Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 2517602, 12 pages
https://doi.org/10.1155/2022/2517602

Research Article

A Hybrid Interpolation Method for Fractional PDEs and Its Applications to Fractional Diffusion and Buckmaster Equations

Ihtisham Ul Haq,[1] Nigar Ali,[1] Shabir Ahmad,[1] and Tayyaba Akram[2,3]

[1]*Department of Mathematics, University of Malakand, Chakdara, Dir Lower, Khyber Pakhtunkhwa, Pakistan*
[2]*Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok, Thailand*
[3]*School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia*

Correspondence should be addressed to Shabir Ahmad; shabirahmad2232@gmail.com

Received 30 April 2022; Accepted 26 May 2022; Published 23 June 2022

Academic Editor: Ibrahim Mahariq

This study presents a novel numerical method to solve PDEs with the fractional Caputo operator. In this method, we apply the Newton interpolation numerical scheme in Laplace space, and then, the solution is returned to real space through the inverse Laplace transform. The Newton polynomial provides good results as compared to the Lagrangian polynomial, which is used to construct the Adams–Bashforth method. This procedure is used to solve fractional Buckmaster and diffusion equations. Finally, a few numerical simulations are presented, ensuring that this strategy is highly stable and quickly converges to an exact solution.

1. Introduction

Recently, partial differential equations (PDEs) have been successfully used to handle many real problems in engineering and mathematical physics such as viscoelastic materials, seismic analysis, and viscous damping [1–3]. Over the past few decades, fractional calculus has been used for better analysis of mathematical models [4, 5]. Fractional operators have been implemented to study nonlocal behaviour of the physical model. Khan et al. investigated hidden attractors in four-dimensional dynamical systems with power law kernel [6]. Khan et al. analyzed the imperfect testing infection disease model using the fractional operator [7]. Some other applications of fractional calculus are listed in [8, 9]. Also, fractional PDEs have attracted the scientists due to their wide range of applications in various areas of science. For instance, Saifullah et al. investigated the Klein–Gordon equation with power law kernel [10]. Gulalai et al. analyzed nonlinear modified KdV equation under Atangana–Baleanu–Caputo derivative [11]. Fractional PDEs have memory terms; they are not the same as classical PDEs, and evaluating FPDEs numerically is much difficult than solving PDEs. So, it is essential to establish effective numerical techniques to deal with PFDEs. There are several analytical and numerical methods for solving fractional PDEs. Akram et al. studied applied extended cubic B-spline collocation scheme for time fractional subdiffusion equation, cable equation, and Klein–Gordan equation [12–14]. Ahmad et al. used Yang transfrom coupled with the homotopy perturbation method (HPM) to study fractional KdV and Fornberg–Whitham equations [15]. The Legendre wavelets method has been utilized to investigate parabolic fractional PDEs [16]. A biorthogonal Hermite cubic spline Galerkin method has been used to solve fractional Riccati equation [17]. Iqbal et al. applied the Elzaki transformation and the new iteration technique to study fractional-order Kaup–Kupershmidt equation [18]. Shah et al. utilized the Haar wavelet collocation method to analyze the fractional-order COVID-19 model [19]. Many other approaches have been used for computing solution of fractional PDEs [20, 21].

The Adams–Bashforth method has been recognized as a powerful numerical scheme for solving nonlinear partial differential equations with integer and noninteger derivatives. Gnitchogna and Atangana proposed a new numerical scheme which can be seen as an extension of the Adams–Bashforth one for partial differential equations using Lagrangian interpolation [22]. There is a need to expand Newton interpolation numerical schemes to PFDEs because of their accuracy and efficiency in integer-order and fractional derivatives [23]. To handle PFDE with Caputo derivative, we develop a numerical scheme that combines the Newton interpolation numerical scheme and Laplace transform. By removing one variable and converting a PFDE to an FDE using the Laplace transform, the newly generated FDE can be easily solved in Laplace space, and the result may then be returned to real space using the inverse transform. Finally, some examples are provided to assess the efficacy of the developed method.

The study is organized as follows: Section 2 provides the basic concepts of fractional calculus. Section 3 deals with numerical approximation of fractional PDEs using Newton polynomial. Sections 4 and 5 are the applications of the suggested scheme. Section 6 gives the numerical simulations of the proposed results. Last, the conclusion is provided in Section 7.

2. Basic Concepts of Caputo Operator

Here, we recall important definitions of Laplace transform and fractional derivatives [24].

Definition 1. The Caputo-type fractional derivative is given as

$$ {}_{0}^{c}D_{t}^{\mu_1}\mathscr{Y}(t) = \frac{1}{\Gamma(p-\mu_1)}\int_0^t (t-\xi)^{p-\mu_1-1}\frac{d}{d\xi}\mathscr{Y}(\xi)d\xi, \quad (1) $$

where $p-1<\mu_1<p\in\mathscr{N}$.

Definition 2. The Laplace transform (LT) of the function $\mathscr{Y}(t)$ is

$$ \mathscr{Y}(s) = \mathscr{L}\{\mathscr{Y}(t)\} = \int_0^\infty e^{-st}\mathscr{Y}(t)dt. \quad (2) $$

Definition 3. The LT of $(c/0)D_t^{\mu_1}\mathscr{Y}(t)$ is

$$ \mathscr{L}\left\{{}_{0}^{c}D_{t}^{\mu_1}\mathscr{Y}(t)\right\}(s) = S^{\mu_1}\mathscr{Y}(s) - \sum_{p=0}^{p-1}S^{k-\mu_1-1}\mathscr{Y}^{p}(0), \quad (3) $$

where $p-1<\mu_1<p\in\mathscr{N}$.

3. Numerical Approximations

Consider a general fractional nonlinear PDE as

$$ \frac{\partial^{\mu_1}\mathscr{Y}(\eta,t)}{\partial t^{\mu_1}} = \mathbb{L}\mathscr{Y}(\eta,t) + \mathbb{N}\mathscr{Y}(\eta,t), \quad (4) $$

where $\mathbb{N}$ is a nonlinear operator and $\mathbb{L}$ is a linear operator. Applying LT to (4), we get

$$ \mathscr{L}\left(\frac{\partial^{\mu_1}\mathscr{Y}(\eta,t)}{\partial t^{\mu_1}}\right) = \mathscr{L}(\mathbb{L}\mathscr{Y}(\eta,t)) + \mathscr{L}(\mathbb{N}\mathscr{Y}(\eta,t)). \quad (5) $$

Then,

$$ {}_{0}^{c}D_{t}^{\mu_1}\mathscr{Y}(s,t) = \mathscr{L}(\mathbb{L}\mathscr{Y}(\eta,t)) + \mathscr{L}(\mathbb{N}\mathscr{Y}(\eta,t)). \quad (6) $$

Let $\mathscr{Y}(s,t)=\mathscr{Y}(t)$ and $\mathscr{G}(t,\mathscr{Y}(t)) = \mathscr{L}(\mathbb{L}\mathscr{Y}(\eta,t)) + \mathscr{L}(\mathbb{N}\mathscr{Y}(\eta,t))$. Then, (6) becomes

$$ \begin{cases} {}_{0}^{c}D_{t}^{\mu_1}\mathscr{Y}(t) = \mathscr{G}(t,\mathscr{Y}(t)), \\ 0<t\le T, \quad 0<\mu_1\le 1,\ \mathscr{Y}(0)=\mathscr{Y}_0. \end{cases} \quad (7) $$

Using the Caputo fractional integral operator, we get

$$ \mathscr{Y}(t) = \mathscr{Y}_0 + \frac{1}{\Gamma(\mu_1)}\int_0^t (t-\xi)^{\mu_1-1}\mathscr{G}(\xi,\mathscr{Y}(\xi))d\xi. \quad (8) $$

For t_{p+1}, one gets

$$ \mathscr{Y}_{p+1} = \mathscr{Y}_0 + \frac{1}{\Gamma(\mu_1)}\int_0^{t_{p+1}} \left(t_{p+1}-\xi\right)^{\mu_1-1}\mathscr{G}(\xi,\mathscr{Y}(\xi))d\xi. \quad (9) $$

Next, we approximate the above equation as

$$ \mathscr{Y}_{p+1} = \mathscr{Y}_0 + \frac{1}{\Gamma(\mu_1)}\sum_{m=0}^{p}\int_{t_m}^{t_{m+1}} \left(t_{p+1}-\xi\right)^{\mu_1-1}\mathscr{G}(\xi,\mathscr{Y}(\xi))d\xi, \quad (10) $$

where $t_0=0$. So, the kernel $\mathscr{G}(\xi,\mathscr{Y}(\xi))$ might be approximated with in $[t_p,t_{p+1}]$ via Newton's polynomial. So,

$$ P_m(\xi) = \begin{cases} \mathscr{G}(t_m,\mathscr{Y}_m) + \dfrac{(\mathscr{G}(t_{m+1},\mathscr{Y}_{m+1}) - \mathscr{G}(t_m,\mathscr{Y}_m))}{\Delta t}(\xi-t_m) \quad \text{if } m=0, \\ \mathscr{G}(t_{m+1},\mathscr{Y}_{m+1}) + \dfrac{(\mathscr{G}(t_{m+1},\mathscr{Y}_{m+1}) - \mathscr{G}(t_m,\mathscr{Y}_m))}{\Delta t}(\xi-t_{m+1}) \quad \text{if } m\in\{1,2,\ldots,p\}, \\ +\dfrac{\mathscr{G}(t_{m+1},\mathscr{Y}_{m+1}) - 2\mathscr{G}(t_m,\mathscr{Y}_m) + \mathscr{G}(t_{m-1},\mathscr{Y}_{m-1})}{2(\Delta t)^2}(\xi-t_m)(\xi-t_{m+1}). \end{cases} \quad (11) $$

Substituting equation (11) in (10), we get

$$\begin{aligned}\mathscr{Y}_{p+1} &= \mathscr{Y}_0 + \frac{1}{\Gamma(\mu_1)}\int_0^{t_1}\left(\mathscr{G}_0 + \frac{(\mathscr{G}_1 - \mathscr{G}_1)}{\Delta t}\xi\right)\left(t_{p+1}-\xi\right)^{\alpha-1}\mathrm{d}\xi \\ &+ \frac{1}{\Gamma(\mu_1)}\sum_{m=1}^{p}\int_{t_m}^{t_{1+m}}\left(\mathscr{G}_m + \frac{(-\mathscr{G}_m + \mathscr{G}_{1+m})}{\Delta t}(\xi - t_{1+m}) + \frac{\mathscr{G}_{m+1} - 2\mathscr{G}_m + \mathscr{G}_{-1+m}}{2(\Delta t)^2}(\xi - t_m)(\xi - t_{1+m})\right) \\ &\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi.\end{aligned} \tag{12}$$

Using the linearity property of integral, we have

$$\begin{aligned}\mathscr{Y}_{p+1} &= \mathscr{Y}_0 + \frac{1}{\Gamma(\mu_1)}\mathscr{G}_0\int_0^{t_1}\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi \\ &+ \frac{1}{\Gamma(\mu_1)}\left(\frac{(\mathscr{G}_1 - \mathscr{G}_0)}{\Delta t}\right)\int_0^{t_1}\xi\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi \\ &+ \frac{1}{\Gamma(\mu_1)}\sum_{m=1}^{p}\mathscr{G}_{m+1}\int_{t_m}^{t_{m+1}}\left(t_{p+1}-\xi\right)^{\alpha-1}\mathrm{d}\xi \\ &+ \frac{1}{\Gamma(\mu_1)}\sum_{m=1}^{p}\frac{(\mathscr{G}_{m+1} - \mathscr{G}_m)}{\Delta t}\int_{t_m}^{t_{m+1}}(\xi - t_{m+1})\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi \\ &+ \frac{1}{(\mu_1)}\sum_{m=1}^{p}\left(\frac{\mathscr{G}_{m+1} - 2\mathscr{G}_m + \mathscr{G}_{m-1}}{2(\Delta t)^2}\right)\int_{t_m}^{t_{m+1}}(\xi - t_m)(\xi - t_{m+1})\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi.\end{aligned} \tag{13}$$

Now, we evaluate the integrals in (13) as

$$\int_0^{t_1}\xi\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi = \frac{(\Delta t)^{\mu_1+1}}{\mu_1(\mu_1+1)}\left[-(\mu_1 + p + 1)^{\mu_1} + (p+1)^{\mu_1+1} - p^{\mu_1+1}\right], \tag{14}$$

$$\int_{t_m}^{t_{m+1}}\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi = \frac{(\Delta t)^{\mu_1}}{\mu_1}\left((p+1-m)^{\mu_1} - (-m+p)^{\mu_1}\right), \tag{15}$$

$$\int_{t_m}^{t_{m+1}}(\xi - t_{m+1})\left(t_{p+1}-\xi\right)^{\mu_1-1}\mathrm{d}\xi = \frac{(\Delta t)^{\mu_1}}{\mu_1(\mu_1+1)}\left[(p+\mu_1-m)(p+1-m)^{\mu_1} - (-m+p)^{\mu_1+1}\right], \tag{16}$$

$$\int_{t_m}^{t_{m+1}}(\xi - t_m)(\xi - t_{m+1})\left(t_{p+1}-\xi\right)\mathrm{d}\xi = \frac{(\Delta t)^{\mu_1+2}}{\mu_1(\mu_1+1)(\mu_1+2)}\begin{bmatrix}(p+1-m)^{\mu_1}\left[2(-m+p)^2 - \mu_1(-m+1+p) + 2(-m+p)\right] \\ -(-m+p)^{\mu_1}\left[2(-m+p)^2 - \mu_1(-m+p) + 2(p-m)\right]\end{bmatrix}. \tag{17}$$

By substituting these integrals values into (13), we get

$$\begin{aligned}\mathcal{Y}_{p+1} &= \mathcal{Y}_0 + \frac{1}{\Gamma(\mu_1)}\mathcal{G}_0\left[\frac{(\Delta t)^{\mu_1}}{\mu_1}\left((p+1)^{\mu_1} - (p)^{\mu_1}\right)\right] \\ &+ \frac{1}{\Gamma(\mu_1)}\left(\frac{(\mathcal{G}_1 - \mathcal{G}_0)}{\Delta t}\right)\left[\frac{(\Delta t)^{\mu_1+1}}{\mu_1(\mu_1+1)}\left[(p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1+1)p^{\mu_1}\right]\right] \\ &+ \frac{1}{\Gamma(\mu_1)}\sum_{m=1}^{p}\frac{(\mathcal{G}_{m+1} - \mathcal{G}_m)}{\Delta t}\int_{t_m}^{t_{m+1}}(\xi - t_{m+1})\left(t_{p+1} - \xi\right)^{\mu_1-1}\mathrm{d}\xi \\ &+ \frac{1}{\Gamma(\mu_1)}\sum_{m=1}^{p}\mathcal{G}_{m+1}\left[\frac{(\Delta t)^{\mu_1}}{\mu_1}\left((-m+p+1)^{\mu_1} - (-m+p)^{\mu_1}\right)\right] \\ &+ \frac{1}{\Gamma(\mu_1)}\sum_{m=1}^{p}\left(\frac{\mathcal{G}_{m+1} - 2\mathcal{G}_m + \mathcal{G}_{m-1}}{2(\Delta t)^2}\right)\left[\frac{(\Delta t)^{\mu_1+2}}{\mu_1(\mu_1+1)(\mu_1+2)}\right. \\ &\left.\left[\begin{array}{l}(p+1-m)^{\mu_1}\left[2(-m+p)^2 - \mu_1(p+1-m) + 2(-m+p)\right] \\ -(-m+p)^{\mu_1}\left[2(-m+p)^2 - \mu_1(-m+p) + 2(-m+p)\right]\end{array}\right]\right].\end{aligned} \tag{18}$$

Rearranging the terms and simplifying, we have

$$\begin{aligned}\mathcal{Y}_{p+1} = \mathcal{Y}_0 &+ \frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\mathcal{G}_0\left[(p+1)^{\mu_1} - p^{\mu_1}\right]\frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}(\mathcal{G}_1 - \mathcal{G}_0)\left[(p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1-1)p^{\mu_1}\right] \\ &+ \frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\sum_{m=1}^{p}\mathcal{G}_{m+1}\left((-m+p+1)^{\mu_1} - (-m+p)^{\mu_1}\right) \\ &+ \frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}\sum_{m=1}^{p}(\mathcal{G}_{m+1} - \mathcal{G}_m)\left[(p+\mu_1-m)(p+1-m)^{\mu_1} - (-m+p)^{\mu_1+1}\right] \\ &+ \frac{(\Delta t)^{\mu_1}}{2\Gamma(\mu_1+3)}\sum_{m=1}^{p}(\mathcal{G}_{m+1} - 2\mathcal{G}_m + \mathcal{G}_{m-1})\left[\begin{array}{l}(p+1-m)^{\mu_1}\left[2(-m+p)^2 - \mu_1(p+1-m) + 2(-m+p)\right] \\ -(-m+p)^{\mu_1}\left[2(-m+p)^2 + 2(-m+p) - \mu_1(-m+p)\right]\end{array}\right].\end{aligned} \tag{19}$$

The inverse Laplace transform is used to get the solution as

$$\mathcal{Y}(\eta, t_{p+1}) = \mathcal{L}^{-1}\left[\begin{array}{c}\mathcal{Y}_0 + \frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\mathcal{G}_0\left[(p+1)^{\mu_1} - p^{\mu_1}\right]\frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}(\mathcal{G}_1 - \mathcal{G}_0)\left[(p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1-1)p^{\mu_1}\right] \\ + \frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\sum_{m=1}^{p}\mathcal{G}_{m+1}\left((p+1-m)^{\mu_1} - (-m+p)^{\mu_1}\right) + \frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}\sum_{m=1}^{p}(\mathcal{G}_{m+1} - \mathcal{G}_m)\left[(p+\mu_1-m)(p+1-m)^{\mu_1} - (-m+p)^{\mu_1+1}\right] \\ + \frac{(\Delta t)^{\mu_1}}{2\Gamma(\mu_1+3)}\sum_{m=1}^{p}(\mathcal{G}_{m+1} - 2\mathcal{G}_m + \mathcal{G}_{m-1})\left[\begin{array}{l}(p+1-m)^{\mu_1}\left[2(-m+p)^2 - \mu_1(p+1-m) + 2(-m+p)\right] \\ -(-m+p)^{\mu_1}\left[2(-m+p)^2 - \mu_1(-m+p) + 2(-m+p)\right]\end{array}\right]\end{array}\right]. \tag{20}$$

Furthermore, we can discretize equation (16) as follows:

$$\mathscr{Y}(\eta_m, t_{p+1}) = \mathscr{Y}_0 + \mathscr{L}^{-1}\left[\begin{array}{c} \frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\mathscr{G}_0\left[(p+1)^{\mu_1} - p^{\mu_1}\right]\frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}(\mathscr{G}_1 - \mathscr{G}_0)\left[(p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1-1)p^{\mu_1}\right] \\ +\frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\mathscr{G}_{m+1}^{p}\left((p+1-m)^{\mu_1} - (-m+p)^{\mu_1}\right) \\ +\frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}\left(\mathscr{G}_{m+1}^{p} - \mathscr{G}_{m}^{p}\right)\left[(p+\mu_1-m)(p+1-m)^{\mu_1} - (-m+p)^{\mu_1+1}\right] \\ +\frac{(\Delta t)^{\mu_1}}{2\Gamma(\mu_1+3)}\left(\mathscr{G}_{m+1}^{p} - 2\mathscr{G}_{m}^{p} + \mathscr{G}_{m-1}^{p}\right)\left[\begin{array}{c}(p+1-m)^{\mu_1}\left[2(-m+p)^2 - \mu_1(p+1-m) + 2(-m+p)\right] \\ -(-m+p)^{\mu_1}\left[2(-m+p)^2 - \mu_1(-m+p) + 2(-m+p)\right]\end{array}\right]\end{array}\right], \tag{21}$$

where $g(\eta_m, t_p) = g_m^p$. Equation (17) gives the numerical approximation of the fractional PDEs using Newton polynomial.

In the following sections, this novel method will be applied to solve a fraction diffusion problem and a fractional Buckmaster equation.

4. Applications to Fractional Diffusion Equation

Diffusion equation is parabolic PDE having many applications in real world phenomena. In physics, it represents the macroscopic behaviour of multiple microparticles in Brownian motion, which is caused by the particles' random motions and collisions. Here, we evaluate the accuracy and effectiveness of the newly proposed numerical method by applying it to a fractional diffusion model. Consider

$$\frac{\partial^{\mu_1} g(z,t)}{\partial t^{\mu_1}} = C\frac{\partial^2 g(z,t)}{\partial z^2}. \tag{22}$$

Using Laplace transform, we get

$${}_0^cD_t^{\mu_1} g(s,t) = C\left\{-g(0,t) - sg(0,t) + s^2 g(s,t)\right\}. \tag{23}$$

Let $g(s,t) = g(t)$ and $\mathscr{G}(t, g(t)) = \{s^2 g(s,t) - sg(0,t) - g(0,t)\}$.

Equation (19) can be rewritten as

$${}_0^cD_t^{\mu_1} g(t) = C\mathscr{G}(t, g(t)). \tag{24}$$

We already established that the solution to equation (22) in the Laplace space is given in equation (15). So,

$$\begin{aligned} g(t_{p+1}) = {} & Cg(t_0) + \frac{C(\Delta t)^{\mu_1}}{(\mu_1+1)}\mathscr{G}_0(t)\left[(p+1)^{\mu_1} - p^{\mu_1}\right]\frac{C(\Delta t)^{\mu_1-1}}{(\mu_1+2)}(\mathscr{G}_1(t) - \mathscr{G}_0(t))\left[(p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1-1)p^{\mu_1}\right] \\ & + \frac{C(\Delta t)^{\mu_1}}{(\mu_1+1)}\sum_{m=1}^{p}\mathscr{G}_{m+1}(t)\left((p+1-m)^{\mu_1} - (-m+p)^{\mu_1}\right) \\ & + \frac{(\Delta t)^{\mu_1-1}}{(\mu_1+2)}\sum_{m=1}^{p}C(\mathscr{G}_{m+1}(t) - \mathscr{G}_m(t))\left[-(-m+p)^{\mu_1+1} + (p+\mu_1-m)(p+1-m)^{\mu_1}\right] \\ & + \frac{(\Delta t)^{\mu_1}}{2(\mu_1+3)}\sum_{m=1}^{p}C(\mathscr{G}_{m+1}(t) + \mathscr{G}_{m-1}(t) - 2\mathscr{G}_m(t))\left[\begin{array}{c}(-m+p+1)^{\mu_1}\left[2(-m+p)^2 - \mu_1(p+1-m) + 2(-m+p)\right] \\ -(-m+p)^{\mu_1}\left[2(-m+p)^2 - \mu_1(-m+p) + 2(-m+p)\right]\end{array}\right]. \end{aligned} \tag{25}$$

Let us define the expression to simplify the formulas as

$$
\begin{aligned}
g\left(z_m, t_{p+1}\right) = {} & Cg\left(t_0\right) + \frac{C(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\delta_1^p \mathcal{G}_0 + \frac{C(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}\left(\mathcal{G}_1 - \mathcal{G}_0\right)\delta_2^p \\
& + \frac{p(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\mathcal{G}_{m+1}^p \delta_3^p + \frac{C(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)} C\left(\mathcal{G}_{m+1}^p - \mathcal{G}_m^p\right)\delta_4^p \\
& + \frac{(\Delta t)^{\mu_1}}{2\Gamma(\mu_1+3)} C\left(\mathcal{G}_{m+1}^p - 2\mathcal{G}_m^p + \mathcal{G}_{m-1}^p\right)\delta_5^p,
\end{aligned}
\tag{26}
$$

where $\delta_1^p = [(p+1)^{\mu_1} - p^{\mu_1}]$, $\delta_2^p = (p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1-1)p^{\mu_1}, \delta_3^p = (p+1-m) \quad \mu_1 - (-m+p)^{\mu_1}$, $\delta_4^p = (\mu_1 - m + p)(-m+p+1)^{\mu_1} - (-m+p)^{\mu_1+1}, \delta_5^p = 2(-m+p)^2 - \mu_1(p+1-m) + 2(-m+p), -(-m+p)^{\mu_1}(2(-m+p) + 2(-m+p)^2 - \mu_1 \ (-m+p))$.

The inverse Laplace transform is used to get the solution as

$$
\begin{aligned}
g\left(t_{p+1}\right) = {} & Cg\left(t_0\right) + \frac{C(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\delta_1^p \frac{\partial^2 g(z,t_0)}{\partial z^2} \\
& + \frac{\delta_2^p C(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}\left(\frac{\partial^2 g(z,t_1)}{\partial z^2} - \frac{\partial^2 g(z,t_0)}{\partial z^2}\right) \\
& + \frac{\delta_3^p C(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\frac{\partial^2 g(z,t_{m+1})}{\partial z^2} + \frac{\delta_4^p C(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)} \\
& \cdot\left(\frac{\partial^2 g(z,t_{m+1})}{\partial z^2} - \frac{\partial^2 g(z,t_m)}{\partial z^2}\right) \\
& + \frac{\delta_5^p C(\Delta t)^{\mu_1}}{2\Gamma(\mu_1+3)}\left(\frac{\partial^2 g(z,t_{m+1})}{\partial z^2} - 2\frac{\partial^2 g(z,t_m)}{\partial z^2}\right. \\
& \left. + \frac{\partial^2 g(z,t_{m-1})}{\partial z^2}\right).
\end{aligned}
\tag{27}
$$

Discretizing in the space variable, we get

$$
\begin{aligned}
g\left(t_{p+1}\right) = {} & Cg\left(t_0\right) + \frac{C(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\delta_1^p\left[g\left(z_m,t_0\right) + \frac{\left(g\left(z_{m+1},t_0\right) - g\left(z_m,t_0\right)\right)}{\Delta t}\right] + \frac{\delta_2^p C(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)} \\
& \cdot\left[\begin{array}{c}
\left(g\left(z_{m+1},t_1\right) + \dfrac{\left(g\left(z_{m+1},t_1\right) - g\left(z_m,t_1\right)\right)}{\Delta t} + \dfrac{g\left(z_{m+1},t_1\right) - 2g\left(z_m,t_1\right) + g\left(z_{m-1},t_1\right)}{2(\Delta t)^2}\right) \\
-\left(g\left(z_m,t_0\right) + \dfrac{\left(g\left(z_{m+1},t_0\right) - g\left(z_m,t_0\right)\right)}{\Delta t}\right)
\end{array}\right] \\
& + \frac{\delta_3^p C(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\left[g\left(z_{m+1},t_1\right)\frac{\left(g\left(z_{m+1},t_{p+1}\right) - g\left(z_m,t_{p+1}\right)\right)}{\Delta t} + \frac{g\left(z_{m+1},t_{p+1}\right) - 2g\left(z_m,t_{p+1}\right) + g\left(z_{m-1},t_{p+1}\right)}{2(\Delta t)^2}\right] \\
& + \frac{\delta_4^p C(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}\left[\begin{array}{c}
\left(g\left(z_{m+1},t_{p+1}\right) + \dfrac{\left(g\left(z_{m+1},t_{p+1}\right) - g\left(z_m,t_{p+1}\right)\right)}{\Delta t} + \dfrac{g\left(z_{m+1},t_{p+1}\right) - 2g\left(z_m,t_{p+1}\right) + g\left(z_{m-1},t_{p+1}\right)}{2(\Delta t)^2}\right) \\
-\left(g\left(z_{m+1},t_p\right) + \dfrac{\left(g\left(z_{m+1},t_p\right) - g\left(z_m,t_p\right)\right)}{\Delta t} + \dfrac{g\left(z_{m+1},t_p\right) - 2g\left(z_m,t_p\right) + g\left(z_{m-1},t_p\right)}{2(\Delta t)^2}\right)
\end{array}\right] \\
& + \frac{C\delta_5^p(\Delta t)^{\mu_1}}{2\Gamma(\mu_1+3)}\left[\begin{array}{c}
\left(g\left(z_{m+1},t_{p+1}\right) + \dfrac{\left(g\left(z_{m+1},t_{p+1}\right) + g\left(z_m,t_{p+1}\right)\right)}{\Delta t} + \dfrac{g\left(z_{m+1},t_{p+1}\right) - 2g\left(z_m,t_{p+1}\right) + g\left(z_{m-1},t_{p+1}\right)}{2(\Delta t)^2}\right) \\
-2\left(g\left(z_{m+1},t_p\right) + \dfrac{\left(g\left(z_{m+1},t_p\right) - g\left(z_m,t_p\right)\right)}{\Delta t} + \dfrac{g\left(z_{m+1},t_p\right) - 2g\left(z_m,t_p\right) + g\left(z_{m-1},t_p\right)}{2(\Delta t)^2}\right) \\
+\left(g\left(z_{m+1},t_{p-1}\right) + \dfrac{\left(g\left(z_{m+1},t_{p-1}\right) - g\left(z_m,t_{p-1}\right)\right)}{\Delta t} + \dfrac{g\left(z_{m+1},t_{p-1}\right) - 2g\left(z_m,t_{p-1}\right) + g\left(z_{m-1},t_{p-1}\right)}{2(\Delta t)^2}\right)
\end{array}\right].
\end{aligned}
\tag{28}
$$

Letting $g(z_m, t_p) = g_m^p$ and $\Delta t = h$, equation (24) becomes

$$g(t_{p+1}) = Cg(t_0) + \frac{\delta_1^p C h^{\mu_1}}{\Gamma(\mu_1+1)}\left[g_m^0 + \frac{g_{m+1}^1 - g_m^0}{h}\right] + \frac{\delta_2^p C h^{\mu_1-1}}{\Gamma(\mu_1+2)}\left[\left(g_{m+1}^1 + \frac{(g_{m+1}^1 - g_m^1)}{h} + \frac{g_{m+1}^1 - 2g_m^1 + g_{m-1}^1}{2h^2}\right) - \left(g_m^0 + \frac{g_{m+1}^1 - g_m^0}{h}\right)\right]$$
$$+ \frac{\delta_3^p C(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\left[g_{m+1}^1 + \frac{(g_{m+1}^{p+1} - g_{m+1}^p)}{h} + \frac{g_{m+1}^{p+1} - 2g_m^{p+1} + g_{m-1}^{p+1}}{2h^2}\right]$$
$$+ \frac{\delta_4^p C h^{\mu_1-1}}{\Gamma(\mu_1+2)}\left[\left(g_{m+1}^{p+1} + \frac{(g_{m+1}^{p+1} - g_{m+1}^p)}{h} + \frac{g_{m+1}^{p+1} - 2g_m^{p+1} + g_{m-1}^{p+1}}{2h^2}\right) - \left(g_{m+1}^p + \frac{(g_{m+1}^p - g_m^p)}{h} + \frac{g_{m+1}^p - 2g_m^p + g_{m-1}^p}{2h^2}\right)\right] + \frac{C\delta_5^p h^{\mu_1}}{2\Gamma(\mu_1+3)}$$
$$\left[\begin{array}{c}\left(g_{m+1}^{p+1} + \frac{(g_{m+1}^{p+1} - g_m^{p+1})}{h} + \frac{g_{m+1}^{p+1} - 2g_m^{p+1} + g_{m-1}^{p+1}}{2h^2}\right) - 2\left(g_{m+1}^p + \frac{(g_{m+1}^p - g_m^p)}{h} + \frac{g_{m+1}^p - 2g_m^p + g_{m-1}^p}{2h^2}\right)\\ +\left(g_{m+1}^{p-1} + \frac{(g_{m+1}^{p-1} - g_m^{p-1})}{h} + \frac{g_{m+1}^{p-1} - 2g_m^{p-1} + g_{m+1}^{p-1}}{2h^2}\right)\end{array}\right]. \tag{29}$$

Thus, equation (25) is the required numerical results for fractional diffusion equation using Newton polynomial.

5. Application to the Fractional Buckmaster Equation

The Buckmaster equation is a nonlinear PDE that was developed in 1977 by John D. Buckmaster. The surface of a thin sheet of viscous liquid is represented by this equation. In this section, we provide another application of the proposed scheme to show the its validity. We consider the fractional Buckmaster equation as

$$ {}_0^c D_t^{\mu_1} y(z,t) = \frac{\partial y^4}{\partial z^2} + \frac{\partial y^3}{\partial z}. \tag{30}$$

Taking the Laplace transform of (29), we get

$$\mathscr{L}\left\{{}_0^c D_t^{\mu_1} y(z,t)\right\} = \mathscr{L}\left\{\frac{\partial y^4}{\partial z^2} + \frac{\partial y^3}{\partial z}\right\}, \tag{31}$$

$$ {}_0^c D_t^{\mu_1} y(s,t) = \mathscr{L}\left\{\frac{\partial y^4}{\partial z^2} + \frac{\partial y^3}{\partial z}\right\}. \tag{32}$$

Let $y(s,t) = y(t)$ and $\mathscr{L}\{(\partial y^4/\partial z^2) + (\partial y^3/\partial z)\} = \mathscr{L}\{(y^4)_{zz} + (y^3)_z\} = \mathscr{Y}(y,t)$. Now, consider

$$ {}_0^c D_t^{\mu_1} y(t) = \mathscr{Y}(y,t). \tag{33}$$

We already established that the solution to (28) in Laplace space is represented by (15). So,

$$y_{p+1} = y_0 + \frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\mathscr{Y}_0\left[(k+1)^{\mu_1} - k^{\mu_1}\right]\frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}(\mathscr{Y}_1 - \mathscr{Y}_0)\left[(p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1-1)p^{\alpha}\right]$$
$$+ \frac{(\Delta t)^{\mu_1}}{\Gamma(\mu_1+1)}\mathscr{Y}_{m+1}^p\left((-m+p+1)^{\mu_1} - (-m+p)^{\mu_1}\right)$$
$$+ \frac{(\Delta t)^{\mu_1-1}}{\Gamma(\mu_1+2)}\left(\mathscr{Y}_{m+1}^p - \mathscr{Y}_m^p\right)\left[-(-m+p)^{\mu_1+1} + (p+\mu_1-m)(p+1-m)^{\mu_1}\right] \tag{34}$$
$$+ \frac{(\Delta t)^{\alpha}}{2\Gamma(\mu_1+3)}\left(\mathscr{Y}_{m+1}^p - 2\mathscr{Y}_m^p + \mathscr{Y}_{m-1}^p\right)\left[\begin{array}{c}(p+1-m)^{\mu_1}\left[2(-m+p)^2 - \mu_1(p+1-m)2(-m+p)\right]\\ -(-m+p)^{\mu_1}\left[2(-m+p)^2 - \mu_1(-m+p) + 2(-m+p)\right]\end{array}\right].$$

Taking the inverse Laplace transform, we get

$$
\begin{aligned}
g\left(z, t_{p+1}\right) = {} & g\left(t_0\right) + \frac{(\Delta t)^{\mu_1}}{\Gamma\left(\mu_1+1\right)} \delta_1^p \left[(y)_{zz}^4 + (y)_z^3\right]\left(t_0\right) \\
& + \frac{\delta_2^k (\Delta t)^{\mu_1-1}}{\Gamma\left(\mu_1+2\right)} \left(\left[(y)_{zz}^4 + (y)_z^3\right]\left(t_1\right) - \left[(y)_{zz}^4 + (y)_z^3\right]\left(t_0\right)\right) + \frac{\delta_3^p (\Delta t)^{\mu_1}}{\Gamma\left(\mu_1+1\right)} \left[(y)_{zz}^4 + (y)_z^3\right]\left(t_{m+1}\right) \\
& + \frac{\delta_4^p (\Delta t)^{\mu_1-1}}{\Gamma\left(\mu_1+2\right)} \left(\left[(y)_{zz}^4 + (y)_z^3\right]\left(t_{m+1}\right) - \left[(y)_{zz}^4 + (y)_z^3\right]\left(t_m\right)\right) \\
& + \frac{\delta_5^p (\Delta t)^{\mu_1}}{2\Gamma\left(\mu_1+3\right)} \left(\left[(y)_{zz}^4 + (y)_z^3\right]\left(t_{m+1}\right) - 2\left[(y)_{zz}^4 + (y)_z^3\right]\left(t_m\right) + \left[(y)_{zz}^4 + (y)_z^3\right]\left(t_{m-1}\right)\right),
\end{aligned} \tag{35}
$$

where $\delta_1^p = [(p+1)^{\mu_1} - p^{\mu_1}]$, $\delta_2^p = (p+1)^{\mu_1+1} - p^{\mu_1+1} - (\mu_1-1)p^{\mu_1}$, $\delta_3^p = (-m+p+1)^{\mu_1} - (-m+p)^{\mu_1}$, $\delta_4^p = (p+\mu_1-m)(p+1-m)^{\mu_1} - (-m+p)^{\mu_1+1}$, $\delta_5^p = 2(-m+p) + 2(-m+p)^2 - \mu_1(p+1-m), -(-m+p)^{\mu_1} \quad (2(-m+p)^2 + 2\ (-m+p) - \mu_1(-m+p))$.

Discretizing in the space variable and substituting $\Delta t = h$, we get

$$
\begin{aligned}
g\left(z_m, t_{p+1}\right) = {} & g\left(t_0\right) + \frac{\delta_1^p h^{\mu_1}}{\Gamma\left(\mu_1+1\right)} \left[y^4\left(z_m, t_0\right) + \frac{\left(y^4\left(z_{m+1}, t_0\right) - y^4\left(z_m, t_0\right)\right)}{h} + y^3\left(z_m, t_0\right) + \frac{\left(y^3\left(z_{m+1}, t_0\right) - y^3\left(z_m, t_0\right)\right)}{h}\right] \\
& + \frac{\delta_2^p h^{\alpha-1}}{\Gamma(\alpha+2)} \left[\begin{array}{c} \left(y^4\left(z_{m+1}, t_1\right) + \frac{\left(y^4\left(z_{m+1}, t_1\right) - y^4\left(z_m, t_1\right)\right)}{h} + \frac{y^4\left(z_{m+1}, t_1\right) - 2y^4\left(z_m, t_1\right) + y^4\left(z_{m-1}, t_1\right)}{2h^2} + y^3\left(z_{m+1}, t_1\right) + \frac{\left(y^3\left(z_{m+1}, t_1\right) - y^3\left(z_m, t_1\right)\right)}{h} + \frac{y^3\left(z_{m+1}, t_1\right) - 2y^3\left(z_m, t_1\right) + y^3\left(z_{m-1}, t_1\right)}{2h^2}\right) \\ y^4\left(z_m, t_0\right) + \frac{\left(y^4\left(z_{m+1}, t_0\right) - y^4\left(z_m, t_0\right)\right)}{h} + y^3\left(z_m, t_0\right) + \frac{\left(y^3\left(z_{m+1}, t_0\right) - y^3\left(z_m, t_0\right)\right)}{h} \end{array}\right] \\
& + \frac{\delta_3^p h^{\mu_1}}{\Gamma\left(\mu_1+1\right)} \left[\begin{array}{c} y^4\left(z_{1+m}, t_{1+m}\right) \frac{\left(y^4\left(z_{1+m}, t_{1+m}\right) - y^4\left(z_m, t_{m+1}\right)\right)}{h} + \frac{y^4\left(z_{1+m}, t_{1+m}\right) - 2y^4\left(z_m, t_{m+1}\right) + y^4\left(z_{m-1}, t_{m+1}\right)}{2h^2} + y^3\left(z_{1+m}, t_{1+m}\right) \\ + \frac{\left(y^3\left(z_{1+m}, t_{1+m}\right) - y^3\left(z_m, t_{m+1}\right)\right)}{h} + \frac{y^3\left(z_{1+m}, t_{1+m}\right) - 2y^3\left(z_m, t_{m+1}\right) + y^3\left(z_{m-1}, t_{m+1}\right)}{2h^2} \end{array}\right] \\
& + \frac{\delta_4^p h^{\mu_1-1}}{\Gamma\left(\mu_1+2\right)} \left[\begin{array}{c} \left(\begin{array}{c} y^4\left(z_{1+m}, t_{1+m}\right) + \frac{\left(y^4\left(z_{1+m}, t_{1+m}\right) - y^4\left(z_m, t_{m+1}\right)\right)}{h} + \frac{y^4\left(z_{1+m}, t_{1+m}\right) - 2y^4\left(z_m, t_{m+1}\right) + y^4\left(z_{m-1}, t_{m+1}\right)}{2h^2} y^3\left(z_{1+m}, t_{1+m}\right) + \frac{\left(y^3\left(z_{1+m}, t_{1+m}\right) - y^3\left(z_m, t_{m+1}\right)\right)}{h} \\ + \frac{y^3\left(z_{1+m}, t_{1+m}\right) - 2y^3\left(z_m, t_{m+1}\right) + y^3\left(z_{m-1}, t_{m+1}\right)}{2h^2} \end{array}\right) \\ -\left(\begin{array}{c} y^4\left(z_{m+1}, t_m\right) + \frac{\left(y^4\left(z_{m+1}, t_m\right) - y^4\left(z_m, t_m\right)\right)}{h} + \frac{y^4\left(z_{m+1}, t_m\right) - 2y^4\left(z_m, t_m\right) + y^4\left(z_{m-1}, t_m\right)}{2h^2} + y^3\left(z_{m+1}, t_m\right) + \frac{\left(y^3\left(z_{m+1}, t_m\right) - y^3\left(z_m, t_m\right)\right)}{h} \\ + \frac{y^3\left(z_{m+1}, t_m\right) - 2y^3\left(z_m, t_m\right) + y^3\left(z_{m-1}, t_m\right)}{2h^2} \end{array}\right) \end{array}\right] \\
& + \frac{\delta_5^p h^{\mu_1}}{2\Gamma(\alpha+3)} \left[\begin{array}{c} \left(\begin{array}{c} y^4\left(z_{1+m}, t_{1+m}\right) + \frac{\left(y^4\left(z_{1+m}, t_{1+m}\right) - y^4\left(z_m, t_{m+1}\right)\right)}{\Delta t} + \frac{y^4\left(z_{1+m}, t_{1+m}\right) - 2y^4\left(z_m, t_{m+1}\right) + y^4\left(z_{m-1}, t_{m+1}\right)}{2h^2} \\ y^3\left(z_{1+m}, t_{1+m}\right) + \frac{\left(y^3\left(z_{1+m}, t_{1+m}\right) - y^3\left(z_m, t_{m+1}\right)\right)}{h} + \frac{y^3\left(z_{1+m}, t_{1+m}\right) - 2y^3\left(z_m, t_{m+1}\right) + y^3\left(z_{m-1}, t_{m+1}\right)}{2h^2} \end{array}\right) \\ -2\left(\begin{array}{c} y^4\left(z_{m+1}, t_m\right) + \frac{\left(y^4\left(z_{m+1}, t_m\right) - y^4\left(z_m, t_m\right)\right)}{h} + \frac{y^4\left(z_{m+1}, t_m\right) - 2y^4\left(z_m, t_m\right) + y^4\left(z_{m-1}, t_m\right)}{2h^2} \\ y^3\left(z_{m+1}, t_m\right) + \frac{\left(y^3\left(z_{m+1}, t_m\right) - y^3\left(z_m, t_m\right)\right)}{h} + \frac{y^3\left(z_{m+1}, t_m\right) - 2y^3\left(z_m, t_m\right) + y^3\left(z_{m-1}, t_m\right)}{2h^2} \end{array}\right) \\ \left(\begin{array}{c} y^4\left(z_{m+1}, t_{m-1}\right) + \frac{\left(y^4\left(z_{m+1}, t_{m-1}\right) - y^4\left(z_m, t_{m-1}\right)\right)}{h} + \frac{y^4\left(z_{m+1}, t_{m-1}\right) - 2y^4\left(z_m, t_{m-1}\right) + y^4\left(z_{m-1}, t_{m-1}\right)}{2h^2} \\ y^3\left(z_{m+1}, t_{m-1}\right) + \frac{\left(y^3\left(z_{m+1}, t_{m-1}\right) - y^3\left(z_m, t_{m-1}\right)\right)}{h} + \frac{y^3\left(z_{m+1}, t_{m-1}\right) - 2y^3\left(z_m, t_{m-1}\right) + y^3\left(z_{m-1}, t_{m-1}\right)}{2h^2} \end{array}\right) \end{array}\right].
\end{aligned} \tag{36}
$$

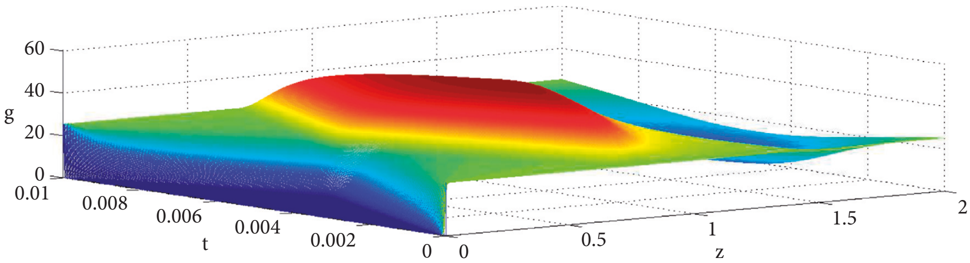

Figure 1: Numerical simulation of problem 1 for $\mu_1 = 0.85$.

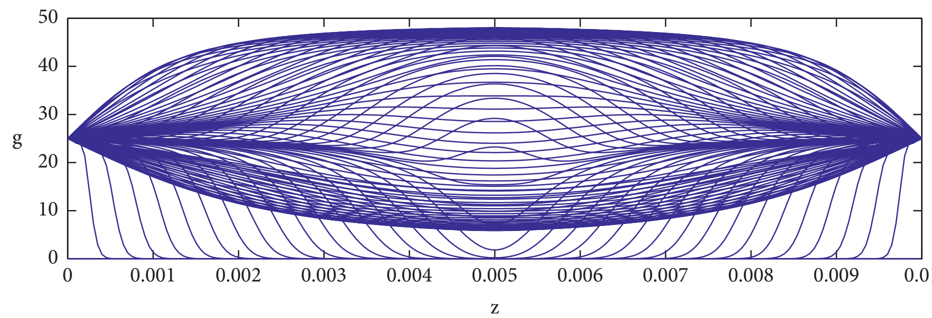

Figure 2: Contour map of numerical solution of problem 1 for $\mu_1 = 0.85$.

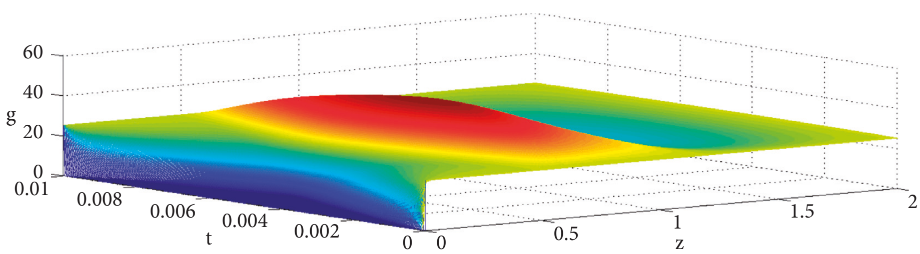

Figure 3: Numerical simulation of problem 1 for $\mu_1 = 0.95$.

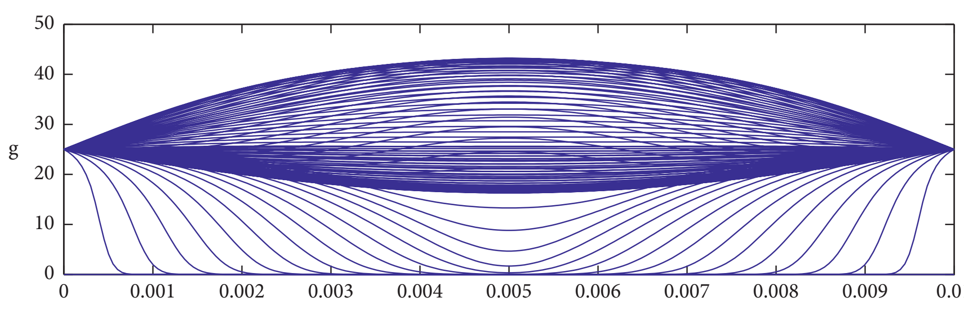

Figure 4: Contour map of numerical solution of problem 1 for $\mu_1 = 0.95$.

Thus, equation (31) gives the numerical solution of the proposed fractional Buckmaster equation.

6. Numerical Simulations

In this section, we demonstrate numerical solutions to diffusion and Buckmaster equations for various fractional orders. Figures 1–6 show the simulation of the numerical solution of the fractional diffusion equation for fractional order $\mu_1 = 0.85, 0.95, 1$. At $\mu_1 = 1$, the solution of the proposed problem converges to the exact solution. The numerical simulations and their associated contour plot are presented to show the efficiency and stability of the numerical scheme, which can be observed from Figures 1–6. The numerical simulation of the obtained numerical solution of the Buckmaster problem is shown in Figures 7–12 for fractional order $\mu_1 = 0.85, 0.95, 1$. The last two figures reveal that the solution directly converges to the exact solution of the proposed problem. Thus, the efficiency

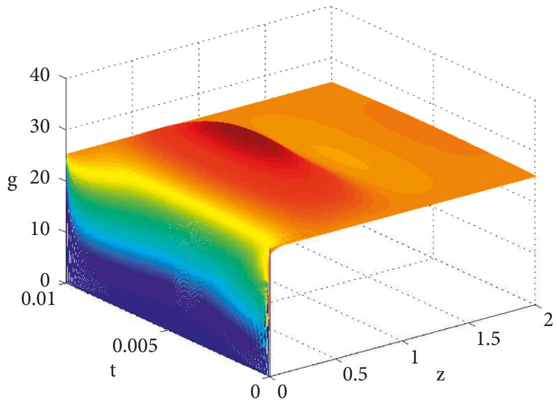

Figure 5: Numerical simulation of problem 1 for $\mu_1 = 1$.

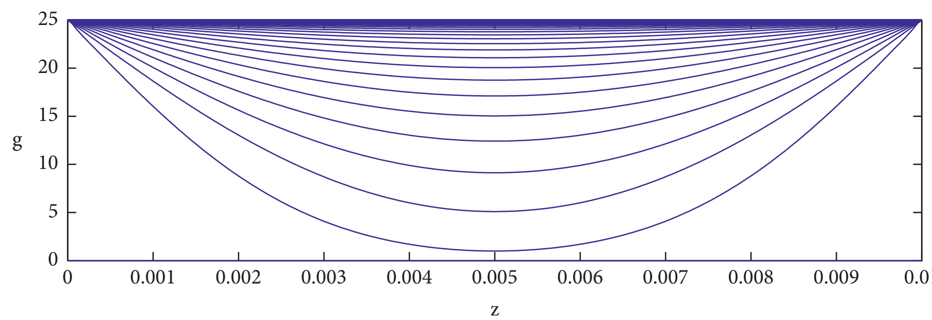

Figure 6: Contour map of numerical solution of problem 1 for $\mu_1 = 1$.

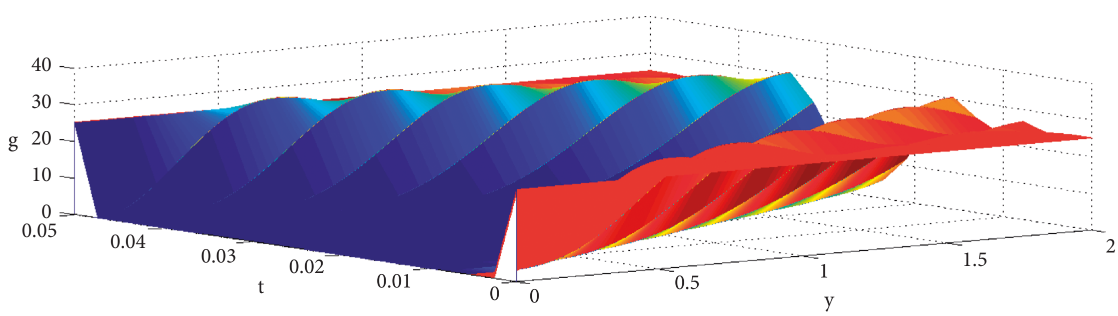

Figure 7: Numerical simulation of problem 2 for $\mu_1 = 0.85$.

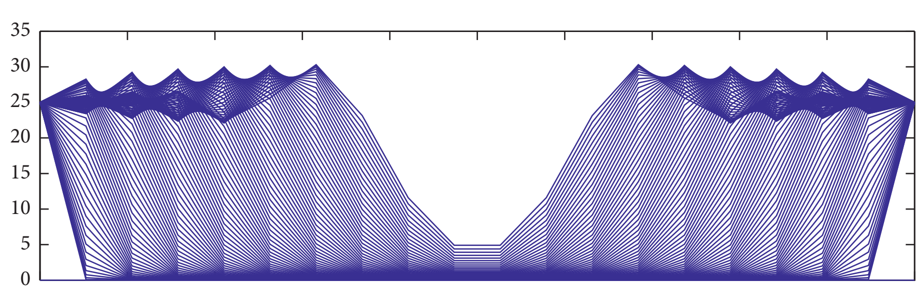

Figure 8: Contour map of numerical solution of problem 2 for $\mu_1 = 0.85$.

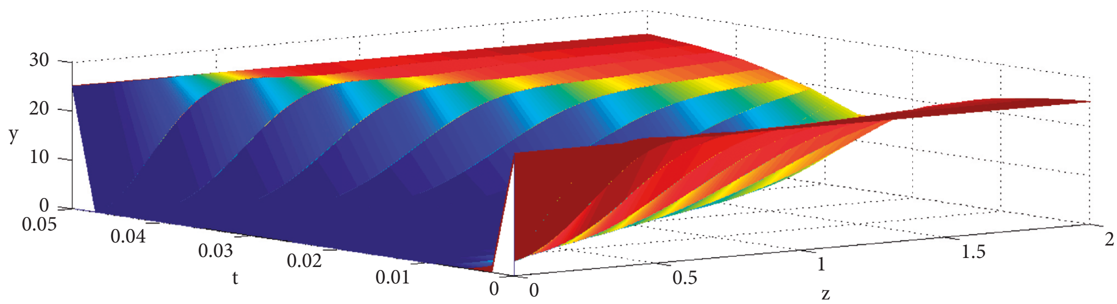

Figure 9: Numerical simulation of problem 2 for $\mu_1 = 0.95$.

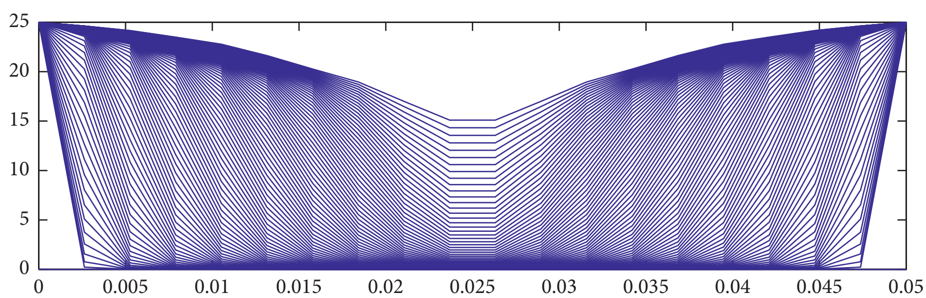

Figure 10: Contour map of numerical solution of problem 2 for $\mu_1 = 0.95$.

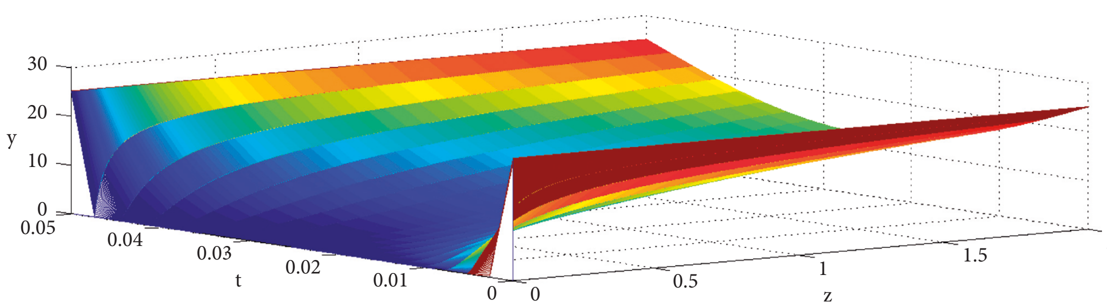

Figure 11: Numerical simulation of problem 2 for $\mu_1 = 1$.

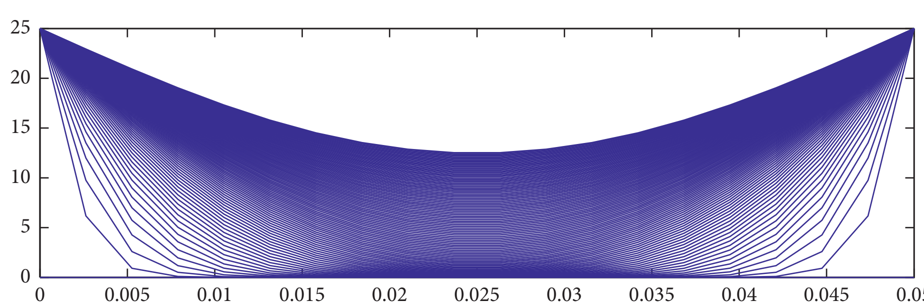

Figure 12: Contour map of numerical solution of problem 2 for $\mu_1 = 1$.

and applicability of the suggested hybrid interpolation method is verified in these two applications.

7. Conclusion

In this study, we have devised a hybrid Newton polynomial interpolation approach to solve fractional PDEs. We have used the Caputo fractional operator. This technique combines Newton interpolation with the Laplace transform. We have derived a general numerical scheme for solution of fractional PDE using Laplace transform and Newton interpolation. The approach was verified by solving two PDEs, including fractional diffusion and Buckmaster equations. All the results have been validated through numerical simulations. In future, this method can be used to solve more difficult nonlinear fractional PDEs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Khaliq, A. Ullah, S. Ahmad, A. Akgül, A. Yusuf, and T. A. Sulaiman, "Some novel analytical solutions of a new extented $(2+1)$-dimensional Boussinesq equation using a novel method," *J. Ocean Eng. Sci.*, in Press, 2022.

[2] S. Y. Lou and X. Hao, "Linear superposition in the general heavenly equation," *Physics Letters A*, vol. 443, Article ID 128203, 2022.

[3] R. Wu, X. Shen, and J. Zhao, "Conforming finite element methods for two-dimensional linearly elastic shallow shell and clamped plate models," *Applied Mathematics and Computation*, vol. 430, Article ID 127259, 2022.

[4] S. Ahmad, A. Ullah, M. Partohaghighi, S. Saifullah, A. Akgül, and F. Jarad, "Oscillatory and complex behaviour of Caputo-Fabrizio fractional order HIV-1 infection model," *AIMS Mathematics*, vol. 7, no. 3, pp. 4778–4792, 2021.

[5] A. Khan, H. M. Alshehri, T. Abdeljawad, Q. M. Al-Mdallal, and H. Khan, "Stability analysis of fractional nabla difference COVID-19 model," *Results in Physics*, vol. 22, Article ID 103888, 2021.

[6] Z. A. Khan, J. Khan, S. Saifullah, and A. Ali, "Dynamics of hidden attractors in four-dimensional dynamical systems with power law," *Journal of Function Spaces*, vol. 2022, Article ID 3675076, 22 pages, 2022.

[7] Z. A. Khan, A. Khan, T. Abdeljawad, and H. Khan, *Computational Analysis of Fractional Order Imperfect Testing Infection Disease Model*, Fractals, NewYork, NY, USA, 2022.

[8] A. Devi, A. Kumar, T. Abdeljawad, and A. Khan, "Stability analysis of solutions and existence theory of fractional Lagevin equation," *Alexandria Engineering Journal*, vol. 60, no. 4, pp. 3641–3647, 2021.

[9] P. Kumam, A. Seadawy, G. Zaman, and Z. Shah, "Study of mathematical model of Hepatitis B under Caputo-Fabrizo derivative," *AIMS Mathematics*, vol. 6, no. 1, pp. 195–209, 2021.

[10] S. Saifullah, A. Ali, M. Irfan, and K. Shah, "Time-fractional Klein–Gordon equation with solitary/shock waves solutions," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6858592, 15 pages, 2021.

[11] S. Gulalai, F. A. Ahmad, F. A. Rihan, A. U. Rihan, Q. M. Al-Mdallal, and A. Ali, "Nonlinear analysis of a nonlinear modified KdV equation under Atangana Baleanu Caputo derivative," *AIMS Mathematics*, vol. 7, no. 5, pp. 7847–7865, 2022.

[12] T. Akram, M. Abbas, and A. I. Ismail, "An extended cubic B-spline collocation scheme for time fractional sub-diffusion equation," in *Innovation*, Article ID 060017, Penang, Malaysia, December 2018.

[13] T. Akram, M. Abbas, and A. I. Ismail, "Numerical solution of fractional cable equation via extended cubic B-spline," in *Proceedings of tand Analytics Conference & Exhibition (IACE 2019)*, Article ID 030004, Kedah, Malaysia, March 2019.

[14] T. Akram, M. Abbas, M. B. Riaz, A. I. Ismail, and N. M. Ali, "Development and analysis of new approximation of extended cubic B-spline to the non-linear time fractional Klein-Gordon equation," *Fractals*, vol. 28, no. 08, Article ID 2040039, 2020.

[15] S. Ahmad, A. Ullah, A. Akgül, and F. Jarad, "A hybrid analytical technique for solving nonlinear fractional order PDEs of power law kernel: application to KdV and Fornberg-Witham equations," *AIMS Mathematics*, vol. 7, no. 5, pp. 9389–9404, 2022.

[16] S. S. Ray and A. K. Gupta, "Numerical solution of fractional partial differential equation of parabolic type with Dirichlet boundary conditions using two-dimensional Legendre wavelets method," *Journal of Computational and Nonlinear Dynamics*, vol. 11, no. 1, Article ID 011012, 2016.

[17] H. Bin Jebreen and I. Dassios, "A biorthogonal hermite cubic spline Galerkin method for solving fractional riccati equation," *Mathematics*, vol. 10, no. 9, 1461 pages, 2022.

[18] N. Iqbal, H. Yasmin, A. Rezaiguia, J. Kafle, A. O. Almatroud, and T. S. Hassan, "Analysis of the fractional-order kaup–kupershmidt equation via novel transforms," *Journal of Mathematics*, vol. 2021, Article ID 2567927, 13 pages, 2021.

[19] K. Shah, Z. A. Khan, A. Ali, R. Amin, H. Khan, and A. Khan, "Haar wavelet collocation approach for the solution of fractional order COVID-19 model using Caputo derivative," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3221–3231, 2020.

[20] N. A. Obeidat and D. E. Bentil, "Novel technique to investigate the convergence analysis of the tempered fractional natural transform method applied to diffusion equations," *Journal of Ocean Engineering and Science*, 2022.

[21] R. P. Agarwal, F. Mofarreh, R. Shah, W. Luangboon, and K. Nonlaopon, "An analytical technique, based on natural transform to solve fractional-order parabolic equations," *Entropy*, vol. 23, no. 8, 1086 pages, 2021.

[22] R. Gnitchogna and A. Atangana, "New two step Laplace Adam-Bashforth method for integer a noninteger order partial differential equations," *Numerical Methods for Partial Differential Equations*, vol. 34, no. 5, pp. 1739–1758, 2017.

[23] A. Atangana and S. I. Araz, *New Numerical Scheme with Newton Polynomial Theory, Methods, and Applications*, Elsevier, Amsterdam, Netherlands, 2021.

[24] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional calculus: models and numerical methods*, World Scientific, Singapore, 2012.