_0 + c_1(\varphi(\rho) - \varphi(0)) + \frac{1}{\Gamma(\theta)} \int_0^{\rho} \varphi'(\varsigma)(\varphi(\rho) - \varphi(\varsigma))^{\theta-1} g(\varsigma) d\varsigma, \tag{16}$$

where $c_0, c_1 \in \mathbb{R}$. By conditions of (14), we get $c_0 = v_0$ and

$$v_{\varphi}'(\rho) = \frac{v\prime(\rho)}{\varphi'(\rho)} = c_1 + \frac{1}{\Gamma(\theta-1)} \int_0^{\rho} \varphi'(\varsigma)(\varphi(\rho) - \varphi(\varsigma))^{\theta-2} g(\varsigma) d\varsigma, \tag{17}$$

which implies

$$c_1 = v_1 - \frac{1}{\Gamma(\theta-1)} \int_0^{\rho} \varphi'(\varsigma)(\varphi(\rho) - \varphi(\varsigma))^{\theta-2} g(\varsigma) d\varsigma. \tag{18}$$

Substituting the value of c_0 and c_1 in (16), we get

$$\begin{aligned} v(\rho) = v_0 + (\varphi(\rho) - \varphi(0))\Bigg[v_1 - \frac{1}{\Gamma(\theta-1)} \\ \cdot \int_0^{\rho} \varphi'(\varsigma)(\varphi(\rho) - \varphi(\varsigma))^{\theta-2} g(\varsigma) d\varsigma\Bigg] \\ + \frac{1}{\Gamma(\theta)} \int_0^{\rho} \varphi'(\varsigma)(\varphi(\rho) - \varphi(\varsigma))^{\theta-1} g(\varsigma) d\varsigma, \end{aligned} \tag{19}$$

which is identical to (15). □

Theorem 1 (see [33]). *Let X is a Banach space. The map $P + Q$ is a λ-set contraction with $0 \le \lambda < 1$, and thus also condensing, if (i) P, Q: $D \subseteq X \longrightarrow X$ are operators on X; (ii) P is λ contractive; (iii) Q is compact.*

To apply Sadovskii's and Banach's fixed point theorems, we will suffice herewith reference to [34, 35].

3. The Linear Variant of FDPs (1.4) and (1.5)

This section deals with the linear variant of FDPs (1.4) and (1.5). For simpleness, we denote ${}^{C}\mathfrak{D}_{0^+}^{\alpha;\varphi}$ and $\mathfrak{I}_{0^+}^{\alpha;\varphi}$ by ${}^{C}\mathfrak{D}^{\alpha;\varphi}$ and $\mathfrak{I}^{\alpha;\varphi}$, respectively.

Lemma 5. *Let $1 < \kappa \le 2$, $0 < \alpha \le 1$, and $f \in C(\mho, \mathbb{R})$. Then, v is a solution of the following linear Langevin-type FDP:*

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa;\varphi} + \mu\right) v(\rho) = f(\rho), \\ v(0) = v_{\varphi}'(0) = v(1) = 0, \end{cases} \tag{20}$$

if and only if v satisfies

$$\begin{aligned}
v(\rho) &= \frac{1}{\Gamma(\alpha+\kappa)} \int_0^\rho \varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^{\alpha+\kappa-1} f(\varsigma)d\varsigma \\
&\quad - \frac{\mu}{\Gamma(\kappa)} \int_0^\rho \varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^{\kappa-1} v(\varsigma)d\varsigma \\
&\quad + \frac{\mathcal{X}_\varphi^\kappa(\rho)}{\mathcal{X}_\varphi^\kappa(1)} \left(\frac{\mu}{\Gamma(\kappa)} \int_0^1 \varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\kappa-1} v(\varsigma)d\varsigma \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha+\kappa)} \int_0^1 \varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\alpha+\kappa-1} f(\varsigma)d\varsigma \right),
\end{aligned} \tag{21}$$

where $\mathcal{X}_\varphi^\kappa(\rho) = [\varphi(\rho)-\varphi(0)]^\kappa$.

Proof. Applying the operator $\mathfrak{I}^{\alpha;\varphi}$ on both sides of the first (20) and using Lemma 2, we get

$$\left({}^C\mathfrak{D}^{\kappa;\varphi} + \mu\right) v(\rho) = \mathfrak{I}^{\alpha;\varphi} f(\rho) + c_0, \tag{22}$$

where $c_0 \in \mathbb{R}$.

Next, applying the operator $\mathfrak{I}^{\kappa;\varphi}$ on both sides of (22) and using Lemma 2, again, we obtain

$$v(\rho) = c_1 + c_2 \mathcal{X}_\varphi^1(\rho) + \mathfrak{I}^{\alpha+\kappa;\varphi} f(\rho) - \mu \mathfrak{I}^{\kappa;\varphi} v(\rho) + \frac{c_0}{\Gamma(\kappa+1)} \mathcal{X}_\varphi^\kappa(\rho), \tag{23}$$

where $c_1, c_2 \in \mathbb{R}$. In view of (23), we have

$$v_\varphi'(\rho) = \frac{v\prime(\rho)}{\varphi'(\rho)} = c_2 \mathcal{X}_\varphi^1(\rho) + \mathfrak{I}^{\alpha+\kappa-1;\varphi} f(\rho) - \mu \mathfrak{I}^{\kappa-1;\varphi} v(\rho) + \frac{c_0}{\Gamma(\kappa)} \mathcal{X}_\varphi^{\kappa-1}(\rho), \tag{24}$$

where we used the fact that

$$\begin{aligned}
\left(\mathfrak{I}^{\theta;\varphi} f(\rho)\right)_\varphi' &= \left(\frac{1}{\varphi'(\rho)} \frac{d}{d\rho}\right) \left(\mathfrak{I}^{\theta;\varphi} f(\rho)\right) \\
&= \left(\frac{1}{\varphi'(\rho)} \frac{d}{d\rho}\right) \frac{1}{\Gamma(\theta)} \int_0^\rho \mathcal{X}_\varphi^{\theta-1}(\rho,\varsigma) f(\varsigma)d\varsigma \\
&= \frac{1}{\Gamma(\theta-1)} \int_0^\rho \mathcal{X}_\varphi^{\theta-2}(\rho,\varsigma) f(\varsigma)d\varsigma \\
&= \mathfrak{I}^{\theta-1;\varphi} f(\rho),
\end{aligned} \tag{25}$$

where $\mathcal{X}_\varphi^\theta(\rho,\varsigma) = \varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^\theta$. Using the initial conditions of (23), we find that $c_1 = c_2 = 0$ and

$$c_0 = \frac{\Gamma(\kappa+1)}{\mathcal{X}_\varphi^\kappa(1)} \left[\mu \mathfrak{I}^{\kappa;\varphi} v(1) - \mathfrak{I}^{\alpha+\kappa;\varphi} f(1)\right]. \tag{26}$$

Substituting the values of c_0, c_1, and c_2 in (23), we obtain the solution given by (21), where

$$\begin{aligned}
\mathfrak{I}^{r;\varphi} h(\eta) &= \frac{1}{\Gamma(r)} \int_0^\rho \varphi'(\varsigma)(\varphi(\eta)-\varphi(\varsigma))^{r-1} h(\varsigma)d\varsigma, \\
& r \in \{\kappa, \alpha+\kappa\}, \quad h \in \{f, v\}, \ \eta \in \{\rho, 1\}.
\end{aligned} \tag{27}$$

The converse follows by direct calculation. Hence, the proof is achieved. □

Lemma 6. *Let* $1 < \kappa \leq 2$, $0 < \alpha \leq 1$, $1 \leq \kappa - \kappa_i < 2$, *and* $f \in C(\mho, \mathbb{R})$. *Then,* v *is a solution of the following linear Langevin-type multiterm FDP:*

$$\left\{ {}^C\mathfrak{D}^{\alpha;\varphi}\left({}^C\mathfrak{D}^{\kappa;\varphi} + \mu\right) v(\rho) - \sum_{i=1}^m \xi_i {}^C\mathfrak{D}^{\alpha;\varphi}\left({}^C\mathfrak{D}^{\kappa_i;\varphi} + \mu_i\right) v(\rho) = f(\rho), v(0) = v_\varphi'(0) = v(1) = 0, \right. \tag{28}$$

if and only if v *satisfies*

$$\begin{aligned}
v(\rho) =& \sum_{i=1}^{m}\frac{\xi_i}{\Gamma(\kappa-\kappa_i)}\int_0^\rho \varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^{\kappa-\kappa_i-1}v(\varsigma)d\varsigma \\
&+\frac{1}{\Gamma(\alpha+\kappa)}\int_0^\rho \varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^{\alpha+\kappa-1}f(\varsigma)d\varsigma \\
&-\frac{\sigma}{\Gamma(\kappa)}\int_0^\rho \varphi'(\varsigma)(\varphi(\rho)-\varphi(\varsigma))^{\kappa-1}v(\varsigma)d\varsigma \\
&+\frac{\mathcal{X}_\varphi^\kappa(\rho)}{\mathcal{X}_\varphi^\kappa(1)}\left(\frac{\sigma}{\Gamma(\kappa)}\int_0^1 \varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\kappa-1}v(\varsigma)d\varsigma-\sum_{i=1}^{m}\frac{\xi_i}{\Gamma(\kappa-\kappa_i)}\int_0^1 \varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\kappa-\kappa_i-1}v(\varsigma)d\varsigma\right. \\
&\left.-\frac{1}{\Gamma(\alpha+\kappa)}\int_0^1 \varphi'(\varsigma)(\varphi(1)-\varphi(\varsigma))^{\alpha+\kappa-1}f(\varsigma)d\varsigma\right),
\end{aligned} \tag{29}$$

where $\sigma=\mu-\sum_{i=1}^{m}\xi_i\mu_i$.

Proof. Applying the operator $\mathfrak{J}^{\alpha;\varphi}$ on both sides of the first (28) and using Lemma 2, we get

$$\begin{aligned}
&\mathfrak{J}^{\alpha;\varphi}\left[{}^C D^{\alpha;\varphi}\left({}^C\mathfrak{D}^{\kappa;\varphi}+\mu\right)v(\rho)\right]-\sum_{i=1}^{m}\xi_i\mathfrak{J}^{\alpha;\varphi}\left[{}^C\mathfrak{D}^{\alpha;\varphi}\left({}^C\mathfrak{D}^{\kappa_i;\varphi}+\mu_i\right)v(\rho)\right]=\mathfrak{J}^{\alpha;\varphi}f(\rho), \\
&\left({}^C\mathfrak{D}^{\kappa;\varphi}+\mu\right)v(\rho)-\sum_{i=1}^{m}\xi_i\left({}^C\mathfrak{D}^{\kappa_i;\varphi}+\mu_i\right)v(\rho)=\mathfrak{J}^{\alpha;\varphi}f(\rho)-\sigma v(\rho)+c_0,
\end{aligned} \tag{30}$$

where $c_0\in\mathbb{R}$.

Next, applying the operator $\mathfrak{J}^{\kappa;\varphi}$ on both sides of (30) and using Lemma 2, again, we obtain

$$\begin{aligned}
v(\rho)=&\,c_1+c_2[\varphi(\rho)-\varphi(0)]+\sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}[v(\rho)-v(0)] \\
&+\mathfrak{J}^{\alpha+\kappa;\varphi}f(\rho)-\sigma\mathfrak{J}^{\kappa;\varphi}v(\rho)+\frac{c_0}{\Gamma(\kappa+1)}\mathcal{X}_\varphi^\kappa(\rho),
\end{aligned} \tag{31}$$

where $c_1,c_2\in\mathbb{R}$. In view of (31), we have

$$\begin{aligned}
v_\varphi'(\rho)=&\frac{v\prime(\rho)}{\varphi'(\rho)}=c_2\mathcal{X}_\varphi^1(\rho)+\sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i-1;\varphi}[v(\rho)-v(0)] \\
&+\mathfrak{J}^{\alpha+\kappa-1;\varphi}f(\rho)-\sigma\mathfrak{J}^{\kappa-1;\varphi}v(\rho)+\frac{c_0}{\Gamma(\kappa)}\mathcal{X}_\varphi^{\kappa-1}(\rho).
\end{aligned} \tag{32}$$

Using the condition $v_\varphi'(0)=0$ in (32) and $v(0)=0$ and $v(1)=0$ in (31), we find that $c_1=c_2=0$ and

$$0=\sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}v(1)+\mathfrak{J}^{\alpha+\kappa;\varphi}f(1)-\sigma\mathfrak{J}^{\kappa;\varphi}v(1)+\frac{c_0}{\Gamma(\kappa+1)}\mathcal{X}_\varphi^\kappa(1), \tag{33}$$

which implies

$$c_0=\frac{\Gamma(\kappa+1)}{\mathcal{X}_\varphi^\kappa(1)}\left[\sigma\mathfrak{J}^{\kappa;\varphi}v(1)-\sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}v(1)-\mathfrak{J}^{\alpha+\kappa;\varphi}f(1)\right]. \tag{34}$$

Substituting the values of c_0, c_1, and c_2 in (31), we obtain the solution given by (29). The converse follows by direct calculation. Hence, the proof is achieved. □

4. Qualitative Analysis

4.1. Existence and Uniqueness Results. This subsection proves the existence and uniqueness results for FDPs (4) and (5) by applying Sadovskii's theorem [34] and Banach's theorem [35].

Theorem 2 (existence). *Let $g\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. We assume that*

(H1): there exists a function $p \in C(\mho, \mathbb{R}^+)$ such that $|g(\rho, x)| \le p(\rho)$, for $(\rho, x) \in \mho \times \mathbb{R}$.

(H2): $\gamma_1 := 2(\sum_{i=1}^m |\xi_i| \ \mathscr{X}_\varphi^{\kappa-\kappa_i}(1)/\Gamma(\kappa-\kappa_i+1) + |\sigma|\mathscr{X}_\varphi^{\kappa}(1)/\Gamma(\kappa+1)) < 1$.

Then, the FDP (1.5) has at least one solution on $\mho$.

Proof. Let $\mathscr{E}_r$ be a closed bounded and convex subset of $C(\mho, \mathbb{R})$, where r is a fixed constant. By virtue of Lemma 6, we define an operator $f\colon C(\mho, \mathbb{R}) \longrightarrow C(\mho, \mathbb{R})$ as follows:

$$
\begin{aligned}
f v(\rho) = {} & \sum_{i=1}^m \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi} v(\rho) + \mathfrak{I}^{\alpha+\kappa;\varphi} g(\rho, v(\rho)) - \sigma \mathfrak{I}^{\kappa;\varphi} v(\rho) \\
& + \frac{\mathscr{X}_\varphi^{\kappa}(\rho)}{\mathscr{X}_\varphi^{\kappa}(1)} \left(\sigma \mathfrak{I}^{\kappa;\varphi} v(1) - \sum_{i=1}^m \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi} v(1) - \mathfrak{I}^{\alpha+\kappa;\varphi} g(1, v(1)) \right),
\end{aligned}
\tag{35}
$$

for $\rho \in \mho$. Let us define two operators $F_1, F_2\colon C(\mho, \mathbb{R}) \longrightarrow C(\mho, \mathbb{R})$ by

$$
\begin{aligned}
(f_1 v)(\rho) = {} & \sum_{i=1}^m \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi} v(\rho) - \sigma \mathfrak{I}^{\kappa;\varphi} v(\rho) \\
& + \frac{\mathscr{X}_\varphi^{\kappa}(\rho)}{\mathscr{X}_\varphi^{\kappa}(1)} \left(\sigma \mathfrak{I}^{\kappa;\varphi} v(1) - \sum_{i=1}^m \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi} v(1) \right), \\
(F_2 v)(\rho) = {} & \mathfrak{I}^{\alpha+\kappa;\varphi} g(\rho, v(\rho)) - \frac{\mathscr{X}_\varphi^{\kappa}(\rho)}{\mathscr{X}_\varphi^{\kappa}(1)} \mathfrak{I}^{\alpha+\kappa;\varphi} g(1, v(1)).
\end{aligned}
\tag{36}
$$

We observe that $(Fv)(\rho) = (F_1 v)(\rho) + (F_2 v)(\rho)$, $\rho \in \mho$.

In order to show that $F_1 + F_2$ has a fixed point, we prove that F_1 and F_2 satisfy the hypotheses of Sadovskii's theorem. This will be provided in several steps: □

Step 1. $F\mathscr{E}_r \subset \mathscr{E}_r$.

Let us select $r \ge \gamma_2/1 - \gamma_1$, where $\gamma_2 := 2\|p\|\mathscr{X}_\varphi^{\alpha+\kappa}(1)/\Gamma(\alpha+\kappa+1)$. For any $v \in \mathscr{E}_r$, we have

$$
\begin{aligned}
\|Fv\| \le {} & \sup_{\rho \in \mho} \Bigg\{ \sum_{i=1}^m \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi} |v(\rho)| + |\sigma| \mathfrak{I}^{\kappa;\varphi} |v(\rho)| + \frac{\mathscr{X}_\varphi^{\kappa}(\rho)}{\mathscr{X}_\varphi^{\kappa}(1)} \left(\sigma \mathfrak{I}^{\kappa;\varphi} v(1) + \sum_{i=1}^m |\xi_i| \mathfrak{I}^{\kappa-\kappa_i;\varphi} v(1) \right) \\
& + \mathfrak{I}^{\alpha+\kappa;\varphi} |g(\rho, v(\rho))| + \frac{\mathscr{X}_\varphi^{\kappa}(\rho)}{\mathscr{X}_\varphi^{\kappa}(1)} \mathfrak{I}^{\alpha+\kappa;\varphi} |g(1, v(1))| \Bigg\} \\
\le {} & 2\|v\| \left(\sum_{i=1}^m |\xi_i| \frac{(\varphi(1) - \varphi(0))^{\kappa-\kappa_i}}{\Gamma(\kappa - \kappa_i + 1)} + |\sigma| \frac{(\varphi(1) - \varphi(0))^{\kappa}}{\Gamma(\kappa+1)} \right) + 2\|p\| \frac{(\varphi(1) - \varphi(0))^{\alpha+\kappa}}{\Gamma(\alpha+\kappa+1)} \\
\le {} & 2r \left(\sum_{i=1}^m |\xi_i| \frac{\mathscr{X}_\varphi^{\kappa-\kappa_i}(1)}{\Gamma(\kappa - \kappa_i + 1)} + |\sigma| \frac{\mathscr{X}_\varphi^{\kappa}(1)}{\Gamma(\kappa+1)} \right) + 2\|p\| \frac{\mathscr{X}_\varphi^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)} \\
= {} & r\gamma_1 + \gamma_2 \le r\gamma_1 + r(1 - \gamma_1) = r,
\end{aligned}
\tag{37}
$$

which implies that $(F_1 + F_2)\mathscr{E}_r \subset \mathscr{E}_r$.

Step 2. F_2 is compact. We note that F_2 is uniformly bounded from Step 1. Let $\rho_1, \rho_2 \in \mho$ with $\rho_1 \leq \rho_2$ and $v \in \mathscr{E}_r$. Then, we obtain

$$\begin{aligned}
&\left|(F_2 v)(\rho_2) - (F_2 v)(\rho_1)\right| \\
&= \left| \frac{1}{\Gamma(\alpha+\kappa)} \int_0^{\rho_2} \mathscr{X}_\varphi^{\alpha+\kappa-1}(\rho_2, s) g(s, v(s)) ds - \frac{\mathscr{X}_\varphi^\kappa(\rho_2)}{\mathscr{X}_\varphi^\kappa(1)} \frac{1}{\Gamma(\alpha+\kappa)} \int_0^1 \mathscr{X}_\varphi^{\alpha+\kappa-1}(1, s) g(s, v(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha+\kappa)} \int_0^{\rho_1} \mathscr{X}_\varphi^{\alpha+\kappa-1}(\rho_1, s) g(s, v(s)) ds + \frac{\mathscr{X}_\varphi^\kappa(\rho_1)}{\mathscr{X}_\varphi^\kappa(1)} \frac{1}{\Gamma(\alpha+\kappa)} \int_0^1 \mathscr{X}_\varphi^{\alpha+\kappa-1}(1, s) g(s, v(s)) ds \right| \\
&\leq \frac{\mathscr{X}_\varphi^\kappa(\rho_1) - \mathscr{X}_\varphi^\kappa(\rho_2)}{\mathscr{X}_\varphi^\kappa(1)} \frac{1}{\Gamma(\alpha+\kappa)} \int_0^1 \mathscr{X}_\varphi^{\alpha+\kappa-1}(1, s) g(s, v(s)) ds \\
&\quad + \frac{1}{\Gamma(\alpha+\kappa)} \int_0^{\rho_1} \left[\mathscr{X}_\varphi^{\alpha+\kappa-1}(\rho_1, s) - \mathscr{X}_\varphi^{\alpha+\kappa-1}(\rho_2, s)\right] |g(s, v(s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha+\kappa)} \int_{\rho_1}^{\rho_2} \mathscr{X}_\varphi^{\alpha+\kappa-1}(\rho_2, s) |g(s, v(s))| ds \\
&\leq \frac{(\varphi(\rho_1) - \varphi(0))^\kappa - (\varphi(\rho_2) - \varphi(0))^\kappa}{\Gamma(\alpha+\kappa+1)} (\varphi(1) - \varphi(0))^{\alpha+\kappa} \|p\| \\
&\quad + \frac{2\|p\|}{\Gamma(\alpha+\kappa+1)} (\varphi(\rho_2) - \varphi(\rho_1))^{\alpha+\kappa} \\
&\leq \frac{2\|p\|}{\Gamma(\alpha+\kappa+1)} (\varphi(\rho_2) - \varphi(\rho_1))^{\alpha+\kappa}.
\end{aligned} \tag{38}$$

From the continuity of φ, $|(F_2 v)(\rho_2) - (F_2 v)(\rho_1)| \longrightarrow 0$ as $\rho_2 \longrightarrow \rho_1$. Thus, F_2 is equicontinuous. So, by the Arzela–Ascoli theorem, $F_2(\mathscr{E}_r)$ is a relatively compact set.

Step 3. F_1 is γ contractive. Let $v_1, v_2 \in \mathscr{E}_r$. Then, we have

$$\begin{aligned}
\|F_1 v_1 - F_1 v_2\| &= \sup_{\rho \in \mho} \left\{ \left| \sum_{i=1}^m \xi_i \mathfrak{I}^{\kappa - \kappa_i;\varphi} [v_1(\rho) - v_1(\rho)] - \sigma \mathfrak{I}^{\kappa;\varphi} [v_1(\rho) - v_1(\rho)] \right. \right. \\
&\quad \left. \left. + \frac{\mathscr{X}_\varphi^\kappa(\rho)}{\mathscr{X}_\varphi^\kappa(1)} \left(\sigma \mathfrak{I}^{\kappa;\varphi} [v_1(1) - v_1(1)] - \sum_{i=1}^m \xi_i \mathfrak{I}^{\kappa - \kappa_i;\varphi} [v_1(1) - v_1(1)] \right) \right| \right\} \\
&\leq \sum_{i=1}^m \xi_i \left(\mathfrak{I}^{\kappa - \kappa_i;\varphi}(s)\right)(1) \|v_1 - v_2\| + |\sigma| \left(\mathfrak{I}^{\kappa;\varphi}(s)\right)(1) \|v_1 - v_2\| \\
&\quad + \sigma \left(\mathfrak{I}^{\kappa;\varphi}(s)\right)(1) \|v_1 - v_2\| + \sum_{i=1}^m |\xi_i| \left(\mathfrak{I}^{\kappa - \kappa_i;\varphi}(s)\right)(1) \|v_1 - v_2\| \\
&\leq 2 \left(\sum_{i=1}^m |\xi_i| \frac{\mathscr{X}_\varphi^{\kappa - \kappa_i}(1)}{\Gamma(\kappa - \kappa_i + 1)} + |\sigma| \frac{\mathscr{X}_\varphi^\kappa(1)}{\Gamma(\kappa+1)} \right) \|v_1 - v_2\| = \gamma_1 \|v_1 - v_2\|,
\end{aligned} \tag{39}$$

which is γ contractive since $\gamma_1 < 1$.

Step 4. F is condensing.

Due to the fact that F_1 is continuous, a γ is a contraction and F_2 is compact, it follows from Lemma 1 that $F\colon \mathscr{E}_r \longrightarrow \mathscr{E}_r$ with $F = F_1 + F_2$ is a condensing on $\mathscr{E}_r$. From the previous arguments, we conclude through Sadovskii's theorem that F has a fixed point. As a result, FDP (5) has a solution on $\mho$.

The second result is based on Banach's fixed point theorem.

Theorem 3 (uniqueness). *Let $g\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and there exists $L_g > 0$ such that*

$$\left|g(\rho, x) - g(\rho, x^*)\right| \le L_g \left|x - x^*\right|, \quad \rho \in \mho, x, x^* \in \mathbb{R}. \tag{40}$$

If

$$\sigma := \gamma_1 + \frac{2\mathscr{X}_{\varphi}^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)} L_g < 1, \tag{41}$$

then the FDP (5) has a unique solution on $\mho$, where γ_1 is defined in Theorem 2.

Proof. We apply Banach's theorem to prove that F defined by (35) has a fixed point. For this end, we show that F is a contraction. Let $v, v^* \in C(\mho, \mathbb{R})$ and $\rho \in \mho$. Then,

$$\begin{aligned}
\left|Fv(\rho) - Fv^*(\rho)\right| &\le \sum_{i=1}^{m} \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi} \left|v(\rho) - v^*(\rho)\right| \\
&\quad + \mathfrak{I}^{\alpha+\kappa;\varphi} \left|g(\rho, v(\rho)) - g(\rho, v^*(\rho))\right| + |\sigma| \mathfrak{I}^{\kappa;\varphi} \left|v(\rho) - v^*(\rho)\right| \\
&\quad + \frac{\mathscr{X}_{\varphi}^{\kappa}(\rho)}{\mathscr{X}_{\varphi}^{\kappa}(1)} \left(\sigma \mathfrak{I}^{\kappa;\varphi} \left|v(1) - v^*(1)\right| + \sum_{i=1}^{m} |\xi_i| \mathfrak{I}^{\kappa-\kappa_i;\varphi} \left|v(1) - v^*(1)\right| \right) \\
&\quad + \frac{\mathscr{X}_{\varphi}^{\kappa}(\rho)}{\mathscr{X}_{\varphi}^{\kappa}(1)} \mathfrak{I}^{\alpha+\kappa;\varphi} \left|g(1, v(1)) - g(1, v^*(1))\right| \\
&\le 2\left(\sum_{i=1}^{m} |\xi_i| \frac{\mathscr{X}_{\varphi}^{\kappa-\kappa_i}(1)}{\Gamma(\kappa-\kappa_i+1)} + |\sigma| \frac{\mathscr{X}_{\varphi}^{\kappa}(1)}{\Gamma(\kappa+1)} \right) \left\|v - v^*\right\| + 2\left(\frac{\mathscr{X}_{\varphi}^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)} \right) L_g \left\|v - v^*\right\| \\
&= \left(\gamma_1 + \frac{2\mathscr{X}_{\varphi}^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)} L_g \right) \left\|v - v^*\right\|,
\end{aligned} \tag{42}$$

which implies

$$\left\|Fv - Fv^*\right\| \le \sigma \left\|v - v^*\right\|. \tag{43}$$

As $\sigma < 1$, it follows that F is a contraction. As a result of Banach's theorem, there is a unique fixed point $v \in C(\mho, \mathbb{R})$ such that $Fv = v$. Therefore, the FDP (5) has a unique solution on $\mho$. □

Corollary 1 (existence). *Let $g\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and there exists a function $p \in C(\mho, \mathbb{R}^+)$ such that $|g(\rho, x)| \le p(\rho)$, for $(\rho, x) \in \mho \times \mathbb{R}$. If $\gamma_3 < 1$, where $\gamma_3 = 2|\mu| \mathscr{X}_{\varphi}^{\kappa}(1)/\Gamma(\kappa+1)$,*

then the FDP (4) has at least one solution on $\mho$.

Corollary 2 (uniqueness). *Let $g\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and there exists $M_g > 0$ such that*

$$\left|g(\rho, x) - g(\rho, x^*)\right| \le M_g \left|x - x^*\right|, \quad \rho \in \mho, x, x^* \in \mathbb{R}. \tag{44}$$

If $\Lambda_2 < 1$, where $\Lambda_2 = \gamma_3 + 2\mathscr{X}_{\varphi}^{\alpha+\kappa}(1)/\Gamma(\alpha+\kappa+1)L_g$, then the FDP (4) has a unique solution on $\mho$, where γ_3 is defined in Corollary 1.

Remark 1. Proofs of Corollaries 1 and 2 can be obtained easily using the same arguments as in the previous theorems. Thus, we omit the details.

Remark 2. If $\varphi(\rho) = \rho$, then the obtained results on FDP (1.5) include the results of Laadjal et al. [32].

Remark 3. The Langevin FDPs (4) and (5) are new to the literature on Langevin-type multiterm FDPs and include many problems, as a special case, for $\varphi(\rho) = \log \rho$ and

$\varphi(\rho) = \rho^{\rho}$; our problems are reduced to Hadamard-type problems and Katugampola problems, respectively.

4.2. UH Stability Analysis. In this subsection, we discuss the UH stability of the considered problem.

Definition 4. FDP (5) is UH stable if there exists a constant $\Upsilon_f > 0$ such that for each $\varepsilon > 0$ and every solution $\omega \in C(\mho, \mathbb{R})$ of the inequalities

$$\left| {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa;\varphi} + \mu\right)\omega(\rho) - \sum_{i=1}^{m} \xi_i {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa_i;\varphi} + \mu_i\right)\omega(\rho) - g(\rho, \omega(\rho))\right| \leq \varepsilon, \quad \rho \in \mho, \tag{45}$$

there exists a solution $\upsilon \in C(\mho, \mathbb{R})$ of FDP (5) that satisfies

$$|\omega(\rho) - \upsilon(\rho)| \leq \Upsilon_f \varepsilon. \tag{46}$$

Remark 4. $\omega \in C(\mho, \mathbb{R})$ satisfies the inequality (45) if and only if there exists a function $\Pi \in C(\mho, \mathbb{R})$ with

(i) $|\Pi(\rho)| \leq \varepsilon$, $\rho \in \mho$

(ii) For all $\rho \in \mho$

$$ {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa;\varphi} + \mu\right)\omega(\rho) - \sum_{i=1}^{m} \xi_i {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa_i;\varphi} + \mu_i\right)\omega(\rho) = g(\rho, \omega(\rho)) + \Pi(\rho). \tag{47}$$

Lemma 7. *We suppose that* $1 < \kappa \leq 2$, $0 < \alpha \leq 1$ *and* $1 \leq \kappa - \kappa_i < 2$, *and* $\omega \in C(\mho, \mathbb{R})$ *is a solution of the inequality (45). Then,* ω *satisfies*

$$\left|\omega(\rho) - \mathscr{W}(\rho) - \mathfrak{I}^{\alpha+\kappa;\varphi} g(\rho, \omega(\rho))\right| \leq 2\varepsilon \frac{\mathscr{X}_{\varphi}^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)}, \tag{48}$$

where

$$\begin{aligned} \mathscr{W}(\rho) &= \sum_{i=1}^{m} \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi}\omega(\rho) - \sigma \mathfrak{I}^{\kappa;\varphi}\omega(\rho) \\ &\quad + \frac{\mathscr{X}_{\varphi}^{\kappa}(\rho)}{\mathscr{X}_{\varphi}^{\kappa}(1)}\left(\sigma \mathfrak{I}^{\kappa;\varphi}\omega(1) + \sum_{i=1}^{m} \xi_i \mathfrak{I}^{\kappa-\kappa_i;\varphi}\omega(1) + \mathfrak{I}^{\alpha+\kappa;\varphi} g(1, \omega(1))\right). \end{aligned} \tag{49}$$

Proof. We suppose that ω is a solution of (45). By Lemma 6 and (ii) of Remark 4, we have

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa;\varphi} + \mu\right)\omega(\rho) - \sum_{i=1}^{m} \xi_i {}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\kappa_i;\varphi} + \mu_i\right)\omega(\rho) = g(\rho, \omega(\rho)) + \Pi(\rho), \\ \omega(0) = \omega_{\varphi}'(0) = \omega(1) = 0. \end{cases} \tag{50}$$

Then, the solution of FDP (50) is

$$\omega(\rho) = \sum_{i=1}^{m} \xi_i \mathfrak{J}^{\kappa-\kappa_i;\varphi}\omega(\rho) + \mathfrak{J}^{\alpha+\kappa;\varphi} g(\rho,\omega(\rho)) - \sigma\mathfrak{J}^{\kappa;\varphi}\omega(\rho) + \frac{\mathcal{X}_\varphi^\kappa(\rho)}{\mathcal{X}_\varphi^\kappa(1)}\left(\sigma\mathfrak{J}^{\kappa;\varphi}\omega(1) - \sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}\omega(1) - \mathfrak{J}^{\alpha+\kappa;\varphi}g(1,\omega(1))\right) + \mathfrak{J}^{\alpha+\kappa;\varphi}\Pi(\rho) - \frac{\mathcal{X}_\varphi^\kappa(\rho)}{\mathcal{X}_\varphi^\kappa(1)}\mathfrak{J}^{\alpha+\kappa;\varphi}\Pi(1). \tag{51}$$

Again by (i) of Remark 4, it is implied that

$$\left|\omega(\rho) - \mathcal{W}(\rho) - \mathfrak{J}^{\alpha+\kappa;\varphi}g(\rho,\omega(\rho))\right| \le \mathfrak{J}^{\alpha+\kappa;\varphi}|\Pi(\rho)| + \frac{\mathcal{X}_\varphi^\kappa(\rho)}{\mathcal{X}_\varphi^\kappa(1)}\mathfrak{J}^{\alpha+\kappa;\varphi}|\Pi(1)| \le \varepsilon\frac{\mathcal{X}_\varphi^{\alpha+\kappa}(\rho)}{\Gamma(\alpha+\kappa+1)}\left(1+\frac{\mathcal{X}_\varphi^\kappa(\rho)}{\mathcal{X}_\varphi^\kappa(1)}\right) \le 2\varepsilon\frac{\mathcal{X}_\varphi^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)}. \tag{52}$$

□

Theorem 4. *Under the hypotheses of Theorem 3, the solution of the FDP (5) is UH stable.*

Proof. We suppose that $\omega \in C(\mho,\mathbb{R})$ is a solution of the inequality (45) and $v \in C(\mho,\mathbb{R})$ is a unique solution of FDP (5). From Lemma 6, we obtain $v(\rho) = \mathcal{V}(\rho) + \mathfrak{J}^{\alpha+\kappa;\varphi}g(\rho,v(\rho))$, where

$$\mathcal{V}(\rho) = \sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}v(\rho) + \mathfrak{J}^{\alpha+\kappa;\varphi}g(\rho,v(\rho)) - \sigma\mathfrak{J}^{\kappa;\varphi}v(\rho) + \frac{\mathcal{X}_\varphi^\kappa(\rho)}{\mathcal{X}_\varphi^\kappa(1)}\left(\sigma\mathfrak{J}^{\kappa;\varphi}v(1) - \sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}v(1) - \mathfrak{J}^{\alpha+\kappa;\varphi}g(1,v(1))\right). \tag{53}$$

Clearly, if $v(0)=\omega(0)$, $v'_\varphi(0)=\omega'_\varphi(0)$, $v(1)=\omega(1)$, then

$$\begin{aligned}\sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}v(\rho) &= \sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}\omega(\rho),\\ \mathfrak{J}^{\alpha+\kappa;\varphi}g(\rho,v(\rho)) &= \mathfrak{J}^{\alpha+\kappa;\varphi}g(\rho,\omega(\rho)),\\ \mathfrak{J}^{\kappa;\varphi}v(\rho) &= \mathfrak{J}^{\kappa;\varphi}\omega(\rho),\\ \sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}v(1) &= \sum_{i=1}^{m}\xi_i\mathfrak{J}^{\kappa-\kappa_i;\varphi}\omega(1).\end{aligned} \tag{54}$$

Hence, we get $\mathcal{V}(\rho) = \mathcal{W}(\rho)$.

Using Lemma 7 and due to the fact that $|x+y| \le |x| + |y|$, for any $\rho \in \mho$, we have

$$\begin{aligned}|\omega(\rho) - v(\rho)| &= \left|\omega(\rho) - \mathcal{V}(\rho) - \mathfrak{J}^{\alpha+\kappa;\varphi}g(\rho,v(\rho))\right|\\ &\le \left|\omega(\rho) - \mathcal{W}(\rho) - \mathfrak{J}^{\alpha+\kappa;\varphi}g(\rho,\omega(\rho))\right| + \mathfrak{J}^{\alpha+\kappa;\varphi}|g(\rho,\omega(\rho)) - g(\rho,v(\rho))|\\ &\le 2\varepsilon\frac{\mathcal{X}_\varphi^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)} + \mathfrak{J}^{\alpha+\kappa;\varphi}L_g|\omega(\rho) - v(\rho)|\\ &\le 2\varepsilon\frac{\mathcal{X}_\varphi^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)} + \frac{\mathcal{X}_\varphi^{\alpha+\kappa}(1)L_g}{\Gamma(\alpha+\kappa+1)}|\omega(\rho) - v(\rho)|,\end{aligned} \tag{55}$$

which implies

$$\left(1 - \frac{\mathcal{X}_\varphi^{\alpha+\kappa}(1)L_g}{\Gamma(\alpha+\kappa+1)}\right)|\omega(\rho) - v(\rho)| \le 2\varepsilon\frac{\mathcal{X}_\varphi^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1)}. \tag{56}$$

From (41), we get $\mathcal{X}_\varphi^{\alpha+\kappa}(1)L_g/\Gamma(\alpha+\kappa+1) < 1$. It follows that

$$|\omega(\rho) - v(\rho)| \le \Upsilon_f\varepsilon, \tag{57}$$

where

$$\Upsilon_f = \frac{2\mathcal{X}_\varphi^{\alpha+\kappa}(1)}{\Gamma(\alpha+\kappa+1) - \mathcal{X}_\varphi^{\alpha+\kappa}(1)L_g}. \tag{58}$$

Hence, the FDP (5) is UH stable in $C(\mho,\mathbb{R})$. □

Example 1. We consider the generalized Caputo-type Langevin FDP:

$$\begin{cases} {}^{C}\mathfrak{D}^{\frac{1}{4};\varphi}\left({}^{C}\mathfrak{D}^{\frac{7}{4};\varphi}+\frac{5}{6}\right)v(\rho)-\frac{1}{3}{}^{C}\mathfrak{D}^{\alpha;\varphi}\left({}^{C}\mathfrak{D}^{\frac{1}{5};\varphi}+\frac{1}{2}\right)v(\rho)=\frac{1}{100}\left(\cos\rho+\tan^{-1}v(\rho)\right), \quad \rho\in[0,1], \\ v(0)=v_{\varphi}'(0)=v(1)=0, \end{cases} \tag{59}$$

where $\alpha = 1/4, \kappa = 7/4, \mu = 5/6, m = 1, \xi_1 = 1/3, \kappa_1 = 1/5, \mu_1 = 1/2,\ 1 \le \kappa - \kappa_1 = 31/20 < 2, \varphi(\rho) = \rho/3, \mho = [0,1]$, and

$$g(\rho, v) = \cos\rho + \tan^{-1} v. \tag{60}$$

We observe that

$$\begin{aligned} |g(\rho,v_1)-g(\rho,v_2)| &\le \frac{1}{100}\left|\tan^{-1}v_1-\tan^{-1}v_2\right| \le \frac{1}{100}|v_1-v_2| = L_g|v_1-v_2|, \\ \mathcal{X}_{\varphi}^{\kappa}(1) = [\varphi(1)-\varphi(0)]^{\kappa} &= \left[\frac{1}{3}\right]^{\frac{7}{4}}, \quad \sigma=\mu-\xi_1\mu_1=\frac{2}{3}, \end{aligned} \tag{61}$$

$$\sigma = \frac{2}{9\times 3^{11/20}\,\Gamma(51/20)} + \frac{4}{9\times 3^{3/4}\Gamma(11/7)} + \frac{1}{900} < 1. \tag{62}$$

Clearly, all the hypotheses of Theorem 3 hold, and hence, FDP (59) has a unique solution on $[0,1]$. Furthermore, we have

$$|g(\rho,v)| \le \cos\rho + 1 \coloneqq p(\rho) \in C\left([0,1],\mathbb{R}^{+}\right), \tag{63}$$

which satisfies the assumption (H1) of Theorem 2. Moreover, we can find that

$$\gamma_1 = \frac{2}{9\times 3^{11/20}\Gamma(51/20)} + \frac{4}{9\times 3^{3/4}\Gamma(11/7)} < 1. \tag{64}$$

Consequently, by Theorem 1, the FDP (59) has at least one solution on $[0,1]$. Moreover, we have

$$\Upsilon_f = \frac{200}{1799} > 0. \tag{65}$$

From Theorem 3, the FDP (59) is UH stable on $[0,1]$.

5. Conclusions

In this paper, we have given some results dealing with the existence and stability of solutions for two types of fractional boundary value problems for multiterm Langevin equations with generalized Caputo fractional operators of different orders. As an initial step, we obtained the equivalent solutions associated with linear problems by applying the instruments of advanced fractional calculus and characterizing a fixed point problem. Once the fixed point problem is available, the existence and uniqueness theorems are established via Banach's and Sadovskii's fixed point techniques, whereas the guarantee of the existence of solutions has been shown by the vigorous techniques, such as UH stability.

We do not apply any significant bearing to the complex transformations, and our outcomes are characteristic of the integral operators' theory of such kind. Indeed, our methodology is straightforward and can without much of a stretch be applied to an assortment of real-world problems. For the justification of the main results, we have given an example. As a special case, the reported results are new, and we have generalized many results with various values of φ function.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest related to this work.

Acknowledgments

The authors are grateful for the support provided by their institutions.

References

[1] K. B. Oldham and J. Spanier, "The fractional calculus: theory and applications of differentiation and integration to arbitrary order," *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1974.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and applications of fractional differential equations," *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, Netherlands, 2006.

[3] G. A. Anastassiou, "On right fractional calculus," *Chaos, Solitons & Fractals*, vol. 42, no. 1, pp. 365–376, 2009.

[4] A. Atangana, "Convergence and stability analysis of a novel iteration method for fractional biological population equation," *Neural Computing & Applications*, vol. 25, no. 5, pp. 1021–1030, 2014.

[5] W. G. Glöckle and T. F. Nonnenmacher, "A fractional calculus approach to self-similar protein dynamics," *Biophysical Journal*, vol. 68, no. 1, pp. 46–53, 1995.

[6] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[7] D. Kumar, J. Singh, and D. Baleanu, "On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 1, pp. 443–457, 2020.

[8] D. Baleanu, J. H. Asad, and A. Jajarmi, "New aspects of the motion of a particle in a circular cavity," *Proceedings of the Romanian Academy - Series A: Mathematics, Physics, Technical Sciences, Information Science*, vol. 19, no. 2, pp. 361–367, 2018.

[9] H. Baghani, "Existence and uniqueness of solutions to fractional Langevin equations involving 440 two fractional orders," *J. Fixed point Theory Appl*, vol. 20, no. 2, pp. 1–7, 2013.

[10] B. Ahmad, J. J. Nieto, A. Alsaedi, and M. El-Shahed, "A study of nonlinear Langevin equation involving two fractional orders in different intervals," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 2, pp. 599–606, 2012.

[11] O. Baghani, "On fractional Langevin equation involving two fractional orders," *Communications in Nonlinear Science and Numerical Simulation*, vol. 42, pp. 675–681, 2017.

[12] B. Li, S. Sun, and Y. Sun, "Existence of solutions for fractional Langevin equation with in nite-point boundary conditions," *J. Appl. Math. Comput.,* vol. 53, no. 1-2, pp. 683–692, 2017.

[13] H. Fazli and J. J. Nieto, "Fractional Langevin equation with anti-periodic boundary conditions," *Chaos, Solitons & Fractals*, vol. 114, pp. 332–337, 2018.

[14] X.-J. Yang, F. Gao, and H. M. Srivastava, "A new computational approach for solving nonlinear local fractional PDEs," *Journal of Computational and Applied Mathematics*, vol. 339, pp. 285–296, 2018.

[15] X. J. Yang, F. Gao, and H. M. Srivastava, "New rheological models within local fractional derivative," *Romanian Reports in Physics*, vol. 69, no. 3, p. 113, 2017.

[16] X. J. Yang, J. A. Machado, and J. J. Nieto, "A new family of the local fractional PDEs," *Fundamenta Informaticae*, vol. 151, no. 1-4, pp. 63–75, 2017.

[17] T. J. Osler, "Leibniz rule for fractional derivatives generalized and an application to infinite series," *SIAM Journal on Applied Mathematics*, vol. 18, no. 3, pp. 658–674, 1970.

[18] R. Almeida, "A Caputo fractional derivative of a function with respect to another function," *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460–481, 2017.

[19] F. Jarad and T. Abdeljawad, "Generalized fractional derivatives and Laplace transform," *Discrete & Continuous Dynamical Systems-S*, vol. 13, no. 3, 2000.

[20] A. Seemab, M. Rehman, J. Adjabi, and Y. Abjabi, "Langevin equation with nonlocal boundary conditions involving a $ \psi $-Caputo fractional operators of different orders," *AIMS Mathematics*, vol. 6, no. 7, pp. 6749–6780, 2021.

[21] H. A. Wahash, M. S. Abdo, A. M. Saeed, and S. K. Panchal, "Singular fractional differential equations with ψ-Caputo operator and modified Picard's iterative method," *Applied Mathematics E-Notes*, vol. 20, pp. 215–229, 2020.

[22] H. A. Wahash, M. S. Abdo, and S. K. Panchal, "Existence and Ulam-Hyers stability of the implicit fractional boundary value problem with %ψ-Caputo fractional derivative," *Journal of Applied Mathematics and Computational Mechanics*, vol. 19, no. 1, 2020.

[23] C. Derbazi, Z. Baitiche, M. S. Abdo, and T. Abdeljawad, "Qualitative analysis of fractional relaxation equation and coupled system with ψ-Caputo fractional derivative in Banach spaces," *AIMS Math*, vol. 6, pp. 2486–2509, 2021.

[24] A. Boutiara, M. S. Abdo, M. A. Alqudah, and T. Abdeljawad, "On a class of Langevin equations in the frame of Caputo function-dependent-kernel fractional derivatives with antiperiodic boundary conditions," *AIMS Mathematics*, vol. 6, no. 6, pp. 5518–5534, 2021.

[25] A. Suechoei and P. S. Ngiamsunthorn, "Existence uniqueness and stability of mild solutions for semilinear ψ-Caputo fractional evolution equations," *Advances in Difference Equations*, vol. 2020, no. 1, 28 pages, Article ID 114, 2020.

[26] L. Lin, Y. Liu, and D. Zhao, "Controllability of impulsive ψ-caputo fractional evolution equations with nonlocal conditions," *Mathematics*, vol. 9, no. 12, 2021.

[27] D. Yang and C. Bai, "Existence of solutions for anti-periodic fractional differential inclusions involving ψ-Riesz-Caputo fractional derivative," *Fractional Differential Equations, Inclusions and Inequalities with Applications*, vol. 7, no. 7, 2020.

[28] J. V. d. C. Sousa, F. G. Rodrigues, and E. Capelas de Oliveira, "Stability of the fractional Volterra integro-differential equation by means of ψ -Hilfer operator," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 9, pp. 3033–3043, 2019.

[29] D. Vivek, K. Kanagarajan, and E. M. Elsayed, "Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions," *Mediterranean Journal of Mathematics*, vol. 15, no. 1, pp. 1–21, 2018.

[30] R. Almeida, A. B. Malinowska, and M. T. T. Monteiro, "Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 1, pp. 336–352, 2018.

[31] B. Ahmad and J. J. Nieto, "Solvability of nonlinear Langevin equation involving two fractional orders with Dirichlet boundary conditions," *International Journal of Differential Equations*, vol. 2010, Article ID 649486, 10 pages, 2010.

[32] Z. Laadjal, B. Ahmad, and N. Adjeroud, "Existence and uniqueness of solutions for multi-term fractional Langevin equation with boundary conditions," *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, vol. 27, pp. 339–350, 2020.

[33] E. Zeidler, "Nonlinear functional analysis and its applications," *Fixed-point Theorems*, Springer, New York, NY, USA, 1986.

[34] B. N. Sadovskii, "On a fixed point principle," *Functional Analysis and Its Applications*, vol. 1, pp. 74–76, 1967.

[35] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, NY, USA, 2003.