A Novel Second-Order and Unconditionally Energy Stable Numerical Scheme for Allen–Cahn Equation

Shimin Lin, Fangying Song, Tao Sun, and Jun Zhang

1 School of Science, Jimei University, Xiamen, China
2 School of Mathematics and Statistics, Fuzhou University, Fuzhou, Fujian, China
3 School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China
4 Guizhou Key Laboratory of Big Data Statistics Analysis, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, China

Correspondence should be addressed to Jun Zhang; zj654440@163.com

Received 24 December 2021; Revised 24 February 2022; Accepted 10 March 2022; Published 29 April 2022

1. Introduction

The Allen–Cahn (AC) equation is an important model for phase field simulation in materials science. It originally describes the phase transitions process of binary alloys at a certain temperature [1]. Its important applications can be found in the fields of image analysis, crystal growth, mean curvature flows, and so on [2–6]. As we all know, the AC equation can be derived from energy variational formula, that is, $L^2$ gradient flow, which will produce a highly nonlinear function.

In order to solve the AC equation more efficiently, two problems need to be solved. Firstly, the nonlinear term should be discretized appropriately. It is well known in numerical analysis that the implicit methods [7] have usually no time step restrictions, but they require the solution of a nonlinear system. On the other hand, the explicit time integrator [8, 9] does not require a solution of linear systems, but small step sizes are taken due to the stability restrictions. Secondly, the constructed numerical scheme is expected to be energy dissipative because this model is also energy dissipative in nature. Nowadays, many numerical methods are applied to solve AC equations. Hwang et al. [10] presented benchmark problems for the numerical methods of the phase field equations. In [11], Choi et al. proposed an unconditionally gradient stable scheme to solve the AC equation with a binary mixture. The pointwise boundedness of the numerical solution was obtained. Based on operator splitting techniques, Li et al. [12] presented a second-order hybrid numerical scheme to solve the AC equation with antiphase domain. They proved that the constructed numerical scheme can achieve second-order accuracy in time and space direction. Feng and Prohl [6] constructed some semi-discrete and fully discrete schemes for solving the AC equation. They also obtained some a priori error estimate results for the proposed numerical schemes.
However, for the phase field model, it is still a challenge to construct a linear, second-order, unconditionally energy stable numerical scheme. The popular invariant energy quadratization (IEQ) [13–16] and scalar auxiliary variable (SAV) [17, 18] techniques can generate linear and unconditional energy numerical schemes. Based on the Runge–Kutta formula and SAV method, Akrivis et al. [19] constructed a linear and arbitrarily high-order numerical scheme for solving AC and Cahn–Hilliard equations. In fact, both IEQ and SAV methods need to assume that the nonlinear term is bounded from below. Moreover, the unconditional energy stability of the two methods is in agreement with the modified energy, but not the original free energy.

In this work, a linear and second-order time-stepping scheme is developed to approximate the AC equation. Different from IEQ or SAV method, our new scheme can achieve unconditional energy stability without making any assumptions about nonlinear terms or introducing auxiliary functions. In addition, the discrete energy strictly corresponds to the original free energy. We also obtain the uniform boundedness of the numerical solution and prove the second-order accuracy of the numerical method in time direction. Finally, some numerical examples are performed to show the effectiveness of the numerical scheme.

The rest of the article is organized as follows. In Section 2, we will briefly introduce the constructed AC equation. In Section 3, unconditional stability, uniqueness, and convergence of the time-stepping scheme are studied in detail. In Section 4, some numerical experiments are performed to demonstrate the accuracy and unconditional stability of the time-stepping scheme. The conclusion remark of this paper will be given in the last section.

2. Energy Dissipation Property of AC Equations

Denote the total free energy

\[ E(\phi) = \int_{\Omega} \frac{\varepsilon^2}{2} |\nabla \phi|^2 + \gamma \phi^2 + \left(\phi^2 - \frac{\gamma + 2}{2}\right)^2 \, dx - \left(\frac{\gamma^2}{4} + \gamma\right) |\Omega|, \]

The above free energy can be rewritten as follows:

\[ \frac{d}{dt} E(\phi) = -\|\phi_t\|^2. \]

3. Allen–Cahn Equation

In this section, we will develop a second-order and linear discrete scheme for AC (3). The unconditional stability and uniqueness of time-discrete scheme are proved.

3.1. Unconditional Stability and Uniqueness of Time-Discrete Scheme. Given \( \phi^n, \phi^{n-1} \), we calculate \( \phi^{n+1} \) as follows:

\[ \frac{\phi^{n+1} - \phi^{n-1}}{2\delta t} - \varepsilon^2 \Delta \phi^{n+1} + \phi^{n+1} + \frac{\gamma}{2} \left( \phi^{n+1} + \phi^{n-1} \right) + 2\left( \phi^n \phi^{n+1} + \phi^{n-1} \phi^n - (\gamma + 2) \right) \phi^n = 0. \]  

Remark 1. The above numerical scheme is different from IEQ or SAV method. We use the leapfrog formula to discretize the time direction and adopt the implicit-explicit method to deal with the nonlinear term. In order to obtain the energy stability, the IEQ and SAV methods need to assume that the free energy functions are bounded from below. But we can prove the unconditional stability of the numerical method without making any assumptions about the nonlinear term.

\[ \frac{\phi^1 - \phi^0}{\delta t} - \varepsilon^2 \Delta \phi^1 + 2\gamma \phi^1 + \left(4(\phi^0)^3 - (2\gamma + 4)\phi^0 \right) = 0. \]  

Remark 2. To initiate the second-order leapfrog scheme (6), we need the initial value \( \phi^1 \), which can be calculated by the following first-order Euler scheme: 

\[ \frac{\phi^1 - \phi^0}{\delta t} - \varepsilon^2 \Delta \phi^1 + 2\gamma \phi^1 + \left(4(\phi^0)^3 - (2\gamma + 4)\phi^0 \right) = 0. \]
For the above time-discrete scheme, we have the following energy stability results.

**Lemma 1.** The time-discrete scheme (6) satisfies the energy dissipation as follows:

\[
E(\phi^{n+1}, \phi^n) = E(\phi^n, \phi^{n-1}) - \frac{1}{4\delta t}\|\phi^{n+1} - \phi^{n-1}\|^2, \quad (8)
\]

where

\[
E(\phi^{n+1}, \phi^n) = \frac{\epsilon^2}{4} \left(\|\nabla \phi^{n+1}\|^2 + \|\nabla \phi^n\|^2\right) + \frac{\gamma}{2} \left(\|\phi^{n+1}\|^2 + \|\phi^n\|^2\right) + \frac{\gamma^2}{2} \left(\|\phi^{n+1}\|^2 - \|\phi^{n-1}\|^2\right)
\]

\[
+ \left\|\phi^{n+1} \phi^n - \frac{\gamma + 2}{2}\right\|^2 - \left(\frac{\gamma^2}{4} + \gamma\right)\Omega. \quad (9)
\]

**Proof.** Computing the inner product of (6) with \(2(\phi^{n+1} - \phi^{n-1})\), and using the following identifies

\[
(a^2(b + c) - (\gamma + 2)a) \times (b - c) = \left(a^2b^2 - (\gamma + 2)ab + \frac{\gamma + 2}{2}\right) - \left(a^2c^2 - (\gamma + 2)ac + \frac{\gamma + 2}{2}\right).
\]

\[
 = \left(ab - \frac{\gamma + 2}{2}\right)^2 - \left(ac - \frac{\gamma + 2}{2}\right)^2. \quad (10)
\]

Then, we find

\[
-\frac{1}{\delta t}\|\phi^{n+1} - \phi^{n-1}\|^2 = \epsilon^2 \left(\|\nabla \phi^{n+1}\|^2 + \|\nabla \phi^n\|^2\right) + 2\gamma \left(\|\phi^{n+1}\|^2 - \|\phi^{n-1}\|^2\right) + 4 \left(\|\phi^{n+1} \phi^n - \frac{\gamma + 2}{2}\right)^2 - \left(\frac{\gamma^2}{4} + \gamma\right)\Omega. \quad (11)
\]

Divide both sides by 4, and we can derive (8). \(\square\)

**Theorem 1.** The time-discrete scheme (6) is uniquely solvable.

**Proof.** First, we can rewrite (6) as follows:

\[
\phi^{n+1} - \epsilon^2 \delta t \Delta \phi^{n+1} + 2\gamma \delta t \phi^{n+1} + 4\delta t (\phi^n)^2 \phi^{n+1} = \eta_1, \quad (12)
\]

The above linear system can be rewritten as

\[
(L \phi, \phi) = (\zeta_1, \phi), \quad (15)
\]

with \(L = 1 - \epsilon^2 \delta t \Delta + 4\delta t + 4\delta t (\phi^n)^2\).

Next, we will show that linear system (15) has a unique solution. One can find that

\[
(\phi, \phi) + \epsilon^2 \delta t (\nabla \phi, \nabla \phi) + 2\gamma \delta t (\phi, \phi) + 4\delta t (\phi^n \phi, \phi^n \phi) = (\zeta_1, \phi), \phi \in H^1(\Omega). \quad (14)
\]

\[
(L \phi, \phi) = \|\phi\|^2 + \epsilon^2 \delta t \|\nabla \phi\|^2 + 2\gamma \delta t \|\phi\|^2 + 4\delta t \|\phi^n \phi\|^2 \geq c_0 \|\phi\|^2_{L^2}, \quad (16)
\]

where \(c_0\) is a constant that depends on \(\epsilon\) and \(\delta t\). Moreover, note that

\[
(\phi, \phi) \leq c_0 \|\phi\|_{L^2} \|\phi\|_{L^2}, \quad (17)
\]

\[
\zeta_1 = \phi^{n+1} + \epsilon^2 \delta t \Delta \phi^{n+1} - 2\gamma \delta t \phi^{n+1} - 4\delta t (\phi^n)^2 \phi^{n+1} + 4(\gamma + 2)\delta t \phi^n. \quad (13)
\]

Thus, one can solve \(\phi^{n+1}\) directly from (12). One can obtain the weak form of (12): find \(\phi \in H^1(\Omega)\), such that
where \( c_1 \) depends on \( \varepsilon, \delta t, \) and \( \| \phi^n \|_{L^\infty} \).

Furthermore, \((\mathcal{L}, \phi, \varphi) = (\mathcal{L}, \phi, \varphi)\). Thus, the bilinear form \((\mathcal{L}, \phi, \varphi)\) is coercive, bounded, and symmetric. Then, we conclude that linear system (15) admits a unique solution by using the Lax–Milgram theorem. Meanwhile, one may check that \((\mathcal{L}, \phi, \varphi) \geq 0\), and \((\mathcal{L}, \phi, \varphi) = 0 \iff \phi = 0\). This means that the bilinear form \((\mathcal{L}, \phi, \varphi)\) is positive definite. 

\[ \phi(t_n, t_{n+1}) - \phi(t_{n-1}, t_n) - \varepsilon^2 \Delta \phi(t_n) + \phi(t_{n-1}) + \frac{\Delta}{2} \phi(t_{n+1}) + \phi(t_{n-1}) + \frac{\Delta}{2} \phi(t_n) + \gamma(\phi(t_n) + \phi(t_{n-1})) = r_n^n, \]

where \( r_n^n \) satisfies

\[ \| r_n^n \| \leq C \delta t^2. \]  

Second, the pointwise error function can be denoted as

\[ \frac{E_{\phi}^{n+1} - E_{\phi}^{n-1}}{2\delta t} - \varepsilon^2 \Delta \phi_{\phi}^{n+1} + \phi_{\phi}^{n-1} + \phi_{\phi}^n + \chi_n = r_n^n, \]

with

\[ \chi_n = 2\phi(t_n)\phi(t_{n+1}) + \phi(t_{n-1})\phi(t_n) - (2 + \gamma)\phi(t_n) \]

\[ - 2(\phi_{\phi}^{n+1} + \phi_{\phi}^{n-1} + \phi_{\phi}^n - (2 + \gamma)\phi_{\phi}^n). \]

In order to obtain consistent results, the following lemmas will be used.

**Lemma 2.** There is a constant \( C_0 > 0 \), such that

\[ \max_{n \in \mathbb{K}} \| \phi(t_n) \|_{L^\infty} \| \phi^n \|_{L^\infty} \leq C_0. \]  

Then, we have

\[ |\chi_n| \leq 2\phi_{\phi}^2(t_n)\phi(t_{n+1}) - (\phi_{\phi}^2)^{n+1} + 2\phi_{\phi}^2(t_{n-1})\phi(t_n) - (\phi_{\phi}^2)^{n-1} + |E_\varphi^n| \]

\[ \leq 2\left( |\phi_{\phi}^2(t_n) - (\phi_{\phi}^2)|^2 \phi(t_{n+1}) + (\phi_{\phi}^2)^2 |\phi_{\phi}^{n+1}| \right) \]

\[ + 2\left( |\phi_{\phi}^2(t_{n-1}) - (\phi_{\phi}^2)|^2 \phi(t_n) + (\phi_{\phi}^2)^2 |\phi_{\phi}^{n-1}| \right) + |E_\varphi^n|. \]  

3.2. Consistency and Convergence Analysis. Here, we will analyze the uniform boundedness and error estimate of numerical solutions. First, using Taylor expansion, one can derive the error equation as follows:
Lemma 3. Let $\{\psi^n\}_{n=0}^{K-1}$ be sequences of discrete functions on $\Omega$. We find

$$|\psi^{n+1}| \leq \sum_{m=1}^{n} |\psi^{m+1} + \psi^{m-1}| + |\psi^0|. \quad (27)$$

Proof. We will apply mathematical induction to prove the above conclusion. For $n = 1$, we find $|\psi^2| - |\psi^1| \leq |\psi^2 + \psi^1|$. Assume

$$|\psi^n| - |\psi^{n-1}| \leq \sum_{m=1}^{n-1} |\psi^{m+1} + \psi^{m-1}|, \text{ for } n \leq k - 1. \quad (28)$$

When $n = k$, we find

$$|\psi^{k+1}| - |\psi^k| = |\psi^{k+1} - |\psi^k| - |\psi^k| - |\psi^{k-1}| - |\psi^k| \leq \sum_{m=1}^{k-1} |\psi^{m+1} + \psi^{m-1}| + |\psi^{k+1} - |\psi^{k-1}| - |\psi^{k-1}| \leq \sum_{m=1}^{k} |\psi^{m+1} + \psi^{m-1}|. \quad (29)$$

This ends the proof.

To analyze the consistency results, we denote $\rho$, such that

$$\rho = \max_{0 \leq t \leq T} \|\psi(\cdot, t)\|_{L^\infty} + 1. \quad (30)$$

It should be mentioned that Shen et al. [9] also made a similar assumption.

For simplicity of analysis, we set $\varepsilon = \gamma = 1$. The following lemma will show the $L^\infty$ uniform boundedness results of numerical solution.

Lemma 4. Assume the exact solution of the AC equation is smooth enough (at least 2-order differentiable in time and $W^{2,\infty}$ bound in space direction). There is a constant $\tau_0 > 0$; if $\delta t < \tau_0$, we can get the uniform boundedness results as follows:

$$\|\psi^n\|_{L^\infty} \leq \rho, k = 0, 1, \ldots, \frac{T}{\delta t}. \quad (31)$$

Remark 3. For the assumption of continuous solution of the AC equation, we can find some similar hypotheses in the following works [9, 13, 16].

Proof. In order to prove the above conclusion, mathematical induction will be used. For the first step, $k = 0$. It is easy to find that $\|\psi^0\|_{L^\infty} \leq \rho$. Suppose that the numerical solution has an $L^\infty$ bound at $t^n$.

Thus, for $n \leq k$, we find

$$\|\chi^n\| \leq C_1 \left( \|E_\phi^n\| + \|E_{\phi}^{n+1}\| + \|E_\phi^{n-1}\| \right). \quad (26)$$

\[\square\]

Then, we will check that $\|\phi^{n+1}\|_{L^\infty} \leq \rho$ is still valid. By the assumption of exact solution, we find

$$\|\phi(\cdot, t_n)\|_{L^\infty} \leq \bar{C}, \|\nabla \phi(\cdot, t_n)\|_{L^\infty} \leq \bar{C}. \quad (33)$$

Taking $L^2$ inner product of (21) with $2(E_{\phi}^{n+1} - E_{\phi}^{n-1})$ gives

$$\frac{1}{\delta t} \|E_{\phi}^{n+1} - E_{\phi}^{n-1}\|^2 + \left( \|\nabla E_{\phi}^{n+1}\|^2 - \|\nabla E_{\phi}^{n-1}\|^2 \right)$$

$$+ 2 \left( \|E_{\phi}^{n+1}\|^2 - \|E_{\phi}^{n-1}\|^2 \right) \quad (34)$$

By using Young’s inequality, we find

$$2(E_{\phi}^{n+1} - E_{\phi}^{n-1}) \leq 4\delta t \|E_{\phi}^{n+1}\|^2 + \frac{1}{4\delta t} \|E_{\phi}^{n+1} - E_{\phi}^{n-1}\|^2, \quad (35)$$

$$-2(E_{\phi}^{n+1} - E_{\phi}^{n-1}) \leq 4\delta t \|E_{\phi}^{n+1}\|^2 + \frac{1}{4\delta t} \|E_{\phi}^{n+1} - E_{\phi}^{n-1}\|^2. \quad (36)$$

Using Lemma 2, we arrive at

$$\|\chi^n\| \leq C \left( \|E_{\phi}^n\| + \|E_{\phi}^{n+1}\| + \|E_{\phi}^{n-1}\| \right). \quad (37)$$

Combining (34)–(37), we obtain

$$\left( \|\nabla E_{\phi}^{n+1}\|^2 - \|\nabla E_{\phi}^{n-1}\|^2 \right) + 2 \left( \|E_{\phi}^{n+1}\|^2 - \|E_{\phi}^{n-1}\|^2 \right)$$

$$+ \frac{1}{2\delta t} \|E_{\phi}^{n+1} - E_{\phi}^{n-1}\|^2 \leq C\delta t^5 + C\delta t \left( \|E_{\phi}^{n+1}\|^2 + \|E_{\phi}^{n+1}\|^2 + \|E_{\phi}^{n-1}\|^2 \right). \quad (38)$$

Summing up the above inequality for $n = 1, \ldots, k$, we have

$$E_{\phi}^{k+1} + \frac{1}{2\delta t} \sum_{n=1}^{k} \|E_{\phi}^{n+1} - E_{\phi}^{n-1}\|^2 \leq E_0 + C\delta t^4 + C\delta t \sum_{n=1}^{k} E_{\phi}^{n+1}, \quad (39)$$

where

$$E_{\phi}^{k+1} = \left( \|\nabla E_{\phi}^{n+1}\|^2 + \|\nabla E_{\phi}^{n}\|^2 \right) + 4 \left( \|E_{\phi}^{n+1}\|^2 + \|E_{\phi}^{n}\|^2 \right). \quad (40)$$

For the first step, we note that

$$E_0 \leq C\delta t^4. \quad (41)$$

Applying discrete Gronwall’s inequality in (39) gives

$$E_{\phi}^{k+1} + \frac{1}{2\delta t} \sum_{n=1}^{k} \|E_{\phi}^{n+1} - E_{\phi}^{n-1}\|^2 \leq C\delta t^4. \quad (42)$$
From (21), we have

\[
\| \Delta (E_{\phi}^{n+1} + E_{\phi}^{n-1}) \| \leq \frac{3}{\delta t} \| E_{\phi}^{n+1} - E_{\phi}^{n-1} \| + \| E_{\phi}^{n+1} + E_{\phi}^{n-1} \| + \| \rho \| .
\] (43)

Summing up for \( n = 1, \ldots, k \) and using Lemma 3, we have

\[
\| \Delta E_{\phi}^{k+1} \| \leq \sum_{n=1}^{k} \| \Delta (E_{\phi}^{n+1} + E_{\phi}^{n-1}) \| \leq C\delta t.
\] (44)

Note that \( \| \phi^{k+1} \|_{L^\infty} \leq \| E_{\phi}^{n+1} \|_{L^\infty} + \| \phi(t_n+1) \|_{L^\infty} \)

\[
\leq C \| E_{\phi}^{n+1} \|_{H^1} \| E_{\phi}^{n-1} \|_{H^1} + \| \phi(t_n+1) \|_{L^\infty}.
\] (45)

If \( C\delta t^{3/2} \leq 1 \), we get

\[
\| \phi^{k+1} \|_{L^\infty} \leq 1 + \| \phi(t_n+1) \|_{L^\infty} \leq \rho.
\] (46)

Then, we obtain (31).

**Theorem 2.** Let \( \phi \) be the solution of (4) and (5) and \( \{ \phi^{k} \}_{k=0}^{K} \) be the solution of (6). Under the assumption of Lemma 4, as \( \delta t \rightarrow 0 \), the following error estimate holds:

\[
\| \phi(t_n) - \phi^{k} \|_{H^1} \leq C\delta t^2, k = 0, 1, \ldots, K.
\] (47)

**Proof.** Note that in Lemma 4, we prove the uniform boundedness results:

\[
p_1 = \log_2 \left( \frac{\| u_{n+1}^{2\Delta t} - u_{n}^{2\Delta t} \|_{L^1}}{\| u_{n+1}^{2\Delta t} - u_{n+1}^{4\Delta t} \|_{L^1}} \right), p_2 = \log_2 \left( \frac{\| u_{n+1}^{2\Delta t} - u_{n+1}^{2\Delta t} \|_{H^1}}{\| u_{n+1}^{2\Delta t} - u_{n+1}^{4\Delta t} \|_{H^1}} \right).
\] (51)

The numerical results indicate that the time-discrete scheme is 2-order convergent in time direction. This is consistent with the theoretical results in Theorem 2.

Second, we fix \( \delta t = 0.001, N_x = N_y = 128, \) set \( f = 0 \) in (49), and present the discrete energies of the time-discrete scheme for \( \varepsilon = 0.1, \varepsilon = 0.25 \) in Figures 1 and 2. Numerical results show that discrete energy decays, which is consistent with our proof.

In order to make a comparison with Choi et al. [11], we choose the same initial value as

\[
\phi(x, y, 0) = \tanh \frac{0.25 - \sqrt{(x - 0.5)^2 + (y - 0.5)^2}}{\sqrt{\varepsilon}}.
\] (52)

Following the process of (34)–(42), we can get the conclusion of the theorem. \( \square \)

4. Numerical Experiments

In this section, we will propose several numerical examples to show the accuracy, convergence, and unconditional energy stability of the time-discrete scheme.

Let \( \gamma = 2, \) and let us consider the following AC equation:

\[
\phi_t - \varepsilon^2 \Delta \phi + (4\phi^3 - 8\phi) + 4\phi = f(x, y, t). \] (49)

First, we will test the accuracy of time-discrete scheme by choosing a suitable source term \( f(x, y, t) \) such that the exact solution is

\[
\phi(x, y, t) = \exp(-t)\cos(x)\cos(y).
\] (50)

Let \( T = 1, \delta = 0.1. \) The \( L^2 \)-error, \( H^1 \)-error, and convergence order of time direction are presented in Table 1. The spatial discretization of the time-discrete scheme is handled by using the Fourier pseudo-spectral method with Fourier modes \( N_x = 128, N_y = 128. \)

We calculate the errors by the following quantities.

Let \( \varepsilon = 0.035, \) \( \delta t = 0.01, N = 256, \gamma = 1. \) Figures 3(a)–3(d) show the evolution process of initial concentration with time. We also observe that the circle gradually shrinks, which is consistent with the theoretical prediction [11]. We still choose the same parameters as above. Figures 4(a)–4(d) show the result behaviours of the proposed method with respect to \( \varepsilon. \)

At last, the dynamic evolution of numerical solutions is also studied. Set \( \phi_0 = \cos(x)\cos(y), \) \( \delta t = 0.001, N_x = N_y = 128. \) The snapshots of phase separation with different times are presented in Figure 5. This results show that our numerical method can capture the process of phase separation.
Table 1: Numerical error and convergence order for 2D AC equation.

<table>
<thead>
<tr>
<th>δt</th>
<th>$L^2$-error</th>
<th>Order</th>
<th>$H^1$-error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>6.6813E-02</td>
<td>-</td>
<td>2.8594E-01</td>
<td>-</td>
</tr>
<tr>
<td>0.02</td>
<td>2.9463E-03</td>
<td>1.9394</td>
<td>1.3413E-02</td>
<td>1.9009</td>
</tr>
<tr>
<td>0.01</td>
<td>7.1127E-04</td>
<td>2.0504</td>
<td>3.2606E-03</td>
<td>2.0404</td>
</tr>
<tr>
<td>0.002</td>
<td>2.7408E-05</td>
<td>2.0232</td>
<td>1.2630E-04</td>
<td>2.0199</td>
</tr>
<tr>
<td>0.001</td>
<td>6.8161E-06</td>
<td>2.0075</td>
<td>3.1430E-05</td>
<td>2.0066</td>
</tr>
<tr>
<td>0.0002</td>
<td>2.7147E-07</td>
<td>2.0026</td>
<td>1.2524E-06</td>
<td>2.0023</td>
</tr>
<tr>
<td>0.0001</td>
<td>6.7832E-08</td>
<td>2.0007</td>
<td>3.1296E-07</td>
<td>2.0006</td>
</tr>
<tr>
<td>0.00002</td>
<td>2.7109E-09</td>
<td>2.0005</td>
<td>1.2507E-08</td>
<td>2.0005</td>
</tr>
</tbody>
</table>

Figure 1: The discrete energies for AC equation with $\varepsilon = 0.1$.

Figure 2: The discrete energies for AC equation with $\varepsilon = 0.05$.

Figure 3: Continued.
Figure 3: The evolution process of initial concentration with time: (a) \( t = 0 \); (b) \( t = 62 \); (c) \( t = 125 \); (d).

Figure 4: The result behaviours of the proposed method with respect to \( \epsilon \): (a) \( \epsilon = 0.05 \); (b) \( \epsilon = 0.02 \); (c) \( \epsilon = 0.01 \); (d).

Figure 5: Numerical solutions of \( \phi \) for the AC equation using the full discrete scheme (6). Snapshots are taken at \( t = 0, 1, 40, 80, 100, 300 \), respectively.
5. Conclusion

In this paper, by skillfully dealing with the nonlinear function, we proposed a novel second-order and linear scheme to solve AC equation. Uniqueness and unconditional energy stability of the numerical scheme are proved. Moreover, by constructing an appropriate auxiliary function, we prove the uniform boundedness of the numerical solution. Based on the uniform boundedness result, we get the error estimate of time direction. Finally, several numerical examples are presented to demonstrate the accuracy, stability, and efficiency of the numerical scheme, and the dynamic evolution of the AC equation is also discussed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The work of Shimin Lin is supported by National Natural Science Foundation of China (No. 11901237). The research of Fangying Song was supported by the NSFC (No. 11901100). The work of Tao Sun is supported in part by NSFC (No. 12171141). The work of Jun Zhang is supported by the National Natural Science Foundation of China (Nos. 11901132 and 62062018), Science and Technology Program of Guizhou Province (Nos. ZK[2022]006 and ZK[2022]031), Natural Science Research Projects of Education Department of Guizhou Province (No. KY[2021]015), and Guizhou Key Laboratory of Big Data Statistics Analysis (No. BDSA20200102).

References