Research Article

Anti-Plane Wave Propagation in the Functionally Graded Hybrid Structure under an External Impulsive Force: A Green’s Function Approach

Uma Bharti and Pramod Kumar Vaishnav

School of Mathematics, Thapar Institute of Engineering and Technology, Patiala 147001, India

Correspondence should be addressed to Pramod Kumar Vaishnav; pvaishnav.ism@gmail.com

Received 10 June 2022; Revised 24 July 2022; Accepted 3 August 2022; Published 9 September 2022

1. Introduction

Anti-plane wave propagation through layered schematic gives immense knowledge about the earth’s interior for possible natural minerals. The dispersion relation of an anti-plane wave depends on the chemical properties (fluid-solid interaction, strength, hardness, etc.) of the composite materials. Consequently, obtaining the dispersion curve of the Love wave to understand the characteristics of the materials present in the earth’s interior is intriguing. The propagation of Love waves in reinforced-orthotropic heterogeneous composite materials under the influence of impulsive forces and uneven interfaces is a new study field in mineral exploration, civil, mining and structural engineering, hydrology, geophysics, etc.

Immense studies on Love wave propagation in composite materials have been carried out by many researchers like Lamb wave, Love wave, and other elastic waves in isotropic and anisotropic composite materials that have been studied in [1–4]. The effect of the reinforcement, different heterogeneities (viz., quadratic, linear, exponential, and hyperbolic), initial stress, several irregular boundaries (viz., parabolic, periodic, and rectangular), rigid and soft mountain surfaces, and viscosity and porosity on the phase velocity of the Love wave in several composite materials (including fiber-reinforced, self-reinforced, void porous, and orthotropic) have been investigated in [5–15]. Mal [16] demonstrated the influence of rectangular irregularity on the phase velocity of the Love wave at the interface of two distinct materials. Furthermore, the perturbation approach was used to solve the non-vanishing governing equation of motion. Because of the existence of impulse forces (body forces associated with composite materials) in the earth’s interior, the governing equation of motion becomes non-homogeneous, making it harder to solve. Subsequently, the Green’s function technique was used to solve the
nonhomogeneous wave equation. In the extant literature, there are very few publications on using Green’s function to address elastic wave propagation issues. Chattopadhyay et al. [17], Kundu et al. [18], and Manna et al. [19, 20] employed the Green’s function approach to solve the elastic wave propagation in the elastic medium with regular boundaries. Singh et al. [21, 22] later employed the aforementioned approach to solve the Love wave propagation in piezoelectric and piezomagnetic structures. Recently, Deliktas and Teymur [23] examined the elastic wave propagation in an irregular layer with an effect of a nonlinear surface. Rayleigh wave propagation in irregular subspace of the porous medium has been explained by Xiao et al. [24]. Kumari et al. [25] studied the influence of the imperfect boundaries and abrupt thickening on the phase velocity of the surface waves. SH waves characteristics in Cosserat isotropic medium with scattering have been studied by Chaki and Singh [26]. Mei et al. [27] examined the band gaps of SH waves of metamaterials.

Apart from existing research, we considered a superficial layer of reinforced materials of finite thickness $H$ resting over an initially stressed heterogeneous orthotropic substrate as a medium of anti-plane wave propagation. At the interface of these materials, a rectangular plate of length $h$ and depth $h_1$ is installed to investigate the effect of irregularity on the phase velocity of the Love wave, we focused on the behavior of the phase velocity of the Love wave only. Impulse forces due to point source are associated along the depth of the substrate to examine the behavior of the phase velocity of the Love wave. Moreover, the sine hyperbolic function is considered as a variation in the rigidities, densities, and initial stress which represent the diversity of the materials. The dispersion relation of the Love wave is derived, the governing equations of motion are deduced by Fourier transformation and solved analytically by Green’s function technique and its well-known properties. However, a number of methods have been developed for solving the problem of wave propagation analytically in different mediums with irregular interfaces such as the integral method and the power series method. In spite of that, the Green’s function is a useful mathematical tool to solve the equations of motion analytically in the presence of point source phenomena. The obtained dispersion relation is fairly matched with the conventional form of the Love wave dispersion which validates the present study.

To the best of the author’s knowledge, no study has been made so far to analyze the anti-plane wave propagation in the proposed schematic. The effect of varying parameters, initial stress, irregularity, and impulse forces on the phase velocity of the Love wave are studied numerically and manifested graphically. The mechanism of the wave propagation and the numerical data for the elastic constants of the materials are taken from [28–32]. Our finding observed that the phase velocity of the Love wave was affected significantly in the presence of irregularity, heterogeneity, body forces (impulse forces due to point source), initial stress, and magnify parameters. Moreover, it is also noticed that the change in the length of the installed rectangular plate at the interface of the schematic affects the phase velocity of the Love wave significantly.

This paper is organized as follows: in Section 2, we present the schematic of the problem, discuss the mechanism of irregularity at the interface of materials and describe the hyperbolic variation in the orthotropic half-space. Section 3, highlights the dynamics of reinforced materials and orthotropic materials under impulsive force. In this section, we deduce the governing equations of motion by using Fourier transformation for the materials. Section 4 presents the boundary conditions and the solution of the problem. Stability of the earth model is discussed in Section 5. Numerical results and discussion are provided in Section 6. Finally, the main conclusions are reported in Section 7.

2. Schematic of the Problem

Hybridization in the propagation medium is considered with the hyperbolic heterogeneity in an anisotropic elastic substrate with the irregular interface of two mediums (layer and half-space). Anti-plane wave propagates in a hybrid structure with the velocity $c$ and the wave number $k$ along $x$–direction, the mechanical displacement of the particle is observed in the $y$–direction only. A two-dimensional earth model has been described in the Cartesian coordinate system as shown in Figure 1.

A plate of the rectangular shape has been installed at the contact interface of the guiding layer and half-space to measure the effect on the phase velocity and wave numbers. Length and depth of the plate are taken as $h_1$ and $h$, respectively. Source of energy disturbance (impulsive point source) has been considered along the $z$–direction in the orthotropic half-space.

Interface of the two mediums has been formulated mathematically as:

$$z = \eta f(x) - H,$$

where

$$f(x) = \begin{cases} h, & |x| < \frac{h}{2} \\ 0, & |x| \geq \frac{h}{2} \end{cases}$$ (1)

$\eta = h_1/h$ is a perturbation parameter which is a very small positive quantity ($\eta << 1$).

Apart from the established results, we will take the hyperbolic heterogeneity in the orthotropic medium with irregular interface and an impulsive point source effect into account when dealing the anti-plane wave propagation. The shear moduli of the orthotropic medium has been assumed as

\[
\begin{align*}
L &= L_1 + \varepsilon \sinh b(z - H), \\
N &= N_1 + \varepsilon \sinhb(z - H), \\
T &= T_1 + \varepsilon \coshb(z - H), \\
\rho &= \rho_1 + \varepsilon \coshb(z - H).
\end{align*}
\] (2)
3. Dynamics of the Materials

3.1. Dynamics for the Upper Layer. The governing equation of motion with the source of disturbance in the medium is

\[ \tau^{(i,j)} + F^{(i)} = \rho \frac{\partial^2 u^{(j)}}{\partial t^2}, \]  

where \( \tau^{(i,j)} \) represents the stress components in \( j \)th direction \( (j = 1, 2, 3) \), \( \rho \) represents the density of the material found in the earth’s interior layer, and \( F^{(i)} \) are the body forces.

As wave is propagating horizontally along the \( x \)-axis: \( u = w = 0 \), \( v = v(x, z, t) \) and \( \partial / \partial y = 0 \).

Considering the impact of the point source in the reinforced medium, the equation of motion is

\[ P \frac{\partial^2 v^{(i)}}{\partial x^2} + Q \frac{\partial^2 v^{(i)}}{\partial x \partial z} + R \frac{\partial^2 v^{(i)}}{\partial z^2} - \rho \frac{1}{2} \frac{\partial^2 v^{(i)}}{\partial t^2} = 4\pi \sigma^{(1)}(r, t), \]  

where

\[ P = \mu_T + (\mu_L - \mu_T)\alpha_3^2, \quad Q = (\mu_T + \mu_L - \mu_T)\alpha_1^2, \quad R = \alpha_1 \alpha_3 (\mu_L - \mu_T). \]  

\( \mu_T \) is transverse shear modulus and \( \mu_L \) is longitudinal shear modulus considered in a preferred direction. \( \sigma^{(1)}(r, t) \) is the force density disturbance with the effect of the point source. Force is acting at a distance of \( r \) from origin at a time \( t \).

\( \alpha_1, \alpha_2, \alpha_3 \) are the components of \( a^r \) in the direction of reinforcement such that \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \).

Considering

\[ v^{(1)}(x, z, t) = v^{(1)}(x, z)e^{i\omega t}, \quad \sigma^{(1)}(r, t) = \sigma^{(1)}(r)e^{i\omega t}, \]  

in (4), we obtain

\[ P \frac{\partial^2 v^{(1)}}{\partial z^2} + Q \frac{\partial^2 v^{(1)}}{\partial x \partial z} + R \frac{\partial^2 v^{(1)}}{\partial x^2} + \rho \frac{1}{2} \frac{\partial v^{(1)}}{\partial t} (x, z) \omega^2 = 4\pi \sigma^{(1)}(r), \]  

where \( \omega = k \) is the angular frequency.

Here the impulsive force \( \sigma^{(1)}(r) \) cause some disturbances which can be represented as \( \sigma^{(1)}(r) = \delta(x) \delta(z - H) \).

So the above equation of motion is

\[ P \frac{\partial^2 v^{(1)}}{\partial z^2} + Q \frac{\partial^2 v^{(1)}}{\partial x \partial z} + R \frac{\partial^2 v^{(1)}}{\partial x^2} + \rho \frac{1}{2} \frac{\partial v^{(1)}}{\partial t} (x, z) \omega^2 = 4\pi \delta(x) \delta(z - H). \]  

Defining the Fourier transform \( \tilde{v}^{(r)}(\xi, z) \) of \( v^{(r)}(x, z) \) as

\[ \tilde{v}^{(r)}(\xi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v^{(r)}(x, z)e^{i\xi x} dx, \]  

and inverse Fourier transform as

\[ v^{(r)}(x, z) = \int_{-\infty}^{\infty} \tilde{v}^{(r)}(\xi, z)e^{-i\xi x} d\xi. \]

Imposing Fourier transform on (9), we get

\[ \frac{d^2 \tilde{v}^{(1)}}{dz^2} + 2M \frac{d \tilde{v}^{(1)}}{dz} + \beta^{(1)} \tilde{v}^{(1)} = \frac{2}{P} \delta(z - H), \]  

where

\[ \beta^{(1)} = \frac{\rho}{P} \omega^2 - \frac{Q \xi^2}{P}, \quad M = \frac{R \xi}{P}. \]

In order to make the equation free of a first derivative term, set \( \tilde{v}^{(1)} = \tilde{w}^{(1)}e^{-Mz} \) in (12), it becomes

\[ \frac{d^2 \tilde{w}^{(1)}}{dz^2} - a^2 \tilde{w}^{(1)} = \frac{2}{P} e^{Mz} \delta(z - H), \]  

where

\[ a^2 = M^2 - \beta^{(1)}. \]

3.2. Dynamics for the Orthotropic Layer. The equation of motion for the anisotropic medium of infinite depth is

\[ \frac{\partial}{\partial x} \left( N \frac{\partial v^{(2)}}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial v^{(2)}}{\partial z} \right) - T \frac{\partial}{\partial x} \left( \frac{\partial v^{(2)}}{\partial x} \right) - \rho \frac{\partial^2 v^{(2)}}{\partial t^2} = 0, \]  

where \( N \) and \( L \) are directional rigidity, \( T \) represents initial stress, and \( \rho \) represents density of the half-space. Using (2), we get
\[ v^{(2)}(x, z, t) = v^{(2)}(x, z)e^{i\omega t}, \]  

in (16), and we get

\[ \left( N_1 - \frac{T_1}{2} \right) \frac{\partial^2 v^{(2)}}{\partial x^2} + L \frac{\partial^2 v^{(2)}}{\partial z^2} + \rho_1 \omega^2 v^{(2)}, \]

\[ = -c \left\{ \sinh(z - H) - \frac{1}{2} \cosh(z - H) \right\} \frac{\partial^2 v^{(2)}}{\partial x^2} + \sinh(z - H) \frac{\partial^2 v^{(2)}}{\partial z^2} + \omega^2 \cosh(z - H)v^{(2)}. \]  

(17)

\[ m^2 = \frac{\xi^2}{L_1} \left( N_1 - \frac{T_1}{2} \right) - \frac{\rho_1}{L_1} \omega^2, \]

\[ 4\pi\sigma^{(2)}(z) = \frac{\xi}{L_1} \left[ \begin{array}{c} -\xi^2 \sinh(z - H) + \frac{\xi^2}{2} \cosh(z - H) + \omega^2 \cosh(z - H) \vspace{1em} \\ + \cosh(z - H) \frac{d\sigma^{(2)}}{dz} + \sinh(z - H) \frac{d^2\sigma^{(2)}}{dz^2} \end{array} \right] \]  

(19)

4. Boundary Conditions and Solution of the Problem

Our aim is to get the displacement relation for both the media from (14) and (19) by the Green’s function method. These equations also satisfy the boundary conditions at the prescribed boundaries at \( z = 0 \) and at \( z = \eta f(x) - H \).

The boundary conditions are as follows:

(i) The surface \( z = 0 \) is stress free.

\[ R \frac{\partial \sigma^{(1)}}{\partial x} + P \frac{\partial \sigma^{(1)}}{\partial z} = 0. \]  

(20)

(ii) The stress is continuous at the common interface \( (z = \eta f(x) - H) \).

\[ R \frac{\partial \sigma^{(1)}}{\partial x} + P \frac{\partial \sigma^{(1)}}{\partial z} = L \frac{\partial \sigma^{(2)}}{\partial z}. \]  

(21)

(iii) The displacement is continuous at \( z = \eta f(x) - H \).

\[ \bar{w}^{(1)}(z) = \bar{w}^{(2)}(z). \]  

(22)

Let \( G_1(z/z_0) \) be Green’s function for the reinforced medium satisfying the conditions \( dG_1/dz = 0 \) at \( z = 0 \) and \( z = \eta f(x) - H \). Equation (14) satisfied by \( G_1(z/z_0) \) is

\[ \frac{d^2G_1}{dz^2} - m^2 G_1 = \delta(z - z_0). \]  

(23)

With Fourier transform on \( \tau^{(2)}(x, z) \), (18) takes the form of ODE as

\[ \frac{d^2\tau^{(2)}}{dz^2} - m^2 \tau^{(2)} = 4\pi\sigma^{(2)}(z), \]  

(18)

where

\[ \frac{d^2G_1}{dz^2} - \alpha^2 G_1 = \delta(z - z_0). \]  

(24)

Multiplying (14) by \( G_1(z/z_0) \) and (25) by \( \bar{w}^{(1)}(z) \), subtracting and integrating from 0 to \( \eta f(x) - H \), then replacing \( z_0 \) by \( z \) and using the symmetric property \( G(\eta f(x) - z_0) = G(\eta f(x) - H/z) \), we get

\[ \bar{w}^{(1)}(z) = \frac{2}{P} \frac{M(\eta f(x) - H)}{\bar{G}_1(\eta f(x) - H)} \left[ \frac{z}{\eta f(x) - H} \right] - \frac{G_1(z)}{\eta f(x) - H} \left[ \frac{d\bar{w}^{(1)}}{dz} \right]_{z = \eta f(x) - H}. \]  

(25)

For half-space of the orthotropic medium, \( G_2(z/z_0) \) be Green’s function that satisfies (19), then \( G_2(z/z_0) \) is the solution given by

\[ \frac{d^2G_2}{dz^2} - m^2 G_2 = \delta(z - z_0). \]  

(26)
Applying boundary condition (24), (26), and (28) collectively implies

\[ \frac{2}{Pe^M} \left( \frac{\eta f(x) - H}{\eta f(x) - H} \right) G_1 \left( \frac{\eta f(x) - H}{\eta f(x) - H} \right) 
- G_1 \left( \frac{\eta f(x) - H}{\eta f(x) - H} \right) \left[ \frac{d\bar{w}^{(1)}}{dz} \right]_{z_0}^{\eta f(x) - H}, \]

\[ = G_2 \left( \frac{\eta f(x) - H}{\eta f(x) - H} \right) \left[ \frac{d\bar{w}^{(2)}}{dz} \right]_{z_0}^{\eta f(x) - H} + \int_{\eta f(x) - H}^{\infty} \frac{4\pi \sigma^{(2)}(z_0) G_2 \left( \frac{z}{z_0} \right)}{d\bar{w}(z_0)} dz_0. \]

Using (23), (29) gives us

\[ \left[ \frac{d\bar{w}^{(2)}}{dz} \right]_{z_0}^{\eta f(x) - H} = \frac{2/Pe^M(\eta f(x) - H) G_1(\eta f(x) - H/\eta f(x) - H) - \int_{\eta f(x) - H}^{\infty} 4\pi \sigma^{(2)}(z_0) G_2(\eta f(x) - H/\eta f(x) - H) dz_0}{G_2(\eta f(x) - H/\eta f(x) - H) + A_1/PG_1(\eta f(x) - H/\eta f(x) - H)}, \]

where \( A_1 \) is defined in the appendix. According to (23), (29) yields

\[ \left[ \frac{d\bar{w}^{(1)}}{dz} \right]_{z_0}^{\eta f(x) - H} = \frac{(2/P)e^M(\eta f(x) - H) G_1(\eta f(x) - H/\eta f(x) - H) - \int_{\eta f(x) - H}^{\infty} 4\pi \sigma^{(2)}(z_0) G_2(\eta f(x) - H/\eta f(x) - H) dz_0}{G_1(\eta f(x) - H/\eta f(x) - H) + (P/A_1)G_2(\eta f(x) - H/\eta f(x) - H) - \int_{\eta f(x) - H}^{\infty} \frac{4\pi \sigma^{(2)}(z_0) G_2(\eta f(x) - H/\eta f(x) - H)}{d\bar{w}(z_0)} dz_0}. \]

With the aid of (31), the value of \( [d\bar{w}^{(1)}/dz] \) and (21), and the value of \( 4\pi \sigma^{(2)}(z_0) \), (26) results in

\[ \bar{w}^{(1)}(z) = \frac{2/A_1 e^M(\eta f(x) - H)}{G_1(\eta f(x) - H/\eta f(x) - H) + P/A_1 G_2(\eta f(x) - H/\eta f(x) - H)} \]

\[ - \frac{\epsilon/L G_1(\eta f(x) - H/\eta f(x) - H) + P/A_1 G_2(\eta f(x) - H/\eta f(x) - H)}{G_1(\eta f(x) - H/\eta f(x) - H) + P/A_1 G_2(\eta f(x) - H/\eta f(x) - H)} \]

\[ \times \int_{\eta f(x) - H}^{\infty} \left\{ -\frac{\xi^2}{2} \sinh(z - H) + \frac{\xi^2}{2} \cosh(z - H) + \omega^2 \cosh(z - H) \right\} \frac{d\bar{w}^{(2)}}{dz} + \sinh(z - H) \frac{d^2\bar{w}^{(2)}}{dz^2} dz_0. \]

Applying (30) in (28), we have
\[ \varphi^{(2)}(z) = \frac{2Pe^{M(\eta f(x)-H)}}{G_1(\eta f(x) - H/\eta f(x) - H)G_2(z/\eta f(x) - H)} + \int_{\eta f(x)-H}^{\infty} \left\{ -\frac{\xi^2}{2}\sinh(z_0 - H) + \frac{\xi^2}{2}\cosh(z_0 - H) + \omega^2 \cosh(z - H) \right\} \varphi^{(2)}(z) \]

\[ + \frac{bcosh(z - H)}{dx_0} + \sinh(z_0 - H) \frac{d^2\varphi^{(2)}}{dz_0^2} \]

\[ G_1(\eta f(x) - H/\eta f(x) - H)G_2(z/\eta f(x) - H) \]

\[ G_2(\eta f(x) - H/\eta f(x) - H) + A_1/PG_1(\eta f(x) - H/\eta f(x) - H) \]

\[ \times \int_{\eta f(x)-H}^{\infty} \left\{ -\frac{\xi^2}{2}\sinh(z_0 - H) + \frac{\xi^2}{2}\cosh(z_0 - H) + \omega^2 \cosh(z_0 - H) \right\} \varphi^{(2)}(z) \]

\[ + \frac{bcosh(z_0 - H)}{dx_0} + \sinh(z_0 - H) \frac{d^2\varphi^{(2)}}{dz_0^2} \]

\[ G_1\left(\frac{\eta f(x) - H}{z_0}\right) dz_0. \]

The value of \( \varphi^{(2)}(z) \) can be obtained from (33) by the method of successive approximations. Taking the first approximation of \( \varphi^{(2)}(z) \) and neglecting the higher powers of \( \epsilon \), (33) reduces to

\[ \varphi^{(2)}(z) = \frac{2Pe^{M(\eta f(x)-H)}}{G_1(\eta f(x) - H/\eta f(x) - H)G_2(z/\eta f(x) - H)} + \int_{\eta f(x)-H}^{\infty} \left\{ -\frac{\xi^2}{2}\sinh(z_0 - H) + \frac{\xi^2}{2}\cosh(z_0 - H) + \omega^2 \cosh(z_0 - H) \right\} \varphi^{(2)}(z) \]

\[ + \frac{bcosh(z_0 - H)}{dx_0} + \sinh(z_0 - H) \frac{d^2\varphi^{(2)}}{dz_0^2} \]

\[ G_1\left(\frac{\eta f(x) - H}{z_0}\right) dz_0. \]

This is the displacement at any point in the half-space. Using (34) in (32), we obtain

\[ \varphi^{(2)}(z) = \frac{2Pe^{M(\eta f(x)-H)}}{G_1(\eta f(x) - H/\eta f(x) - H)G_2(z/\eta f(x) - H)} + \frac{2\epsilon/PLG_1(z/\eta f(x) - H)G_2(\eta f(x) - H/\eta f(x) - H)}{G_1(\eta f(x) - H/\eta f(x) - H) + A_1/PG_1(\eta f(x) - H/\eta f(x) - H)} \]

\[ \times \int_{\eta f(x)-H}^{\infty} \left\{ -\frac{\xi^2}{2}\sinh(z_0 - H) + \frac{\xi^2}{2}\cosh(z_0 - H) + \omega^2 \cosh(z_0 - H) \right\} \varphi^{(2)}(z) \]

\[ + \frac{bcosh(z_0 - H)}{dx_0} + \sinh(z_0 - H) \frac{d^2\varphi^{(2)}}{dz_0^2} \]

\[ G_1\left(\frac{\eta f(x) - H}{z_0}\right) dz_0. \]

The value of \( \varphi^{(1)}(z) \) can be evaluated by first determining the values of \( G_1(z_0/\eta f(x) - H) \) and \( G_2(z_0/\eta f(x) - H) \). For this, we will find the solution satisfying (25). Let us consider two independent solutions for the following equation:

\[ \frac{d^2G_1(z/z_0)}{dz^2} - \alpha^2 G_1\left(\frac{z}{z_0}\right) = 0, \]

which vanish at \( z = -\infty \) and \( z = \infty \) as

\[ \frac{E_1(z)\epsilon^{az}}{W} \text{ for } z < z_0, \]

\[ \frac{E_1(z_0)\epsilon^{az}}{W} \text{ for } z > z_0, \]

where

\[ \frac{E_1(z)\epsilon^{az}}{W} \text{ for } z < z_0, \]

\[ \frac{E_1(z_0)\epsilon^{az}}{W} \text{ for } z > z_0, \]
\[ W = E_1(z)E_2'(z) - E_2(z)E_1'(z) = -2\alpha. \] (37)

So, we can express the solution of (36) as
\[ G_1 \left( \frac{z}{z_0} \right) = \frac{e^{-a|z-z_0|}}{2\alpha} \] (38)

The solution of (25) can be written as
\[ G_1 \left( \frac{z}{z_0} \right) = \frac{e^{-a|z-z_0|}}{2\alpha} + Ae^{az} + Be^{-az}. \] (39)

Since \( G_1(z/z_0) \) satisfy the condition \( dG_1/dz = 0 \) at \( z = 0 \) and at \( z = \eta f(x) - H \). Using this, evaluating \( A \) and \( B \), we obtain
\[ G_1 \left( \frac{z}{z_0} \right) = \frac{e^{-a|z-z_0|}}{2\alpha} + \frac{e^{az} + e^{-az}}{2a(e^{\eta f(x)-H} - e^{-\alpha(\eta f(x)-H)})} \times \left( e^{-a\eta f(x)-Hz}e^{H+z_0-\eta f(x)} \right) \]
\[ -e^{\alpha(\eta f(x)-H)z_0}. \] (40)

In view of (42), the values of Green’s function for the reinforced medium are as follows:
\[ G_1 \left( \frac{z}{\eta f(x) - H} \right) = \frac{e^{\alpha z} + e^{-az}}{a(e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)})}. \] (41)

Adopting the same procedure, we find the solution of (27) as
\[ G_2 \left( \frac{z}{z_0} \right) = \frac{e^{-m|z-z_0|}}{2m} + \frac{e^{-m(z_0+z-2(\eta f(x)-H))}}{2m}, \] (43)

Substituting the values from (43), (44), and (45) in (35), it may be deduced to

\[ \bar{w}^{(1)}(z) = \frac{-2/mA_1 \left( e^{M(z-\eta f(x)+H)}(e^{az} + e^{-az}/a(e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)}) \right)}{M_0 + N_0} \times \left[ 1 + \frac{(e/aL_1)(e^{\alpha(\eta f(x)-H)} + e^{-\alpha(\eta f(x)-H)}e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)})Fm}{M_0 + N_0} \right]. \] (44)

where \( M_0 = (\alpha(e^{\alpha(\eta f(x)-H)} + e^{-\alpha(\eta f(x)-H)})/e^{az} - e^{-az}) \) and \( N_0 = P/ma_1 \), and \( F \) is given in appendix.

Neglecting higher powers of \( e \), we obtain
\[ \bar{w}^{(1)}(z) = \frac{-2/mA_1 \left( e^{M(z-\eta f(x)+H)}(e^{az} + e^{-az}/a(e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)}) \right)}{(M_0 + N_0) - (e/aL_1)(e^{\alpha(\eta f(x)-H)} + e^{-\alpha(\eta f(x)-H)}e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)})Fm} \] (45)

Applying inverse Fourier transform on (47) defined in (11), we have
\[ w^{(1)}(z) = -2 \int_{-\infty}^{\infty} \frac{e^{-M(z-\eta f(x)+H)}(1/mA_1)(e^{az} + e^{-az}/a(e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)}) \right)}{(M_0 + N_0) - (e/aL_1)(e^{\alpha(\eta f(x)-H)} + e^{-\alpha(\eta f(x)-H)}e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)})Fm} e^{-ix} \, dx. \] (46)

Setting \( w^{(1)}(z) = v^{(1)}(z)e^{Mz} \) in (48), we get
\[ v^{(1)}(z) = -2 \int_{-\infty}^{\infty} \frac{e^{-M(z-\eta f(x)+H)}(1/mA_1)(e^{az} + e^{-az}/a(e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)}) \right)}{(M_0 + N_0) - (e/aL_1)(e^{\alpha(\eta f(x)-H)} + e^{-\alpha(\eta f(x)-H)}e^{\alpha(\eta f(x)-H)} - e^{-\alpha(\eta f(x)-H)})Fm} e^{-ix} \, dx. \] (47)
The value of the integral in (49) depends on the pole of the integrand. So, here it needs to equate the denominator to zero.

\[
(M_0 + N_0) - \left( \frac{e}{a L_1} \right) \left( e^{\alpha (q f (x) - H)} + e^{-\alpha (q f (x) - H)} \right) F m = 0.
\]

Putting \( \alpha = ik \sqrt{R^2/P^2 + c^2/P^1 c_1^2 - Q/P} \), and \( m = k \sqrt{N_1/L_1 - T_1/2L_1 - c^2/c_2^2} \), and replacing \( \xi \) by \( k \), equation (50) may be simplified to

\[
\tan \left( kH - \eta \frac{kH}{2} \right) = \frac{A_2}{P \sqrt{A_3}} \frac{eA_2}{P \sqrt{A_3}} \left[ \left( \frac{-1}{A_2} + 1 \right) \left( 2A_5 \sqrt{A_2} + bA_4 \right) \right] + \frac{1}{2A_2} \frac{c^2}{A_2} \frac{b}{k \sqrt{A_2}} \left( 2A_4 + \frac{bA_5}{k} \right) + \frac{e \sinh bA_5}{P \sqrt{A_3}} \]

(49)

where \( A_2, A_3, A_4, A_5, \) and \( P_1 \) are shown in the appendix.

This is the dispersion relation of propagation of the Love wave in the reinforced medium lying over orthotropic medium under the impact of point source and irregularity.

### 5. Stability of the Model

These are special cases considered to validate the dispersion relation of Love wave propagation under the influence of the point source and irregularity obtained in (51) with the dispersion relation for propagation of classical Love-wave.

**Case 1.** When there is no reinforcement \( N_1 \rightarrow L_1 \rightarrow \mu_2 \) and \( \mu_2 \rightarrow \mu_1 \rightarrow \mu_1 \), we get

\[
\tan kH - \eta \frac{kH}{2} = \frac{\mu_2}{\mu_1} \sqrt{\frac{1-c^2/c_2^2}{c_1^2-1}} - \frac{e}{\mu_1} \sqrt{\frac{1-c^2/c_2^2}{c_1^2-1}} \left[ \left( \frac{-1}{A_2} + 1 \right) \left( 2A_5 \sqrt{A_2} + bA_4 \right) \right] + \frac{1}{2A_2} \frac{c^2}{A_2} \frac{b}{k \sqrt{A_2}} \left( 2A_4 + \frac{bA_5}{k} \right) + \frac{e \sinh bA_5}{P \sqrt{A_3}} \]

(50)

**Case 2.** When the surfaces are smooth, there is no irregularity present at the interface of the mediums, i.e., \( \eta = 0 \), \( A_4 = 1 \), and \( A_5 = -2b/k(H) \), (52) becomes

\[
\tan kH \left[ \frac{c^2}{c_1} - 1 \right] = \frac{\mu_2}{\mu_1} \left( \frac{1-c^2/c_2^2}{c_1^2-1} \right) - \frac{e}{\mu_1} \sqrt{\frac{1-c^2/c_2^2}{c_1^2-1}} \left[ \left( \frac{-1}{A_2} + 1 \right) \left( 2A_5 \sqrt{A_2} + bA_4 \right) \right] + \frac{1}{2A_2} \frac{c^2}{A_2} \frac{b}{k \sqrt{A_2}} \left( 2A_4 + \frac{bA_5}{k} \right) \]

(51)

**Case 3.** If heterogeneity becomes homogeneity in the medium, then (52) is obtained as

\[
\tan kH \left[ \frac{c^2}{c_1} - 1 \right] = \frac{\mu_2}{\mu_1} \left( \frac{1-c^2/c_2^2}{c_1^2-1} \right) \]

(52)

So, (52) verifies the dispersion relation of this problem with the classical Love-wave dispersion relation.

### 6. Numerical Computations and Discussions

Numerical computations are performed for (51) using the data provided in Table 1.

Dispersion curve (51) reveals the chemical properties of the reinforced and orthotropic materials with frequency and phase velocity of the wave. Figures 2–9 demonstrate the relation of frequency and wave number for different chemical properties of the materials under the consideration of irregularity in the materials, while Figures 10–13 demonstrated in the absence of irregularity at the interface of the materials. Figure 2 shows the influence of irregularity (between the materials) on the phase velocity \( c^2/c_2^2 \) with wave number \( kH \). According to this graph, the phase velocity of the wave increases as the length of the rectangular plate (irregularity) grows \((kh/2 = 0.2, 0.4, 0.6, 0.8)\). The phase velocity has been found to be influenced by the size of the interfacial irregularity, and hence the chemical characteristics of the materials are also influenced. One of the most important factors in understanding the chemical processes of orthotropic materials during anti-plane wave propagation is inhomogeneity. Figure 3 demonstrates the effect of functionally graded elastic properties of orthotropic materials on propagation speed. The phase velocity of the wave falls with respect to wave numbers as the magnitude
of the parameter $b/k$ grows. It has been visually examined in Figures 4 and 10 that the influence of the heterogeneity parameter $\varepsilon/\mu_T$ on the phase velocity remains invariant in the presence and absence of the irregularity. In each figure, the phase velocity increases for low frequency and reduces for high frequency. Figures 5 and 11 demonstrate the effects of initial stress parameter $T_1/2L_1$ on the phase velocity of the wave. The moderate effect of the parameter $T_1/2L_1$ is seen in these figures, phase velocity increases

---

**Table 1:** Rigidity and density of the anisotropic materials gubbins [32].

<table>
<thead>
<tr>
<th></th>
<th>Reinforced</th>
<th>Orthotropic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigidity</td>
<td>$\mu_T = 5.66 \times 10^{10} N/m^2$</td>
<td>$N_1 = 5.82 \times 10^{10} N/m^2$</td>
</tr>
<tr>
<td></td>
<td>$\mu_T = 2.46 \times 10^{10} N/m^2$</td>
<td>$L_1 = 3.99 \times 10^{10} N/m^2$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho^{(1)} = 7.8 \times 10^3 kg/m^2$</td>
<td>$\rho_1 = 4.5 \times 10^3 kg/m^2$</td>
</tr>
</tbody>
</table>
slightly as the magnitude of the initial stress increases. In the absence and presence of rectangular irregularity at the materials’ interface, the influence of the parameter $T_{1}/2L_{1}$ remains invariant, according to our findings.

Figures 6 and 12 show the major effect of chemical characteristics of reinforce materials under the effect of irregularity and without irregularity. The wave’s phase velocity increases rapidly as the reinforce parameter $a_{2}$ is increased. Geologists can use this relationship between reinforcing parameters and phase velocity to determine the composition of natural minerals. The fluctuation of phase velocity with respect to frequency is depicted in Figures 7 and 13. With rising values of $a_{1}$ and lowering values of $a_{3}$, the reinforced parameters have a significant effect on phase velocity, i.e., (0.75, 0.25), (0.65, 0.35), (0.55, 0.45), and (0.45, 0.55) are the values of $(a_{1}, a_{3})$ for curve 1, curve 2, curve 3, and curve 4, respectively. For these settings, the phase velocity declines significantly for $kH \leq 3.5$ and then decreases monotonically with rising frequency values. The phase velocity for the lowest value of $a_{1}$ and the greatest value of $a_{3}$ declines quicker than for the other values.

Figure 8 is an attempt to study the impact of irregularity on phase velocity of the Love wave. In this case,
Phase velocity increases slightly as the amount of the irregularity parameter $\eta$ increases. Figure 9 renders the effect of heterogeneity parameter $b/k$ in the absence of the irregularity ($\eta = 0$). Effect of the parameter $b/k$ on the phase velocity is significantly influenced by the irregularity parameter. In the presence of irregularity, the phase velocity decreases slightly as the magnitude of the parameter $b/k$ grows, while the phase velocity increases as the value of $b/k$ increases in the absence of the irregularity parameter. It has been noticed that, the irregularity between the materials must be considered during the exploration of the natural minerals.

7. Conclusions

Anti-plane wave propagation in a hybrid earth structure consisting of a reinforced layer and functionally graded orthotropic half-space with irregularity and external impulsive forces is theoretically investigated in this paper. External impulsive force is taken in terms of Dirac delta function and the nonhomogeneous equation of motion has been solved by using the Green’s function approach. The dispersion relation of anti-plane wave propagation has been derived and deduced into the standard form. A discussion of
the dispersion relation in the absence and presence of the irregularity was carried out in detail. Graphical results indicate that the chemical properties of the materials really influenced by irregularity of the interface. Reinforced parameters ($a_1^2$ and $a_2^2$), initial stress ($T_1/2L_1$), coefficient of heterogeneity ($e/\mu_0$), and heterogeneity parameter ($b/k$) have been observed in the presence and absence of the irregularity. It has been noticed that ($b/k$) has a profound effect on the phase velocity with and without irregularity. The performance of the parameter ($b/k$) on the phase velocity is recorded different in case of irregular and regular interface of the materials, whereas the performance of other parameters on the phase velocity is moderate in both cases.

It is also pointed out that the irregularity between materials should not be ignored when analyzing the chemical properties of materials through dispersion relation of the anti-plane wave propagation in the hybrid structure. Due to computational complexity, the group velocity of the wave is not discussed in this study. The obtained results of the current study are fundamental and can provide the sufficient path for exploring the possible natural minerals in such hybrid structures.

Appendix

\[ A_1 \]

\[ A_3 = \frac{N_1}{L_1} - \frac{T_1}{2L_1} - \frac{c^2}{c_T^2}, \]

\[ A_4 = \frac{R^2}{P^2} + \frac{c^2}{P_1} - \frac{Q}{P}, \]

\[ A_5 = \frac{b}{\kappa} \eta f(x)k(kH), \]

\[ A_5 = \frac{b}{\kappa} \eta f(x)k - 2\left(\frac{b}{\kappa}\right)(kH), \]

\[ P_1 = 1 + \left(\frac{\mu_L}{\mu_T} - 1\right) a_3^3, \]

\[ F = \frac{1}{4m^2 - b^2} \left[ \left( \frac{k^2}{m^2} + 1 \right) [2m(b\eta f(x) - 2bH) \right. \]

\[ + b(1 - 2b^2\eta f(x)H) \bigg] \left. + \left( \frac{k^2}{2m^2} + \frac{\omega^2}{m^2} - \frac{b}{m} \right) [2m(1 - 2b^2\eta f(x)H) \right. \]

\[ + b(bH \eta f(x) - 2bH)] \bigg]. \]

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported (in terms of facilities only) by the DST-FIST (project grant no. SR/FST/MS-I/2017/13) of School of Mathematics, the Thapar Institute of Engineering and Technology.

References


