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Research Article

Prime Decomposition of Zero Divisor Graph in a Commutative Ring

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Let R be a commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of a commutative ring R, whose vertices are nonzero zero divisors of Z_n, and such that the two vertices u, v are adjacent if n divides uv. In this paper, we introduce the concept of prime decomposition of zero divisor graph in a commutative ring and also discuss some special cases of $\Gamma(Z_{3p})$, $\Gamma(Z_{5p})$, $\Gamma(Z_{7p})$, and $\Gamma(Z_{pq})$.

1. Introduction

Let us consider the finite, simple, and undirected graphs to discuss about the prime decomposition in the form of zero divisor graphs [1].

Here, complete graph with n vertices is represented by K_n and the complete bipartite graph is represented by $K_{m,n}$. Also, a path of length k is by S_k. (ie) $S_k = K_{1,k}$. Let K^{β}_{1,m_i} be the star graph of β copies where m be the prime number with i tupes. Let L be the family of all subgraph and L is an edge disjoint decomposition with α_i copies of H_i, where $i \in \{1, 2, \ldots, r\}$ and α is a positive integer. Furthermore, if each H_i $(i \in \{1, 2, 3, \ldots, r\})$ is isomorphic to a graph H, then we say that G has an $H-$ decomposition.

The zero divisor graphs play an important role in algebraic properties and algebraic structures such as a commutative ring. The concept of a zero divisor graph is a commutative ring was proposed by Beck's [2]. The general terminology and notation everything based on the papers [[3–7]]. In this paper, we investigate the prime decomposition of $\Gamma(Z_{pq})$ into K_{1,m_i} star graph with m_i edges and obtain the following results. We already investigated the concept of decomposition of zero divisor graph for some special cases $\Gamma(Z_n)$ where $n = pq, p^2, p^2q$ [8].

Let R be the ring and $G(R)$ be the graph of the ring. If two distinct vertices $r.s \in G(R)$ are adjacent then we can represent it as $rs = 0$. Beck's initiated his work with a chromatic number of the graph. In this paper, we discuss the prime decomposition of Z_{pq} into K_{1,m_i} star graph with m_i edges.

2. Preliminaries

Definition 1 (see [4]). Let R be a commutative ring (with 1) and let Z (R) be its set of zero-divisors. We associate a (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisor of R, and for distinct $x, y \in Z(R)^*$ the vertices x and y are adjacent if and only if $xy = 0$. Thus, $\Gamma(R)$ is the empty graph if and only if R is an integral domain.

Definition 2. A graph G is decomposable into $H_1, H_2, H_3, \ldots, H_k$ if G has subgraphs $H_1, H_2, H_3, \ldots, H_k$ such that

(1) Each edge of G belongs to one of the $H_{i'}s$ for some $i = 1, 2, 3, ..., k$
(2) If $i \neq j$, then H_i and H_j have no edges in common.

3. Prime Decomposition of Zero Divisor Graph

Theorem 1. *If p is any prime number, $p > 3$ and $r \geq 0$ then the graph $\Gamma(Z_{3p})$ admits a $\{K_{1,m_i}^{2|p-1/m_i|}, K_{1,r}^{2}\}-$ prime decomposition if and only if $p-1 \equiv r\,(\mathrm{mod}\, m_i)$ where m_i is any prime less than p.*

Proof 1. Suppose that p is any prime number, $p > 3$ and $r \geq 0$, we have, $\Gamma(Z_{3p})$ be the nonzero zero divisor graph. The vertex set of $\Gamma(Z_{3p})$ is $\{3, 6, 9, \ldots, 3(p-1), p, 2p\}$.

Case 1. Let us consider $p-1|m_i$. If the graph $\Gamma(Z_{3p})$ is prime decomposition into $2\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i} then there exists $p-1 \equiv r\,(\mathrm{mod}\, m_i)$ where m_i is any prime number less than p.

Case 2. Let us consider $p-1|/m_i$ then their exists a remainder r. If the graph $\Gamma(Z_{3p})$ is prime decomposition into $2\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i} and (copies of $K_{1,r} \cong r$ copies of $K_{1,2}$) then there exists $p-1 \equiv r\,(\mathrm{mod} m_i)$ where m_i is any prime number less than p. Above-given case 1 and case 2 clearly show that the prime decomposition of $\Gamma(Z_{3p})$ into $\{K_{1,m_i}^{2|p-1/m_i|}, K_{1,r}^{2}\}$ then $p-1 \equiv r\,(\mathrm{mod}\, m_i)$.

Conversely, suppose that $p-1 \equiv r\,(\mathrm{mod}\, m_i)$ where m is any prime less than p.

Let us consider $p = 5$ and $m = 2, 3$. If $4 \equiv 0\,(\mathrm{mod} 2)$ then there exists 4 copies of $K_{1,2}$. If $4 \equiv 1\,(\mathrm{mod} 3)$ then there exists 2 copies of $K_{1,3}$ and (2 copies of $K_{1,1} \cong$ one copy of $K_{1,2}$).

Let us take $p = 7$ and $m = 2, 3, 5$. If $6 \equiv 0\,(\mathrm{mod} 2)$ then there exists 6 copies of $K_{1,2}$. If $6 \equiv 0\,(\mathrm{mod} 3)$ then there exists 4 copies of $K_{1,3}$. If $6 \equiv 1\,(\mathrm{mod} 5)$ then there exists 2 copies of $K_{1,5}$ and (2 copies of $K_{1,1} \cong$ one copy of $K_{1,2}$).

In general, take p is any prime number $p > 3$ and m_i is prime numbers less than p. Clearly, If $p-1 \equiv r\,(\mathrm{mod} m_i)$ then $\Gamma(Z_{3p})$ is a prime decomposition into $2\lfloor p-1/m_i \rfloor$ copies of $K_{1,m}$ and (2 copies of $K_{1,r} \cong r$ copies of $K_{1,2}$). Hence the proof (see Figure 1).

Example 1. Let us take $p = 7$ and $q = 11$ the graph $\Gamma(Z_{15})$ as the example of Theorem 1.

Theorem 2. *If pis any prime number, $p > 5$ and $r \geq 0$ then the graph $\Gamma(Z_{5p})$ admits a $\left\{K_{1,m_i}^{4\lfloor p-1/m_i \rfloor}, K_{1,r}^{4}\right\}-$ prime decomposition if and only if $p-1 \equiv r\,(\mathrm{mod}\, m_i)$ where m_i is any prime less than p.*

Proof 2. Suppose that p is any prime number, $p > 5$, we have, $\Gamma(Z_{5p})$ be the nonzero zero divisor graph with isomorphic to $K_{4,p-1}$. The vertex set of $\Gamma(Z_{5p})$ is $\{5, 10, 15, \ldots, 5(p-1), p, 2p, 3p, 4p\}$.

Case 3. Let us take $p-1|m_i$. If the prime decomposition of $\Gamma(Z_{5p})$ into $4\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i} then there exists $p-1 \equiv r\,(\mathrm{mod} m_i)$ where m_i is any prime numbers less than p.

Case 4. Let us take $p-1|/m_i$. If the prime decomposition of $\Gamma(Z_{5p})$ into $4\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i}. Clearly, the

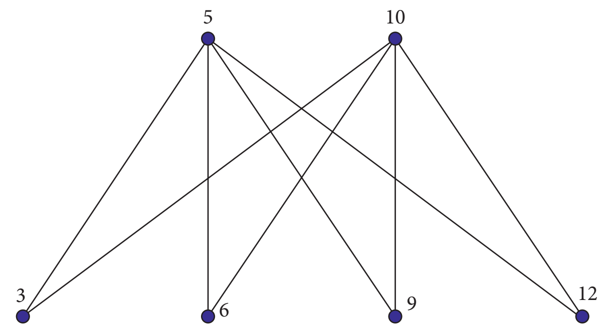

Figure 1: $\Gamma(Z_{15})$.

remaining edges are (r copies of $K_{1,4} \cong$ 4 copies of $K_{1,r}$) where r is the remainder of $\{p-1, m_i\}$. Then, there exists $p-1 \equiv r\,(\mathrm{mod} m_i)$. Clearly, shows that the above cases prime decomposition of $\Gamma(Z_{5p})$ into $\left\{K_{1,m_i}^{4\lfloor p-1/m_i \rfloor}, K_{1,r}^{4}\right\}$ then $p-1 \equiv r\,(\mathrm{mod}\, m_i)$.

Conversely, suppose that $p-1 \equiv r\,(\mathrm{mod} m_i)$ where m is any prime numbers less than p.

Let us take $p = 7$ and $m_i = 2, 3, 5$. If $6 \equiv 0\,(\mathrm{mod} 2)$ then 12 copies of $K_{1,2}$. If $6 \equiv 0\,(\mathrm{mod} 3)$ then 8 copies of $K_{1,3}$. If $6 \equiv 1\,(\mathrm{mod} 5)$ then 4 copies of $K_{1,5}$ and (4 copies of $K_{1,1} \cong$ one copy of $K_{1,4}$).

Let us take $p = 11$ and $m_i = 2, 3, 5, 7$. If $10 \equiv 0\,(\mathrm{mod} 2)$ then 20 copies of $K_{1,2}$. If $10 \equiv 1\,(\mathrm{mod} 3)$ then 12 copies of $K_{1,3}$ and (4 copies of $K_{1,1} \cong$ one copy of $K_{1,4}$. If $10 \equiv 0\,(\mathrm{mod} 5)$ then 8 copies of $K_{1,5}$. If $10 \equiv 3\,(\mathrm{mod} 7)$ then 4 copies of $K_{1,7}$ and (4 copies of $K_{1,3} \cong$ 3 copies of $K_{1,4}$).

In general, take p is any prime number $p > 5$ and m_i is prime numbers less than p. Clearly, If $p-1 \equiv r\,(\mathrm{mod} m_i)$ then $\Gamma(Z_{5p})$ is a prime decomposition into $4\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i} and (4 copies of $K_{1,r} \cong r$ copies of $K_{1,4}$). Hence the proof (see Figure 2).

Example 2. Let us take $p = 7$ and $q = 11$ the graph $\Gamma(Z_{15})$ as the example of Theorem 2

Theorem 3. *Ifp is any prime number, $p > 7$ and $r \geq 0$ then the graph $\Gamma(Z_{7p})$ admits a $\left\{K_{1,m_i}^{6\lfloor p-1/m_i \rfloor}, K_{1,r}^{6}\right\}-$ prime decomposition if and only if $p-1 \equiv r\,(\mathrm{mod} m_i)$ where m_i is any prime less than p.*

Proof 3. Suppose that p is any prime number, $p > 7$, we have, $\Gamma(Z_{7p})$ be the nonzero zero divisor graph with isomorphic to $K_{6,p-1}$. The vertex set of $\Gamma(Z_{6p})$ is $\{7, 14, 21, \ldots, 7(p-1), p, 2p, 3p, 4p, 5p, 6p\}$.

Case 5. Let us take $p-1|m_i$ the prime decomposition of $\Gamma(Z_{7p})$ into $6\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i} then there exists $p-1 \equiv r\,(\mathrm{mod}\, m_i)$ where m_i is any prime numbers less than p.

Case 6. Let us take $p-1|/m_i$ the prime decomposition of $\Gamma(Z_{7p})$ into $6\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i}. Clearly, the remaining edges are (r copies of $K_{1,6} \cong$ 6 copies of $K_{1,r}$) where r is the remainder of $\{p-1, m_i\}$. Then, there exists $p-1 \equiv r\,(\mathrm{mod}\, m_i)$. Clearly, shows that the above-given

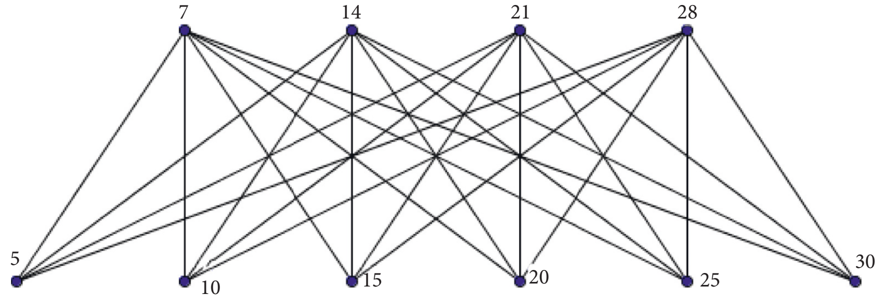

Figure 2: $\Gamma(Z_{35})$.

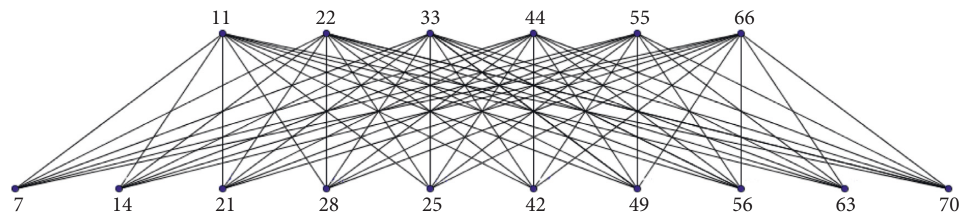

Figure 3: $\Gamma(Z_{77})$.

cases prime decomposition of $\Gamma(Z_{7p})$ into $\left\{K_{1,m_i}^{6\lfloor p-1/m_i \rfloor}, K_{1,r}^{6}\right\}$ then $p-1 \equiv r \,(\mathrm{mod}\, m_i)$.

Conversely, suppose that $p-1 \equiv r\,(\mathrm{mod}\, m_i)$ where m_i is any prime numbers less than p. Let us take $p = 11$ and $m_i = 2, 3, 5, 7$. If $10 \equiv 0\,(\mathrm{mod} 2)$ then 30 copies of $K_{1,2}$. If $10 \equiv 1\,(\mathrm{mod} 3)$ then 18 copies of $K_{1,3}$ and (6 copies of $K_{1,1} \cong$ one copy of $K_{1,6}$). If $10 \equiv 0\,(\mathrm{mod} 5)$ then 12 copies of $K_{1,5}$. If $10 \equiv 3\,(\mathrm{mod} 7)$ then 6 copies of $K_{1,7}$ and (6 copies of $K_{1,3} \cong$ 3 copies of $K_{1,6}$). Let us take $p = 13$ and $m_i = 2, 3, 5, 7, 11$. If $12 \equiv 0\,(\mathrm{mod} 2)$ then 36 copies of $K_{1,2}$. If $12 \equiv 0\,(\mathrm{mod} 3)$ then 24 copies of $K_{1,3}$. If $12 \equiv 2\,(\mathrm{mod} 5)$ then 12 copies of $K_{1,5}$ and (6 copies of $K_{1,2} \cong$ 2 copies of $K_{1,6}$). If $12 \equiv 5\,(\mathrm{mod} 7)$ then 6 copies of $K_{1,7}$ and (6 copies of $K_{1,5} \cong$ 5 copies of $K_{1,6}$). If $12 \equiv 1\,(\mathrm{mod} 11)$ then 6 copies of $K_{1,11}$ and (6 copies of $K_{1,1} \cong$ one copy of $K_{1,6}$). In general, take p is any prime number $p > 7$ and m_i is prime numbers less than p. Clearly, If $p - 1 \equiv r\,(\mathrm{mod} m_i)$ then $\Gamma(Z_{7p})$ is a prime decomposition into $6\lfloor p-1/m_i \rfloor$ copies of K_{1,m_i} and 6 copies of $K_{1,r} \cong r$ copies of $K_{1,6}$. Hence the proof (see Figure 3).

Example 3. Let us take $p = 7$ and $q = 11$ the graph $\Gamma(Z_{15})$ as the example of Theorem 3,

Theorem 4. *If p and q are any distinct prime numbers, $q > p$ and $r \geq 0$ then the graph $\Gamma(Z_{pq})$ admits a $\left\{K_{1,m_i}^{p-1\lfloor q-1/m_i \rfloor}, K_{1,r}^{p-1}\right\}$ – prime decomposition if and only if $q - 1 \equiv r\,(\mathrm{mod} m_i)$ where m_i is any prime less than p.*

Proof 4. Suppose that p and q are any distinct prime numbers, $q > p$, we have, $\Gamma(Z_{pq})$ be the nonzero zero divisor graph with isomorphic to $K_{p-1,q-1}$. The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, 3p, \ldots, p(q-1), q, 2q, 3q, \ldots, q(p-1)\}$.

Case 7. Let us take $q - 1 | m_i$. If prime decomposition of $\Gamma(Z_{pq})$ into $(p-1)\lfloor q-1/m_i \rfloor$ copies of K_{1,m_i} then there exists $q - 1 \equiv r\,(\mathrm{mod}\, m_i)$ where m_i is any prime numbers less than p.

Case 8. Let us take $q - 1 \nmid m_i$ the prime decomposition of $\Gamma(Z_{pq})$ into $(p-1)\lfloor q-1/m_i \rfloor$ copies of K_{1,m_i}. Clearly, the remaining edges are (r copies of $K_{1,q-1} \cong q-1$ copies of $K_{1,r}$) where r is the remainder of $\{q-1, m_i\}$. Then, there exists $q - 1 \equiv r\,(\mathrm{mod}\, m_i)$. Clearly, shows that the above cases prime decomposition of $\Gamma(Z_{pq})$ into $\left\{K_{1,m_i}^{(p-1)\lfloor q-1/m_i \rfloor}, K_{1,r}^{p-1}\right\}$ then $q - 1 \equiv r\,(\mathrm{mod}\, m_i)$.

Conversely, suppose that $q - 1 \equiv r\,(\mathrm{mod} m_i)$ where m_i is any prime numbers less than p.

Let us take p and q are any distinct prime numbers and m_i is any prime less than q. Clearly, the above-given theorems show. If $q - 1 \equiv r\,(\mathrm{mod} m_i)$ then $\Gamma(Z_{pq})$ is a prime decomposition into $(p-1)\lfloor q-1/m_i \rfloor$ copies of K_{1,m_i} and $p-1$ copies of $K_{1,r} \cong r$ copies of $K_{1,p-1}$. Hence the proof.

4. Conclusion

In this paper, we have defined the Prime Decomposition of the Zero Divisor Graph of a commutative ring. Also, some special cases of $\Gamma(Z_{3p})$, $\Gamma(Z_{5p})$, $\Gamma(Z_{7p})$, and $\Gamma(Z_{pq})$ are established. In the future, we will study some more properties and applications of Prime Decomposition of Zero Divisor Graph. [9].

Data Availability

The data utilized for the model development of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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