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Research Article

Bivariate Kumaraswamy Distribution Based on Conditional Hazard Functions: Properties and Application

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A new class of bivariate distributions is deduced by specifying its conditional hazard functions (hfs) which are Kumaraswamy distribution. The interest of this model is positively, negatively, or zero correlated. Properties and local measures of dependence of the bivariate Kumaraswamy conditional hazard (BKCH) distribution are studied. The estimation of type parameters is considered by used the maximum likelihood and pseudolikelihood of the new class. A simulation study was performed to inspect the bias and mean squared error of the maximum likelihood estimators. Finally, an application is obtained to clarify our results with the maximum likelihood and pseudolikelihood. Also, the results are used to compare BKCH distribution with bivariate exponential conditionals (BEC) and bivariate Lindley conditionals hazard (BLCH) distributions.

1. Introduction

The problem of constructing new families of continuous bivariate distributions is one of the important and continue active research problem in statistics. This is due from one side to the limitations of the existing distributions and its lack in modeling some stochastic phenomena and from the other side to meet the needs of modern and developed applied field. Balakrishnon et al. [1] obtained a new technology to build the bivariate probability density functions (pdfs) with specified conditional hfs, and the development of the models determines many of the several properties, including survival function (sf), hazard bivariate function, the Clayton–Oakes measure, and the measure conditional pdfs and hfs of the marginal and conditional pdfs. Moreover, Navarro and Sarabia [2] studied the reliability properties for the series and parallel systems with component lifetimes having these dependence models.

Therefore, the authors introduce a new form of the bivariate distributions with Kumaraswamy conditional hfs, and we studied some of the characteristics and discussed the estimation procedures of the parameters of this distribution. Moreover, we estimate the parameters of this distribution by two methods, e.g., maximum likelihood estimation (MLE) and maximum pseudolikelihood estimation (MPLE) procedures. Finally, this distribution is compared with other distributions considering a real dataset collected from a secondary source (p. 374 of Johnson and Wichern [3]).

A continuous random variable is Kumaraswamy distribution if its pdf (Kumaraswamy [4] is

$$f_X(x) = abx^{a-1}\left(1-x^a\right)^{b-1}, 0<x<1,\ a>0,\ b>0. \tag{1}$$

The corresponding sf is

$$\overline{F}(x) = \left(1-x^a\right)^b, 0<x<1,\ a>0,\ b>0, \tag{2}$$

and the hf is

$$\mathrm{r}(x) = \frac{abx^{a-1}}{1-x^{a}}, 0<x<1,\ a>0,\ b>0. \quad (3)$$

We note that

$$\overline{F}(x) = exp\left(-\int_{0}^{x}\mathrm{r}(u)du\right), \quad (4)$$

$$f_X(x) = \mathrm{r}(x)\,exp\left(-\int_{0}^{x}\mathrm{r}(u)du\right). \quad (5)$$

This distribution is applicable to some scientific, practical, and hydrological applications (Dey et al. [5]; Ishaq et al. [6]).

2. BKCH Model

Suppose that the conditional hfs (Kumaraswamy) are distributed as

$$\mathrm{r}_1(x|y) = \frac{a_1b_1(\mathrm{y})x^{a_1-1}}{1-x^{a_1}}, 0<x,\ y<1,\ a_1\ posative\ constant,$$
$$\mathrm{r}_2(y|x) = \frac{a_2b_2(x)y^{a_2-1}}{1-y^{a_2}}, 0<x,\ y<1,\ a_2\ posative\ constant, \quad (6)$$

where $b_1(\mathrm{y})$ and $b_2(x)$ are function of y and x, respectively.

Then, the bivariate distribution from (6) will be called a bivariate BKCH distribution.

Theorem 1. *The general bivariate distribution with conditional hfs as $r_1(x|y)$ and $r_2(x|y)$ is*

$$f_{X,Y}(x,y) = [N(\Theta)]^{-1}x^{(a_1-1)}y^{(a_2-1)}\mathrm{Exp}\{\theta_1\mathrm{Log}[1-y^{a_2}] + (\theta_2+\theta_3\mathrm{Log}[1-y^{a_2}])(\mathrm{Log}[1-x^{a_1}])\}, 0<x,\ y<1, -\infty<\theta_i<\infty,\ i=1,2,3, \quad (7)$$

where $N(\Theta)=\{\theta_1,\theta_2,\theta_3\}$ is the normalizing constant such that $f_{X,Y}(x,y)$ integrates to 1.

Proof. The conditional pdfs from (6) using (5) are

$$f_{X|Y}(x|y) = \frac{a_1b_1(\mathrm{y})x^{a_1-1}}{1-x^{a_1}}exp\left(-\int_0^x\frac{a_1b_1(\mathrm{y})u^{a_1-1}}{1-u^{a_1}}du\right), \quad (8)$$

$$f_{Y|X}(y|x) = \frac{a_2b_2(x)\mathrm{y}^{a-1}}{1-\mathrm{y}^{a_2}}exp\left(-\int_0^y\frac{a_2b_2(x)u^{a_2-1}}{1-u^{a_2}}du\right). \quad (9)$$

Then, the identity $f_Y(y)\,f_{X|Y}(x|y) = f_X(x)\,f_{Y|X}(y|x)$ yields the following relation:

$$f_Y(y)\frac{a_1b_1(y)(1-\mathrm{y}^{a_2})}{\mathrm{y}^{a_2-1}}(1-x^{a_1})^{b_1(y)-1} = f_X(x)\frac{a_2b_2(x)(1-x^{a_1})}{x^{a_1-1}}(1-\mathrm{y}^{a_2})^{b_2(x)-1}, \quad (10)$$

where $f_Y(y)$ and $f_X(x)$ denote the marginal pdfs.

Denoting

$$g(y) = log\left[f_Y(y)\frac{b_1(y)(1-\mathrm{y}^{a_2})}{\mathrm{y}^{a_2-1}}\right], \quad (11)$$

$$h(x) = \log\left[f_X(x)\frac{b_2(x)(1-x^{a_1})}{x^{a_1-1}}\right], \quad (12)$$

then taking logarithms in (9) and using (10) and (11) WITH then taking logarithms in (10), using (11) and (12)

$$g(y) + (b_1(y)-1)\,log(1-x^{a_1}) - h(x) - (b_2(x)-1)log(1-\mathrm{y}^{a_2}) = 0, \quad (13)$$

which is a functional equation of the form $\sum_{k=1}^{n} f_k(x)g_k(y) = 0$, whose most general solution is given by Aczel [7] as

$$b_1(y) - 1 = \theta_2 + \theta_3 log(1-\mathrm{y}^{a_2}),$$
$$b_2(x) - 1 = \theta_1 + \theta_3 log(1-x^{a_1}). \quad (14)$$

From (11) and (12), we have the corresponding marginal pdfs given by

$$f_X(x) = \frac{[N(\Theta)]^{-1}x^{a_1-1}(1-x^{a_1})^{\theta_2}}{a_2(\theta_1+\theta_3 log(1-x^{a_1})+1)},\ 0<x,\ y<1, -\infty<\theta_i<\infty,\ i=1,2,3, \quad (15)$$

and

$$f_Y(y) = \frac{[N(\Theta)]^{-1}\mathrm{y}^{a_2-1}(1-\mathrm{y}^{a_2})^{\theta_1}}{a_1(\theta_2+\theta_3 log(1-\mathrm{y}^{a_2})+1)},\ 0<x,\ y<1, -\infty<\theta_i<\infty,\ i=1,2,3. \quad (16)$$

Equation (7) describes the complete class of BKCH distribution that has the three parameters θ_1, θ_2, and θ_3.

Figure 1 shows the BKCH given by (7) for the special cases for any θ_1, θ_2, and θ_3.

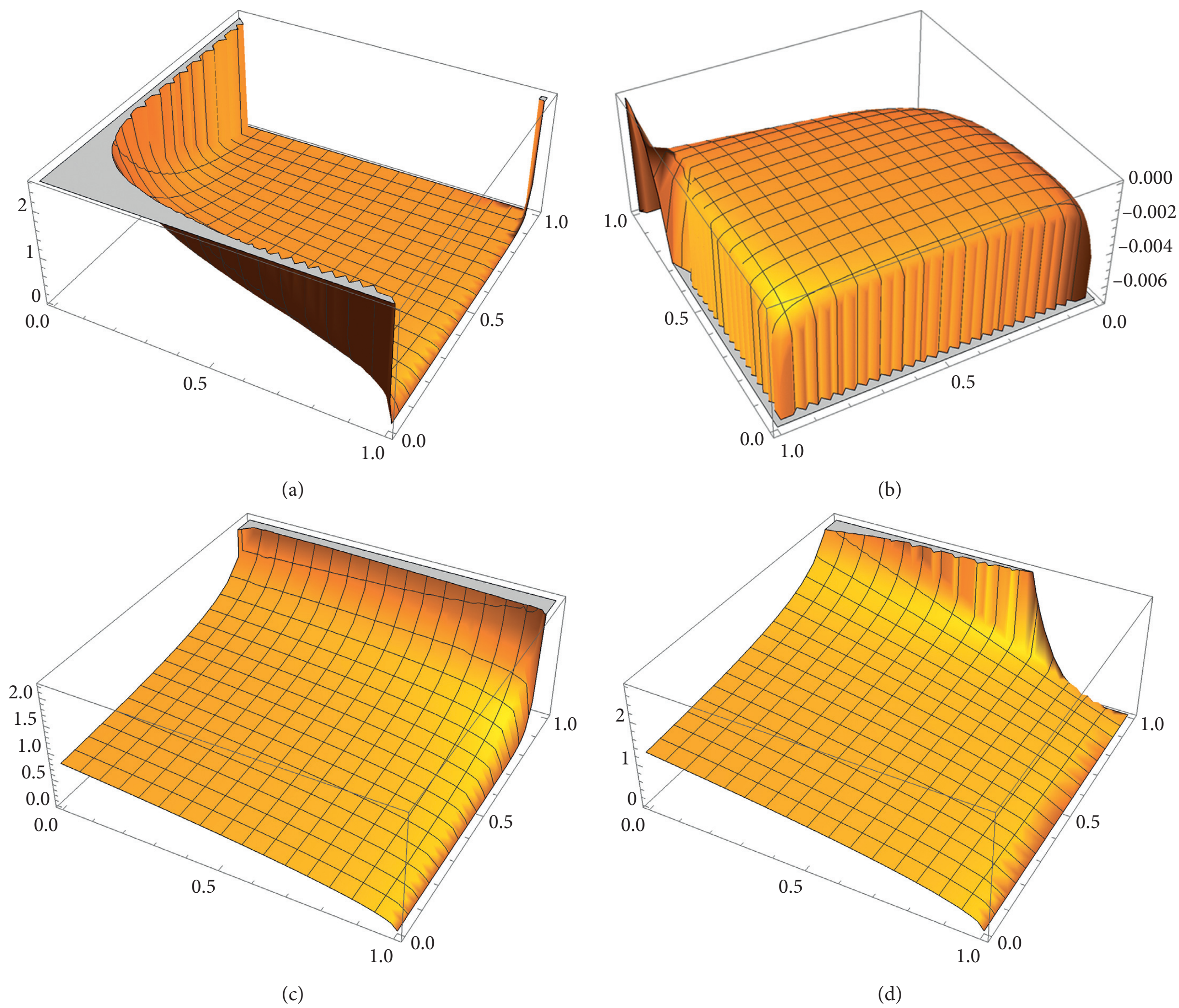


FIGURE 1: The pdf $f_{X,Y}(x,y)$ of the BKCH distribution for selected values of θ_1, θ_2, and θ_3. (a) $\theta_1 = 1.3, \theta_2 = .5, \theta_3 = .2; a_1 = a_2 = 0.5$. (b) $\theta_1 = -1.3, \theta_2 = -.5, \theta_3 = -.2; a_1 = a_2 = 0.5$. (c) $\theta_1 = -.3, \theta_2 = .5, \theta_3 = 0.2; a_1 = a_2 = 1$. (d) $\theta_1 = -.3, \theta_2 = .5, \theta_3 = -0.2; a_1 = a_2 = 1$.

The conditional hfs for the BKCH distribution are

$$r_1(x|y) = \frac{a_1 x^{a_1-1}(\theta_2 + \theta_3 log(1 - y^{a_2}) + 1)}{1 - x^{a_1}}, \quad 0 < x,\ y < 1, -\infty < \theta_i < \infty, \quad i = 1, 2, 3, \tag{17}$$

$$r_2(y|x) = \frac{a_2 y^{a_2-1}(\theta_1 + \theta_3 log(1 - x^{a_1}) + 1)}{1 - y^{a_2}}, \quad 0 < x,\ y < 1, -\infty < \theta_i < \infty, \quad i = 1, 2, 3. \tag{18}$$

The compatibility of (17) and (18), Balakrishnan et al. [1] secures the existence of BKCH distribution. Figure 2 shows hf $r_1(x|y)$ of the BKCH given by (7) for the special cases for θ_2 and θ_3. Similarly, $r_2(y|x)$. □

3. BKCH Properties

3.1. The Conditional Distributions. The conditional pdfs of the BKCH distribution are

$$f_{X|Y}(x|y) = a_1 x^{a_1-1}(\theta_2 + \theta_3 \, log(1 - y^{a_2}) + 1) \cdot (1 - x^{a_1})^{(\theta_2 + \theta_3 log(1 - y^{a_2}))}, \tag{19}$$

$$f_{Y|X}(y|x) = a_2 y^{a_2-1}(\theta_1 + \theta_3 log(1 - x^{a_1}) + 1) \cdot (1 - y^{a_2})^{(\theta_1 + \theta_3 log(1 - x^{a_1}))}, \tag{20}$$

i.e.,

$$X|Y = y \sim kumaraswamy(\theta_2 + \theta_3 log(1 - y^{a_2})), \quad Y|X = x \sim kumaraswamy(\theta_1 + \theta_3 log(1 - x^{a_1})). \tag{21}$$

When $\theta_1 = \theta_2$, the conditional pdfs are identical.

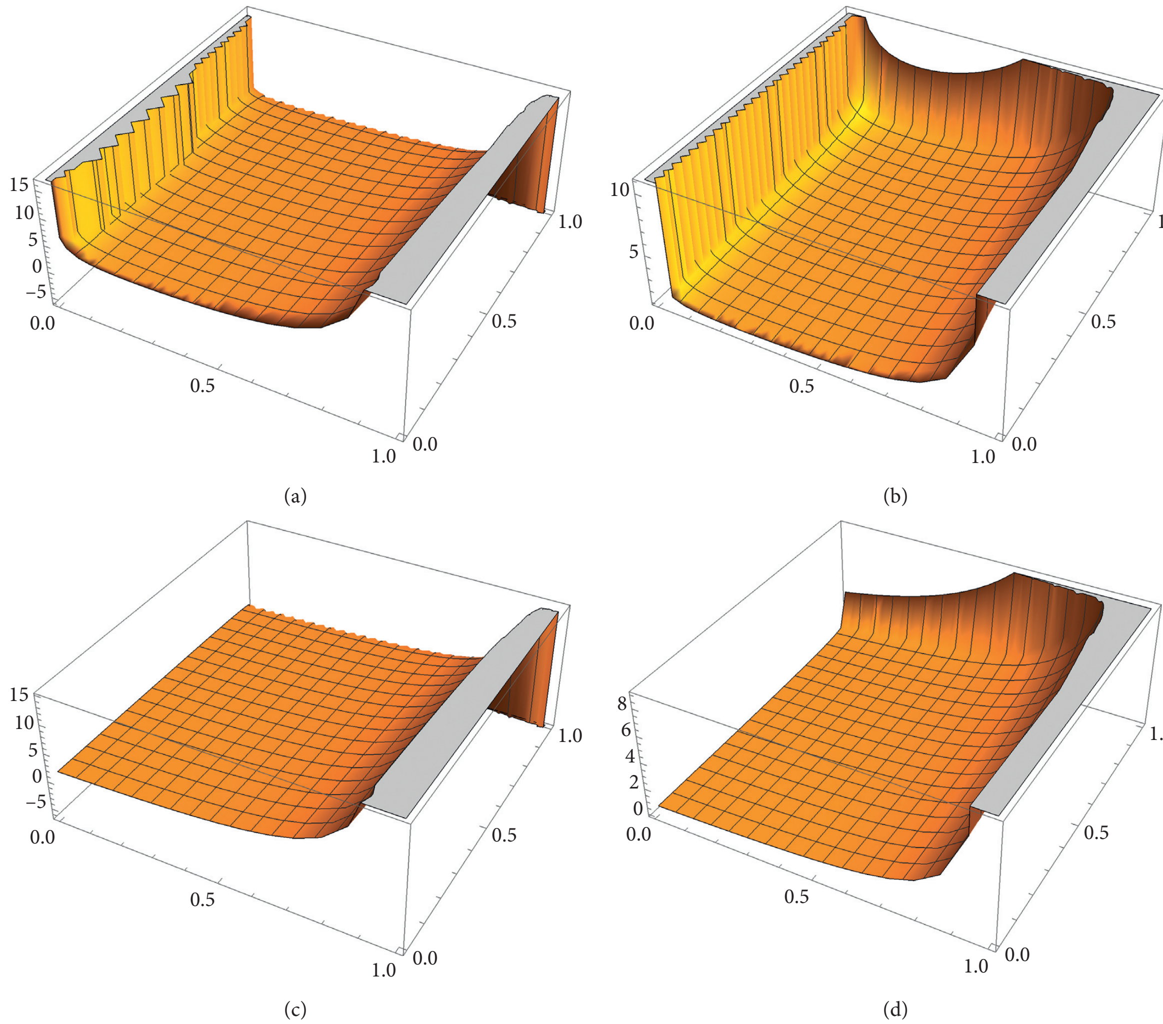


Figure 2: The hf $r_1(x|y)$ of the BKCH distribution for selected values of θ_2 and θ_3. (a) $\theta_2 = .5$, $\theta_3 = .2$; $a_1 = a_2 = 0.5$. (b) $\theta_2 = -.5$, $\theta_3 = -.2$; $a_1 = a_2 = 0.5$. (c) $\theta_2 = .5$, $\theta_3 = .2$; $a_1 = a_2 = 1$. (d) $\theta_2 = -.5$, $\theta_3 = -.2$; $a_1 = a_2 = 1$.

$f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ given by (19) and (20) satisfy the compatibility conditions studied by Arnold et al. [8] and secure existence of BKCH distribution.

3.2. Regression Function. Now, using (19) and (20), the conditional k^{th} moments are

$$E(X_k|Y=y) = \frac{\text{Gamma}[a_1 + k/a_1]\text{Gamma}[2+\theta_2+\text{Log}[1-y^{a_2}]\theta_3](1+\theta_2+\text{Log}[1-y^{a_2}]\theta_3)}{\text{Gamma}[3+k/a_1+\theta_2+\text{Log}[1-y^{a_2}]\theta_3]}, \quad (22)$$

$$E(Y_k|X=x) = \frac{\text{Gamma}[a_2 + k/a_2]\text{Gamma}[2+\theta_1+\text{Log}[1-x^{a_1}]\theta_3](1+\theta_1+\text{Log}[1-x^{a_1}]\theta_3)}{\text{Gamma}[3+k/a_2+\theta_1+\text{Log}[1-x^{a_1}]\theta_3]}. \quad (23)$$

For k = 1, we have

$$E(X|Y=y) = \frac{\text{Gamma}[a_1 + 1/a_1]\text{Gamma}[2+\theta_2+\text{Log}[1-y^{a_2}]\theta_3](1+\theta_2+\text{Log}[1-y^{a_2}]\theta_3)}{\text{Gamma}[3+1/a_1+\theta_2+\text{Log}[1-y^{a_2}]\theta_3]}, \quad (24)$$

and

$$E(Y|X=x)=\frac{\text{Gamma}[a_2+1/a_2]\text{Gamma}[2+\theta_1+\text{Log}[1-x^{a_1}]\theta_3](1+\theta_1+\text{Log}[1-x^{a_1}]\theta_3)}{\text{Gamma}[3+1/a_2+\theta_1+\text{Log}[1-x^{a_1}]\theta_3]}. \tag{25}$$

We notice that the regression function for $X|Y$ is nonlinear increasing for $\theta_1,\theta_3<0$ and decreasing for $\theta_1,\theta_3>0$ (Figure 3). Additionally, the regression function for $X|Y$ is nonlinear increasing for $\theta_2,\theta_3<0$ and decreasing for $\theta_2,\theta_3>0$ (Figure 4).

3.3. Stochastic Ordering. Let $f_1(x,y)$ and $f_2(x,y)$ be two-bivariate BKCH distribution, and $\Delta f_{12}(x,y)=f_1(x,y)-f_2(x,y)$ such that $\Delta f_{12}(x,y)\geq-\Delta f_{12}(y,x)$ and $\Delta f_{12}(x,y)\geq 0$, i.e., $f_1(x,y)\geq f_2(x,y)$ and $\Delta f_{12}(x,y)$ increasing in x, for all $x\geq y$. Also, $\Delta f_{12}(x,y)$ decreases in y, for all $y\leq x$. Based on the results obtained by Shanthikumar [9], then we obtain likelihood ratio order ($\mathbf{X}\geq_{\mathbf{lr}}\mathbf{Y}$), hazard rate order ($\mathbf{X}\geq_{\mathbf{hr}}\mathbf{Y}$), and stochastic order ($\mathbf{X}\geq_{\mathbf{st}}\mathbf{Y}$) $\Leftrightarrow$ $E[f_1(x,y)]\geq E[f_2(x,y)]$, for all $f_1(x,y)$ and $f_2(x,y)$.

4. Local Measures of Dependence

Holland and Wang [10, 11] introduced a local dependence function given by the following formula:

$$\gamma_f(x,y)=\frac{\partial^2}{\partial_x\partial_y}\ln f_{X,Y}(x,y), \tag{26}$$

where $f_{X,Y}(x,y)$ is the joint pdf of x and y.

Theorem 2. *The BKCH distribution given by (8) is TP₂ (totally positive of order 2) iff $\theta_3<0$ and TN₂ (totally negative of order 2) iff $\theta_3>0$.*

Proof. From (7), we obtain

$$\gamma_f(x,y)=-\frac{x^{a_1-1}y^{a_2-1}}{(1-x^{a_1})(1-y^{a_2})}a_1a_2\theta_3. \tag{27}$$

Hence, $\gamma_f(x,y)>0$. If $\theta_3<0$, then BKCH is TP_2, and $\gamma_f(x,y)<0$. If $\theta_3>0$, then BKCH is TN_2. □

Remark 1

(1) It follows from theorem (2) that if the BKCH distribution, given by (7), is TP_2 or TN_2, then X and Y are positively or negatively, respectively, correlated.

(2) The conditional pdfs in (19) and (20) have the local measures of dependence:

$$\gamma_{f_{x|y}}(x,y)=\gamma_{f_{x|y}}(x,y)=\gamma_f(x,y). \tag{28}$$

(3) $\theta_3=0$ iff $\gamma_f(x,y)=0$; furthermore, the two random variables X and Y are independent iff $\theta_3=0$ Balakrishnan and Lai [12].

The class has two parameters only by choosing $\theta_3=0$; the joint pdf (7) can be written as

$$f_{X,Y}(x,y)=[N(\Theta)]^{-1}x^{a_1-1}y^{a_2-1}\text{Exp}\{\theta_1\text{Log}[1-y^{a_2}]+\theta_2\text{Log}[1-x^{a_1}]\},\ x>0,\ y>0,-\infty<\theta_i<\infty,\quad i=1,2,\ \ a_1,a_2>0. \tag{29}$$

Figure 5 shows the joint pdf given by (29) for the special cases for θ_1 and θ_2.

The conditional hfs for the joint pdf given by (29) are

$$r_1(y|x)=\frac{a_1x^{a_1-1}(-\theta_2+1)}{1-x^{a_1}},\ r_2(y|x)=\frac{a_2y^{a_2-1}(-\theta_1+1)}{1-y^{a_2}}, \tag{30}$$

and the conditional pdfs are

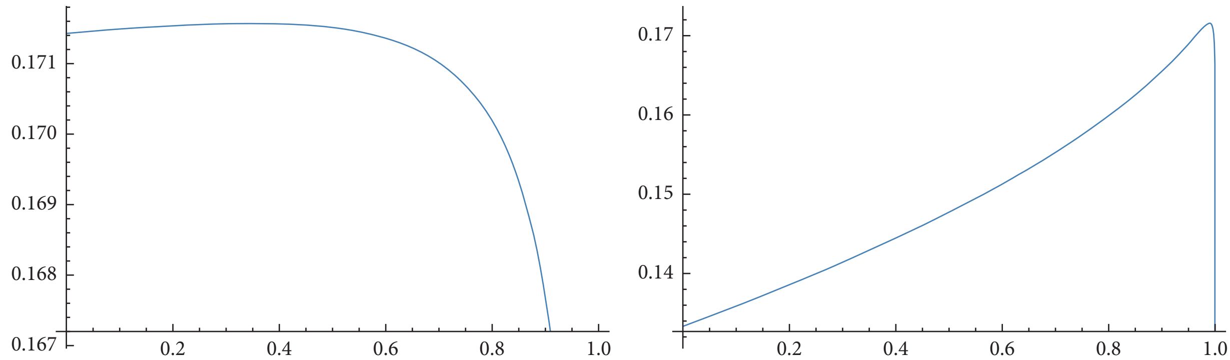
$$f_{X|Y}(X|Y)=a_1\left(x^{a_1-1}\right)(1-\theta_2)exp(-(\theta_2+1)log(1-x^{a_1})),$$
$$f_{Y|X}(y|x)=a_2\left(y^{a_2-1}\right)(1-\theta_1)exp(-(\theta_1+1)log(1-y^{a_2})). \tag{31}$$

Now, the marginal pdfs of X and Y using (29) are

$$f_X(x)=\frac{[N(\Theta)]^{-1}x^{a_1-1}}{a_2(1-\theta_1)}exp(-\theta_2log(1-x^{a_1})),$$
$$f_Y(y)=\frac{[N(\Theta)]^{-1}y^{a_2-1}}{a_2(1-\theta_2)}exp(-\theta_1log(1-y^{a_2})). \tag{32}$$

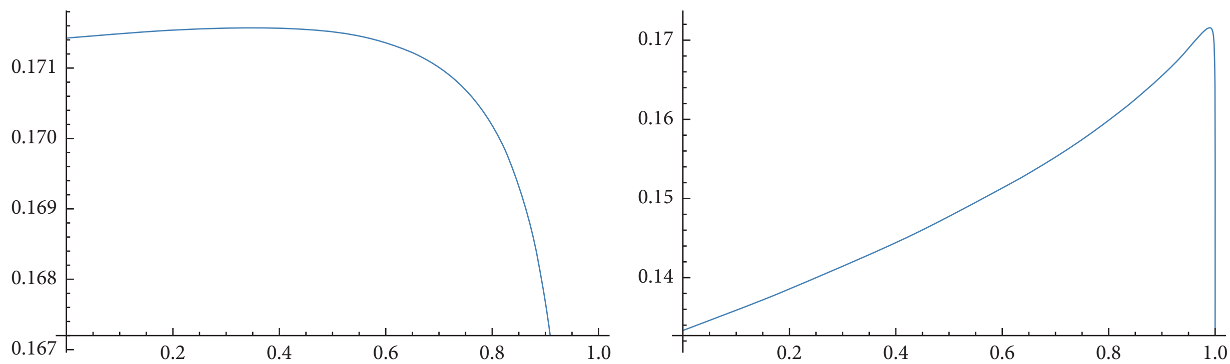
And the moment of joint pdf given by (29) is

$$E(XY)=\frac{[N(\Theta)]^{-1}\text{Gamma}[1/a]\text{Gamma}[1/b]\text{Gamma}[1-\theta_1]\text{Gamma}[1-\theta_2]}{a^2b^2\text{Gamma}[2+1/b-\theta_1]\text{Gamma}[2+1/a-\theta_2]}. \tag{33}$$



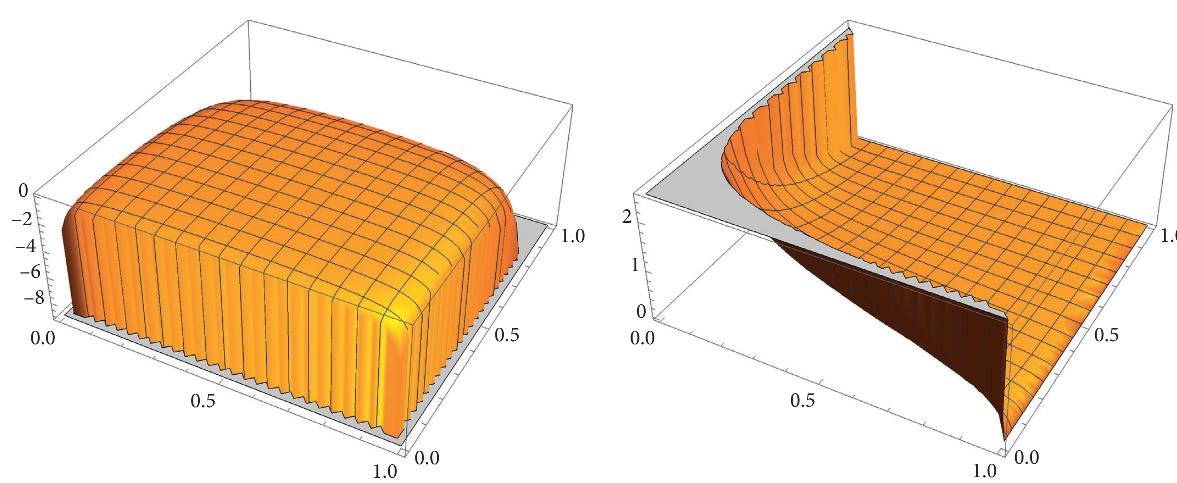
(a) (b)

Figure 3: $E(X|Y=y)$ of the BKCH distribution for selected values. (a) $\theta_1 = 0.5$, $\theta_3 = .2; a_1 = a_2 = 1$. (b) $\theta_1 = -0.5$, $\theta_3 = -.2; a_1 = a_2 = 1$.



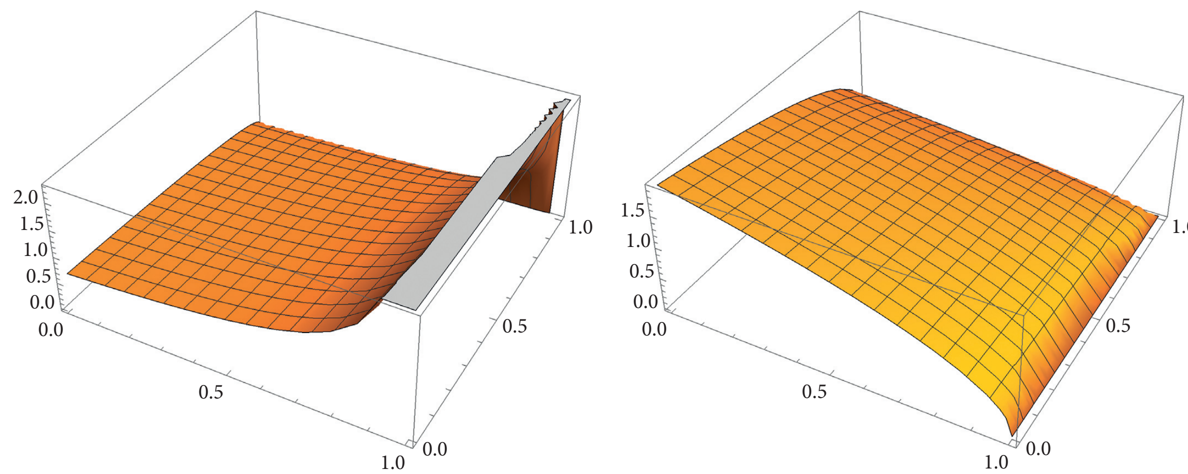
(a) (b)

Figure 4: $E(Y|X=x)$ of the BKCH distribution for selected values of θ_2 and θ_3. (a) $\theta_2 = 0.5$, $\theta_3 = .2; a_1 = a_2 = 1$. (b) $\theta_2 = -0.5$, $\theta_3 = -.2; a_1 = a_2 = 1$.



(a) (b)

Figure 5: Continued.



(c) (d)

FIGURE 5: The pdf $f_{X,Y}(x,y)$ given by (19) for selected values of θ_1 and θ_2. (a) $\theta_1 = 1.3, \theta_2 = .5; a_1 = a_2 = 0.5$. (b) $\theta_1 = -1.3; \theta_2 = -.5; a_1 = a_2 = 0.5$. (c) $\theta_1 = -.3, \theta_2 = .5; a_1 = a_2 = 1$. (d) $\theta_1 = -.3, \theta_2 = -.5; a_1 = a_2 = 1$.

5. Estimation of BKCH Parameters

Suppose that (x_i, y_i), $(i = 1, 2, \ldots, n)$ are observed from BKCH distribution.

5.1. MLE of BKCH Parameters and Asymptotic Confidence Intervals. From the BKCH distribution, the log-likelihood $\mathbf{l}(\Theta)$ function for the $(x_i, y_i)(i = 1, 2, \ldots, n)$ sample is

$$\begin{aligned} l(\Theta) = {} & -nlog(N(\Theta)) + (a_1 - 1)\sum_{i=1}^{n} log\, x_i + (a_2 - 1)\sum_{i=1}^{n}\log\, y_i \\ & + \theta_1 \sum_{i=1}^{n}\log(1 - y_i^{a_2}) + \theta_2 \sum_{i=1}^{n}\log(1 - x_i^{a_1}) \\ & + \theta_3 \sum_{i=1}^{n}\log(1 - x_i^{a_1})\, log(1 - y_i^{a_2}). \end{aligned} \tag{34}$$

The likelihood equations are

$$\begin{aligned} \frac{\partial N(\Theta)/\partial\theta_1}{N(\Theta)} &= \frac{1}{n}\sum_{i=1}^{n}\log(1 - y_i^{a_2}), \\ \frac{\partial N(\Theta)/\partial\theta_2}{N(\Theta)} &= \frac{1}{n}\sum_{i=1}^{n}\log(1 - x_i^{a_1}), \\ \frac{\partial N(\Theta)/\partial\theta_3}{N(\Theta)} &= \frac{1}{n}\sum_{i=1}^{n}\log(1 - x_i^{a_1})\, log(1 - y_i^{a_2}). \end{aligned} \tag{35}$$

The MLE $\widehat{\Theta} = \{\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}\}_3$ can be obtained numerically using systems (35). For the asymptotic confidence interval (CI), the normal approximation of the MLE can be used to construct asymptotic CIs for the parameters Θ when the sample size is large enough. A two-sided $(1 - \alpha)100\%$ CIs for Θ are $(\widehat{\Theta} \pm Z_{\alpha/2}\sqrt{Var(\widehat{\Theta})})$, where $Var(\widehat{\Theta})$ are the asymptotic variances of $\widehat{\Theta}$.

5.2. MPLE of BKCH Parameters. In [13, 14], the study of alternative estimation technique is not based on normalizing constant. The MPLE for joint probability mass function (pmf) or joint pdf is based on the maximization of the product conditional pmfs or conditional pdfs, respectively. It is clear from (19) and (20). The log pseudlikelihood (log $PL(\Theta)$) function is

$$\begin{aligned} logPL(\Theta) = {} & n\log(a_1 a_2) + \sum_{i=1}^{n}\log\left(x_i^{(a_1-1)}\, y_i^{(a_2-1)}\right) \\ & + \sum_{i=1}^{n}\log(\theta_2 + \theta_3 log(1 - y_i^{a_2}) + 1) \\ & + \sum_{i=1}^{n}(\theta_2 + \theta_3 log(1 - y_i^{a_2}))log(1 - x_i^{a_1}) \\ & + \sum_{i=1}^{n}\log(\theta_1 + \theta_3 log(1 - x_i^{a_1}) + 1) \\ & + \sum_{i=1}^{n}(\theta_1 + \theta_3 log(1 - x_i^{a_1}))log(1 - y_i^{a_2}). \end{aligned} \tag{36}$$

Table 1: The bias and MSE values for the BKCH (1.5, 3.5, 0.05).

Method	Size N	θ_1 Bias	θ_1 MSE	θ_2 Bias	θ_2 MSE	θ_3 Bias	θ_3 MSE
MLE	50	0.0792	0.0089	−0.2901	0.1248	0.7355	0.4865
	100	0.0789	0.0085	−0.2865	0.1079	0.7358	0.4887
	200	0.0781	0.0038	0.0074	0.0938	0.7368	0.4345
	300	0.0781	0.0028	−0.2936	0.0879	0.7343	0.4912
	400	0.0800	0.0028	−0.2852	0.0784	0.7341	0.3365
	500	−0.0868	0.0025	−0.2921	0.0004	0.6123	0.4549

Table 2: The bias and MSE values for the BKCH (1.5, 2.5, -0.05).

Method	Size N	θ_1 Bias	θ_1 MSE	θ_2 Bias	θ_2 MSE	θ_3 Bias	θ_3 MSE
MLE	50	−0.0359	0.0506	−0.0206	1.097	0.4267	0.1860
	100	−0.2389	0.0041	−0.0188	0.0007	0.4291	0.1851
	200	−0.0387	0.0013	−1.018	0.0004	0.4276	0.1818
	300	−0.2386	0.0011	−0.0188	0.0003	0.4271	0.1817
	400	−0.0397	0.0019	−0.0173	0.0001	0.3279	0.1539
	500	−0.0382	0.0008	−0.0397	0.0001	0.4317	0.1069

The MSEs decrease to zero as $n \longrightarrow \infty$. This shows the consistency of the estimators.

The MPLE of BKCH parameters can be obtained by solving the equations:

$$\frac{\partial \log \mathrm{PL}(\Theta)}{\partial \theta_1} = \sum_{i=1}^{n} \frac{1}{\theta_1 + \theta_3 log\left(1 - x_i^{a_1}\right) + 1} + \sum_{i=1}^{n} \log\left(1 - y_i^{a_2}\right),$$
$$\frac{\partial \log \mathrm{PL}(\Theta)}{\partial \theta_2} = \sum_{i=1}^{n} \frac{1}{\theta_2 + \theta_3 log\left(1 - y_i^{a_2}\right) + 1} + \sum_{i=1}^{n} \log\left(1 - x_i^{a_1}\right),$$
$$\frac{\partial \log \mathrm{PL}(\Theta)}{\partial \theta_3} = \sum_{i=1}^{n} \frac{log\left(1 - y_i^{a_2}\right)}{\theta_2 + \theta_3 log\left(1 - y_i^{a_2}\right) + 1} + \sum_{i=1}^{n} \frac{log\left(1 - x_i^{a_1}\right)}{\theta_1 + \theta_3 log\left(1 - x_i^{a_1}\right) + 1} + 2\sum_{i=1}^{n} log\left(1 - y_i^{a_2}\right) log\left(1 - x_i^{a}\right). \quad (37)$$

6. Simulation Study

In this section, the MLE and MPLE approaches are used to estimate the parameters θ_2, θ_2, and θ_3 of the BKCH distribution. The population parameters are generated using software Mathematica package. The sampling distributions are obtained for different sample sizes $n = 50$, 100, 200, 300, 400, and 500 from $N = 500$ repetitions. This study presents an assessment of the properties for both MLE and MPLE techniques in terms of bias and mean square error (MSE). A general form to generate x from one marginal $f_X(x)$ and then simulate a corresponding a bivariate vector using the conditional density is $f_{Y|X}(y|x)$. The MLEs are reported in Table 1 for BKCH (1.5, 3.5, 0.05) and Table 2BKCH (1.5, 2.5, -0.05) with $a_1 = a_2 = 0.5$.

7. Application of Real Data

Suppose that X and Y are Dominant Ulna and Ulna bones. Then, the bivariate data in Table 3 represent the bone mineral density (BMD); after one year of birth, measure in gm/cm^2 for 24 kids [3].

The statistic measures for the given data are $\mu_x = 0.71267$, $\sigma_x^2 = 0.01103$, $\mu_y = 0.68904$, $\sigma_y^2 = 0.01242$, and $\rho_{x,y} = 0.62813$. The results in Table 4 represent the MLE and MPLE of BKCH parameters.

The joint pdf of bivariate exponential conditionals $BPC\ (\lambda_1, \lambda_2, \lambda_3)$ distribution is [15]

$$f_{X,Y}(x, y) = \exp\left(C - \lambda_1 y - \lambda_2 x + \lambda_3 xy\right), \quad x > 0, y > 0, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 \leq 0, \quad (38)$$

where C is the normalizing constant.

The joint pdf of bivariate Lindley conditionals hazard $BLCH\ (\lambda_1, \lambda_2, \lambda_3)$ distribution is [16]

$$f_{X,Y}(x, y) = \left[N\left(\lambda_1, \lambda_2, \lambda_3\right)\right]^{-1} (1 + x)(1 + y)\exp\left(\lambda_1 y - \left(\lambda_2 + \lambda_3 y\right)x\right), \ x > 0, y > 0, \lambda_1 < 0, \lambda_2 > 0, \lambda_3 \geq 0, \quad (39)$$

where $N(\lambda_1, \lambda_2, \lambda)_3$ is the normalized constant. $BLCH\ (\lambda_1, \lambda_2, \lambda_3)$ is TN_2; also, the conditional pdfs X|Y = y and Y|X = x are Lindley distribution with parameter $\lambda_2 + \lambda_3 y$ and $-\lambda_1 + \lambda_3 x$, respectively. Table 5 includes log-likelihood and AIC and BIC of BKCH with other models.

Table 3: The bivariate dataset of (X_i) and (Y_i).

Children	1	2	3	4	5	6	7	8
X_i	0.869	0.602	0.765	0.761	0.551	0.753	0.708	0.687
Y_i	0.964	0.689	0.738	0.698	0.619	0.515	0.787	0.715
Children	9	10	11	12	13	14	15	16
X_i	0.844	0.869	0.654	0.692	0.670	0.823	0.746	0.656
Y_i	0.656	0.789	0.726	0.526	0.580	0.773	0.729	0.506
Children	17	18	19	20	21	22	23	24
X_i	0.693	0.883	0.577	0.802	0.540	0.804	0.570	0.585
Y_i	0.740	0.785	0.627	0.769	0.498	0.779	0,634	0.640

Table 4: MLE and MPLE of parameters of the BKCH.

Parameters	MLE	MSE	MPLE	MSE
θ_1	0.4791	0.0488	−0.5383	0.0015
θ_2	0.4236	1.158	0.2688	0.0109
θ_3	2.466	0.9331	0.5283	0.0008

Table 5: Log-likelihood and AIC and BIC of BKCH, BLCH, and BEC distributions.

Model	Parameter	MLE	Log-likelihood	AIC	BIC
	θ_1	0.4791			
BKCH	θ_2	0.4236	51.075	45.075	46.308
	θ_3	2.466			
	λ_1	−0.3624			
BLCH	λ_2	1.0253	39.416	33.416	34.649
	λ_3	1.497			
	λ_1	−0.0631			
BEC	λ_2	0.0811	−123.799	−129.796	−128.563
	λ_3	0.000496			

We note that AIC and BIC of the BKCH model more than the corresponding of the BLCH and BEC models which means that BKCH distribution is better to fit for the given data. The approximate 95% two-sided CI of the parameters θ_1, θ_2, and θ_3 are given, respectively, as [−0.7548, 1.7128], [−0.381, 1.2282], and [1.394, 3.5379].

8. Conclusion

In this study, we introduced a BKCH distribution based on specified conditional hfs of Kumaraswamy distribution and its local measures of dependence. In addition, the methods of MLE and MPLE of BKHC parameters are present. In view of results, Theorem 2, the interest of BKHC is positively, negatively, or zero correlated; this indicates the generality of the distribution. Furthermore, the simulation results showed that MLE operates quite uniformly, and it can be used to estimate the BKHC parameters. In particular, in Table 4, the MPLE is better than MLE because the MPLE technique uses the conditional pdfs which does not contain the normalizing constant. Therefore, Table 5 shows the BKCH distribution is a better fit for the given data compared to the BLCH and BEC distributions.

Data Availability

All data are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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