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Research Article

The Continuous and Discrete Stability Characterization of Hepatitis B Deterministic Model

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The hepatitis B infection is a global epidemic disease which is a huge risk to the public health. In this paper, the transmission dynamics of hepatitis B deterministic model are presented and studied. The basic reproduction number is attained and by applying it, the local as well as global stability of disease-free and endemic equilibria of continuous hepatitis B deterministic model are discussed. To better understand the dynamics of the disease, the discrete nonstandard finite difference (NSFD) scheme is produced for the continuous model. Different criteria are employed to check the local and global stability of disease-free and endemic equilibria for the NSFD scheme. Our findings demonstrate that the NSFD scheme is convergent for all step sizes and consequently reasonable in all respect for the continuous deterministic epidemic model. All the aforementioned properties and their effects are also proved numerically at each stage to show their mathematical as well as biological feasibility. The theoretical and numerical findings used in this paper can be employed as a helpful tool for predicting the transmission of other infectious diseases.

1. Introduction

Hepatitis is a general term that means inflammation of the liver. This disease can cause both acute and chronic infections. The acute stage is usually defined as the first six months of virus infection. During this phase, the immune system is capable to manipulate the infection of human body. The two primary indications of the acute stage are feeling sick and having a high temperature, which subsides after few weeks due to the immune system. Chronic disease affects the liver ability to perform life-sustaining processes such as removing dangerous transmitted substances from the blood, collecting sugar levels, and converting it to useable energy forms [1]. Hepatitis B virus (HBV) is one of the world most serious health problems [2]. HBV has a large rate of deaths, both from acute and chronic infection [3]. HBV is spread by blood transfusion and gets transmitted to the newly born child during pregnancy from affected mother. Vaccination is the most enchanting and efficient process in newly born children to decrease the occurrence of HBV [4].

HBV can induce chronic infection which can lead to death from cirrhosis and liver cancer if not treated properly. The HBV fatality rates are among the higher causes of universal deaths [5]. Some pharmacological therapies for chronic HBV have been proposed, including alpha interferon, lamivudine, pegylated interferon, tenofovir disoproxil, entecavir, and telbivudine [6]. During treatment, the viral load is decreased, which reduces virus-related reproduction in the liver [7]. The vaccine against HBV is available since 1982, but still its transmission continues to rise [8, 9]. According to Sheikhan and Ghoreishi [6], HBV can also live beyond the mortal body. HBV can survive on the outer part

of the body for at least seven days, and it can be transmitted to any unimmunized human body during this time.

The use of mathematical modeling helps us to concentrate on the procedure by which an infectious disease spreads throughout an area. Many mathematical models are constructed by researchers from all around the world to understand different types of infectious diseases and their dynamic characteristics. In [10–13], the mathematical models of fractional order derivatives have been employed to investigate and evaluate the transmission of various infectious diseases. The authors not only examined the precise qualitative characteristics of the formulated models but also offered numerical simulations to verify the obtained theoretical findings. In [14, 15], the authors presented vaccination effects on HBV transmission with control strategies by using different age structures in the population. In [16–20], several specialized models of HBV transmission dynamics have been focused on the impact of commitment and control measures like vaccination and antiviral therapy. Din et al. [21] performed a detailed analysis of stability, showing that the reproduction number determines the entire dynamic activities of the system. Recently, in [22], the author discussed and analyzed the stochastic SACR model for HBV transmission and left the deterministic model unsolved. The author investigated the analytical results, including the stability of disease-free and endemic equilibria only for the continuous stochastic model. The purpose of the present work is continuous and discrete characterization of the hepatitis B deterministic model. Different criteria are used to discuss the local as well as global stability of disease-free and endemic equilibria for the continuous deterministic model. The discrete NSFD scheme is constructed for the continuous model to display its sustainability and biological suitability. The NSFD scheme constructed for the model is dynamically consistent with the original system for any step size. Our theoretical and numerical findings indicate that the NSFD scheme retains the essential qualitative characteristics of the continuous model. Consequently, this scheme is not only realistic but also verifies various features of the continuous model. The results acquired through this scheme are very precise and accurate.

The paper is structured as follows: In Section 2, the HBV epidemic model is presented and associated parameters are explained. The existing equilibria and reproduction number are established for the deterministic model in Section 3. By using the reproduction number, the local and global stability of disease-free and endemic equilibria for the continuous model are discussed in Section 4. The discrete NSFD scheme is constructed in Section 5 to analyze the convergence and divergence of disease-free and endemic equilibria for the proposed model. Our calculations show that the NSFD scheme is an effective and powerful technique that presents a clear portrait of the continuous model. The numerical simulations are also provided which strengthen our theoretical results. Finally, a brief conclusion is presented in the last section.

2. Mathematical Model for HBV

In order to define the stochastic HBV disease model with variable population environment, it is required to put some conditions on the epidemic model. It is assumed that the total population $N(t)$ at time t is divided into four classes, i.e., susceptible $S(t)$, acutely infected $A(t)$, chronically infected $C(t)$, and recovered $R(t)$ where $N(t) = S(t) + A(t) + C(t) + R(t)$. The second supposition is that all state variables and parameters of the proposed model are non-negative. The function $\Omega \longrightarrow \Omega + \Omega B$ with $\Omega^2 > 0$ denotes the concentration of white noise, where $B(t)$ shows the normal which satisfy $B(0) = 0$. Based on all above information, the stochastic hepatitis B epidemic model [22] illustrated by the system of four stochastic differential equations is defined as follows:

$$\begin{aligned} dS(t) &= (p - \beta S(t)C(t) - (\tau + \mu_1)S(t))dt - \Omega S(t)A(t)dB(t), \\ dA(t) &= (\beta S(t)C(t) - (\gamma + \mu_1 + \omega)A(t))dt + \Omega S(t)A(t)dB(t), \\ dC(t) &= (\omega A(t) - (\gamma_2 + \mu_1 + \varepsilon)C(t))dt, \\ dR(t) &= (\tau S(t) + \gamma_2 C(t) + \omega A(t) - \mu_1 R(t))dt. \end{aligned} \tag{1}$$

By putting $\Omega = 0$, the model (1) deduces into the following deterministic model:

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= p - \beta S(t)C(t) - (\tau + \mu_1)S(t), \\ \frac{\mathrm{d}A(t)}{\mathrm{d}t} &= \beta S(t)C(t) - (\gamma + \mu_1 + \omega)A(t), \\ \frac{\mathrm{d}C(t)}{\mathrm{d}t} &= \omega A(t) - (\gamma_2 + \mu_1 + \varepsilon)C(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \tau S(t) + \gamma_2 C(t) + \omega A(t) - \mu_1 R(t), \\ \frac{\mathrm{d}C(t)}{\mathrm{d}t} &= \omega A(t) - (\gamma_2 + \mu_1 + \varepsilon)C(t). \end{aligned} \tag{2}$$

Our main aim is continuous and discrete characterization of model (2). It is assumed that all the parameters are positive constants where the parameters and their explanations are provided in Table 1.

As the total population N is denoted by

$$N = S + A + C + R. \tag{3}$$

So, by employing model (2), we attain

$$\frac{dN}{dt} = p - \mu_1 S - \gamma A - \mu_1 A - \mu_1 C + \omega A - \mu_1 R. \tag{4}$$

From previous equation, we can write that

$$\frac{dN}{dt} \leq p - \mu_1 S. \tag{5}$$

Table 1: Parameters included in model (2) and their explanations.

Parameter	Parameter description
p	The constant birth rate
β	The transmission rate
τ	The vaccination rate
μ_1	Natural death rate
γ	The constant recovery rate for acutely infected individuals
ω	The moving rate of acutely infected individuals to chronic stage
γ_2	The disease induced death rate
ε	The constant recovery rate for chronically infected individuals

$$\lim_{t\longrightarrow\infty} \sup N \leq \frac{p}{\mu_1}. \tag{6}$$

Therefore, the feasible region for model (2) becomes

$$K = \left\{(S, A, C, R) \in R^4, N \leq \frac{p}{\mu_1}\right\}. \tag{7}$$

3. Equilibria of the Model and Basic Reproduction Number

3.1. Equilibria of the Model. The following two equilibria exist for model (2):

3.1.1. Disease Free Equilibrium (DFE) Point. To find DFE point, we take all other classes equal to zero except the susceptible class, i.e., if $A(t) = 0, C(t) = 0, R(t) = 0$, then we get $S(t) = p/\tau + \mu_1$. Therefore, the DFE point denoted by $E^0(S^0, A^0, C^0, R^0)$ becomes $E^0 = (p/t + \mu_1, 0, 0, 0)$.

3.1.2. Disease Endemic Equilibrium (DEE) Point. To find DEE point, we simultaneously solve the proposed model (2) for S, A, C and R. If the DEE point is denoted by $E^*(S^*, A^*, C^*, R^*)$, then from model (2), we get

$$\begin{aligned} S^*(t) &= \frac{(\gamma + \mu_1 + \omega)A^*(t)}{\beta(\gamma_2 + \mu_1 + \varepsilon)C^*(t)}, \\ A^*(t) &= \frac{(\gamma_2 + \mu_1 + \varepsilon)C^*(t)}{\omega}, \\ C^*(t) &= \frac{\omega A^*(t)}{(\gamma_2 + \mu_1 + \varepsilon)}, \\ R^*(t) &= \frac{p}{(\tau + \mu_1)\mu_1}. \end{aligned} \tag{8}$$

3.2. The Basic Reproduction Number ($\mathbf{R_0}$). The quantity R_0 is the most crucial threshold related to any infectious disease. It assists to find out whether an infectious disease will transmit through population or not [23]. If $R_0 < 1$ throughout its infectious period, then infection does not grow. On the other hand, if $R_0 > 1$ then infection grows and disease remains in the population. To obtain R_0, we employ transmission and translation matrices $F(x)$ and $V(x)$, respectively. The previously discussed matrices can be demonstrated as

$$F(x) = \begin{bmatrix} \beta S(t)C(t) \\ 0 \end{bmatrix} \text{and } \mathscr{V}(x) = \begin{bmatrix} -(\gamma + \mu_1 + \omega)A(t) \\ \omega A(t) - (\gamma_2 + \mu_1 + \varepsilon)C(t) \end{bmatrix}. \tag{9}$$

From the abovementioned matrices, we get

$$\begin{aligned} F &= \begin{bmatrix} 0 & \dfrac{\beta p}{\tau + \mu_1} \\ 0 & 0 \end{bmatrix}, \\ V &= \begin{bmatrix} (\gamma + \mu_1 + \omega) & 0 \\ -\omega & (\gamma_2 + \mu_1 + \varepsilon) \end{bmatrix}. \end{aligned} \tag{10}$$

As we know that

$$R_0 = \rho\left(FV^{-1}\right). \tag{11}$$

Therefore, using F and V, we obtain

$$R_0 = \frac{\beta p \omega}{(\gamma + \mu_1 + \omega)(\gamma_2 + \mu_1 + \varepsilon)(\tau + \mu_1)}. \tag{12}$$

4. Local and Global Stability of DFE and DEE Points for the Deterministic Model

In the following section, we first discuss the local and global stability of DFE point for the deterministic HBV disease model (2):

4.1. Local and Global Stability of DFE Point. To discuss the local stability, we assume

$$\begin{aligned} F &= p - \beta S(t)C(t) - (\tau + \mu_1)S(t), \\ G &= \beta S(t)C(t) - (\gamma + \mu_1 + \omega)A(t), \\ H &= \omega A(t) - (\gamma_2 + \mu_1 + \varepsilon)C(t), \\ I &= \tau S(t) + \gamma_2 C(t) + \omega A(t) - \mu_1 R(t). \end{aligned} \tag{13}$$

In the following theorem, we first discuss the local stability of DFE point by using Routh–Hurwitz criterion [24, 25]:

Theorem 1. *The DFE point E^0 of model (2) is locally asymptotically stable whenever $R_0 < 1$.*

Proof. Let us take the Jacobian matrix as follows:

$$J = \begin{bmatrix} \frac{\partial F}{\partial S} & \frac{\partial F}{\partial A} & \frac{\partial F}{\partial C} & \frac{\partial F}{\partial R} \\ \frac{\partial G}{\partial S} & \frac{\partial G}{\partial A} & \frac{\partial G}{\partial C} & \frac{\partial G}{\partial R} \\ \frac{\partial H}{\partial S} & \frac{\partial H}{\partial A} & \frac{\partial H}{\partial C} & \frac{\partial H}{\partial R} \\ \frac{\partial I}{\partial S} & \frac{\partial I}{\partial A} & \frac{\partial I}{\partial C} & \frac{\partial I}{\partial R} \end{bmatrix}. \tag{14}$$

We first find all the derivatives included in (14) as follows:

$$\begin{aligned} &\frac{\partial F}{\partial S} = -\beta C(t) - (\tau + \mu_1), \frac{\partial F}{\partial A} = 0, \frac{\partial F}{\partial C} = -\beta S(t), \frac{\partial F}{\partial R} = 0, \frac{\partial G}{\partial S} = \beta C(t), \frac{\partial G}{\partial A} = -(\gamma + \mu_1 + \omega), \\ &\frac{\partial G}{\partial C} = \beta S(t), \frac{\partial G}{\partial R} = 0, \frac{\partial H}{\partial S} = 0, \frac{\partial H}{\partial A} = \omega, \frac{\partial H}{\partial C} = -(\gamma_2 + \mu_1 + \varepsilon), \frac{\partial H}{\partial R} = 0, \frac{\partial I}{\partial S} = \tau, \frac{\partial I}{\partial A} = \omega, \frac{\partial I}{\partial C} = \gamma_2, \\ &\frac{\partial I}{\partial R} = -\mu_1. \end{aligned} \tag{15}$$

By replacing all the derivatives in (14), we get

$$J = \begin{bmatrix} -\beta C(t) - (\tau + \mu_1) & 0 & -\beta S(t) & 0 \\ \beta C(t) & -(\gamma + \mu_1 + \omega) & \beta S(t) & 0 \\ 0 & \omega & -(\gamma_2 + \mu_1 + \varepsilon) & 0 \\ \tau & \omega & \gamma_2 & -\mu_1 \end{bmatrix}. \tag{16}$$

By putting DFE point $E^0 = (p/t + \mu_1, 0, 0, 0)$, we get

$$J(E^0) = \begin{bmatrix} -(\tau + \mu_1) & 0 & -\frac{\beta p}{(\tau + \mu_1)} & 0 \\ 0 & -(\gamma + \mu_1 + \omega) & \frac{\beta p}{(\tau + \mu_1)} & 0 \\ 0 & \omega & -(\gamma_2 + \mu_1 + \varepsilon) & 0 \\ \tau & \omega & \gamma_2 & -\mu_1 \end{bmatrix}. \tag{17}$$

In order to find eigenvalues, we consider

$$\begin{vmatrix} -(\tau+\mu_1)-\lambda & 0 & -\dfrac{\beta p}{(\tau+\mu_1)} & 0 \\ 0 & -(\gamma+\mu_1+\omega)-\lambda & \dfrac{\beta p}{(\tau+\mu_1)} & 0 \\ 0 & \omega & -(\gamma_2+\mu_1+\varepsilon)-\lambda & 0 \\ \tau & \omega & \gamma_2 & -\mu_1-\lambda \end{vmatrix} = 0. \tag{18}$$

The characteristic equation for the abovementioned equation becomes

$$\left(-(\tau+\mu_1)-\lambda\right)(-\mu_1-\lambda)\left(\lambda^2+\lambda P+Q\right)=0, \tag{19}$$

where

$$\begin{aligned} P &= (\gamma+\mu_1+\omega)+(\gamma_2+\mu_1+\varepsilon), \\ Q &= (\gamma+\mu_1+\omega)(\gamma_2+\mu_1+\varepsilon)(1-R_0). \end{aligned} \tag{20}$$

The two negative roots of (19) are $\lambda_1 = -\mu_1$ and $\lambda_2 = -(\tau+\mu_1)$. Also, it is clear that $P>0$ and $Q>0$ whenever $R_0<1$. So, by using Routh–Hurwitz criterion, the other two roots of $\lambda^2+\lambda P+Q=0$ must have negative real parts. Therefore, we deduce that E^0 is locally asymptotically stable for $R_0<1$. This completes the proof. □

Theorem 2. *The DFE point E^0 of model (2) is globally asymptotically stable whenever $R_0 \le 1$.*

Proof. In order to demonstrate the global stability of DFE point E^0 of model (2), we construct the Lyapunov function as

$$L = (S-S_0)+A+C+R. \tag{21}$$

From (21), the following can easily be obtained:

$$\frac{dL}{dt} = \frac{dS}{dt}+\frac{dA}{dt}+\frac{dC}{dt}+\frac{dR}{dt}. \tag{22}$$

After simple calculations, we get

$$\begin{aligned} \frac{dL}{dt} &= p-(\tau+\mu_1)S-(\gamma+\mu_1)A-(\gamma_2+\mu_1+\varepsilon)C-\mu_1 R(t), \\ \frac{dL}{dt} &= -(\tau+\mu_1)(S-S_0)-(\gamma+\mu_1)A-(\gamma_2+\mu_1+\varepsilon)C-\mu_1 R(t), \\ \frac{dL}{dt} &= -\left((\tau+\mu_1)(S-S_0)+(\gamma+\mu_1)A+(\gamma_2+\mu_1+\varepsilon)C+\mu_1 R(t)\right)\le 0. \end{aligned} \tag{23}$$

Thus, $dL/dt \le 0$ for $R_0 \le 1$. Also note that $dL/dt = 0$ if and only if $S = S_0$ and $A = C = R = 0$. Hence, Labzai et al. [26] imply that E^0 is globally asymptotically stable, as shown in Figure 1(a). □

4.2. Local and Globally Stability of DEE Point

Theorem 3. *The DEE point E^* of model (2) is locally asymptotically stable whenever $R_0 > 1$.*

Proof. In the similar way as in Theorem 1, the Jacobian matrix can be written as

$$J = \begin{bmatrix} -\beta C(t)-(\tau+\mu_1) & 0 & -\beta S(t) & 0 \\ \beta C(t) & -(\gamma+\mu_1+\omega) & \beta S(t) & 0 \\ 0 & \omega & -(\gamma_2+\mu_1+\varepsilon) & 0 \\ \tau & \omega & \gamma_2 & -\mu_1 \end{bmatrix}. \tag{24}$$

By putting DEE point E^*, we get

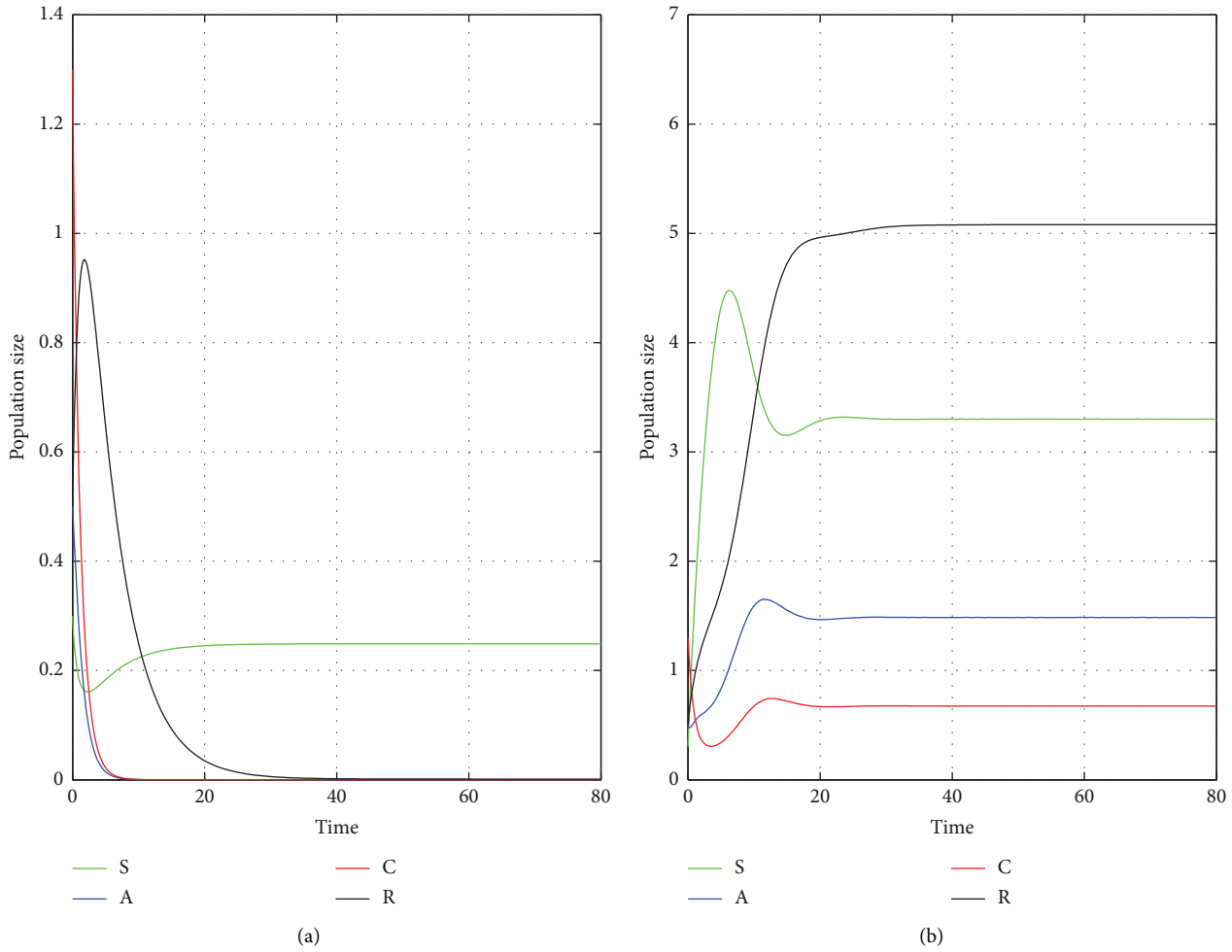

FIGURE 1: The solutions of HBV model (2) obtained through ODE-45. (a) Stable DFE point whenever $R_0 \leq 1$ and $p = 0.05$, (b) stable DEE point whenever $R_0 \geq 1$ and $p = 2$. Other parameters remain fixed, as $\beta = 0.60, \gamma = 0.2, \omega = 0.5, \tau = 0.001, \mu_1 = 0.2, \gamma_2 = 0.4, \varepsilon = 0.5$.

$$J(E^*) = \begin{bmatrix} \dfrac{-\beta\omega A^*(t)}{(\gamma_2+\mu_1+\varepsilon)} - (\tau+\mu_1) & 0 & -\dfrac{\beta(\gamma+\mu_1+\omega)A^*(t)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)} & 0 \\ \dfrac{\beta\omega A^*(t)}{(\gamma_2+\mu_1+\varepsilon)} & -(\gamma+\mu_1+\omega) & \dfrac{\beta(\gamma+\mu_1+\omega)A^*(t)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)} & 0 \\ 0 & \omega & -(\gamma_2+\mu_1+\varepsilon) & 0 \\ \tau & \omega & \gamma_2 & -\mu_1 \end{bmatrix}. \tag{25}$$

In order to find the eigenvalues, we consider

$$\begin{vmatrix} \frac{-\beta\omega A^*(t)}{(\gamma_2+\mu_1+\varepsilon)} - (\tau+\mu_1) - \lambda & 0 & -\frac{\beta(\gamma+\mu_1+\omega)A^*(t)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)} & 0 \\ \frac{\beta\omega A^*(t)}{(\gamma_2+\mu_1+\varepsilon)} & -(\gamma+\mu_1+\omega) - \lambda & \frac{\beta(\gamma+\mu_1+\omega)A^*(t)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)} & 0 \\ 0 & \omega & -(\gamma_2+\mu_1+\varepsilon) - \lambda & 0 \\ \tau & \omega & \gamma_2 & -\mu_1 - \lambda \end{vmatrix} = 0. \tag{26}$$

The characteristic equation of abovementioned equation becomes $(-\mu_1 - \lambda)\ (\lambda^3 + B\lambda^2 + D\lambda + E) = 0$.

The abovementioned equation gives one negative eigenvalue $\lambda_1 = -\mu_1$. The other eigenvalues can be obtained from

$$\lambda^3 + B\lambda^2 + D\lambda + E = 0, \tag{27}$$

where

$$\begin{aligned} B &= \frac{\beta\omega A^*(t)}{(\gamma_2+\mu_1+\varepsilon)} + (\tau+\mu_1) + (\gamma+\mu_1+\omega) + (\gamma_2+\mu_1+\varepsilon), \\ D &= \frac{\beta\omega A^*(t)(\gamma+\mu_1+\omega)}{(\gamma_2+\mu_1+\varepsilon)} + \beta\omega A^*(t) + (\tau+\mu_1)(\gamma+\mu_1+\omega) \\ &\quad + (\tau+\mu_1)(\gamma_2+\mu_1+\varepsilon) + (\gamma+\mu_1+\ \omega)(\gamma_2+\mu_1+\varepsilon) + \frac{\beta\omega(\gamma+\mu_1+\omega)A^*(t)(\tau+\mu_1)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)}. \end{aligned} \tag{28}$$

$$E = \frac{\beta\omega A^*(t)(\tau+\mu_1)(\gamma+\mu_1+\omega)A^*(t)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)}\left(1 - \frac{(\gamma+\mu_1+\omega)A^*(t)}{p}R_0\right) > 0. \tag{29}$$

It is clear that $B, D, E > 0$ whenever $R_0 > 1$. Also,

$$\begin{aligned} BD - E &= \left(\frac{\beta\omega A^*(t)}{(\gamma_2+\mu_1+\varepsilon)} + (\tau+\mu_1) + (\gamma+\mu_1+\omega) + (\gamma_2+\mu_1+\varepsilon)\right)\left(\frac{\beta\omega A^*(t)(\gamma+\mu_1+\omega)}{(\gamma_2+\mu_1+\varepsilon)} + \beta\omega A^*(t)\right. \\ &\quad \left. + (\tau+\mu_1)(\gamma+\mu_1+\omega) + (\tau+\mu_1)(\gamma_2+\mu_1+\varepsilon) + (\gamma+\mu_1+\omega)(\gamma_2+\mu_1+\varepsilon) + \frac{\beta\omega(\gamma+\mu_1+\omega)A^*(t)(\tau+\mu_1)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)}\right) \\ &\quad - \left(\frac{\beta\omega A^*(t)(\tau+\mu_1)(\gamma+\mu_1+\omega)A^*(t)}{(\gamma_2+\mu_1+\varepsilon)C^*(t)}\left(1 - \frac{(\gamma+\mu_1+\omega)A^*(t)}{p}R_0\right)\right) > 0. \end{aligned} \tag{30}$$

Hence, by applying Routh–Hurwitz criterion, all the solutions of (27) must have negative real parts if and only if $R_0 > 1$. Therefore, E^* is locally asymptotically stable whenever $R_0 > 1$. □

Theorem 4. *The DEE point E^* of model (2) is globally asymptotically stable whenever $R_0 \geq 1$.*

Proof. In order to demonstrate the global stability of DEE point E^* of model (2), we construct the Lyapunov function as

$$\Psi = \frac{1}{2}\left[\left(S - S^*\right) + \left(A - A^*\right) + \left(C - C^*\right) + \left(R - R^*\right)\right]^2. \quad (31)$$

Now, we calculate the derivative with respect to the time of (31) and then using model (2), we get

$$\frac{d\Psi}{dt} = \left[\left(S - S^*\right) + \left(A - A^*\right) + \left(C - C^*\right)\right] * \left[p - (\tau + \mu_1)S - (\gamma + \mu_1)A - (\gamma_2 + \mu_1 + \varepsilon)C - \mu_1 R\right]. \quad (32)$$

If we put $\Phi_1 = (\tau + \mu_1), \Phi_2 = (\gamma + \mu_1), \Phi_3 = (\gamma_2 + \mu_1 + \varepsilon), \Phi_4 = \mu_1$. Then, after simple arrangement, we get

$$\begin{aligned}
\frac{d\Psi}{dt} &= \left[\left(S - S^*\right) + \left(A - A^*\right) + \left(C - C^*\right) + \left(R - R^*\right)\right] * \left[-S^* R_0 \Phi_1 - S\Phi_1 - \Phi_2 A - \Phi_3 C - \Phi_4 R\right], \\
\frac{d\Psi}{dt} &= -\left[\left(S - S^*\right) + \left(A - A^*\right) + \left(C - C^*\right) + \left(R - R^*\right)\right] * \left[S^* R_0 \Phi_1 + S\Phi_1 + \Phi_2 A + \Phi_3 C + \Phi_4 R\right], \\
\frac{d\Psi}{dt} &= -\left[\left(S - S^*\right) + \left(A - A^*\right) + \left(C - C^*\right) + \left(R - R^*\right)\right] * \left[\left(S^* R_0 + S\right)\Phi_1 + \Phi_2 A + \Phi_3 C + \Phi_4 R\right].
\end{aligned} \quad (33)$$

Since the right-hand side of (33) has a negative sign, so the derivative on right hand side is less than or equal to zero, i.e., $d\Psi/dt \leq 0$. Substituting $S = S^*, A = A^*, C = C^*, R = R^*$ in (33), $d\Psi/dt$ yields zero, i.e., $d\Psi/dt = 0$. Therefore, the largest invariant set in $\{(S, A, C, R) \epsilon R^4 : d\Psi/dt = 0\}$ is the singleton invariant set E^*, where E^* is the DEE point. Then, by applying invariant principle of LaSalle et al. [27], it implies that E^* is globally asymptotically stable, as shown in Figure 1(b). □

5. The NSFD Scheme

The main objective of this subsection is to develop a dynamically reliable discrete NSFD scheme for system (2). The NSFD scheme has been taken successfully to a variety of challenges, including ecology [28, 29], epidemiology [30, 31], and population models [32, 33]. To develop the NSFD scheme for system (2), we use S_n, A_n, C_n, and R_n as numerical approximations of $S(t)$, $A(T), C(t)$,and $R(t)$ at $t = nh$, where $n = 0, 1, 2 \ldots$, and h denotes the time-step size. By applying the concept of Mickens [34], we can discretize model (2) as follows:

$$\begin{aligned}
\frac{S_{n+1} - S_n}{h} &= p - \beta S_{n+1}(t) C_n(t) - (\tau + \mu_1) S_{n+1}(t), \\
\frac{A_{n+1} - A_n}{h} &= \beta S_{n+1}(t) C_n(t) - (\gamma + \mu_1 + \omega) A_{n+1}(t), \\
\frac{C_{n+1} - C_n}{h} &= \omega A_n(t) - (\gamma_2 + \mu_1 + \varepsilon) C_{n+1}(t), \\
\frac{R_{n+1} - R_n}{h} &= \tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t) - \mu_1 R_{n+1}(t).
\end{aligned} \quad (34)$$

The discrete NSFD model (10) can be rearranged to get explicit form as

$$\begin{aligned} S_{n+1} &= \frac{S_n + hp}{\left(1 + h\left(\beta C_n(t) + h\left(\tau + \mu_1\right)\right)\right)}, \\ A_{n+1} &= \frac{A_n + h\beta C_n(t) S_{n+1}(t)}{\left(1 + h\left(\gamma + \mu_1 + \omega\right)\right)}, \\ C_{n+1} &= \frac{C_n + h\omega A_n(t)}{\left(1 + h\left(\gamma_2 + \mu_1 + \varepsilon\right)\right)}, \\ R_{n+1} &= \frac{R_n + h\left(\tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t)\right)}{\left(1 + h\mu_1\right)}. \end{aligned} \tag{35}$$

5.1. Local and Global Stability of DFE Point for NSFD Scheme. To obtain the local stability of the DFE point, we assume that

$$\begin{aligned} F_1 &= S_{n+1} = \frac{S_n + hp}{\left(1 + h\left(\beta C_n(t) + h\left(\tau + \mu_1\right)\right)\right)}, \\ F_2 &= A_{n+1} = \frac{A_n + h\beta C_n(t) S_{n+1}(t)}{\left(1 + h\left(\gamma + \mu_1 + \omega\right)\right)}, \\ F_3 &= C_{n+1} = \frac{C_n + h\omega A_n(t)}{\left(1 + h\left(\gamma_2 + \mu_1 + \varepsilon\right)\right)}, \\ F_4 &= R_{n+1} = \frac{R_n + h\left(\tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t)\right)}{\left(1 + h\mu_1\right)}. \end{aligned} \tag{36}$$

Theorem 5. *For all $h > 0$, the DFE point E^0 of the NSFD model (11) is locally asymptotically stable whenever $R_0 < 1$.*

Proof. Let us take the Jacobian matrix as follows:

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial A} & \frac{\partial F_1}{\partial C} & \frac{\partial F_1}{\partial R} \\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial A} & \frac{\partial F_2}{\partial C} & \frac{\partial F_2}{\partial R} \\ \frac{\partial F_3}{\partial S} & \frac{\partial F_3}{\partial A} & \frac{\partial F_3}{\partial C} & \frac{\partial F_3}{\partial R} \\ \frac{\partial F_4}{\partial S} & \frac{\partial F_4}{\partial A} & \frac{\partial F_4}{\partial C} & \frac{\partial F_4}{\partial R} \end{bmatrix}. \tag{37}$$

First, we find all the derivatives of matrix (13) as follows:

$$\begin{aligned} &\frac{\partial \mathrm{F}_1}{\partial \mathrm{S}} = \frac{1}{1 + h(\beta C_n(t) + (\tau + \mu_1))}, \\ &\frac{\partial \mathrm{F}_1}{\partial \mathrm{A}} = 0, \frac{\partial \mathrm{F}_1}{\partial \mathrm{C}} = \frac{-(S_n + hp)h\beta}{\left(1 + h\left(\beta C_n(t) + \tau + \mu_1\right)\right)^2}, \\ &\frac{\partial \mathrm{F}_1}{\partial \mathrm{R}} = 0, \frac{\partial \mathrm{F}_2}{\partial \mathrm{S}} = \frac{h\beta C_n(t)}{1 + h\left(\gamma + \mu_1 + \omega\right)}, \\ &\frac{\partial \mathrm{F}_2}{\partial \mathrm{A}} = \frac{1}{1 + h\left(\gamma + \mu_1 + \omega\right)}, \\ &\frac{\partial \mathrm{F}_2}{\partial \mathrm{C}} = \frac{h\beta S_{n+1}(t)}{1 + h\left(\gamma + \mu_1 + \omega\right)}, \frac{\partial \mathrm{F}_2}{\partial \mathrm{R}} = 0, \\ &\frac{\partial \mathrm{F}_3}{\partial \mathrm{S}} = 0, \\ &\frac{\partial \mathrm{F}_3}{\partial \mathrm{A}} = \frac{h\omega}{1 + h\left(\gamma_2 + \mu_1 + \varepsilon\right)}, \\ &\frac{\partial \mathrm{F}_3}{\partial \mathrm{C}} = \frac{1}{1 + h\left(\gamma_2 + \mu_1 + \varepsilon\right)}, \\ &\frac{\partial \mathrm{F}_3}{\partial \mathrm{R}} = 0, \\ &\frac{\partial \mathrm{F}_4}{\partial \mathrm{S}} = \frac{h\tau}{1 + h\mu_1}, \\ &\frac{\partial \mathrm{F}_4}{\partial \mathrm{A}} = \frac{h\omega}{1 + h\mu_1}, \\ &\frac{\partial \mathrm{F}_4}{\partial \mathrm{C}} = \frac{h\gamma_2}{1 + h\mu_1}, \\ &\frac{\partial \mathrm{F}_4}{\partial \mathrm{R}} = \frac{1}{1 + h\mu_1}. \end{aligned} \tag{38}$$

By replacing all the derivatives in (37), we get

$$J=\begin{bmatrix} \frac{1}{1+h(\beta C_n(t)+(\tau+\mu_1))} & 0 & \frac{-(S_n+hp)h\beta}{(1+h(\beta C_n(t)+\tau+\mu_1))^2} & 0 \\ \frac{h\beta C_n(t)}{1+h(\gamma+\mu_1+\omega)} & \frac{1}{1+h(\gamma+\mu_1+\omega)} & \frac{h\beta S_{n+1}(t)}{1+h(\gamma+\mu_1+\omega)} & 0 \\ 0 & \frac{h\omega}{1+h(\gamma_2+\mu_1+\varepsilon)} & \frac{1}{1+h(\gamma_2+\mu_1+\varepsilon)} & 0 \\ \frac{h\tau}{1+h\mu_1} & \frac{h(\omega)}{1+h\mu_1} & \frac{h\gamma_2}{1+h\mu_1} & \frac{1}{1+h\mu_1} \end{bmatrix}. \tag{39}$$

After putting the DFE point E^0, we get

$$J(E^0)=\begin{bmatrix} \frac{1}{1+h(\tau+\mu_1)} & 0 & \frac{-(p/t+\mu_1+hp)h\beta}{(1+h(\tau+\mu_1))^2} & 0 \\ 0 & \frac{1}{1+h(\gamma+\mu_1+\omega)} & \frac{h\beta p}{(1+h(\gamma+\mu_1+\omega))(\tau+\mu_1)} & 0 \\ 0 & \frac{h\omega}{1+h(\gamma_2+\mu_1+\varepsilon)} & \frac{1}{1+h(\gamma_2+\mu_1+\varepsilon)} & 0 \\ \frac{h\tau}{1+h\mu_1} & \frac{h(\omega)}{1+h\mu_1} & \frac{h\gamma_2}{1+h\mu_1} & \frac{1}{1+h\mu_1} \end{bmatrix}. \tag{40}$$

In order to find the eigenvalues, we consider

$$\begin{vmatrix} \frac{1}{1+h(\tau+\mu_1)}-\lambda & 0 & \frac{-(p/t+\mu_1+hp)h\beta}{(1+h(\tau+\mu_1))^2} & 0 \\ 0 & \frac{1}{1+h(\gamma+\mu_1+\omega)}-\lambda & \frac{h\beta p}{(1+h(\gamma+\mu_1+\omega))(\tau+\mu_1)} & 0 \\ 0 & \frac{h\omega}{1+h(\gamma_2+\mu_1+\varepsilon)} & \frac{1}{1+h(\gamma_2+\mu_1+\varepsilon)}-\lambda & 0 \\ \frac{h\tau}{1+h\mu_1} & \frac{h(\omega)}{1+h\mu_1} & \frac{h(\gamma_2)}{1+h\mu_1} & \frac{1}{1+h\mu_1}-\lambda \end{vmatrix} = 0. \tag{41}$$

The abovementioned equation gives the following characteristic equation:

$$\left(\frac{1}{(1+h\mu_1)}-\lambda\right)\left(\frac{1}{(1+h(\tau)+\mu_1)}-\lambda\right)\left(\lambda^2+L\lambda+M\right)=0, \quad (42)$$

where

$$L=\frac{1}{1+h(\gamma+\mu_1+\omega)}+\frac{1}{1+h(\gamma_2+\mu_1+\varepsilon)},$$
$$M=\frac{h\beta p\omega}{(1+h(\gamma_2+\mu_1+\varepsilon))(1+h(\gamma+\mu_1+\omega))(\tau+\mu_1)}. \quad (43)$$

The two roots of the (42) are $\lambda_1 = 1/1 + h\mu_1 < 1$ and $\lambda_2 = 1/1 + h(\tau + \mu_1) < 1$. Also, it is clear that $L > 0$ and $M > 0$ whenever $R_0 < 1$. So, by using Routh–Hurwitz criterion, the other two roots of $(\lambda^2 + \lambda L + M) = 0$ must have negative real parts. Therefore, we conclude that the DFE point E^0 of the discrete NSFD model (11) is locally asymptotically stable whenever $R_0 < 1$. This completes the proof.

In the following theorem, we now prove the global stability of the DFE point E^0. To prove it, we use the criterion employed by Vaz and Torres [35]. □

Theorem 6. *For all $h > 0$, the DFE point E^0 of the NSFD model (11) is globally asymptotically stable whenever $R_0 \le 1$, as shown in Figures 2(a)–2(d).*

Proof. If we choose $\varepsilon > 0$, then there exists an integer n_0, such that for any $n \ge n_0$, $S_{n+1} < p/(\tau + \mu_1) + \varepsilon$. We consider the sequence $\{T(n)\}$ defined by

$$T(n) = h\beta S_{n+1}C_n + A_n + \frac{\omega}{C_3}C_n + \frac{\mu_1}{C_2}R_n, \quad (44)$$

where $C_2 = (\gamma_2 + \mu_1 + \varepsilon)$, $C_3 = (\gamma + \mu_1 + \omega)$. From abovementioned equation, we can write that

$$\begin{aligned} T(n+1)-T(n) &= h\beta S_{n+2}C_n + A_{n+1} + \frac{\omega}{C_3}C_{n+1} + \frac{\mu_1}{C_2}R_{n+1} - h\beta S_{n+1}C_n - A_n - \frac{\omega}{C_3}C_n - \frac{\mu_1}{C_2}R_n, \\ &= h\beta S_{n+2}C_n + (A_{n+1}-A_n) + \frac{\omega}{C_3}(C_{n+1}-C_n) + \frac{\mu_1}{C_2}(R_{n+1}-R_n). \end{aligned} \quad (45)$$

After simple calculations, we obtain

$$\begin{aligned} &= h\beta S_{n+2}C_n + h\left(\beta S_{n+1}(t)C_n(t) - (\gamma+\mu_1+\omega)A_{n+1}(t)\right) + \frac{\omega}{C_3}h\left(\omega A_n(t) - (\gamma_2+\mu_1+\varepsilon)C_{n+1}(t)\right) \\ &\quad + \frac{\mu_1}{C_2}h\left(\tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t) - \mu_1 R_{n+1}(t)\right) - h\beta S_{n+1}C_n. \end{aligned} \quad (46)$$

$$\begin{aligned} &= h\beta S_{n+2}C_n - h(\gamma+\mu_1+\omega)A_{n+1}(t) + \frac{\omega}{C_3}h\left(\omega A_n(t) - (\gamma_2+\mu_1+\varepsilon)C_{n+1}(t)\right) \\ &\quad + \frac{\mu_1}{C_2}h\left(\tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t) - \mu_1 R_{n+1}(t)\right) \end{aligned} \quad (47)$$

Let $C_1 = (\gamma + \mu_1 + \omega)$, then

$$\begin{aligned} &= h\beta S_{n+2}C_n - hC_1A_{n+1}(t) + \frac{\omega}{C_3}h\left(\omega A_n(t) - (\gamma_2+\mu_1+\varepsilon)C_{n+1}(t)\right) + \frac{\mu_1}{C_2}h\left(\tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t) - \mu_1 R_{n+1}(t)\right) \\ &= h\left(\beta S_{n+2}C_n - C_1A_{n+1}(t) + \frac{\omega}{C_3}\left(\omega A_n(t) - (\gamma_2+\mu_1+\varepsilon)C_{n+1}(t)\right) + \frac{\mu_1}{C_2}\left(\tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t) - \mu_1 R_{n+1}(t)\right)\right) \\ &= h\left(\beta S_{n+2}C_n + (\omega - C_1)A_{n+1}(t) + \left(\frac{\mu_1}{C_2}\gamma_2 - \frac{\omega}{C_3}(\gamma_2+\mu_1+\varepsilon)\right)C_{n+1}(t) + \frac{\mu_1}{C_2}\left(\tau S_{n+1}(t) - \mu_1 R_{n+1}(t)\right)\right). \end{aligned} \quad (48)$$

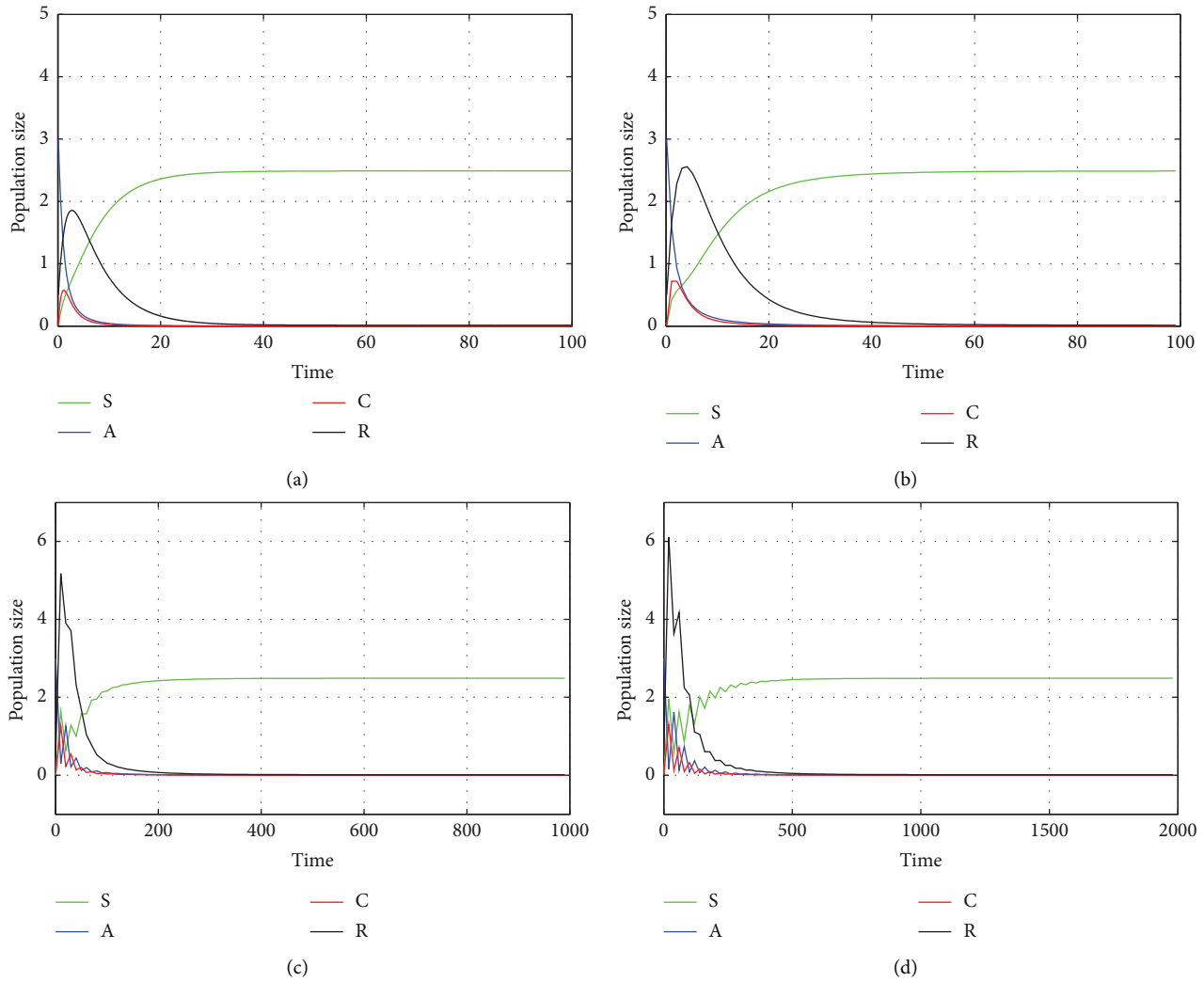

Figure 2: The solutions of the HBV model (2) obtained through the NSFD scheme whenever $R_0 \leq 1$ and (a) $h = 0.01$, (b) $h = 1$, (c) $h = 10$ and (d) $h = 20$. Other parameters remain fixed as $p = 0.5, \beta = 0.60, \gamma = 0.2, \omega = 0.5, \tau = 0.001, \mu_1 = 0.2, \gamma_2 = 0.4, \varepsilon = 0.5$.

If $U = \mu_1\gamma_2C_3 - C_2\omega(\gamma_2 + \mu_1 + \varepsilon)$, then

$$= h\left(\beta S_{n+2}C_n - (\mu_1\gamma_2C_3 - C_2\omega(\gamma_2 + \mu_1 + \varepsilon))C_{n+1}(t) + (\omega - C_1)A_{n+1}(t) + \frac{\mu_1}{C_2}(\tau S_{n+1}(t) - UC_{n+1}(t) - \mu_1R_{n+1}(t))\right). \quad (49)$$

We can select β, a very small positive number such that

$$\beta S_{n+2}C_n \le q\left(A_{n+1} + C_1 C_{n+1} + C_2 R_{n+1}\right). \tag{50}$$

After simple rearrangement, we can write that

$$\begin{aligned}
\beta S_{n+2}C_n &\le q\left(A_{n+1} + C_1 A_{n+1} + C_2 A_{n+1}\right),\\
&\le q\left(\left(A_{n+1} + C_1\frac{\omega}{C_2}A_{n+1} - C_2\left(\frac{U\omega\gamma_2}{C_2} + \frac{\omega\gamma_2}{C_2}\right)A_{n+1}\right) - U\frac{A_{n+1}}{C_3}\right),\\
&\le q\left(A_{n+1}\left(\left(1 + C_1\frac{\omega}{C_2} + C_2\frac{\omega}{C_2}\gamma_2\right) - \frac{U}{C_3}\right)\right),\\
&\le q\left(A_{n+1}\left(1 + \frac{\beta p(\tau+\mu_1)C_1R_0}{\beta p} - (\omega\gamma_2(U+1)) - \frac{U}{C_3}\right)\right).
\end{aligned} \tag{51}$$

If $R_0 \le 1$, and because it is imprecise, we reach the conclusion that $T(n+1) - T(n) \le 0$ and $\lim_{n\longrightarrow\infty} I_n = 0$ for any $n \ge 0$. The sequence $\{T(n)\}_{n=0}^{\infty}$ is a monotonic decreasing and $\lim_{n\longrightarrow\infty} S_n = p/\tau + \mu_1$. Hence, the DFE point E^0 is globally asymptotically stable. □

5.2. Local and Global Stability of DEE Point for NSFD Scheme

Theorem 7. *For all $h > 0$, the DEE point E^* of the NSFD model (11) is locally asymptotically stable whenever $R_0 > 1$.*

Proof. In the similar way as in Theorem 5, the Jacobian matrix can be obtained as

$$J = \begin{bmatrix}
\frac{1}{1+h\left(\beta C_n(t) + (\tau+\mu_1)\right)} & 0 & \frac{-(S_n + \mathrm{hp})h\beta}{\left(1+h\left(\beta C_n(t) + (\tau) + \mu_1\right)\right)^2} & 0\\
\frac{h\beta C(t)}{1+h\left(\gamma+\mu_1+\omega\right)} & \frac{1}{1+h\left(\gamma+\mu_1+\omega\right)} & \frac{h\beta S(t)}{1+h\left(\gamma+\mu_1+\omega\right)} & 0\\
0 & \frac{h\omega}{1+h\left(\gamma_2+\mu_1+\varepsilon\right)} & \frac{1}{1+h\left(\gamma_2+\mu_1+\varepsilon\right)} & 0\\
\frac{h(\tau)}{1+h\mu_1} & \frac{h(\omega)}{1+h\mu_1} & \frac{h(\gamma_2)}{1+h\mu_1} & \frac{1}{1+h\mu_1}
\end{bmatrix}. \tag{52}$$

By putting DEE point E^*, we get

$$J(E^*) = \begin{bmatrix}
\frac{1}{1+h\left(\beta C^* + (\tau)+\mu_1\right)} & 0 & \frac{-(S^* + hp)h\beta}{\left(1+h\left(\beta C_n(t) + (\tau) + \mu_1\right)\right)^2} & 0\\
\frac{h\beta C^*}{\left(1+h\left(\gamma+\mu_1+\omega\right)\right)} & \frac{1}{1+h\left(\gamma+\mu_1+\omega\right)} & \frac{h\beta S^*}{\left(1+h\left(\gamma+\mu_1+\omega\right)\right)} & 0\\
0 & \frac{h\omega}{1+h\left(\gamma_2+\mu_1+\varepsilon\right)} & \frac{1}{1+h\left(\gamma_2+\mu_1+\varepsilon\right)} & 0\\
\frac{h\tau}{1+h\mu_1} & \frac{h\omega}{1+h\mu_1} & \frac{h\gamma_2}{\left(1+h\mu_1\right)} & \frac{1}{1+h\mu_1}
\end{bmatrix}. \tag{53}$$

To find the eigenvalues of (53), we consider

$$\begin{vmatrix} \left(\frac{1}{(1+h(\beta C^{*}+(\tau)+\mu_1))}\right)-\lambda & 0 & \frac{-(S^{*}+hp)h\beta}{(1+h(\beta C_n(t)+(\tau)+\mu_1))^2} & 0 \\ \frac{h\beta C^{*}}{(1+h(\gamma+\mu_1+\omega))} & \frac{1}{(1+h(\gamma+\mu_1+\omega))}-\lambda & \frac{h\beta S^{*}}{(1+h(\gamma+\mu_1+\omega))} & 0 \\ 0 & \frac{h\omega}{(1+h(\gamma_2+\mu_1+\varepsilon))} & \frac{1}{(1+h(\gamma_2+\mu_1+\varepsilon))}-\lambda & 0 \\ \frac{h(\tau)}{(1+h\mu_1)} & \frac{h(\omega)}{(1+h\mu_1)} & \frac{h(\gamma_2)}{(1+h\mu_1)} & \frac{1}{(1+h\mu_1)}-\lambda \end{vmatrix} = 0. \tag{54}$$

The abovementioned equation gives the following deterministic equation:

$$\left(\frac{1}{(1+h\mu_1)}-\lambda\right)\left(\lambda^3+\lambda^2K+\lambda L+M\right)=0. \tag{55}$$

The abovementioned equation provides one eigenvalue $\lambda_1 = 1/(1+h\mu_1) < 1$. The remaining eigenvalues can be obtained from

$$\lambda^3+\lambda^2K+\lambda L+M=0, \tag{56}$$

where

$$\begin{aligned} K &= \frac{1}{(1+h(\gamma+\mu_1+\omega))}+\frac{1}{(1+h(\gamma_2+\mu_1+\varepsilon))}-\frac{1}{(1+h(\beta C^{*}+(\tau)+\mu_1))}, \\ L &= \frac{1}{(1+h(\gamma+\mu_1+\omega))(1+h(\beta C^{*}+(\tau)+\mu_1)}+\frac{1}{(1+h(\gamma_2+\mu_1+\varepsilon))(1+h(\beta C^{*}+(\tau)+\mu_1))} \\ &\quad +\frac{1}{(1+h(\gamma+\mu_1+\omega))(1+h(\gamma_2+\mu_1+\varepsilon))}+\frac{h^2\beta S^{*}\omega}{(1+h(\gamma_2+\mu_1+\varepsilon))}, \\ M &= \frac{h^3\beta^2C^{*}(S^{*}+\mathrm{hp})\omega}{(1+h(\gamma+\mu_1+\omega))(1+h(\gamma_2+\mu_1+\varepsilon))(1+h(\beta C^{*}(t)+\tau+\mu_1))^2} \\ &\quad -\frac{1}{(1+h(\gamma+\mu_1+\omega))(1+h(\gamma_2+\mu_1+\varepsilon))(1+h(\beta C^{*}+(\tau+\mu_1).} \end{aligned} \tag{57}$$

It is clear that $K, L, M > 0$ whenever $R_0 > 1$. Also,

$$\begin{aligned} KL-M &= \left(\frac{1}{(1+h(\gamma+\mu_1+\omega))}+\frac{1}{(1+h(\gamma_2+\mu_1+\varepsilon))}-\frac{1}{(1+h(\beta C^{*}+(\tau+\mu_1)))}\right) \\ &\quad \cdot\left(\frac{1}{(1+h(\gamma+\mu_1+\omega))(1+h(\beta C^{*}+(\tau)+\mu_1))}+\frac{1}{(1+h(\gamma_2+\mu_1+\varepsilon))(1+h(\beta C^{*}+(\tau)+\mu_1))}+\frac{1}{(1+h(\gamma+\mu_1+\omega))(1+h(\gamma_2+\mu_1+\varepsilon))}+\frac{h^2\beta S^{*}\omega}{(1+h(\gamma_2+\mu_1+\varepsilon))}\right) \\ &\quad -\frac{h^3\beta^2C^{*}(S^{*}+hp)\omega}{(1+h(\gamma+\mu_1+\omega))(1+h(\gamma_2+\mu_1+\varepsilon))(1+h(\beta C^{*}(t)+\tau+\mu_1))^2}+\frac{1}{(1+h(\gamma+\mu_1+\omega))(1+h(\gamma_2+\mu_1+\varepsilon))(1+h(\beta C^{*}+(\tau+\mu_1)))}>0. \end{aligned} \tag{58}$$

Therefore, by applying Routh–Hurwitz criterion, all the solutions of the (56) must have negative real parts whenever $R_0 > 1$. Hence, the DEE point E^* of the NSFD scheme is locally asymptotically stable whenever $R_0 > 1$. □

Theorem 8. *For all $h > 0$, the DEE point E^* of the NSFD model (11) is globally asymptotically stable whenever $R_0 \geq 1$, as shown in Figures 3(a)–3(d)).*

Proof. We construct a sequence $\{V(n)\}$ such that

$$V(n) = \frac{1}{hA^*} f\left(\frac{S_n}{S^*}\right) + \frac{1}{hS^*} f\left(\frac{A_n}{A^*}\right) + \frac{\omega C^*}{hC_3 S^* C^*} f\left(\frac{C_n}{C^*}\right) + \frac{\beta R^*}{hC_2 S^* A^*} f\left(\frac{R_n}{R^*}\right). \tag{59}$$

where $C_2 = (\gamma + \mu_1 + \omega)$ and $C_3 = (\gamma_2 + \mu_1 + \varepsilon)$. Let $f(x) = x - 1 - \ln(x), x \in R^+$. Clearly $f(x) \geq 0$, and the equality holds true if $x = 1$. We have

$$\begin{aligned}
f\left(\frac{S_{n+1}}{S^*}\right) - f\left(\frac{S_n}{S^*}\right) &= \frac{S_{n+1} - S_n}{S^*} - \ln\left(\frac{S_{n+1}}{S_n}\right) \leq \frac{(S_{n+1} - S^*)(S_{n+1} - S_n)}{S_{n+1}S^*} \\
&= \frac{(S_{n+1} - S^*)}{S_{n+1}S^*} h\left(p - \beta S_{n+1}(t) C_n(t) - (\tau + \mu_1) S_{n+1}(t)\right) \\
&= \frac{(S_{n+1} - S^*)}{S_{n+1}S^*} h\left(\beta S^*(t(t)) C^* + (\tau + \mu_1) S^*(t) - \beta S_{n+1}(t) C_n(t) - (\tau + \mu_1) S_{n+1}(t)\right) \\
&= h\left((S_{n+1} - S^*)\left(-(\tau + \mu_1)(S_{n+1} - S^*) - \beta C^*(S_{n+1} - S^*)\right)\right) \\
&= \left(-\left(\frac{\tau + \mu_{1)}(S_{n+1} - S^*)^2}{S_{n+1}S^* + \beta h C^*\left(1 - \frac{S^*}{S_{n+1}}\right)\left(\frac{A_n}{A^*}\frac{S_{n+1}}{S^*} - 1\right)}\right)\right.
\end{aligned} \tag{60}$$

Similarly.

$$\begin{aligned}
f\left(\frac{A_{n+1}}{A^*}\right) - f\left(\frac{A_n}{A^*}\right) &= \frac{A_{n+1} - A_n}{A^*} - \ln\left(\frac{A_{n+1}}{A_n}\right) \leq \frac{(A_{n+1} - A^*)}{A_{n+1}A^*} \\
&\leq \frac{(A_{n+1} - A^*)}{A_{n+1}A^*} (\beta S_{n+1}(t) C_n(t) - (\gamma + \mu_1 + \omega) A_{n+1}(t)) \\
&\leq \frac{(A_{n+1} - A^*)}{A_{n+1}A^*} \left(\beta S_{n+1}(t) C_n(t) - \frac{C_{n+1}S^*}{C^*} - \frac{\omega C_{n+1}A^*}{C^*} + \omega A_{n+1}(t)\right) \\
&= \left(1 - \frac{A^*}{A_{n+1}}\right)\left(\frac{\omega h A^*}{C^*}\left(\frac{A_{n+1}}{A^*} - \frac{C_{n+1}}{C^*}\right) + \left(\left(1 - \frac{A^*}{A_{n+1}}\right)\left(\frac{hS^*C^*}{C^*}\left(\frac{\beta C_n S_{n+1}}{S^*} - \frac{C_{n+1}S^*}{C^*}\right)\right)\right)\right)
\end{aligned} \tag{61}$$

$$\begin{aligned}
f\left(\frac{C_{n+1}}{C^*}\right) - f\left(\frac{C_n}{C^*}\right) &= \frac{C_{n+1} - C_n}{C^*} - \ln\left(\frac{C_{n+1}}{C_n}\right) \leq \frac{(C_{n+1} - C^*)(C_{n+1} - C_n)}{C_{n+1}C^*} \\
&\leq \frac{(C_{n+1} - C^*)}{C_{n+1}C^*} (\omega A_n(t) - (\gamma_2 + \mu_1 + \varepsilon) C_n(t)) \\
&\leq \frac{(C_{n+1} - C^*)}{C_{n+1}C^*} \left(\omega A_n(t) - C_3 C_{n+1} = C_3 h\left(1 - \frac{C^*}{C_{n+1}}\right)\left(\frac{A_{n+1}}{A^*} - \frac{C_{n+1}}{C^*}\right).\right.
\end{aligned}$$

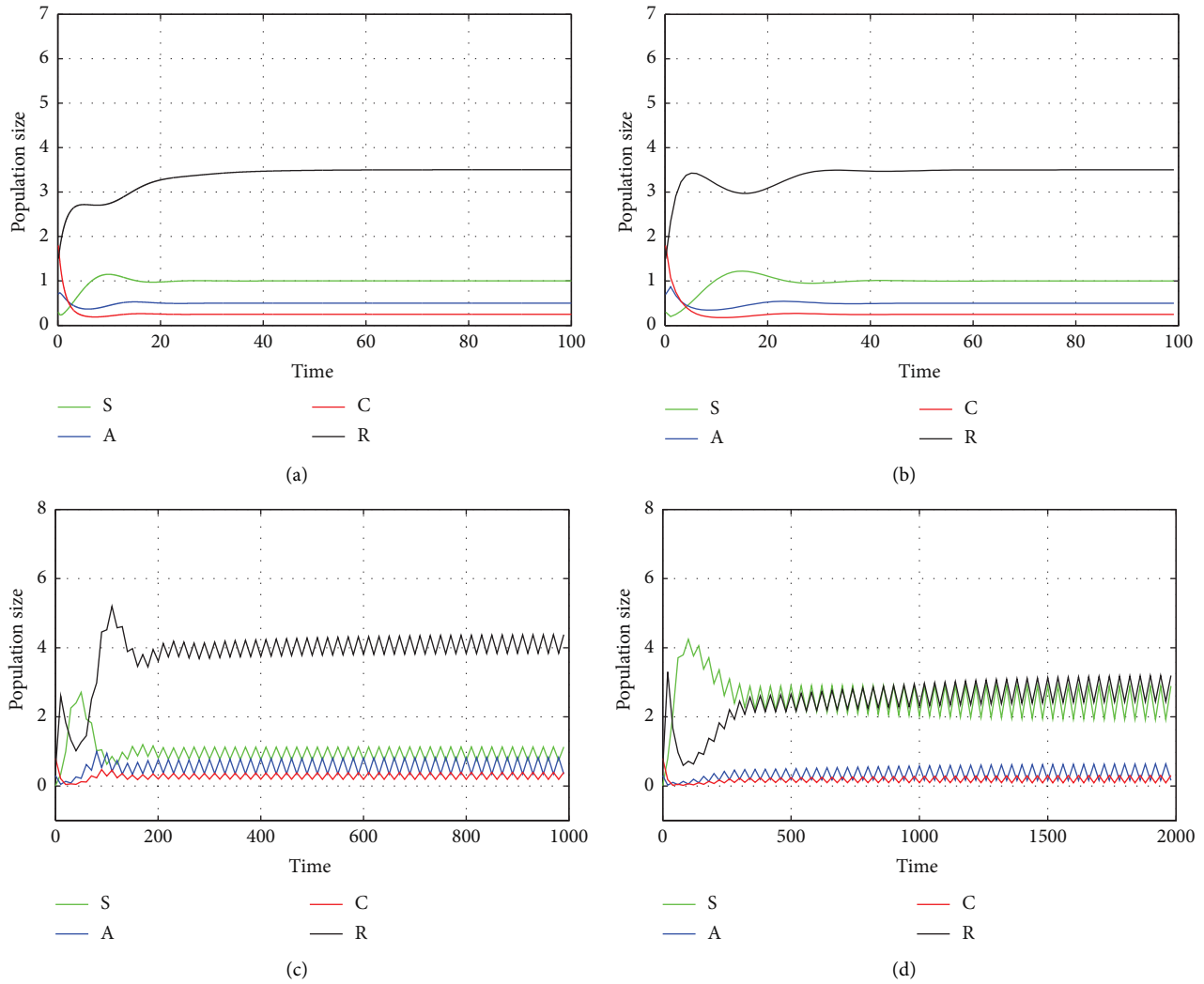

Figure 3: The solutions of HBV model (2) obtained through NSFD scheme whenever $R_0 \geq 1$ and (a) $h = 0.01$, (b) $h = 1$, (c) $h = 10$ and (d) $h = 20$. Other parameters remain fixed as $p = 0.5$, $\beta = 1.60$, $\gamma = 0.2$, $\omega = 0.5$, $\tau = 0.001$, $\mu_1 = 0.1$, $\gamma_2 = 0.4$, $\varepsilon = 0.5$.

$$
\begin{aligned}
f\left(\frac{R_{n+1}}{R^*}\right) - f\left(\frac{R_n}{R^*}\right) &= \frac{R_{n+1} - R_n}{R^*} - \ln\left(\frac{R_{n+1}}{R_n}\right) \leq \frac{\left(R_{n+1} - R^*\right)\left(R_{n+1} - R_n\right)}{R_{n+1}R^*} \\
&\leq \frac{\left(R_{n+1} - R^*\right)}{R_{n+1}R^*}\left(\tau S_{n+1}(t) + \gamma_2 C_{n+1}(t) + \omega A_{n+1}(t) - \mu_1 R_{n+1}(t)\right) \\
&= h\left(1 - \frac{R^*}{R_{n+1}}\right)\left(\frac{A_{n+1}}{A^*} - \frac{R_{n+1}}{R^*}\right).
\end{aligned} \tag{62}
$$

The difference of $T(n)$ satisfies

$$
\begin{aligned}
T(n+1)-T(n) &= \frac{1}{hA^*}\left(f\left(\frac{S_{n+1}}{S^*}\right)-f\left(\frac{S_n}{S^*}\right)+\frac{1}{hS^*}f\left(\frac{A_{n+1}}{A^*}\right)-f\left(\frac{A_n}{A^*}\right)\right),\\
&+\frac{\omega I^*}{hC_3S^*A^*}\left(f\left(\frac{C_{n+1}}{C^*}\right)-f\left(\frac{C_n}{C^*}\right)\right)+\frac{\beta R^*}{hC_2S^*A^*}\left(f\left(\frac{R_{n+1}}{R^*}\right)-f\left(\frac{R_n}{R^*}\right)\right),\\
&\le\left(\frac{-(\tau)+(\mu_1)\left(S_{n+1}-S^*\right)^2}{S_{n+1}S^*}+\beta hC^*\left(1-\frac{S^*}{S_{n+1}}\right)\left(\frac{A_n}{A^*}\frac{S_{n+1}}{S^*}-1\right)\right)-\left(1-\frac{A^*}{A_{n+1}}\right)\left(\frac{\omega hA^*}{C^*}\left(\frac{A_{n+1}}{A^*}-\frac{C_{n+1}}{C^*}\right)\right.\\
&\left.+\left(\left(1-\frac{A^*}{A_{n+1}}\right)\frac{hS^*C^*}{C^*}t\left(\frac{\beta C_nS_{n+1}}{S^*}-\frac{C_{n+1}S^*}{C^*}\right)\right)\right)\\
&-\frac{\omega I^*}{hC_3S^*A^*}\left(\frac{\left(C_{n+1}-C^*\right)}{C_{n+1}C^*}\left(\omega A_n(t)-C_3C_{n+1}\right)\right)-\frac{\beta R^*}{hC_2S^*A^*}\left(\left(1-\frac{R^*}{R_{n+1}}\right)\left(\frac{A_{n+1}}{A^*}-\frac{R_{n+1}}{R^*}\right)\right),\\
&\le\frac{-\beta h\left(S_{n+1}-S^*\right)^2}{S_{n+1}S^*}-\frac{\omega hA^*}{C^*}\left(\frac{C^*}{C_{n+1}}\frac{A_n}{A^*}\frac{S_{n+1}}{S^*}-2-\frac{A_n}{A^*}+\frac{S^*}{S_{n+1}}+\frac{A_{n+1}}{A^*}\right),\\
&-\frac{1}{hS^*}\left(\frac{C^*A_{n+1}}{C_{n+1}C^*}+\frac{A^*C_{n+1}}{A_{n+1}C^*}-2\right)-\frac{\beta R^*}{hC_2S^*A^*}\left(\frac{C^*R_{n+1}}{C_{n+1}R^*}+\frac{R^*C_{n+1}}{R_{n+1}C^*}-2\right),\\
&\le\frac{-\beta h\left(S_{n+1}-S^*\right)^2}{S_{n+1}S^*}-\frac{1}{hA^*}\left(f\left(\frac{S^*}{S_{n+1}}\right)+f\left(\frac{A_{n+1}}{A^*}\right)\right)+f\left(\frac{C^*}{C_{n+1}}\frac{A_n}{A^*}\frac{S_{n+1}}{S^*}\right)-f\left(\frac{A_n}{A^*}\right)-\frac{1}{hS^*}\left(f\left(\frac{A^*C_{n+1}}{A_{n+1}C^*}\right)+f\left(\frac{C^*A_{n+1}}{C_{n+1}A^*}\right)\right)\\
&-\frac{\beta R^*}{hC_2S^*A^*}\left(f\left(\frac{A^*R_{n+1}}{A_{n+1}R^*}\right)+f\left(\frac{R^*A_{n+1}}{R_{n+1}A^*}\right)\right).
\end{aligned}
\tag{63}
$$

Therefore, $\{T(n)\}$ is a monotonic decreasing sequence for any $n \ge 0$. Since $\{T(n)\} \ge 0$ and $\lim_{n\longrightarrow\infty}(T(n+1)-T(n))=0$, we obtain $\lim_{n\longrightarrow\infty}S_{n+1}=S^*$, $\lim_{n\longrightarrow\infty}A_{n+1}=A^*$, $\lim_{n\longrightarrow\infty}C_{n+1}=C^*$ and $\lim_{n\longrightarrow\infty}R_{n+1}=R^*$. Hence, according to Vaz et al. [35], we conclude that when $R_0 \ge 1$, the DEE point E^* is globally asymptotically stable. □

6. Conclusions

In the present paper, we proposed and investigated the deterministic HBV disease model. The aim is to understand the disease dynamics and generate methods for managing the disease transmission among the people. The continuous as well as discrete description of the model is discussed, and different essential properties are developed. Both continuous and discrete outcomes can be used as efficient tool for monitoring the transmission of HBV epidemic diseases. The basic reproduction number R_0 is constructed for the model and by applying it, the local as well as global stability of DFE and DEE points are discussed for the continuous model. The discrete NSFD scheme is developed for the model which is not only unconditionally convergent but also produces precise findings that are mathematically and biologically consistent with the associated continuous model. The local and global stability of both DFE and DEE points are demonstrated for the NSFD scheme by utilizing various criteria and conditions. The qualities of the NSFD scheme are presented, which shows that the results of the NSFD scheme are qualitatively exact and proficient. The NSFD scheme is a simple technique which shows that the continuous and discrete models behave identically, providing mathematically stable and high-quality response. By using the NSFD scheme, we are able to better explain the results that would be valuable to society and to the area of medicine as well. The outcomes provided in this paper can be used as a helpful apparatus for forecasting the development of HBV epidemic diseases. To support our theoretical results, numerical simulations are offered at every stage.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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