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Research Article

Characterizations of Hyperideals and Interior Hyperideals in Ordered Γ-Semihypergroups

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We give some conditions on ordered Γ-semihypergroups under which their interior hyperideal is equal to the hyperideal. In this paper, it is shown that in regular (resp., intraregular, semisimple) ordered Γ-semihypergroups, the hyperideals and the interior hyperideals coincide. To show the importance of these results, some examples and conclusions are provided.

1. Introduction and Preliminaries

Heidari and Davvaz [1] gave the idea of an ordered semihypergroup in 2011. Connection between ordered semihypergroups was studied by Tang et al. [2]. For some works on ordered Γ-semihypergroups, we may refer to Ref. [3].

The general structure of factorizable ordered hypergroupoids is studied in Ref. [4]. Tang et al. [5] and Tipachot and Pibaljommee [6] combined the fuzzy set with ordered hyperstructures and proposed the concept of fuzzy interior hyperideal and proved some results. The notion of hypergroups was initially founded by F. Marty [7] in 1934.

Recently, many authors, for example, those in Refs. [8, 9], have investigated on ordered hyperstructures. The paper given in Ref. [8] is a detailed study of interior hyperfilters in ordered Γ-semihypergroups. In Ref. [9], w-pseudo-orders in ordered (semi)-hyperrings were defined, and some important properties are investigated.

The notion of uni-soft interior Γ-hyperideals is investigated in Ref. [10]. Motivated by these studies, this note investigates the ordered Γ-semihypergroups that their interior hyperideal is equal to the hyperideal. We prove that in regular (resp., intraregular, semisimple) ordered Γ-semihypergroups, the concepts of interior Γ-hyperideals and Γ-hyperideals coincide.

Definition 1 (see [11]). Let H and Γ be two nonempty sets. Then, H is called a Γ-semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on H, i.e., $x\gamma y \subseteq H$ for every $x, y \in H$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in H$, we have $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Let A and B be two nonempty subsets of H. We define

$$A\Gamma B = \cup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\} = \bigcup_{\gamma\in\Gamma} A\gamma B. \tag{1}$$

Definition 2. An ordered Γ-semihypergroup $(H, \Gamma, \leq)$ is a Γ-semihypergroup (H, Γ) together with a partial order relation $\leq$ such that for any $h, h', x \in H$ and $\alpha \in \Gamma$, we have

$$h \leq h' \Rightarrow \begin{cases} x\alpha h \preceq x\alpha h', \\ h\alpha x \preceq h'\alpha x. \end{cases} \tag{2}$$

Here, $C \preceq D$ means that for any $c \in C$, there exists $d \in D$ such that $c \leq d$, where $\emptyset \neq C, D \subseteq H$.

Now, let

$(K] \coloneqq \{h \in H \mid h \leq k \text{ for some } k \in K\}$.

Then, $(H, \Gamma, \leq)$ can be called as follows:

(1) Regular (resp., intraregular) if $K \subseteq (K\Gamma H\Gamma K]$ (resp., $K \subseteq (H\Gamma K\Gamma K\Gamma(H])$) for every $K \subseteq H$

(2) $(H, \Gamma, \leq)$ is called semisimple if $K \subseteq (H\Gamma K\Gamma H\Gamma K\Gamma H]$ for every $K \subseteq H$

A nonempty subset K of H is called a Γ-hyperideal of H if

(1) $H\Gamma K \subseteq K$ and $K\Gamma H \subseteq K$

(2) $(K] \subseteq K$

Definition 3 (see [5]). A sub Γ-semihypergroup K of H is called an interior Γ-hyperideal (briefly, I-Γ-hyperideal) if

(1) $H\Gamma K\Gamma H \subseteq K$

(2) $(K] \subseteq K$

Remark 1. Note that each hyperideal of an ordered hyperstructure H is an I-Γ-hyperideal, but an I-Γ-hyperideal need not be hyperideal.

Example 1. Let $H = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$. Define the hyperoperation γ (as shown in Table 1) and (partial) order relation $\leq$ on H as follows:

$$\leq \; := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, d)\}. \quad (3)$$

Here, $A = \{a, c\}$ is an I-Γ-hyperideal of ordered Γ-semihypergroup H but not a Γ-hyperideal of H. Indeed, as $c\gamma d = \{a, b\}$ and $b \notin A$, A is not a Γ-hyperideal of H.

In this note, we investigate on the ordered Γ-semihypergroups that their interior hyperideal is equal to the hyperideal.

2. Main Results

This section aims to outline sufficient conditions for an I-Γ-hyperideal to be a Γ-hyperideal. We continue our study with the characterization of regular (resp., Intraregular, semisimple) ordered Γ-semihypergroup in terms of I-Γ-hyperideals.

Theorem 1. *Let $(H, \Gamma, \leq)$ be regular. Then, every I-Γ-hyperideal of H is a Γ-hyperideal.*

Proof. Assume that K is an I-Γ-hyperideal of S and $a \in K$. By hypothesis, there exist $h \in H$ and $\mu, \lambda \in \Gamma$ such that $a \leq a\mu h\lambda a$. It means that $K \subseteq (K\Gamma H\Gamma K]$. If $x \in H$ and $\gamma \in \Gamma$, then

$$\begin{aligned} a\gamma x &\leq (a\mu h\lambda a)\gamma x, \\ &= a\mu (h\lambda a)\gamma x, \\ &\subseteq K\Gamma H\Gamma K\Gamma H, \\ &= K\Gamma (H\Gamma K\Gamma H), \\ &\subseteq K\Gamma K, \\ &\subseteq K. \end{aligned} \quad (4)$$

Table 1: Table of γ for Example 1.

γ	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	$\{a, b\}$
d	a	a	$\{a, b\}$	$\{a, b, c\}$

Thus, $K\Gamma H \subseteq (K] = K$. Similarly, $H\Gamma K \subseteq K$. □

Example 2. Consider the Γ-semihypergroup (H, Γ) [12] (see Tables 2 and 3).

Now, we set

$$\leq: \; = \{(a, a), (b, a), (b, b), (b, c), (c, c), (d, d), (d, e)\}. \quad (5)$$

Clearly, $(H, \Gamma, \leq)$ is regular. The only I-Γ-hyperideals of H are $K_1 = \{d, e\}$ and $K_2 = H$. Both the I-Γ-hyperideals are Γ-hyperideal.

Theorem 2. *Let $(H, \Gamma, \leq)$ be intraregular. Then, we get those as follows:*

(1) Every I-Γ-hyperideal of H is a Γ-hyperideal

(2) Every I-Γ-hyperideal of H is idempotent

Proof

(1) Assume that K is an I-Γ-hyperideal of H and $a \in K$. By hypothesis, there exist $h, h' \in H$ and $\mu, \lambda, \delta \in \Gamma$ such that $a \leq h\mu a\lambda a\delta h'$. It means that $K \subseteq (H\Gamma K\Gamma K\Gamma H]$. If $x \in H$ and $\gamma \in \Gamma$, then

$$\begin{aligned} a\gamma x &\leq (h\mu a\lambda a\delta h')\gamma x, \\ &= h\mu a\lambda a\delta (h'\gamma x), \\ &\subseteq H\Gamma K\Gamma K\Gamma H, \\ &\subseteq H\Gamma K\Gamma H, \\ &\subseteq K. \end{aligned} \quad (6)$$

So, $K\Gamma H \subseteq (K] = K$. Similarly, $H\Gamma K \subseteq K$.

(2) Assume that K is an I-Γ-hyperideal of H. Then, we have

$$\begin{aligned} K &\subseteq (H\Gamma K\Gamma K\Gamma H], \\ &\subseteq (H\Gamma (H\Gamma K\Gamma K\Gamma H]\Gamma (H\Gamma K\Gamma K\Gamma H]\Gamma H], \\ &\subseteq (((H\Gamma H)\Gamma K\Gamma (K\Gamma H))]\Gamma ((H\Gamma K)\Gamma K\Gamma (H\Gamma H)], \\ &\subseteq ((H\Gamma K\Gamma H]\Gamma (H\Gamma K\Gamma H]], \\ &\subseteq (K\Gamma K]. \end{aligned} \quad (7)$$

Now, let $a \in \; K\Gamma K$]. Then, $a \leq k\gamma k'$ for some $k, k' \in K$ and $\gamma \in \Gamma$. By hypothesis, there exist $h, h' \in H$ and $\mu, \lambda, \delta \in \Gamma$ such that $a \leq h\mu a\lambda a\delta h'$. We have

Table 2: Table of γ for Example 2.

γ	a	b	c	d	e
a	$\{a,c\}$	b	$\{a,c\}$	$\{d,e\}$	e
b	b	b	b	$\{d,e\}$	e
c	$\{a,c\}$	b	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Table 3: Table of β for Example 2.

β	a	b	c	d	e
a	a	b	$\{a,c\}$	$\{d,e\}$	e
b	b	b	b	$\{d,e\}$	e
c	$\{a,c\}$	b	$\{a,c\}$	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

$$\begin{aligned} a &\preceq h\mu a\lambda a\delta h', \\ &\preceq h\mu (k\gamma k')\lambda (k\gamma k')\delta h', \\ &\subseteq H\Gamma K\Gamma H, \\ &\subseteq K. \end{aligned} \tag{8}$$

Thus, $a \in (K]$ and so $(K\Gamma K]\subseteq K$. □

Example 3. Consider the Γ-semihypergroup (H,Γ) [13] (see Tables 4 and 5).

Now, we set

$$\preceq := \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c), (d,d), (e,e)cs\}. \tag{9}$$

Clearly, $(H,\Gamma,\preceq)$ is an intraregular ordered Γ-semihypergroup. The only I-Γ-hyperideals of H are $K_1=\{d,e\}$ and $K_2=H$. Both the I-Γ-hyperideals are Γ-hyperideal and idempotent.

Theorem 3. *Let $(H,\Gamma,\preceq)$ be a semisimple ordered Γ-semihypergroup. Then, every I-Γ-hyperideal of H is a Γ-hyperideal.*

Proof. Assume that K is an I-Γ-hyperideal of H and $a \in K$. By hypothesis, there exist $x,y,z \in H$ and $\alpha,\beta,\gamma,\delta \in \Gamma$ such that $a \preceq x\alpha a\beta y\gamma a\delta z$. It means that $K\subseteq (H\Gamma K\Gamma H\Gamma K\Gamma H]$. If $h \in H$ and $\lambda \in \Gamma$, then

$$\begin{aligned} a\lambda h &\preceq (x\alpha a\beta y\gamma a\delta z)\lambda h, \\ &= x\alpha a\beta y\gamma a\delta (z\lambda h), \\ &\subseteq H\Gamma K\Gamma H\Gamma K\Gamma (H\Gamma H), \\ &= H\Gamma K\Gamma (H\Gamma K\Gamma H)\Gamma H, \\ &\subseteq H\Gamma K\Gamma K\Gamma H, \\ &\subseteq H\Gamma K\Gamma H, \\ &\subseteq K. \end{aligned} \tag{10}$$

So, $K\Gamma H\subseteq (K] = K$. Similarly, $H\Gamma K\subseteq K$. □

Table 4: Table of γ for Example 3.

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,c\}$	c	$\{d,e\}$	e
b	$\{b,c\}$	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Table 5: Table of β for Example 3.

β	a	b	c	d	e
a	$\{b,c\}$	c	c	$\{d,e\}$	e
b	c	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Theorem 4. *$(H,\Gamma,\preceq)$ is semisimple if and only if every Γ-hyperideal of H is idempotent.*

Proof. (*Necessity*). Let K be a Γ-hyperideal of H. By hypothesis, we have

$$\begin{aligned} K &\subseteq (H\Gamma K\Gamma H\Gamma K\Gamma H], \\ &= ((H\Gamma K)\Gamma H\Gamma (K\Gamma H)], \\ &\subseteq (K\Gamma (H\Gamma K)], \\ &\subseteq (K\Gamma K]. \end{aligned} \tag{11}$$

Also,

$$\begin{aligned} (K\Gamma K] &\subseteq (S\Gamma K], \\ &\subseteq (K], \\ &= K. \end{aligned} \tag{12}$$

So, $K = (K\Gamma K]$, and it completes the proof.

Sufficiency. Let $a \in H$. We denote by $I_H(a)$ the Γ-hyperideal of H generated by a. Then, we get $I_H(a) = (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$.

By hypothesis, we have

$$\begin{aligned} a \in I_H(a) &= (I_H(a)\Gamma I_H(a)], \\ &= ((a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]\Gamma (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]], \\ &\subseteq (H\Gamma a\Gamma H\Gamma a\Gamma H]. \end{aligned} \tag{13}$$

Therefore, H is semisimple. □

Example 4. In Example 2,

$$\preceq := \{(a,a), (a,b), (b,b), (c,b), (c,c), (d,d), (d,e), (e,e)\}, \tag{14}$$

is a partial order relation. Clearly, $(H,\Gamma,\preceq)$ is semisimple. The only I-Γ-hyperideals of H are $K_1=\{d,e\}$ and $K_2=H$. Both the I-Γ-hyperideals are Γ-hyperideal and idempotent.

3. Conclusions

This paper gives some conditions under which the I-Γ-hyperideals are Γ-hyperideals. By Theorems 1–3, we prove that in a regular (resp., intraregular, semisimple) ordered hyperstructure H, every interior hyperideal of H is a hyperideal. By Theorems 3 and 4, H is a semisimple ordered hyperstructure if and only if every interior hyperideal of H is idempotent. Our future work will concentrate on some results which are related with the fuzzy interior hyperideals of ordered hyperstructures.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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