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Research Article

A New Generalized-X Family for Analyzing the COVID-19 Data Set: a Case Study

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Nowadays, researchers in applied sectors are highly motivated to propose and study new generalizations of the existing distributions to provide the best fit to data. To provide a close fit to data in numerous sectors, a series of new distributions have been proposed. In this study, we propose a new family called the new generalized-X (for short, "NG-X") family of distributions. Based on the NG-X method, a novel modification of the Weibull model called the new generalized-Weibull (for short, "NG-Weibull") distribution is studied. The heavy-tailed characteristics of the NG-X distributions are derived. The maximum likelihood estimators of the NG-X distributions are also obtained. Based on the special case (i.e., NG-Weibull) of the NG-X family, a simulation study is provided. The practical performance of the new NG-Weibull model is assessed by analyzing the COVID-19 data set. The fitting results of the NG-Weibull model are compared with three other competing models. Based on certain statistical measures, it is observed that the NG-Weibull model is the best competitive model.

1. Introduction

In the practice of applied statistics, statisticians are highly motivated and attracted to introduce new probability distributions. This motivation has gained popularity in the recent decade. In this credit, the development and introduction of new families of distribution have received much attention. The new families are developed by introducing new parameters [1–4].

Theoretically and practically, some distributions do not have enough flexibility to counter the complex forms of the data sets. For example, the classical models do not provide a reasonable fit to the extreme value data [5]. This is the main attracting reason to go for the development of new statistical distributions. The development of the modified forms of the existing distributions helps to improve the fitting power of the existing distributions [6].

Recently, Chipepa et al. [7] introduced a new generalized family of distributions called the OGHLW-G (odd generalized half-logistic Weibull-G) family of distributions. They have obtained some structural properties of the OGHLW-G family. They have also discussed some special models of the OGHLW-G family to show the importance of the new models. Chipepa et al. [8] also proposed the Burr III-Topp-Leone-G family of distributions for analyzing (i) carbon fibres data, (ii) fracture toughness of silicon nitride data, and (iii) turbocharger failure times data.

Ahmad and Hussain [9] introduced a NEF-Weibull (new extended flexible-Weibull) distribution to model the lifetime data with bathtub-shaped failure rates. Some other modified and extended distributions include the beta-Weibull (B-Weibull) distribution of Famoye et al. [10]; the flexible Weibull (FWEx) distribution of Bebbington et al. [11]; the generalized modified Weibull (GM-Weibull) distribution proposed by Carrasco et al. [12]; the Kumaraswamy Weibull (K-Weibull) distribution proposed by Cordeiro et al. [13]; the beta modified Weibull (BM-Weibull) distribution of Silva et al. [14]; the exponentiated modified Weibull extension (EMWE) distribution introduced by Sarhan and Apaloo [15]; the Gumbel-Lomax (GL) distribution suggested by Tahir et al. [16]; and the Marshall–Olkin inverse Lomax distribution (MO-ILD) studied by Maxwell et al. [17].

Another prominent approach for extending and updating the existing distributions is called the new exponential-X family of Ahmad et al. [5]. They introduced the new exponential-X family by combining the exponential distribution with the T-X approach of Alzaatreh et al. [18].

Eliwa et al. [19] further contributed to the literature by introducing the exponentiated form of the Chen-G family of distributions. For other recent contributions to the development of new approaches for updating the existing distributions, we refer to (i) the truncated burr-XG family [20], (ii) the Fréchet Topp Leone-G family [21], (iii) the transmuted alpha power-G family [22], (iv) the Marshall–Olkin Zubair-G family [23], (v) a new family of continuous (NFC) distributions [24], and (vi) the Topp-Leone Odd Burr III-G (TLOBIII-G) family [25].

In this article, we propose a NG-X family of distributions to obtain the modified and updated versions of the existing distributions. The NG-X method can be implemented to obtain new flexible distributions for modeling data in various sectors such as reliability, survival analysis, healthcare, biomedical engineering, and lifetime studies. The next section is devoted to introduce the proposed NG-X family of distributions.

2. A New Generalized-X Family of Distributions: the Proposed Method

In this section, the probability density function (PDF), cumulative distribution function (CDF), survival function (SF), hazard function (HF), and cumulative HF (CHF) for the NG-X family are computed.

Definition: A random variable X has a NG-X family, if it is CDF $G(x;\alpha,\delta,\Xi)$ is given by

$$G(x;\alpha,\delta,\Xi) = 1 - \left(1 - \frac{(1-\delta)^2 F(x;\Xi)}{[1-\delta F(x;\Xi)]^2}\right)^{\alpha}, \quad \alpha>0,\ \delta\in(0,1),\ x\in\mathbb{R}. \tag{1}$$

The function defined in (1) is a valid DF, if and only if, $\delta\in(0,1)$. Corresponding to $G(x;\alpha,\delta,\Xi)$, the PDF $g(x;\alpha,\delta,\Xi)$ is given by

$$g(x;\alpha,\delta,\Xi) = \frac{\alpha(1-\delta)^2 f(x;\Xi)\{1+\delta F(x;\Xi)\}}{[1-\delta F(x;\Xi)]^{2\alpha+1}} \cdot \left[\overline{F}(x;\Xi)\left\{1-\delta^2 F(x;\Xi)\right\}\right]^{\alpha-1}. \tag{2}$$

Furthermore, in link to $G(x;\alpha,\delta,\Xi)$ and $g(x;\alpha,\delta,\Xi)$, the $\overline{G}(x;\alpha,\delta,\Xi)$, HF $h(x;\alpha,\delta;\Xi)$, and CHF $H(x;\alpha,\delta;\Xi)$ are given by

$$\begin{aligned}\overline{G}(x;\alpha,\delta,\Xi) &= \left(1 - \frac{(1-\delta)^2 F(x;\Xi)}{[1-\delta F(x;\Xi)]^2}\right)^{\alpha},\\ h(x;\alpha,\delta,\Xi) &= \frac{\alpha(1-\delta)^2 f(x;\Xi)\{1+\delta F(x;\Xi)\}}{[1-\delta F(x;\Xi)]}\left[\overline{F}(x;\Xi)\left\{1-\delta^2 F(x;\Xi)\right\}\right]^{-1},\\ H(x;\alpha,\delta,\Xi) &= -\alpha \log\left(1 - \frac{(1-\delta)^2 F(x;\Xi)}{[1-\delta F(x;\Xi)]^2}\right),\end{aligned} \tag{3}$$

respectively.

In this paper, we implement the NG-X family approach to introduce an updated version of the Weibull distribution called the NG-Weibull (new generalized Weibull) distribution. In the next section, we provide the expressions for the PDF, CDF, SF, and HF of the NG-Weibull distribution. Furthermore, the plots for the PDF, CDF, SF, and HF of the NG-Weibull distribution are also provided.

3. A NG-Weibull Distribution: a Special Model

With parameters $\theta>0$ and $\gamma>0$, consider the CDF $F(x;\Xi)$ of the Weibull distribution is given by

$$G(x;\Xi) = 1 - e^{-\gamma x^{\theta}}, \quad x\geq 0. \tag{4}$$

The respective PDF $g(x;\Xi)$, SF $S(x;\Xi)$, and HF $h(x;\Xi)$ are, respectively, given by

$$\begin{aligned} g(x;\Xi) &= \theta\gamma x^{\theta-1} e^{-\gamma x^{\theta}}, \quad x>0,\\ S(x;\Xi) &= e^{-\gamma x^{\theta}}, \quad x>0,\\ h(x;\Xi) &= \theta\gamma x^{\theta-1}, \quad x>0,\end{aligned} \tag{5}$$

where $\Xi = (\alpha, \vartheta)$.

Using (4) in (1), we get the CDF of the NG-Weibull distribution, given by

$$G(x) = 1 - \left(1 - \frac{(1-\delta)^2\left(1-e^{-\gamma x^{\theta}}\right)}{\left[1-\delta\left(1-e^{-\gamma x^{\theta}}\right)\right]^2}\right)^{\alpha}, \quad x\geq 0. \tag{6}$$

The PDF of the NG-Weibull distribution is given by

$$g(x) = \frac{\alpha(1-\delta)^2\theta\gamma x^{\theta-1}e^{-\alpha\gamma x^\theta}\left\{1+\delta\left(1-e^{-\gamma x^\theta}\right)\right\}}{\left[1-\delta\left(1-e^{-\gamma x^\theta}\right)\right]^{2\alpha+1}} \cdot \left[1-\delta^2\left(1-e^{-\gamma x^\theta}\right)\right]^{\alpha-1}. \quad (7)$$

Some possible behaviors for $g(x)$ of the NG-Weibull are presented in Figure 1. These plots are obtained for (i) $\alpha = 2.9, \delta = 0.8, \gamma = 2.5, \theta = 4.9$ (red curve), (ii) $\alpha = 2.7, \delta = 0.7, \gamma = 0.6, \theta = 4.1$ (blue curve), (iii) $\alpha = 3.1, \delta = 0.7, \gamma = 4.6, \theta = 0.9$ (gold curve), (iv) $\alpha = 3.8, \delta = 0.1, \gamma = 3.2, \theta = 3.5$ (green curve), and (iv) $\alpha = 0.8, \delta = 0.4, \gamma = 3.8, \theta = 0.5$ (black curve).

Furthermore, the SF and HF of the NG-Weibull distribution are given by

$$\overline{G}(x) = \left(1 - \frac{(1-\delta)^2\left(1-e^{-\gamma x^\theta}\right)}{\left[1-\delta\left(1-e^{-\gamma x^\theta}\right)\right]^2}\right)^\alpha,$$

$$h(x) = \frac{\alpha(1-\delta)^2\theta\gamma x^{\theta-1}e^{-\alpha\gamma x^\theta}\left\{1+\delta\left(1-e^{-\gamma x^\theta}\right)\right\}}{\left[1-\delta\left(1-e^{-\gamma x^\theta}\right)\right]} \cdot \left[1-\delta^2\left(1-e^{-\gamma x^\theta}\right)\right]^{-1}, \quad (8)$$

respectively.

For $\alpha = 1.2, \delta = 0.99, \gamma = 7.7$, and $\theta = 0.6$, the plots of CDF $G(x)$ and SF $\overline{G}(x) = 1 - G(x)$ of the NG-Weibull distribution are provided in Figure 2.

Furthermore, some possible behaviors for $h(x)$ of the NG-Weibull are presented in Figure 3. These plots are obtained for (i) $\alpha = 0.8, \delta = 0.5, \gamma = 2.8, \theta = 0.5$ (red curve), (ii) $\alpha = 1.8, \delta = 0.2, \gamma = 2.1, \theta = 1.2$ (green curve), and (iii) $\alpha = 5.2, \delta = 0.8, \gamma = 1.1, \theta = 0.2$ (blue curve).

The NG-Weibull distribution is an updated version of the Weibull distribution. It has certain advantages over some other updated versions of the Weibull distribution.

(i) The NG-Weibull distribution has a closed-form CDF, which makes it easier to generate random numbers for the simulation study.

(ii) The NG-Weibull is capable of capturing the bathtub shape of the HF. Statistical models with a bathtub shape of the HF are very useful for modeling the healthcare engineering data sets.

(iii) The NG-Weibull distribution provides a close fit to the COVID-19 data set. Therefore, the implementation of the NG-Weibull distribution could be a useful choice for modeling data in health and other related sectors.

Besides the abovementioned advantages, the NG-Weibull distribution also has certain limitations. For example,

(i) The NG-Weibull distribution is a continuous model, and it is employed to analyze the mortality rates of COVID- 19 infected people. Therefore, it could not be implemented to analyze the data sets that are not continuous in nature, such as (i) the number of COVID-19 daily registered cases, (ii) the number of COVID-19 confirmed deaths, and (iii) the number of COVID-19 recovered cases.

(ii) Since the PDF of the NG-X distributions has a complicated form, more computational work would be required to derive the distributional properties.

4. The HT Characteristics

The statistical distributions that possess HT (heavy-tailed) behavior are very useful for dealing with extreme value phenomena. In this section, we prove mathematically the HT characteristics of the NG-X distributions.

4.1. The Regularly Varying Tail Behavior. This subsection is devoted to proving the RVTB (regularly varying tail behavior) of the NG-X distributions. According to Karamata's theorem [26], in terms of SF $\overline{G}(x; \alpha, \delta; \Xi)$, we have

Theorem If $\overline{F}(x; \Xi)$ is the SF of the regular varying (RVa) distribution, then $\overline{G}(x; \alpha, \delta; \Xi)$ is also an RVa distribution.

Proof. Let assume $\lim_{x\longrightarrow\infty}\overline{F}(\lambda x; \Xi)/\overline{F}(x; \Xi) = f(\lambda)$ is finite but nonzero $\forall\, \lambda > 0$. Using (3), we have

$$\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x; \alpha, \delta; \Xi)}{\overline{G}(x; \alpha, \delta; \Xi)} = \lim_{x\longrightarrow\infty}\left(1 - \frac{(1-\delta)^2F(x;\Xi)}{[1-\delta F(x;\Xi)]^2}\right)^\alpha \Big/ \left(1 - \frac{(1-\delta)^2F(x;\Xi)}{[1-\delta F(x;\Xi)]^2}\right)^\alpha. \quad (9)$$

Using $\alpha = 1$ in (9), we get

$$\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x; \alpha, \delta; \Xi)}{\overline{G}(x; \alpha, \delta; \Xi)} = \lim_{x\longrightarrow\infty}\left(1 - \frac{(1-\delta)^2F(\lambda x;\Xi)}{[1-\delta F(\lambda x;\Xi)]^2}\right) \Big/ \left(1 - \frac{(1-\delta)^2F(x;\Xi)}{[1-\delta F(x;\Xi)]^2}\right). \quad (10)$$

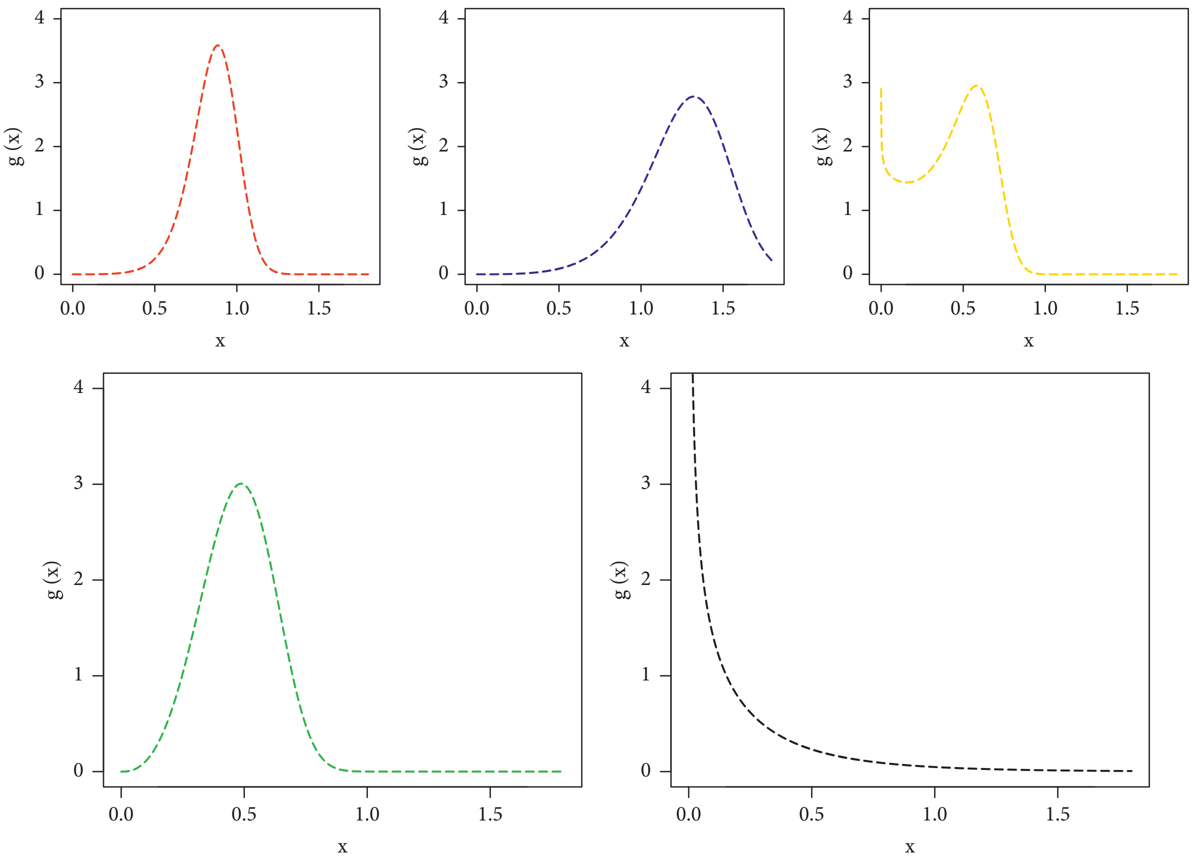

Figure 1: The PDF plots of the NG-Weibull distribution.

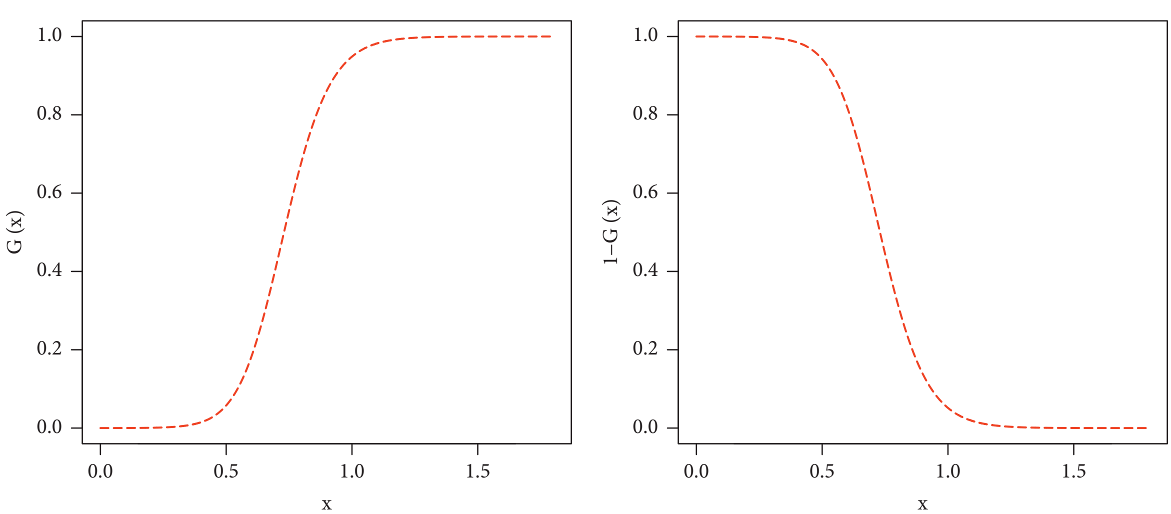

Figure 2: The plots of the CDF and SF of the NG-Weibull distribution.

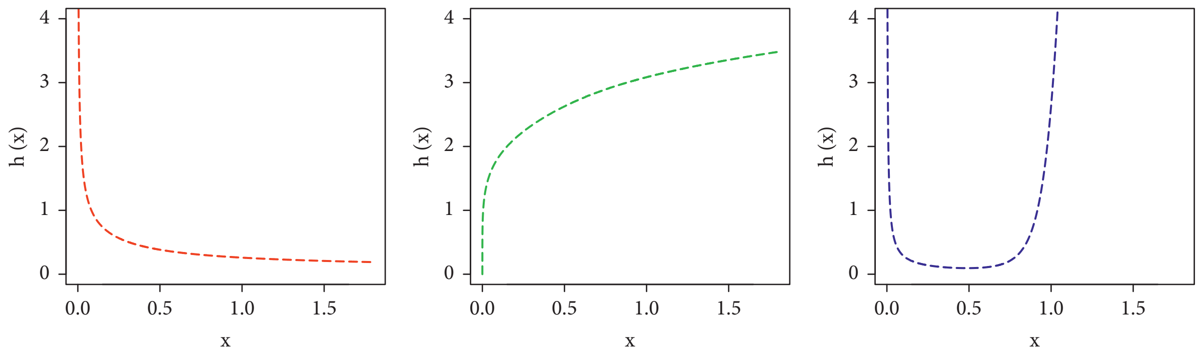

Figure 3: The plots of the HF of the NG-Weibull distribution.

On simplification, we get

$$\begin{aligned}
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]\left[1-\delta^2F(\lambda x;\Xi)\right]}{[1-\delta F(\lambda x;\Xi)]^2}\Big/\frac{[1-F(x;\Xi)]\left[1-\delta^2F(x;\Xi)\right]}{[1-\delta F(x;\Xi)]^2},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]\left[1-\delta^2F(\lambda x;\Xi)\right]}{[1-F(x;\Xi)]\left[1-\delta^2F(x;\Xi)\right]}\,\frac{[1-\delta F(x;\Xi)]^2}{[1-\delta F(\lambda x;\Xi)]^2},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{[1-\delta F(\lambda x;\Xi)]^2\left[1-\delta^2F(x;\Xi)\right]}{[1-\delta F(x;\Xi)]^2\left[1-\delta^2F(\lambda x;\Xi)\right]}.
\end{aligned} \tag{11}$$

Since, $F(x;\Xi)$ is a CDF. Therefore, we have

$$\lim_{x\longrightarrow\infty}F(x;\Xi)=1. \tag{12}$$

Hence, from (11), we get

$$\begin{aligned}
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{[1-\delta]^2\left[1-\delta^2\right]}{[1-\delta]^2\left[1-\delta^2\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= f(\lambda).
\end{aligned} \tag{13}$$

which is finite but nonzero $\forall u>0$; thus, $\overline{G}(x;\alpha,\delta;\Xi)$ is an RVa distribution.

Now, we prove the HT behavior of the special case of the NG-X family. Using (4) in (11), we get

$$\begin{aligned}
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{\left[1-\delta\left(1-e^{-\gamma(\lambda x)^{\theta}}\right)\right]^{2}\left[1-\delta^{2}\left(1-e^{-\gamma x^{\theta}}\right)\right]}{\left[1-\delta\left(1-e^{-\gamma x^{\theta}}\right)\right]^{2}\left[1-\delta^{2}\left(1-e^{-\gamma(\lambda x)^{\theta}}\right)\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{\left[1-\delta\left(1-e^{-\gamma(\lambda.\infty)^{\theta}}\right)\right]^{2}\left[1-\delta^{2}\left(1-e^{-\gamma.\infty^{\theta}}\right)\right]}{\left[1-\delta\left(1-e^{-\gamma.\infty^{\theta}}\right)\right]^{2}\left[1-\delta^{2}\left(1-e^{-\gamma(\lambda.\infty)^{\theta}}\right)\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{[1-\delta(1-e^{-\infty})]^{2}\left[1-\delta^{2}(1-e^{-\infty})\right]}{[1-\delta(1-e^{-\infty})]^{2}\left[1-\delta^{2}(1-e^{-\infty})\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{[1-\delta(1-1/e^{\infty})]^{2}\left[1-\delta^{2}(1-1/e^{\infty})\right]}{[1-\delta(1-1/e^{\infty})]^{2}\left[1-\delta^{2}(1-1/e^{\infty})\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{[1-\delta(1-1/\infty)]^{2}\left[1-\delta^{2}(1-1/\infty)\right]}{[1-\delta(1-1/\infty)]^{2}\left[1-\delta^{2}(1-1/\infty)\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{[1-\delta(1-0)]^{2}\left[1-\delta^{2}(1-0)\right]}{[1-\delta(1-0)]^{2}\left[1-\delta^{2}(1-0)\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]}\times\frac{[1-\delta]^{2}\left[1-\delta^{2}\right]}{[1-\delta]^{2}\left[1-\delta^{2}\right]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= \lim_{x\longrightarrow\infty}\frac{[1-F(\lambda x;\Xi)]}{[1-F(x;\Xi)]},\\
\lim_{x\longrightarrow\infty}\frac{\overline{G}(\lambda x;\alpha,\delta;\Xi)}{\overline{G}(x;\alpha,\delta;\Xi)} &= f(\lambda),\\
\frac{\partial}{\partial\Xi}L(x;\alpha,\delta,\Xi) &= 0.
\end{aligned} \tag{14}$$

□

4.2. A Supportive Application of the RVTB. Consider the distribution of X has the power-law behavior (PLB), then we have

$$\overline{F}(x;\Xi)=\mathbb{P}(X>x)\sim x^{-\lambda}. \tag{15}$$

By incorporating Karamata's theorem, we can write $\overline{F}(x;\Xi)$ as

$$\overline{F}(x;\Xi)=x^{-\lambda}L(x;\Xi), \tag{16}$$

where $L(x;\Xi)$ stands for the slowly varying function. From (3), we have

$$\overline{G}(x;\alpha,\delta;\Xi)=\left(1-\frac{(1-\delta)^{2}F(x;\Xi)}{[1-\delta F(x;\Xi)]^{2}}\right)^{\alpha}. \tag{17}$$

Using $\alpha=1$ in (17), we get

$$\begin{aligned}
\overline{G}(x;\alpha,\delta;\Xi) &= 1-\frac{(1-\delta)^{2}F(x;\Xi)}{[1-\delta F(x;\Xi)]^{2}},\\
\overline{G}(x;\alpha,\delta;\Xi) &= \frac{[1-F(x;\Xi)]\left[1-\delta^{2}F(x;\Xi)\right]}{[1-\delta F(x;\Xi)]^{2}}.
\end{aligned} \tag{18}$$

Since $\overline{F}(x;\Xi)=1-F(x;\Xi)=x^{-\lambda}$. Therefore, we can write (18), as follows:

$$\begin{aligned}
\overline{G}(x;\alpha,\delta;\Xi) &= \frac{x^{-\lambda}\left[1-\delta^{2}F(x;\Xi)\right]}{[1-\delta F(x;\Xi)]^{2}},\\
\overline{G}(x;\alpha,\delta;\Xi) &= x^{-\lambda}L(x;\Xi),
\end{aligned} \tag{19}$$

where

$$L(x;\Xi)=\frac{\left[1-\delta^{2}F(x;\Xi)\right]}{[1-\delta F(x;\Xi)]^{2}}. \tag{20}$$

If $L(x;\Xi)$ is a slowly varying function, then the result obtained in (19) is true. According to Resnick [27], for all $\lambda>0$, we have to prove

$$\lim_{x\longrightarrow\infty}\frac{L(tx;\Xi)}{L(x;\Xi)}=1. \tag{21}$$

By incorporating (19), we get

$$\begin{aligned}
\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}F(tx;\Xi)\right]}{\left[1-\delta F(tx;\Xi)\right]^{2}}\times\frac{\left[1-\delta F(x;\Xi)\right]^{2}}{\left[1-\delta^{2}F(x;\Xi)\right]},\\
\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}\left(1-(tx)^{-\lambda}\right)\right]}{\left[1-\delta\left(1-(tx)^{-\lambda}\right)\right]^{2}}\times\frac{\left[1-\delta\left(1-x^{-\lambda}\right)\right]^{2}}{\left[1-\delta^{2}\left(1-x^{-\lambda}\right)\right]},\\
\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}\left(1-t^{-\lambda}x^{-\lambda}\right)\right]}{\left[1-\delta\left(1-t^{-\lambda}x^{-\lambda}\right)\right]^{2}}\times\frac{\left[1-\delta\left(1-x^{-\lambda}\right)\right]^{2}}{\left[1-\delta^{2}\left(1-x^{-\lambda}\right)\right]},\\
\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}\left(1-t^{-\lambda}/x^{\lambda}\right)\right]}{\left[1-\delta\left(1-t^{-\lambda}/x^{\lambda}\right)\right]^{2}}\times\frac{\left[1-\delta\left(1-x^{-\lambda}\right)\right]^{2}}{\left[1-\delta^{2}\left(1-x^{-\lambda}\right)\right]},\\
\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}\left(1-t^{-\lambda}/x^{\lambda}\right)\right]}{\left[1-\delta\left(1-t^{-\lambda}/x^{\lambda}\right)\right]^{2}}\times\frac{\left[1-\delta\left(1-1/x^{\lambda}\right)\right]^{2}}{\left[1-\delta^{2}\left(1-1/x^{\lambda}\right)\right]}.
\end{aligned} \tag{22}$$

Applying $\lim_{x\longrightarrow\infty}$ to both sides of (22), we get

$$\begin{aligned}
\lim_{x\longrightarrow\infty}\frac{L(tx;\Xi)}{L(x;\Xi)}&=\lim_{x\longrightarrow\infty}\frac{\left[1-\delta^{2}\left(1-t^{-\lambda}/x^{\lambda}\right)\right]}{\left[1-\delta\left(1-t^{-\lambda}/x^{\lambda}\right)\right]^{2}}\times\frac{\left[1-\delta\left(1-1/x^{\lambda}\right)\right]^{2}}{\left[1-\delta^{2}\left(1-1/x^{\lambda}\right)\right]},\\
\lim_{x\longrightarrow\infty}\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}\left(1-t^{-\lambda}/\infty\right)\right]}{\left[1-\delta\left(1-t^{-\lambda}/\infty\right)\right]^{2}}\times\frac{\left[1-\delta(1-1/\infty)\right]^{2}}{\left[1-\delta^{2}(1-1/\infty)\right]},\\
\lim_{x\longrightarrow\infty}\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}(1-0)\right]}{\left[1-\delta(1-0)\right]^{2}}\times\frac{\left[1-\delta(1-0)\right]^{2}}{\left[1-\delta^{2}(1-0)\right]},\\
\lim_{x\longrightarrow\infty}\frac{L(tx;\Xi)}{L(x;\Xi)}&=\frac{\left[1-\delta^{2}\right]}{\left[1-\delta\right]^{2}}\times\frac{\left[1-\delta\right]^{2}}{\left[1-\delta^{2}\right]},\\
\lim_{x\longrightarrow\infty}\frac{L(tx;\Xi)}{L(x;\Xi)}&=1.
\end{aligned} \tag{23}$$

5. Estimation and Simulation Study

In this section, the procedure of the maximum likelihood estimation for the model parameters of the NG-X distributions is implemented. Furthermore, a simulation study for different values of the model parameters is discussed.

5.1. Maximum Likelihood Estimation. In this subsection, the computation of the maximum likelihood estimators (MLEs) for the parameters (α,δ,Ξ) of NG-X is provided. Consider $x_1,x_2,\ldots,x_n$ be a set of observed values of a sample randomly selected from NG-X distributions with parameters α,δ, and Ξ. In link to the PDF of the NG-X distributions given by (2), the LF (likelihood function) is

$$L(x;\alpha,\delta,\Xi)=\prod_{i=1}^{n}\frac{\alpha(1-\delta)^2 f(x_i;\Xi)\{1+\delta F(x_i;\Xi)\}}{[1-\delta F(x_i;\Xi)]^{2\alpha+1}}\left[\overline{F}(x_i;\Xi)\left\{1-\delta^2 F(x_i;\Xi)\right\}\right]^{\alpha-1}. \quad (24)$$

Corresponding to $L(x;\alpha,\delta,\Xi)$, the log LF (LLF) is given by

$$\begin{aligned}L(x;\alpha,\delta,\Xi)=&\, n\log\alpha+2n\log(1-\delta)+\sum_{i=1}^{n}\log f(x_i;\Xi)+\sum_{i=1}^{n}\log\{1+\delta F(x_i;\Xi)\}\\&+(\alpha-1)\sum_{i=1}^{n}\log\left[\overline{F}(x_i;\Xi)\left\{1-\delta^2F(x_i;\Xi)\right\}\right]\\&-(2\alpha+1)\sum_{i=1}^{n}\log[1-\delta F(x_i;\Xi)].\end{aligned} \quad (25)$$

On behalf of the parameters (α,δ,Ξ) of the NG-X distributions, the partial derivative of the LLF are, respectively, given by

$$\begin{aligned}\frac{\partial}{\partial\alpha}L(x;\alpha,\delta,\Xi)=&\,\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[\overline{F}(x_i;\Xi)\left\{1-\delta^2F(x_i;\Xi)\right\}\right]-2\sum_{i=1}^{n}\log[1-\delta F(x_i;\Xi)],\\ \frac{\partial}{\partial\delta}L(x;\alpha,\delta,\Xi)=&-\frac{2n}{(1-\delta)}+\sum_{i=1}^{n}\frac{F(x_i;\Xi)}{\{1+\delta F(x_i;\Xi)\}}+(2\alpha+1)\sum_{i=1}^{n}\frac{F(x_i;\Xi)}{[1-\delta F(x_i;\Xi)]}\\&-2(\alpha-1)\sum_{i=1}^{n}\frac{\delta\overline{F}(x_i;\Xi)F(x_i;\Xi)}{\left[\overline{F}(x_i;\Xi)\left\{1-\delta^2F(x_i;\Xi)\right\}\right]},\\ \frac{\partial}{\partial\Xi}L(x;\alpha,\delta,\Xi)=&\sum_{i=1}^{n}\frac{\partial/\partial\Xi f(x_i;\Xi)}{f(x_i;\Xi)}+\delta\sum_{i=1}^{n}\frac{\partial/\partial\Xi F(x_i;\Xi)}{\{1+\delta F(x_i;\Xi)\}}+\delta(2\alpha+1)\sum_{i=1}^{n}\frac{\partial/\partial\Xi F(x_i;\Xi)}{[1-\delta F(x_i;\Xi)]}\\&+(\alpha-1)\sum_{i=1}^{n}\frac{\left\{1-\delta^2F(x_i;\Xi)\right\}\partial/\partial\Xi\overline{F}(x_i;\Xi)-\delta\overline{F}(x_i;\Xi)\partial/\partial\Xi F(x_i;\Xi)}{\left[\overline{F}(x_i;\Xi)\left\{1-\delta^2F(x_i;\Xi)\right\}\right]}.\end{aligned} \quad (26)$$

The MLEs $(\widehat{\alpha}_{\mathrm{MLE}},\widehat{\delta}_{\mathrm{MLE}},\widehat{\Xi}_{\mathrm{MLE}})$ of the parameters (α,δ,Ξ) can be obtained by solving

$$\begin{aligned}\frac{\partial}{\partial\alpha}L(x;\alpha,\delta,\Xi)&=0,\\ \frac{\partial}{\partial\delta}L(x;\alpha,\delta,\Xi)&=0,\\ \frac{\partial}{\partial\Xi}L(x;\alpha,\delta,\Xi)&=0.\end{aligned} \quad (27)$$

5.2. Simulation Study. In this subsection, the performance of MLEs $(\widehat{\alpha}_{\mathrm{MLE}},\widehat{\delta}_{\mathrm{MLE}},\widehat{\Xi}_{\mathrm{MLE}})$ of (α,δ,Ξ) is evaluated by conducting a simulation study. The process is carried out as follows:

(i) A sequence of RS (random sample), say $X_1,X_2,\ldots,X_n$ of sizes, $n=25,50,\ldots,975$, and 1000 are obtained from the NG-Weibull distribution.

(ii) Two statistical quantities (MSE and bias) are considered to evaluate $\widehat{\alpha}_{\mathrm{MLE}},\widehat{\delta}_{\mathrm{MLE}}$, and $\widehat{\Xi}_{\mathrm{MLE}}$. These quantities are, respectively, given by

$$\begin{aligned}\mathrm{Bias}(\widehat{\nu})&=\frac{1}{1000}\sum_{i=1}^{1000}(\widehat{\nu}-\nu),\\ \mathrm{MSE}(\widehat{\nu})&=\frac{1}{1000}\sum_{i=1}^{1000}(\widehat{\nu}-\nu)^2.\end{aligned} \quad (28)$$

(iii) Two different cases are considered for the four parameters of the NG-Weibull distribution.

Table 1: For $\alpha = 0.4, \delta = 0.5, \theta = 0.6, \gamma = 0.5$, the simulation results of the NG-Weibull model.

n	Parameters	MLEs	MSEs	Biases
25	α	1.3776477	5.3789876	0.9706542
	δ	0.6128757	0.1738765	0.1129876
	θ	0.5867868	0.0912987	0.1398765
	γ	1.2228970	2.6349807	0.7226754
100	α	0.8647665	2.0033452	0.4642937
	δ	0.5660988	0.1513287	0.0963454
	θ	0.5833452	0.0732938	0.1019876
	γ	0.9570987	1.2232093	0.4578769
375	α	0.5406557	1.2409876	0.2409865
	δ	0.4992098	0.1079871	0.0665438
	θ	0.6023452	0.0409090	0.0876548
	γ	0.7062093	0.9398782	0.2062983
525	α	0.4780987	0.7488761	0.0788967
	δ	0.4920909	0.4698762	0.0198764
	θ	0.6012837	0.0387658	0.0209873
	γ	0.6153546	0.1609872	0.1152342
750	α	0.4232341	0.4876754	0.0238972
	δ	0.473–098	0.2609872	0.0098766
	θ	0.607–092	0.0198765	0.0088735
	γ	0.5652789	0.0999876	0.0659897
950	α	0.4127658	0.0550987	0.0120981
	δ	0.4830987	0.0188976	0.0077653
	θ	0.6050956	0.0015098	0.0059887
	γ	0.5682376	0.0876543	0.0498706
1000	α	0.4042833	0.0300895	0.0040892
	δ	0.5010293	0.0029674	0.0038765
	θ	0.6085412	0.0010876	0.0024498
	γ	0.5182398	0.0198763	0.0286710

(iv) One thousand repetitions are done to compute the bias and the MSE for $\hat{\alpha}_{\mathrm{MLE}}, \hat{\delta}_{\mathrm{MLE}}$, and $\hat{\Xi}_{\mathrm{MLE}}$.

The numerical results are provided in Tables 1–3. From the numerical illustration presented in Tables 1–3, it can be observed that as sample size n increases (i) the values of the MLEs $(\hat{\alpha}_{\mathrm{MLE}}, \hat{\delta}_{\mathrm{MLE}}, \hat{\Xi}_{\mathrm{MLE}})$ get closer and closer to the parameters (α, δ, Ξ), and (ii) the values of the MSE and bias of $\hat{\alpha}_{\mathrm{MLE}}, \hat{\delta}_{\mathrm{MLE}}$, and $\hat{\Xi}_{\mathrm{MLE}}$ tend to zero.

6. Analyzing the COVID-19 Data Set

This section is devoted to demonstrating the usefulness of the NG-X method by applying its special model (NG-Weibull distribution) to analyze the COVID-19 data set. The data set represents the mortality rates of COVID-19 infected person in Canada. Corresponding to the COVID-19 data set, the fitting results of the NG-Weibull distribution are compared with three other competing models. These models are given by

(i) Weibull distribution

$$G(x;\Xi) = 1 - e^{-\gamma x^{\theta}}, \quad x \geq 0, \tag{29}$$

where $\theta > 0, \gamma > 0$.

(ii) K-Weibull distribution

$$G(x;a,b,\Xi) = 1 - \left[1 - \left(1 - e^{-\gamma x^{\theta}}\right)^{a}\right]^{b}, \quad x \geq 0, \tag{30}$$

where $a > 0, b > 0, \theta > 0, \gamma > 0$.

(iii) Alpha power transformed Weibull (APT-Weibull) distribution

$$G(x;\alpha_1 \Xi) = \frac{\alpha_1^{\left(1-e^{-\gamma x^{\theta}}\right)} - 1}{\alpha_1 - 1}, \quad 4x \geq 0, \tag{31}$$

where $\alpha_1 > 0, \alpha_1 \neq 1, \theta > 0, \gamma > 0$.

Some commonly used analytical goodness of fit measures are considered to show which distribution better fits the data. These measures are given by

$$\begin{aligned}
\mathrm{AIC} &= 2p - 2\ell(\Xi),\\
\mathrm{BIC} &= p\,\log(n) - 2\ell(\Xi),\\
\mathrm{CBIC} &= \frac{2np}{n-p-1} - 2\ell(\Xi),\\
\mathrm{HQIC} &= 2p\,\log(\log(n)) - 2\ell(\Xi),\\
\mathrm{AD} &= -n - \frac{1}{n}\sum_{u=1}^{n}(2u-1)\\
&\quad \cdot \left[\log G(x_u) + \log\{1 - G(x_{n-u+1})\}\right],\\
\mathrm{CM} &= \frac{1}{12n} + \sum_{u=1}^{n}\left[\frac{2u-1}{2n} - G(x_u)\right]^2.
\end{aligned} \tag{32}$$

Table 2: For $\alpha = 0.5, \delta = 0.5, \theta = 0.6, \gamma = 1$, the simulation results of the NG-Weibull model.

n	Parameters	MLEs	MSEs	Biases
25	α	1.5398875	4.7660923	1.0399876
	δ	0.5399585	0.1751234	0.1295234
	θ	0.5996539	0.3132938	0.1089097
	γ	1.5190987	2.2270984	0.5199870
100	α	0.9698768	2.0767589	0.4698795
	δ	0.521–098	0.1064256	0.1087653
	θ	0.5948764	0.1786567	0.0987609
	γ	1.4513098	1.5681234	0.4517563
375	α	0.6288765	0.6017541	0.1280987
	δ	0.470–098	0.0954378	0.0865309
	θ	0.6093245	0.1398764	0.0554233
	γ	1.1638769	0.5044122	0.2798734
525	α	0.5820987	0.3599872	0.0974076
	δ	0.4726543	0.0654369	0.0680969
	θ	0.6090987	0.1075489	0.0187653
	γ	1.1099875	0.3254165	0.1094234
750	α	0.5342098	0.1350987	0.0347865
	δ	0.4732390	0.0174578	0.0260987
	θ	0.6072760	0.0498763	0.0076538
	γ	1.0491234	0.1721982	0.0493452
950	α	0.5062091	0.0477892	0.0068759
	δ	0.4712342	0.0104568	0.0284327
	θ	0.6082095	0.0026789	0.0064231
	γ	1.0347654	0.1054217	0.0346899
1000	α	0.5050471	0.0278901	0.0047887
	δ	0.4976540	0.0085674	0.0206541
	θ	0.6079876	0.0009563	0.0030982
	γ	1.0198769	0.0680980	0.0156983

Table 3: For $\alpha = 0.2, \delta = 0.5, \theta = 0.6, \gamma = 0.3$, the simulation results of the NG-Weibull model.

n	Parameters	MLEs	MSEs	Biases
25	α	0.6489876	1.9180987	0.4488765
	δ	0.6398765	0.1660989	0.1399780
	θ	0.5706543	0.1265439	0.0892543
	γ	1.1387890	2.8809876	0.8380987
100	α	0.3967654	0.6728765	0.1966543
	δ	0.6150987	0.1296453	0.1158987
	θ	0.5635432	0.1010987	0.0667890
	γ	0.8476709	1.5413458	0.5478654
375	α	0.2497586	0.1057654	0.1094410
	δ	0.5290987	0.0958909	0.0997981
	θ	0.5895799	0.0966549	0.0409783
	γ	0.4578790	0.2116543	0.1578091
525	α	0.2309876	0.0990987	0.0800876
	δ	0.5136987	0.0770990	0.0536543
	θ	0.5930989	0.0327543	0.0166541
	γ	0.4306543	0.1607890	0.1309876
750	α	0.2168761	0.0187654	0.0460987
	δ	0.5167568	0.0220987	0.0267659
	θ	0.5959876	0.0174569	0.0090987
	γ	0.3950987	0.0880987	0.0954569
950	α	0.2139801	0.0106432	0.0139878
	δ	0.5059875	0.0202098	0.0090645
	θ	0.5965432	0.0015789	0.0060989
	γ	0.3295490	0.0516748	0.0595437
1000	α	0.2100561	0.0100658	0.0101432
	δ	0.4910982	0.0091276	0.0028976
	θ	0.5996534	0.0010019	0.0019909
	γ	0.3109755	0.0340987	0.0287654

Table 4: The values of the MLEs along with standard errors in parentheses using the COVID-19 data.

Models	α	δ	γ	θ	a	b	α_1
NG-Weibull	0.7358 (0.5199)	0.9864 (0.0230)	1.5708 (1.1629)	1.0923 (0.4809)	—	—	—
Weibull	—	—	0.0139 (0.0075)	3.3089 (0.3591)	—	—	—
K-Weibull	—	—	0.7011 (1.8876)	1.0555 (1.9107)	13.1558 (40.5978)	2.1563 (6.6188)	—
APT-Weibull	—	—	0.0675 (0.0509)	2.5153 (0.4365)	—	—	7.2916 (7.7550)

Table 5: The analytical measures of the NG-Weibull and other competitive models for the COVID-19 data.

Models	AIC	BIC	CAIC	HQIC	CM	AD
NG-Weibull	102.7614	109.0955	104.0518	104.9722	0.0647	0.3741
Weibull	106.9487	110.1157	107.3123	108.0541	0.1726	0.9904
K-Weibull	104.0038	110.3379	105.2941	106.2145	0.0924	0.5373
APT-Weibull	107.8743	112.6249	108.6243	109.5324	0.1531	0.8723

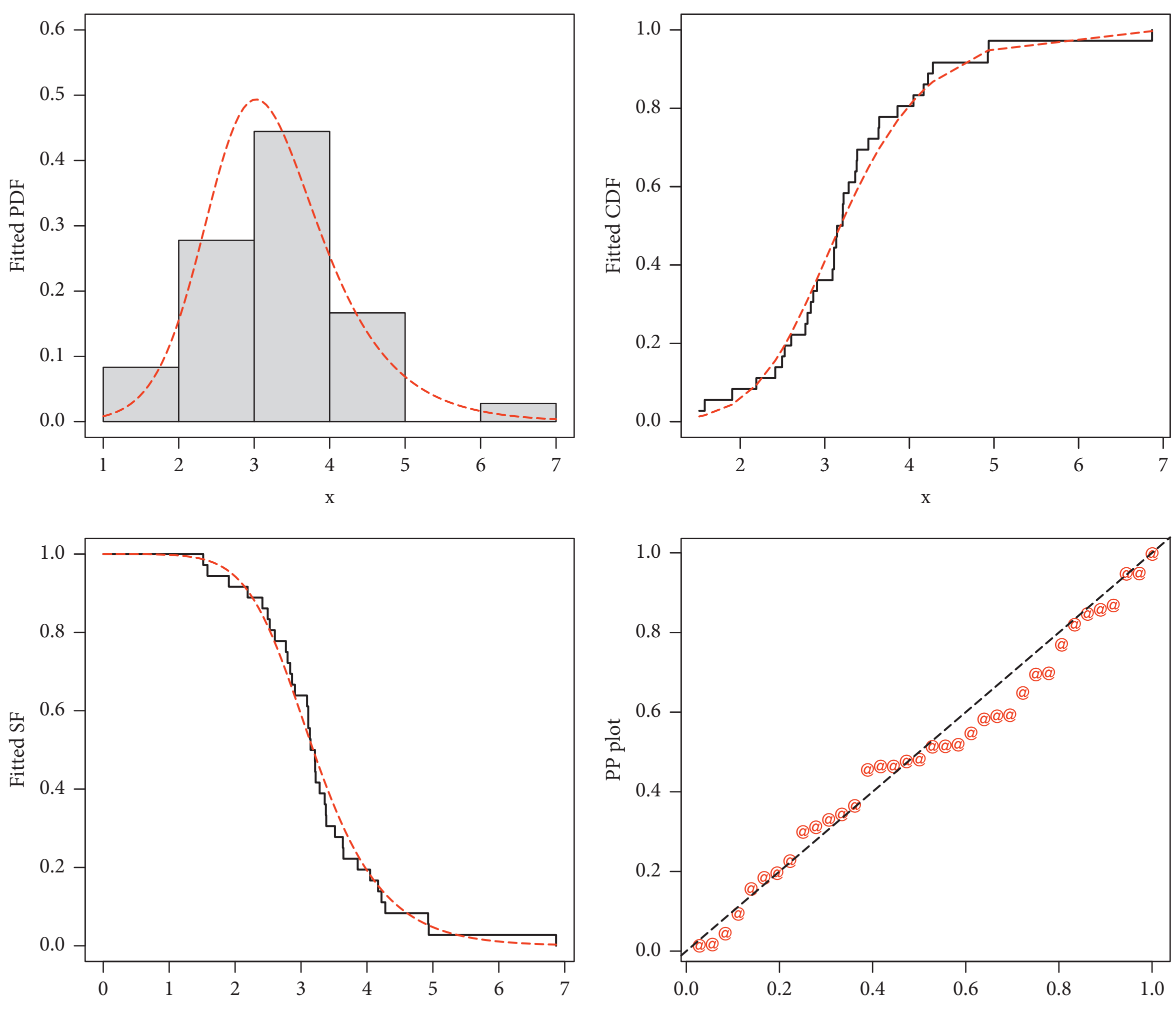

Figure 4: The fitted PDF, CDF, SF, and PP plots of the NG-Weibull distribution.

Corresponding to the COVID-19 data set, Table 4 gives the values of the MLEs with standard errors enclosed in parentheses. However, Table 5 gives the analytical measures of the competing models for the COVID-19 data set.

Based on the results in Table 4, it is clear that the implementation of the NG-Weibull model is a suitable choice to use for modeling the COVID-19 data set. Furthermore, for supporting the numerical results obtained in Table 4, the estimated PDF, CDF, SF, and PP (probability-probability) plots of the NG-Weibull model are presented in Figure 4. The plots in Figure 4 visually confirm the close-fitting of the NG-Weibull distribution.

7. Concluding Remarks

In this paper, a new family of distributions called the NG-X family was introduced. The NG-X method was introduced using the exponential distribution in combination with the T-X approach. The purpose of introducing the NG-X family was to update the distributional flexibility of the classical/modified distributions for modeling data in applied sectors. Based on the NG-X distributions approach, an updated/modified form of the Weibull model called the NG-Weibull was studied in depth. The HT characteristics of the NG-Weibull distribution were proved empirically. Based on the NG-Weibull distribution, a simulation study was also conducted. The simulation study showed that the values of MLEs are quite steady and are close to the true parameters' values as the sample gets increased. Finally, to establish the applicability of the NG-Weibull distribution, a data set related to the mortality rates of COVID-19 infected people was analyzed. The performance of the NG-Weibull distribution was compared with three other competitors. Based on the different six criteria, it is shown that the NG-Weibull distribution outperforms the competitive models [5].

Data Availability

The data are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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