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Research Article

A New Approach to Study h-Hemiregular Hemirings in terms of Bipolar Fuzzy h-Ideals

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This paper provides a generalized form of ideals, that is, h-ideals of hemirings with the combination of a bipolar fuzzy set (BFS). The BFS is an extension of the fuzzy set (FS), which deals with complex and vague problems in both positive and negative aspects. The basic purpose of this paper is to introduce the idea of $(\alpha, \beta)-$bipolar fuzzy h-subhemirings (h-BFSHs), $(\alpha, \beta)-$bipolar fuzzy h-ideals (h-BFIs), and $(\alpha, \beta)-$bipolar fuzzy h-bi-ideals (h-BFbIs) in hemirings by applying the definitions of belongingness $(\in)$ and quasicoincidence (q) of the bipolar fuzzy point. We will also focus on upper and lower parts of the h-product of bipolar fuzzy subsets (BFSSs) of hemirings. In the end, we have characterized the h-hemiregular and h-intrahemiregular hemirings in terms of the $(\in, \in\vee q)-$h-BFIs and $(\in, \in\vee q)-$h-BFbIs.

1. Introduction and Motivation

In 1994, Zhang [1] introduced bipolar fuzzy set theory which is an inflation of fuzzy set theory. Bipolarity is an important idea that is mostly used in our daily life. In a lot of disciplines such as decision making, algebraic structures, graph theory, and medical science, bipolar valued fuzzy sets have become a significant research work. In real life, it is noticed that people may have a different response at a time for the same qualities of an item or a plan. One may have a positive response, and the other one may have a negative response; for example, \$100 is a big amount for a needy person, but at the same time, this amount may have less value for a rich man. Similarly, sweetness and sourness of a food, effects and side effects of medicines, good and bad human behavior, happiness and sadness, thin and thick fluid, and honesty and dishonesty all are two-sided aspects of an object or situation. See [2–10] for examples and results which are relevant to bipolar fuzzy sets.

In 1965, Zadeh [11] introduced the concept of fuzzy set theory which deals with the uncertain and complex problems in decision-making theory, medical science, engineering, automata theory, and graph theory [12–16]. The right place of entries to fuzzy set is indicated by a membership degree. In [0, 1] interval, perimeter point 0 shows no fuzzy set acceptability and 1 shows the fuzzy set acceptance. Also, (0, 1) defines the fuzzy collection to be partially belonging. If the membership degree is any property, then 1 describes that the element satisfies the property and 0 describes that the element does not satisfy the property. Interval (0, 1) shows the midway condition. But, there was a difficulty to deliberate the irrelevancy of data to the fuzzy set. The FS is extended to BFS to tackle such situations.

In 1934, Vandiver [17] firstly familiarized the theory of semiring. In 1935, Von Neumann introduced the idea of regularity in rings and showed that for any nonempty set R, if the semigroup $(R, \cdot)$ is regular, then the ring $(R, +, \cdot)$ is also regular [18]. In 1951, Bourne showed if for all $r \in R$ there exist $x, y \in R$ such that $r + rxr = ryr$, then semiring $(R, +, \cdot)$ is also regular [19]. Hemirings (semirings with zero and commutative addition) are studied in the theory of automata and formal languages [20–22]. Algebraic patterns are very important in mathematics. Hemiring is also a useful algebraic structure. It is very useful in functional analysis, physics, computation, coding, topological space, automata theory, formal language theory, mathematical modelling, and graph theory.

In the structure theory of semirings, ideals play a vital role [23]. Henriksen gave in [24] a restricted class of ideals in semirings, which is k-ideals. Another more restricted class of ideals which is h-ideals has been given in hemirings by Iizuka [25]. However, in an additively commutative semiring, ideals of a semiring coincide with "ideals" of a ring, provided that the semiring is a hemiring [26, 27]. For more applications of h-ideals, see [28, 29].

Ahsan et al. [30] presented applications of fuzzy semirings in automata theory. Since the bipolar fuzzy theory is a development of fuzzy theory, one may expect that bipolar fuzzy semirings will be useful in studying bipolar fuzzy automata theory and bipolar fuzzy languages.

1.1. Related Works. Zadeh's fuzzy set is a much innovative, crucial, and useful set due to its significance in multiple research dimensions. Fuzzy set addresses the ill-defined situations by which we are often encountered. From these ill-defined situations, we can evaluate results by using a degree of membership of the fuzzy set but the bipolar fuzzy set is a much better set to manage uncertainty, vagueness, and impreciseness than the fuzzy set. A bipolar fuzzy set is much valuable due to its degree of membership [−1,1]. Latorre [31] discussed the properties of h-ideals of hemirings and presented some suitable results with respect to hemirings. Zhan and Dudek [28] proposed a model of fuzzy h-ideals with their useful properties. They presented some algebraic properties of prime fuzzy h-hemiregular hemirings. Kumaran et al. [32] discussed some basic properties of (α, β)-level subsets of bipolar valued fuzzy subsemiring of a hemiring and their related results. We extended this useful model and introduced h-hemiregular hemiring in terms of bipolar fuzzy h-ideals.

1.2. Historical Background. The bipolar fuzzy set describes the main idea which lies in the existence of bipolarity (positivity and negativity). In fact, human decision-making consists of double sides on the positive and negative aspects, for example, competition and cooperation, effect and side effect, and hostility and friendship. In traditional Chinese medicine, "Yin" is a negative side of a system and "Yang" is a positive side of a system. The equilibrium and coexistence of the two sides are keys for prosperity and stability of a social system [33].

1.3. Motivation. Bipolarity fuzzy sets have potential impacts on many fields, including information science, neural science, computer science, artificial intelligence, decision science, economics, cognitive science, and medical science [33]. Recently, the bipolarity fuzzy set has been studied in terms of a hemiring a bit increasingly and a bit enthusiastically. That is why we have been encouraged and motivated to introduce and study h-hemiregular hemiring in terms of a bipolar fuzzy set.

1.4. Innovative Contribution. In fuzzy sets, the degree of membership was restricted to [0,1]. In our realistic life, someone may have a negative response and another one may have a positive response at the same time for the same characteristic of an object. In this regard, bipolarity is a very useful concept that is commonly used in our real-world problems. Recently, Shabir et al. presented $(\in, \in\vee q)$-fuzzy h-subhemirings and their related properties [34]. Bhakat and Das [35] firstly introduced the idea of (α, β)−fuzzy subgroups. Dudek et al. [36] worked on (α, β)−fuzzy ideals of hemirings. The idea of finite state machine on bipolar fuzzy theory is given by Jun and Kavikumar in [2]. Lee [37] introduced bipolar valued fuzzy ideals. Ibrar et al. [38] used (α, β)−bipolar fuzzy generalized bi-ideal for characterizations of regular ordered semigroups. In 2019, Shabir et al. [39] used (α, β)−bipolar fuzzy ideals and (α, β)−bipolar fuzzy bi-ideals for the characterizations of the regular and intraregular semiring. Recently, Anjum et al. [27] studied ordered h-ideals in regular semiring. Here, we have extended the study in [39] for (α, β)−h-BFSHs and (α, β)−h-BFIs of hemirings.

1.5. Organization of the Paper. In Section 1, we have familiarized ourselves with the background of bipolar fuzzy hemiring and its characterizations towards regular and intraregular hemiring. Section 2 describes the literature review of h-BFSHs, h-BFIs, and some new important definitions which is used as the basic concept for the major works. In Section 3, the concepts of (α, β)−h-BFSHs and (α, β)−h-BFIs of hemirings are discussed. In Section 4, we have worked on the upper and lower parts of BFSH by h-product. In Section 5, the description of theorems of h-hemi-regular and h-intrahemiregular hemirings in terms of the $(\in, \in\vee q)$−h-BFIs and $(\in, \in\vee q)$−h-BFbIs is given. In Section 6, the comparative study is given, and the last section consists of the conclusions and future plans.

The list of acronyms used here is given in Table 1.

2. Preliminaries

A semiring is an algebraic system $(M, +, \cdot)$ consisting of a nonempty set M together with two binary operations addition and multiplication such that $(M, +)$ and $(M, \cdot)$ are semigroups satisfying for all $u, v, w \in M$, the following distributive laws $u(v + w) = uv + uw$ and $(u + v)w = uw + vw$. By zero, we mean an element $0 \in M$ such that $0 \cdot u = u \cdot 0 = 0$ and $0 + u = u + 0 = u$ for all $u \in M$. A semiring with zero and a commutative semigroup $(M, +)$ is known as hemiring.

A nonempty subset N of a semiring M is called a subhemiring of M if N is a hemiring under the induced operations of addition and multiplication of M. A nonempty subset "N" of a hemiring M is called a left (right) ideal of M if N is closed under "+" and $mr \in N\,(rm \in N)$ for all $m \in M$ and $r \in N$, and N is called a two-sided ideal or simply an ideal of M if it is both a left and a right ideal of M.

A hemiring M is said to be h-hemiregular if, for each $x \in M$, there exist $a_1, a_2, z \in M$ such that

$x + xa_1x + z = xa_2x + z$, and M is said to be h-intra-hemiregular if, for each $x \in M$, there exist $a_i, a'_j, b_i, b'_j, z \in R$ such that $x + \sum_{i=1}^{m} a_ix^2b_i + z = \sum_{j=1}^{n} a'_jx^2b'_j + z$ [29]. Throughout the paper, M is hemiring unless otherwise identified.

Lemma 1 (see [29]). *If M is h-intrahemiregular iff for any left h-ideal L and right h-ideal N, $L\cap N \subseteq \overline{LN}$.*

Lemma 2 (see [29]). *If M is h-hemiregular iff for any right h-ideal L and any left h-ideal N, we have $\overline{LN} = L\cap N$.*

Definition 1 (see [1]). Let $\eta = (M, \eta_n, \eta_p)$ and $\zeta = (M, \zeta_n, \zeta_p)$ be two BFSSs of M. Then, h-product $\eta \circ_h \zeta = (M, \eta_n \circ_h \zeta_n, \eta_p \circ_h \zeta_p)$ is defined as follows:

$$\begin{aligned}
(\eta_n \circ_h \zeta_n)(u) &= \bigwedge_{u+\sum_{i=1}^{m} c_id_i + v = \sum_{j=1}^{n} c'_jd'_j + v} \left\{\eta_n(c_i)\vee\eta_n(c'_j)\vee\zeta_n(d_i)\vee\zeta_n(d'_j)\right\},\\
(\eta_p \circ_h \zeta_p)(u) &= \bigvee_{u+\sum_{i=1}^{m} c_id_i + v = \sum_{j=1}^{n} c'_jd'_j + v} \left\{\eta_p(c_i)\wedge\eta_p(c'_j)\wedge\zeta_p(d_i)\wedge\zeta_p(d'_j)\right\},\\
(\eta_n \circ_h \zeta_n)(u) &= 0,\\
(\eta_p \circ_h \zeta_p)(u) &= 0 \text{ if } u \text{ is not representable as } u + \sum_{i=1}^{m} c_id_i + v = \sum_{j=1}^{n} c'_jd'_j + v.
\end{aligned} \tag{1}$$

Definition 2 (see [1]). If H is an h-subset of M. Then, the bipolar fuzzy characteristic function on H is denoted by $\chi_H = (M; \chi_{nH}, \chi_{pH})$ and is defined by $\chi_{nH}(u) = \begin{cases} -1 \text{ if } u \in H \\ 0 \text{ if } u \notin H \end{cases}$, $\chi_{pH}(u) = \begin{cases} 1 \text{ if } u \in H \\ 0 \text{ if } u \notin H \end{cases}$ for all $u \in M$. If $H = M$, then we have BFSS $M = (M; M_n, M_p)$ defined as $\widetilde{M}_n(u) = -1$ and $\widetilde{M}_p(u) = 1$ for all $u \in M$.

Definition 3 (see [1]). A BFSS $\eta = (M, \eta_n, \eta_p)$ of M is called h-BFSH of M if it satisfies the following:

(a) $\eta_n(0) \leq \eta_n(u)$ and $\eta_p(0) \geq \eta_p(u)$
(b) $\eta_n(u+v) \leq \max\{\eta_n(u), \eta_n(v)\}$ and $\eta_p(u+v) \geq \min\{\eta_p(u), \eta_p(v)\}$
(c) $\eta_n(uv) \leq \max\{\eta_n(u), \eta_n(v)\}$ and $\eta_p(uv) \geq \min\{\eta_p(u), \eta_p(v)\}$
(d) If $u + c + v = d + v$, then $\eta_n(u) \leq \max\{\eta_n(c), \eta_n(d)\}$ and $\eta_p(u) \geq \min\{\eta_p(c), \eta_p(d)\}$ for all $c, d, u, v \in M$

Definition 4. A BFSS $\eta = (M, \eta_n, \eta_p)$ of M is said to be an h-BFI$_{\mathrm{L}}$ (resp., h-BFI$_{\mathrm{R}}$) of M, if it satisfies the following:

(a) $\eta_n(u+v) \leq \max\{\eta_n(u), \eta_n(v)\}$ and $\eta_p(u+v) \geq \min\{\eta_p(u), \eta_p(v)\}$
(b) $\eta_n(uv) \leq \eta_n(v)$ and $\eta_p(uv) \geq \eta_p(v)$ (resp., $\eta_n(uv) \leq \eta_n(u)$ and $\eta_p(uv) \geq \eta_p(u)$)
(c) If $u + c + v = d + v$, then $\eta_n(u) \leq \max\{\eta_n(c), \eta_n(d)\}$ and $\eta_p(u) \geq \min\{\eta_p(c), \eta_p(d)\}$ for all $c, d, u, v \in M$

Example 1. Consider a BFSS η of a hemiring $M = \{0, 1, 2, 3, \ldots\}$ as $\eta_n(u) = \begin{cases} 0 \text{ if } u = 0 \\ -0.5 \text{ otherwise} \end{cases}$ and $\eta_p(u) = \begin{cases} 0 \text{ if } u = 0 \\ 0.5 \text{ otherwise} \end{cases}$.

Then, η is a BFI but not a h-BFI of M because if we take $x = 0, a = 1, b = 1$ and $z = 5$, then for $0 + 1 + 5 = 1 + 5$, $\eta_n(0) = 0 \geq -0.5 = \max\{\eta_n(1), \eta_n(5)\}$ and $\eta_p(0) = 0 \leq 0.5 = \min\{\eta_p(1), \eta_p(5)\}$.

Example 2. Consider a hemiring $M = \{0, a, b, c\}$ under the operations as in Tables 2 and 3.

Define a BFSS $\eta = (M, \eta_n, \eta_p)$ on M as shown in Table 4. Then, it is easy to check that η is a h-BFI of M.

Definition 5. A BFSS $\eta = (M, \eta_n, \eta_p)$ of M is called an h-BFbI of M, if is satisfies the following:

(a) $\eta_n(0) \leq \eta_n(u)$ and $\eta_p(0) \geq \eta_p(u)$
(b) $\eta_n(u+v) \leq \max\{\eta_n(u), \eta_n(v)\}$ and $\eta_p(u+v) \geq \min\{\eta_p(u), \eta_p(v)\}$
(c) $\eta_n(uv) \leq \max\{\eta_n(u), \eta_n(v)\}$ and $\eta_p(uv) \geq \min\{\eta_p(u), \eta_p(v)\}$
(d) $\eta_n(uvw) \leq \max\{\eta_n(u), \eta_n(w)\}$ and $\eta_p(uvw) \geq \min\{\eta_p(u), \eta_p(w)\}$
(e) If $u + c + v = d + v$, then $\eta_n(u) \leq \max\{\eta_n(c), \eta_n(d)\}$ and $\eta_p(u) \geq \min\{\eta_p(c), \eta_p(d)\}$ for all $c, d, u, v, w \in M$

Table 1: List of acronyms.

Acronyms	Representation
FS	Fuzzy set
BFS	Bipolar fuzzy set
BFSS	Bipolar fuzzy subset
h-BFSH	Bipolar fuzzy h-subhemiring
h-BFI	Bipolar fuzzy h-ideal
h-BFIL	Bipolar fuzzy left h-ideal
h-BFIR	Bipolar fuzzy right h-ideal
h-BFbI	Bipolar fuzzy h-bi-ideal
Iff	If and only if

Table 2: Binary addition.

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

Table 3: Binary multiplication.

.	0	a	b	c
0	0	0	0	0
a	0	b	b	b
b	0	b	b	b
c	0	b	b	b

Table 4: Values of BFSS on M.

M	0	a	b	c
η_p	0.5	0.5	0.5	0.2
η_n	−0.5	−0.5	−0.5	−0.1

Lemma 3. *If H is a left h-ideal (resp., right) of M, then $\chi_H = (M;\chi_{nH},\chi_{pH})$ is an h-BFI$_L$ (resp., h-BFI$_R$) of M.*

Proof. We have to prove the following three inequalities for left h-ideal:

(a) $\chi_{nH}(u+v) \le \max\{\chi_{nH}(u),\chi_{nH}(v)\}$ and $\chi_{pH}(u+v) \ge \min\{\chi_{pH}(u),\chi_{pH}(v)\}$

(b) $\chi_{nH}(uv) \le \chi_{nH}(v)$ and $\chi_{pH}(uv) \ge \chi_{pH}(v)$

(c) If $u+c+v = d+v$, then $\chi_{nH}(u) \le \max\{\chi_{nH}(c),\chi_{nH}(d)\}$ and $\chi_{pH}(u) \ge \min\{\chi_{pH}(c),\chi_{pH}(d)\}$ for all $c,d,u,v \in H$

For the proof of parts (a) and (b), see [2]. Here, we just prove part (c). For this, we discuss cases for all $c,d,u,v \in H, u+c+v = d+v$. □

Case 1. If $u,c,d \in H$, then $\chi_{nH}(u) = -1 = \chi_{nH}(v) = \chi_{nH}(c) = \chi_{nH}(d)$ and $\chi_{pH}(u) = 1 = \chi_{pH}(v) = \chi_{pH}(c) = \chi_{pH}(d)$.

Then, $\chi_{nH}(u) = -1 \le \max\{\chi_{nH}(c),\chi_{nH}(d)\}$ and $\chi_{pH}(u) = 1 \ge \min\{\chi_{pH}(c),\chi_{pH}(d)\}$.

Case 2. If $u,c,d \notin H$, then $\chi_{nH}(u) = 0 = \chi_{nH}(v) = \chi_{nH}(c) = \chi_{nH}(d)$ and $\chi_{pH}(u) = 0 = \chi_{pH}(v) = \chi_{pH}(c) = \chi_{pH}(d)$.

Then, $\chi_{nH}(u) = 0 \le \max\{\chi_{nH}(c),\chi_{nH}(d)\}$ and $\chi_{pH}(u) = 0 \ge \min\{\chi_{pH}(c),\chi_{pH}(d)\}$.

Case 3. If one of u, c, and d does not belong to H, say $c \notin H$, then $\chi_{nH}(c) = 0 = \chi_{pH}(c)$, $\chi_{nH}(u) = -1 = \chi_{nH}(v) = \chi_{nH}(d)$, and $\chi_{pH}(u) = 1 = \chi_{pH}(v) = \chi_{pH}(d)$. This implies that $\chi_{nH}(u) = -1 \le 0 = \max\{\chi_{nH}(c),\chi_{nH}(d)\}$ and $\chi_{pH}(u) = 1 \ge 0 = \min\{\chi_{pH}(c),\chi_{pH}(d)\}$.

Case 4. If any two of u, c, and d do not belong to H, say $u,c \notin H$, then $\chi_{nH}(u) = 0 = \chi_{nH}(c)$, $\chi_{pH}(u) = 0 = \chi_{pH}(c)$, $\chi_{nH}(d) = -1$, and $\chi_{pH}(d) = 1$. This implies that $\chi_{nH}(u) = 0 \le \max\{\chi_{nH}(c),\chi_{nH}(d)\}$ and $\chi_{pH}(u) = 0 \ge \min\{\chi_{pH}(c),\chi_{pH}(d)\}$.

Hence, $\chi_H = (M;\chi_{nH},\chi_{pH})$ is an h-BFI$_L$ of M.

3. (α,β) – Bipolar Fuzzy h-Ideals

In this section, we will introduce the concept of (α,β)−h-BFSHs, (α,β)−h-BFIs, and (α,β)−h-BFbIs of hemiring and properties of such kinds of ideals are discussed.

Definition 6 (see [2]). Let $(c,d) \in [-1,0) \times t(0,1]$; then, $t/(c,d)$ is called bipolar fuzzy point in M with BFSS $\eta = (M,\eta_n,\eta_p)$ of M if $\eta_p(u) = \begin{cases} d \text{ if } u = t \\ 0 \text{ if } u \ne t \end{cases}$ and $\eta_n(u) = \begin{cases} c \text{ if } u = t \\ 0 \text{ if } u \ne t \end{cases}$ for all $u,t \in M$.

Definition 7 (see [2]). A bipolar fuzzy point $t/(c,d)$ belongs to (resp., quasicoincident to) a BFSS $\eta = (M,\eta_n,\eta_p)$ as follows:

(a) $t/(c,d) \in \eta$ if $\eta_n(t) \le c$ and $\eta_p(t) \ge d$

(b) $t/(c,d)q\eta$ if $\eta_n(t) + c < -1$ and $\eta_p(t) + d > 1$

(c) $t/(c,d) \in\vee q\eta$ if $t/(c,d) \in \eta$ or $t/(c,d)q\eta$

(d) $t/(c,d) \in\wedge q\eta$ if $t/(c,d) \in \eta$ and $t/(c,d)q\eta$

In this paper, we consider $\alpha,\beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \ne \in\wedge q$. Consider a BFSS $\eta = (R,\eta_n,\eta_p)$ such that $\eta_n(t) \ge -0.5$ and $\eta_p(t) \ge 0.5$ for all $t \in R$ and $t/(c,d) \in\wedge q\eta$. Then, $\eta_n(t) \le c$, $\eta_p(u) \ge d$, $\eta_n(t) + c < -1$, and $\eta_p(u) + d > 1$. It follows that $-1 > \eta_n(t) + c \ge \eta_n(t) + \eta_n(t) = 2\eta_n(t)$ which implies that $\eta_n(t) \le -0.5$, thus a contradiction. So, $t/(c,d)\overline{\in\wedge q}\eta$, and hence, $\alpha \ne \in\wedge q$.

Definition 8. A BFSS $\eta = (M,\eta_n,\eta_p)$ of M is said to be an (α,β)−h-BFSH of M if for all $t,u \in M$ and $(c_1,d_1),(c_2,d_2) \in [-1,0) \times t(0,1]$, it satisfies the following:

(a) $t/(c_1,d_1)\alpha\eta$ and $u/(c_2,d_2)\alpha\eta \longrightarrow t+u/(c_1\vee c_2, d_1\wedge d_2)\beta\eta$

(b) $t/(c_1,d_1)\alpha\eta$ and $u/(c_2,d_2)\alpha\eta \longrightarrow tu/(c_1\vee c_2, d_1\wedge d_2)\beta\eta$

(c) If $c,d,t,u \in M$ such that $t+c+u = d+u$, then $c/(c_1,d_1)\alpha\eta$ and $d/(c_2,d_2)\alpha\eta \longrightarrow t/(c_1\vee c_2, d_1\wedge d_2)\beta\eta$; here, $\alpha \ne \in\wedge q$

Definition 9. A BFSS $\eta = (M, \eta_n, \eta_p)$ of M is known as an (α, β)–h-BFI$_L$ (resp., (α, β)–h-BFI$_R$) of M if for all $t, u \in M$ and $(c_1, d_1), (c_2, d_2) \in [-1, 0) \times t(0, 1]$, it satisfies the following:

(a) $t/(c_1, d_1)\alpha\eta$ and $u/(c_2, d_2)\alpha\eta \longrightarrow t + u/(c_1 \vee c_2, d_1 \wedge d_2)\beta\eta$
(b) $t/(c_1, d_1)\alpha\eta$ and $u \in M \longrightarrow ut/(c_1, d_1)\beta\eta$ (rep. $(tu/(c_1, d_1)\beta\eta)$)
(c) If $c, d, t, u \in M$ such that $t + c + u = d + u$, then $c/(c_1, d_1)\alpha\eta$ and $d/(c_2, d_2)\alpha\eta \longrightarrow t/(c_1 \vee c_2, d_1 \wedge d_2)\beta\eta$; here, $\alpha \neq \in\wedge q$

Definition 10. A BFSS $\eta = (M, \eta_n, \eta_p)$ of M is known as an (α, β)–h-BFbI of M if for all $t, u, y \in M$ and $(c_1, d_1), (c_2, d_2) \in [-1, 0) \times t(0, 1]$, it satisfies the following:

(a) $t/(c_1, d_1)\alpha\eta$ and $u/(c_2, d_2)\alpha\eta \longrightarrow t + u/(c_1 \vee c_2, d_1 \wedge d_2)\beta\eta$
(b) $t/(c_1, d_1)\alpha\eta$ and $u/(c_2, d_2)\alpha\eta \longrightarrow tu/(c_1 \vee c_2, d_1 \wedge d_2)\beta\eta$
(c) $t/(c_1, d_1)\alpha\eta$ and $u/(c_2, d_2)\alpha\eta \longrightarrow tyu/(c_1 \vee c_2, d_1 \wedge d_2)\beta\eta$
(d) If $c, d, t, u \in M$ such that $t + c + u = d + u$, then $c/(c_1, d_1)\alpha\eta$ and $d/(c_2, d_2)\alpha\eta \longrightarrow t/(c_1 \vee c_2, d_1 \wedge d_2)\beta\eta$; here, $\alpha \neq \in\wedge q$

Throughout this paper, our main focus will be on $(\in, \in\vee q)$–h-BFSHs, $(\in, \in\vee q)$–h-BFIs, and $(\in, \in\vee q)$–h-BFBIs of M.

Lemma 4. *For a BFSS $\eta = (M, \eta_n, \eta_p)$ of M, if $c, d, t, u \in M$ such that $t + c + u = d + u$, then $c/(c_1, d_1)\alpha\eta$ and $d/(c_2, d_2)\alpha\eta \longrightarrow t/(c_1 \vee c_2, d_1 \wedge d_2)\beta\eta$; then, $\eta_n(t) \leq \max\{\eta_n(c), \eta_n(d), -0.5\}$ and $\eta_p(t) \geq \min\{\eta_p(c), \eta_p(d), 0.5\}$.*

Proof. Let $c, d, t, u \in M$; then, we have the following four cases:

(i) $\max\{\eta_n(c), \eta_n(d)\} \leq -0.5$ and $\min\{\eta_p(c), \eta_p(d)\} \geq 0.5$
(ii) $\max\{\eta_n(c), \eta_n(d)\} \leq -0.5$ and $\min\{\eta_p(c), \eta_p(d)\} < 0.5$
(iii) $\max\{\eta_n(c), \eta_n(d)\} > -0.5$ and $\min\{\eta_p(c), \eta_p(d)\} \geq 0.5$
(iv) $\max\{\eta_n(c), \eta_n(d)\} > -0.5$ and $\min\{\eta_p(c), \eta_p(d)\} < 0.5$

Contrarily assume for $c, d \in M$, $\eta_n(t) > \max\{\eta_n(c), \eta_n(d), -0.5\}$ or $\eta_p(t) < \min\{\eta_p(c), \eta_p(d), 0.5\}$.

For case-(i): $\eta_n(t) > -0.5$ or $\eta_p(t) < 0.5$; this implies $t/(-0.5, 0.5)\overline{\in}\,\eta$. Now, consider $\max\{\eta_n(c), \eta_n(d)\} \leq -0.5$ and $\min\{\eta_p(c), \eta_p(d)\} \geq 0.5$ and then $c/(-0.5, 0.5) \in \eta$ and $d/(-0.5, 0.5) \in \eta$ but for $t \in M, t/(-0.5, 0.5)\overline{\in}\,\eta$. In the same way, $t/(-0.5, 0.5)\overline{q}\eta$ because $\eta_n(t) - 0.5 > -1$ or $\eta_p(t) + 0.5 < 1$. So, $t/(-0.5, 0.5)\overline{\in\vee q}\eta$ which is a contradiction.

For case-(ii): $\eta_n(t) > -0.5$ or $\eta_p(t) < \min\{\eta_p(c), \eta_p(d)\}$; so for $s \in (0, 0.5)$, $\eta_p(t) < s = \min\{\eta_p(c), \eta_p(d)\}$ implies $t/(-0.5, s)\overline{\in}\,\eta$. Now, consider again $\max\{\eta_n(c), \eta_n(d)\} \leq -0.5$ and $\min\{\eta_p(c), \eta_p(d)\} = s$ and then $c/(-0.5, s) \in \eta$ and $d/(-0.5, s) \in \eta$ but for $t \in M, t/(-0.5, s)\overline{\in}\,\eta$. In the same way, $t/(-0.5, s)\overline{q}\eta$ because $\eta_n(t) - 0.5 > -1$ or $\eta_p(t) + s < 1$. So, $t/(-0.5, s)\overline{\in\vee q}\eta$, and it contradicts our supposition.

For case-(iii): $\eta_n(t) > \max\{\eta_n(c), \eta_n(d)\}$ or $\eta_p(t) < 0.5$. So, for $r \in (-0.5, 0)$, $\eta_n(t) > r = \max\{\eta_n(c), \eta_n(d)\}$ implies $t/(r, 0.5)\overline{\in}\,\eta$. Also, $\max\{\eta_n(c), \eta_n(d)\} = r$ and $\min\{\eta_p(c), \eta_p(d)\} \geq 0.5$ and then we obtain $c/(r, 0.5) \in \eta$ and $d/(r, 0.5) \in \eta$ but for $t \in M, t/(r, 0.5)\overline{\in}\,\eta$. Similarly, $t/(r, 0.5)\overline{q}\eta$ because $\eta_n(t) + r > -1$ or $\eta_p(t) + 0.5 < 1$. Therefore, $t/(r, 0.5)\overline{\in\vee q}\eta$, which is a contradiction.

For case-(iv): $\eta_n(t) > \max\{\eta_n(c), \eta_n(d)\}$ or $\eta_p(t) < \min\{\eta_p(c), \eta_p(d)\}$. For $r \in (-0.5, 0)$ and $s \in (0, 0.5)$, $\eta_n(t) > r = \max\{\eta_n(c), \eta_n(d)\}$ and $\eta_p(t) < s = \min\{\eta_p(c), \eta_p(d)\}$ and then $t/(r, s)\overline{\in}\,\eta$. Also, $\max\{\eta_n(c), \eta_n(d)\} = r$ and $\min\{\eta_p(c), \eta_p(d)\} = s$; then, we obtain $c/(r, s) \in \eta$ and $d/(r, s) \in \eta$ but for $t \in M, t/(r, s)\overline{\in}\,\eta$. Similarly, $t/(r, s)\overline{q}\eta$ because $\eta_n(t) + r > -1$ or $\eta_p(t) + s < 1$. Therefore, $t/(r, s)\overline{\in\vee q}\eta$, which is a contradiction.

Therefore, $\eta_n(t) \leq \max\{\eta_n(c), \eta_n(d), -0.5\}$ and $\eta_p(t) \geq \min\{\eta_p(c), \eta_p(d), 0.5\}$. The reverse of this theorem does not hold in general, shown by the following example. □

Example 3. Consider a BFSS η in Example 2; for $t + c + u = d + u$, η satisfies $\eta_n(t) \leq \max\{\eta_n(c), \eta_n(d), -0.5\}$ and $\eta_p(t) \geq \min\{\eta_p(c), \eta_p(d), 0.5\}$. But if $c/(c_1, d_1) \in \eta$ and $d/(c_2, d_2) \in \eta$, then $t/(c_1 \vee c_2, d_1 \wedge d_2) \in\vee q\eta$ does not hold. Because for $b + 0 + c = a + c$, $0/(-0.5, 0.5) \in \eta$ and $a/(-0.5, 0.5) \in \eta$, but $b/(-0.5, 0.5)\overline{\in\vee q}\eta$ as $\eta_p(b) = 0.2 < 0.5$ and $\eta_n(b) = -0.1 > -0.5$. Also, $\eta_p(b) + 0.5 = 0.2 + 0.5 < 1$ and $\eta_n(b) + (-0.5) = -0.1 + (-0.5) > -1$ implies $b/(-0.5, 0.5)\overline{q}\eta$. So, $b/(-0.5, 0.5)\overline{\in\vee q}\eta$.

Theorem 1. *If a BFSS $\eta = (M, \eta_n, \eta_p)$ of M is an $(\in, \in\vee q)$ -h-BFSH of M, then it fulfills the following conditions for each $c, d, t, u \in M$:*

(a) $\eta_n(c + d) \leq \max\{\eta_n(c), \eta_n(d), -0.5\}$ and $\eta_p(c + d) \geq \min\{\eta_p(c), \eta_p(d), 0.5\}$
(b) $\eta_n(c\, d) \leq \max\{\eta_n(c), \eta_n(d), -0.5\}$ and $\eta_p(c\, d) \geq \min\{\eta_p(c), \eta_p(d), 0.5\}$
(c) For $t + c + u = d + u$, $\eta_n(t) \leq \max\{\eta_n(c), \eta_n(d), -0.5\}$ and $\eta_p(t) \geq \min\{\eta_p(c), \eta_p(d), 0.5\}$

Proof. Follows from Lemma 4 and [2]. □

Theorem 2. *If a BFSS $\eta = (M, \eta_n, \eta_p)$ of M is an $(\in, \in\vee q)$ -h-BFI$_L$ (resp., $(\in, \in\vee q)$ -h-BFI$_R$) of M, then it fulfills the following conditions for each $c, d, t, u \in M$:*

(a) $\eta_n(c+d) \le \max\{\eta_n(c), \eta_n(d), -0.5\}$ *and* $\eta_p(c+d) \ge \min\{\eta_p(c), \eta_p(d), 0.5\}$

(b) $\eta_n(c\,d) \le \max\{\eta_n(d), -0.5\}$ *and* $\eta_p(c\,d) \ge \min\{\eta_p(d), 0.5\}$ *(resp.,* $\eta_n(c\,d) \le \max\{\eta_n(c), -0.5\}$ *and* $\eta_p(c\,d) \ge \min\{\eta_p(c), 0.5\}$*)*

(c) *For* $t+c+u = d+u$, $\eta_n(t) \le \max\{\eta_n(c), \eta_n(d), -0.5\}$ *and* $\eta_p(t) \ge \min\{\eta_p(c), \eta_p(d), 0.5\}$

Proof: Follows from Lemma 4 and [2]. □

Theorem 3. *If a BFSS* $\eta = (M, \eta_n, \eta_p)$ *of M is an* $(\in, \in\vee q)$*-h-BFbI of M, then it fulfills the following conditions for each* $c, d, t, u \in M$:

(a) $\eta_n(c+d) \le \max\{\eta_n(c), \eta_n(d), -0.5\}$ *and* $\eta_p(c+d) \ge \min\{\eta_p(c), \eta_p(d), 0.5\}$

(b) $\eta_n(c\,d) \le \max\{\eta_n(c), \eta_n(d), -0.5\}$ *and* $\eta_p(c\,d) \ge \min\{\eta_p(c), \eta_p(d), 0.5\}$

(c) $\eta_n(cb\,d) \le \max\{\eta_n(c), \eta_n(d), -0.5\}$ *and* $\eta_p(cb\,d) \ge \min\{\eta_p(c), \eta_p(d), 0.5\}$

(d) *For* $t+c+u = d+u$, $\eta_n(t) \le \max\{\eta_n(c), \eta_n(d), -0.5\}$ *and* $\eta_p(t) \ge \min\{\eta_p(c), \eta_p(d), 0.5\}$

Proof: Follows from Lemma 4 and [2]. □

4. Upper and Lower Parts of a Bipolar Fuzzy Set

In the following segment, we have offered the definitions and several stimulating outcomes regarding the upper parts and lower parts of h-BFSS.

Definition 11. (see [2]). If $\eta = (M, \eta_n, \eta_p)$ is a bipolar fuzzy h-set in M, then the upper part $\eta^+ = (M, \eta_n^+, \eta_p^+)$ of η is written as $(\eta_n^+)(t) = (\eta_n)(t)\wedge -0.5$ and $(\eta_p^+)(t) = (\eta_p)(t)\vee 0.5$. The lower part of η is denoted by $\eta^- = (M, \eta_n^-, \eta_p^-)$ and is defined as $(\eta_n^-)(t) = (\eta_n)(t)\vee -0.5$ and $(\eta_p^-)(t) = (\eta_p)(t)\wedge 0.5$ for all $t \in M$.

Lemma 5. *Let* $\eta = (M, \eta_n, \eta_p)$ *and* $\zeta = (M, \zeta_n, \zeta_p)$ *be two BFSHs in M. Then, the following equations hold:*

(a) $(\eta\wedge\zeta)^- = \eta^-\wedge\zeta^-$

(b) $(\eta\vee\zeta)^- = \eta^-\vee\zeta^-$

(c) $(\eta\circ_h\zeta)^- = \eta^-\circ_h\zeta^-$

Proof: Straightforward. □

Lemma 6. *Let* $\eta = (M, \eta_n, \eta_p)$ *and* $\zeta = (M, \zeta_n, \zeta_p)$ *be two BFSHs in M. Then, the following equations hold:*

(a) $(\eta\wedge\zeta)^+ = \eta^+\wedge\zeta^+$

(b) $(\eta\vee\zeta)^+ = \eta^+\vee\zeta^+$

(c) $(\eta\circ_h\zeta)^+ = \eta^+\circ_h\zeta^+$

Proof: Straightforward. □

Lemma 7. *For an* $(\in, \in\vee q)$*–h-BFI*$_R$ $\eta = (M, \eta_n, \eta_p)$ *and* $(\in, \in\vee q)$*–h-BFI*$_L$ $\zeta = (M, \zeta_n, \zeta_p)$, *then we have* $(\eta\circ_h\zeta)^- \le (\eta\wedge\zeta)^-$.

Proof: Let $t \in M$ if $(\eta\circ_h\zeta)(t) = 0$, that is, $(\eta_n\circ_h\zeta_n)(t) = 0$, and then $(\eta_n\circ_h\zeta_n)^-(t) = 0\vee -0.5 = 0 \ge (\eta_n\vee\zeta_n)^-(t)$ and $(\eta_p\circ_h\zeta_p)(t) = 0$ indicates $(\eta_p\circ_h\zeta_p)^-(t) = 0\wedge 0.5 = 0 \le (\eta_p\wedge\zeta_p)^-(t)$. Otherwise,

$$
\begin{aligned}
(\eta_n\circ_h\zeta_n)^-(t) &= (\eta_n\circ_h\zeta_n)(t)\vee -0.5 \\
&= \bigwedge_{t+\sum_{i=1}^m a_ib_i + v = \sum_{j=1}^n c_jd_j + v} \left\{\left\{\bigvee_{i=1}^m \eta_n(a_i)\right\}\vee\left\{\bigvee_{i=1}^m \zeta_n(b_i)\right\}\vee\left\{\bigvee_{j=1}^n \eta_n(c_j)\right\}\vee\left\{\bigvee_{j=1}^n \zeta_n(d_j)\right\}\right\}\vee -0.5 \\
&= \bigwedge_{t+\sum_{i=1}^m a_ib_i + v = \sum_{j=1}^n c_jd_j + v} \left\{\begin{array}{l}\left\{\bigvee_{i=1}^m (\eta_n(a_i)\vee -0.5)\right\}\vee\left\{\bigvee_{i=1}^m (\zeta_n(b_i)\vee -0.5)\right\}\vee \\ \left\{\bigvee_{j=1}^n (\eta_n(c_j)\vee -0.5)\right\}\vee\left\{\bigvee_{j=1}^n (\zeta_n(d_j)\vee -0.5)\right\}\end{array}\right\}\vee -0.5 \\
&\ge \bigwedge_{t+\sum_{i=1}^m a_ib_i + v = \sum_{j=1}^n c_jd_j + v} \left\{\begin{array}{l}\left\{\bigvee_{i=1}^m \eta_n(a_ib_i)\right\}\vee\left\{\bigvee_{i=1}^m \zeta_n(a_ib_i)\right\}\vee \\ \left\{\bigvee_{j=1}^n \eta_n(c_jd_j)\right\}\vee\left\{\bigvee_{j=1}^n \zeta_n(c_jd_j)\right\}\end{array}\right\}\vee -0.5
\end{aligned}
$$

$$\begin{aligned}
&= \bigwedge_{t+\sum_{i=1}^{m} a_i b_i + v = \sum_{j=1}^{n} c_j d_j + v} \left\{ \begin{array}{l} \left\{ \bigvee_{i=1}^{m} (\eta_n (a_i b_i) \vee -0.5) \right\} \vee \left\{ \bigvee_{i=1}^{m} (\zeta_n (a_i b_i) \vee -0.5) \right\} \vee \\ \left\{ \bigvee_{j=1}^{n} (\eta_n (c_j d_j) \vee -0.5) \right\} \vee \left\{ \bigvee_{j=1}^{n} (\zeta_n (c_j d_j) \vee -0.5) \right\} \end{array} \right\} \vee -0.5 \\
&\geq \bigwedge_{t+\sum_{i=1}^{m} a_i b_i + v = \sum_{j=1}^{n} c_j d_j + v} \left\{ \begin{array}{l} \eta_n \left(\sum_{i=1}^{m} a_i b_i \right) \vee \zeta_n \left(\sum_{i=1}^{m} a_i b_i \right) \vee \\ \eta_n \left(\sum_{j=1}^{n} c_j d_j \right) \vee \zeta_n \left(\sum_{j=1}^{n} c_j d_j \right) \end{array} \right\} \vee -0.5 \\
&= (\eta_n \vee \zeta_n)(t) \vee -0.5.
\end{aligned} \tag{2}$$

This implies $(\eta_n \circ_h \zeta_n)^- (t) \geq (\eta_n \vee \zeta_n)^- (t)$.

Similarly, $(\eta_p \circ_h \zeta_p)^- (t) \leq (\eta_p \wedge \zeta_p)^- (t)$.

Hence, $(\eta \circ_h \zeta)^- \leq (\eta \wedge \zeta)^-$. □

Definition 12. If H is an h-subset of M, the upper part $\chi_H^+ = (M; \chi_{nH}^+, \chi_{pH}^+)$ of the bipolar fuzzy characteristic function $\chi_H = (M; \chi_{nH}, \chi_{pH})$ of H is defined by the following:

$$\begin{aligned}
\chi_{nH}^+ (t) &= \begin{cases} -1 \text{ if } t \in H, \\ -0.5 \text{ if } t \notin H, \end{cases} \\
\chi_{pH}^+ (t) &= \begin{cases} 1 \text{ if } t \in H, \\ 0.5 \text{ if } t \notin H. \end{cases}
\end{aligned} \tag{3}$$

Similarly, the lower part $\chi_H^- = (M; \chi_{nH}^-, \chi_{pH}^-)$ of the bipolar fuzzy characteristic function $\chi_H = (M; \chi_{nH}, \chi_{pH})$ of H is defined by the following:

$$\begin{aligned}
\chi_{nH}^- (t) &= \begin{cases} -0.5 \text{ if } t \in H, \\ 0 \text{ if } t \notin H, \end{cases} \\
\chi_{pH}^- (t) &= \begin{cases} 0.5 \text{ if } t \in H, \\ 0 \text{ if } t \notin H. \end{cases}
\end{aligned} \tag{4}$$

For all $t \in M$, if $H = M$, then we have BFSS $M = (M; M_n, M_p)$ defined as $\tilde{M}_n (t) = -1$ and $\tilde{M}_p (t) = 1$ for all $t \in M$.

Lemma 8. *Let E and F be two nonempty h-subsets of M. Then, the following results hold:*

(a) $(\chi_E \wedge \chi_F)^- = \chi_{E \cap F}^-$

(b) $(\chi_E \vee \chi_F)^- = \chi_{E \cup F}^-$

(c) $(\chi_E \circ_h \chi_F)^- = \chi_{\overline{EF}}^-$

Proof: Proofs of (a) and (b) are straightforward. Here is just a proof of (c).

Suppose $t \in M$ and $t \in \overline{EF}$. Then, $\chi_{n\overline{EF}} (t) = -0.5$ and $\chi_{p\overline{EF}} (t) = 0.5$. For $t \in \overline{EF}$, $t + \sum_{i=1}^{m} a_i b_i + v = \sum_{j=1}^{n} c_j d_j + v$ where $a_i, c_j \in E$ and $b_i, d_j \in F$.; then, we have $\chi_{nE} (a_i) = -1$, $\chi_{pE} (c_j) = 1$, $\chi_{nF} (b_i) = -1$, and $\chi_{pF} (d_j) = 1$.

Consider the following:

$$\begin{aligned}
(\chi_{nE} \circ_h \chi_{nF})^- (t) &= \bigwedge_{t+\sum_{i=1}^{m} a_i b_i + v = \sum_{j=1}^{n} c_j d_j + v} \left\{ \begin{array}{l} \left(\bigvee_{i=1}^{m} \chi_{nE} (a_i) \right) \vee \left(\bigvee_{i=1}^{m} \chi_{nF} (b_i) \right) \\ \vee \left(\bigvee_{j=1}^{n} \chi_{nE} (c_j) \right) \vee \left(\bigvee_{j=1}^{n} \chi_{nF} (d_j) \right) \end{array} \right\} \vee -0.5 \\
&= \{(-1) \vee (-1) \vee (-1) \vee (-1)\} \vee -0.5 = -0.5 \\
&= \chi_{n\overline{EF}}^- (t).
\end{aligned} \tag{5}$$

Now,

$$\left(\chi_{pE}\circ_h\chi_{pF}\right)^-(t)=\bigvee_{t+\sum_{i=1}^{m}a_ib_i+v=\sum_{j=1}^{n}c_jd_j+v}\left\{\begin{array}{c}\left(\bigwedge_{i=1}^{m}\chi_{nE}(a_i)\right)\wedge\left(\bigwedge_{i=1}^{m}\chi_{nF}(b_i)\right)\\ \wedge\left(\bigwedge_{j=1}^{n}\chi_{nE}(c_j)\right)\wedge\left(\bigwedge_{j=1}^{n}\chi_{nF}(d_j)\right)\end{array}\right\}\wedge 0.5$$
$$=\{(1)\wedge(1)\wedge(1)\wedge(1)\}\wedge 0.5=0.5$$
$$=\chi^-_{p\overline{EF}}(t). \tag{6}$$

Hence, $\left(\chi_E\circ_h\chi_F\right)^-=\chi^-_{\overline{EF}}$.

If $t\notin\overline{EF}$, then $\chi_{n\overline{EF}}(t)=0$ and also $\chi_{p\overline{EF}}(t)=0$.

Then, $\left(\chi_{nE}\circ_h\chi_{nF}\right)^-(t)=\left(\chi_{nE}\circ_h\chi_{nF}\right)(t)\vee-0.5=0\vee-0.5=0=\chi^-_{n\overline{EF}}(t)$.

And, $\left(\chi_{pE}\circ_h\chi_{pF}\right)^-(t)=\left(\chi_{pE}\circ_h\chi_{pF}\right)(t)\wedge 0.5=0\wedge 0.5=0=\chi^-_{p\overline{EF}}(t)$.

That is, $\left(\chi_E\circ_h\chi_F\right)^-=\chi^-_{\overline{EF}}$. □

Lemma 9. *If H is left (resp., right) h-ideal of M, then $\chi^-_H=(M;\chi^-_{nH},\chi^-_{pH})$ is $(\in,\in\vee q)$–h-BFI$_L$ (resp., $(\in,\in\vee q)$–h-BFI$_R$) of M.*

Proof: Straightforward. □

Lemma 10. *If H is h-bi-ideal of M, then $\chi^-_H=(M;\chi^-_{nH},\chi^-_{pH})$ is $(\in,\in\vee q)$–h-BFbI of M.*

Proof: Straightforward. □

5. h-Hemiregular and h-Intrahemiregular Hemirings

In this section, by using the h-BFIs, theorems on hemiregular and intrahemiregular hemirings are presented.

Theorem 4. *Suppose M is a hemiring, then the following subsequent conditions are identical:*

(a) M is h-hemiregular

(b) $(\eta\wedge\zeta)^-=(\eta\circ_h\zeta)^-$ for all $(\in,\in\vee q)$–h-BFI$_R$ $\eta=(M,\eta_n,\eta_p)$ and $(\in,\in\vee q)$–h-BFI$_L$ $\zeta=(M,\zeta_n,\zeta_p)$ of M

Proof: Let M be an h-hemi-regular hemiring. For each $t\in M$, there exist $x_1,x_2,y\in M$ such that $t+tx_1t+y=tx_2t+y$; then,

$$\left(\eta_n\circ_h\zeta_n\right)^-(t)=\bigwedge_{t+\sum_{i=1}^{m}a_ib_i+v=\sum_{j=1}^{n}c_jd_j+v}\left\{\begin{array}{c}\left(\bigvee_{i=1}^{m}\eta_n(a_i)\right)\vee\left(\bigvee_{i=1}^{m}\zeta_n(b_i)\right)\\ \vee\left(\bigvee_{j=1}^{n}\eta_n(c_j)\right)\vee\left(\bigvee_{j=1}^{n}\zeta_n(d_j)\right)\end{array}\right\}\vee-0.5$$
$$\leq\{\eta_n(tx_1)\vee\zeta_n(t)\vee\eta_n(tx_2)\vee\zeta_n(t)\}\vee-0.5$$
$$\leq\{(\eta_n(t)\vee-0.5)\vee\zeta_n(t)\vee(\eta_n(t)\vee-0.5)\vee\zeta_n(t)\}\vee-0.5$$
$$=\{(\eta_n(t)\vee\zeta_n(t)\}\vee-0.5$$
$$=(\eta_n\vee\zeta_n)^-(t). \tag{7}$$

In the same way, $(\eta_p\circ_h\zeta_p)^-(t)\geq(\eta_p\wedge\zeta_p)^-(t)$.

Thus, $(\eta\wedge\zeta)^-\leq(\eta\circ_h\zeta)^-$. But, we have $(\eta\circ_h\zeta)^-\leq(\eta\wedge\zeta)^-$ by Lemma 7. Hence, $(\eta\wedge\zeta)^-=(\eta\circ_h\zeta)^-$.

Conversely, $(\eta\wedge\zeta)^-=(\eta\circ_h\zeta)^-$ for all $(\in,\in\vee q)$–h-BFI$_R$ $\eta=(M,\eta_n,\eta_p)$ and $(\in,\in\vee q)$–h-BFI$_L$ $\zeta=(M,\zeta_n,\zeta_p)$ of M. Let E and F be right h-ideal and left h-ideal of M correspondingly. Formerly, the lower parts of the bipolar fuzzy characteristic function $\chi^-_E=(M;\chi_{nE},\chi_{pE})$ are an $(\in,\in\vee q)$–h-BFI$_R$ and $\chi^-_F=(M;\chi_{nF},\chi_{pF})$ are an $(\in,\in\vee q)$–h-BFI$_L$ of M. Therefore, by our supposition, $(\chi_E\wedge\chi_F)^-=(\chi_E\circ\chi_F)^-$. This indicates that $\chi^-_{E\cap F}=\chi_{\overline{EF}}$ implies $E\cap F=\overline{EF}$. Therefore, by Lemma 2, M is h-hemiregular. □

Theorem 5. *For a hemiring M, the following subsequent conditions are identical:*

(a) M is h-hemiregular

(b) $(\eta\wedge\zeta\wedge\upsilon)^-\leq(\eta\circ_h\zeta\circ_h\upsilon)^-$ for each $(\in,\in\vee q)$–h-BFI$_R$ $\eta=(M,\eta_n,\eta_p)$, for all $(\in,\in\vee q)$–h-BFI $\zeta=(M,\zeta_n,\zeta_p)$ and for all $(\in,\in\vee q)$–h-BFI$_L$ $\upsilon=(M,\upsilon_n,\upsilon_p)$ of M

Proof: Same as Theorem 4. □

Theorem 6. *Suppose M is hemiring, then the following conditions are equivalent:*

(a) M is h-hemiregular

(b) $(\eta \wedge \zeta)^- \le (\eta \circ_h \zeta)^-$ for all $(\in, \in \vee q)$–h-BFbI $\eta = (M, \eta_n, \eta_p)$ and for all $(\in, \in \vee q)$–h-BFI$_L$ $\zeta = (M, \zeta_n, \zeta_p)$ of M

(c) $(\eta \wedge \zeta)^- \le (\eta \circ_h \zeta)^-$ for all $(\in, \in \vee q)$–h-BFI$_R$ $\eta = (M, \eta_n, \eta_p)$ and for all $(\in, \in \vee q)$–h-BFbI $\zeta = (M, \zeta_n, \zeta_p)$ of M

Proof: (a) $\longrightarrow$ (b): Let M be an h-hemiregular. For each $t \in M$, there exists $x_1, x_2, y \in M$ such that $t + tx_1 t + y = tx_2 t + y$; then,

$$\begin{aligned}
(\eta_n \circ_h \zeta_n)^-(t) &= \bigwedge_{t+\sum_{i=1}^{m} a_i b_i + v = \sum_{j=1}^{n} c_j d_j + v} \left\{ \begin{array}{c} \left(\bigvee_{i=1}^{m} \eta_n(a_i)\right) \vee \left(\bigvee_{i=1}^{m} \zeta_n(b_i)\right) \\ \vee \left(\bigvee_{j=1}^{n} \eta_n(c_j)\right) \vee \left(\bigvee_{j=1}^{n} \zeta_n(d_j)\right) \end{array} \right\} \vee -0.5 \\
&\le \{\eta_n(t) \vee \zeta_n(x_1 t) \vee \eta_n(t) \vee \zeta_n(x_2 t)\} \vee -0.5 \\
&\le \{\eta_n(t) \vee (\zeta_n(t) \vee -0.5) \vee \eta_n(t) \vee (\zeta_n(t) \vee -0.5)\} \vee -0.5 \\
&= \{(\eta_n(t) \vee \zeta_n(t)\} \vee -0.5 \\
&= (\eta_n \vee \zeta_n)^-(t).
\end{aligned} \tag{8}$$

In the same way, $(\eta_p \circ_h \zeta_p)^-(t) \ge (\eta_p \wedge \zeta_p)^-(t)$. Thus, $(\eta \wedge \zeta)^- \le (\eta \circ_h \zeta)^-$.

(b) $\longrightarrow$ (a): Let $\eta = (M, \eta_n, \eta_p)$ be a $(\in, \in \vee q)$–h-BFI$_R$ and $\zeta = (M, \zeta_n, \zeta_p)$ be a $(\in, \in \vee q)$–h-BFI$_L$ of M. Since each $(\in, \in \vee q)$–h-BFI$_R$ is $(\in, \in \vee q)$–h-BFbI of M, by (a), we obtain $(\eta \wedge \zeta)^- \le (\eta \circ_h \zeta)^-$ but by Lemma 7, $(\eta \circ_h \zeta)^- \le (\eta \wedge \zeta)^-$. Hence, $(\eta \wedge \zeta)^- = (\eta \circ_h \zeta)^-$, and by Theorem 4, M is h-hemiregular. Similarly, we can prove (a) $\longrightarrow$ (c) and (c) $\longrightarrow$ (a). □

Theorem 7. *For a hemiring M, the following subsequent conditions are equivalent:*

(a) M is h-intrahemiregular

(b) $(\eta \wedge \zeta)^- \le (\eta \circ_h \zeta)^-$ for all $(\in, \in \vee q)$–h-BFI$_L$ $\eta = (M, \eta_n, \eta_p)$ and for all $(\in, \in \vee q)$–h-BFI$_R$ $\zeta = (M, \zeta_n, \zeta_p)$ of M

Proof: Assume M is h-intrahemiregular hemiring. For every $t \in M$, there exists $x_i, z_i, x'_j, z'_j, y \in M$ such that $t + \sum_{i=1}^{n} x_i t^2 z_i + v = \sum_{j=1}^{m} x'_j t^2 z'_j + v$.

$$\begin{aligned}
(\eta_n \circ_h \zeta_n)^-(t) &= \bigwedge_{t+\sum_{i=1}^{m} a_i b_i + v = \sum_{j=1}^{n} c_j d_j + v} \left\{ \begin{array}{c} \left(\bigvee_{i=1}^{m} \eta_n(a_i)\right) \vee \left(\bigvee_{i=1}^{m} \zeta_n(b_i)\right) \\ \vee \left(\bigvee_{j=1}^{n} \eta_n(c_j)\right) \vee \left(\bigvee_{j=1}^{n} \zeta_n(d_j)\right) \end{array} \right\} \vee -0.5 \\
&\le \{\eta_n(x_i t) \vee \zeta_n(t z_i) \vee \eta_n(x_j t) \vee \zeta_n(t z_j)\} \vee -0.5 \\
&\le \{(\eta_n(t) \vee -0.5) \vee (\zeta_n(t) \vee -0.5) \vee (\eta_n(t) \vee -0.5) \vee (\zeta_n(t) \vee -0.5)\} \vee -0.5 \\
&= \{(\eta_n(t) \vee \zeta_n(t)\} \vee -0.5 \\
&= (\eta_n \vee \zeta_n)^-(t).
\end{aligned} \tag{9}$$

In the same way, $(\eta_p \circ_h \zeta_p)^-(t) \ge (\eta_p \wedge \zeta_p)^-(t)$.

Thus, $(\eta \wedge \zeta)^- \le (\eta \circ_h \zeta)^-$.

Conversely, assume E and F are left and right h-ideal of M, respectively. The lower parts of the bipolar fuzzy characteristic function of E and F, $\chi_E^- = (M, \chi_{nE}^-, \chi_{pE}^-)$ and $\chi_F^- = (M, \chi_{nF}^-, \chi_{pF}^-)$, are $(\in, \in \vee q)$–h-BFI$_L$ and $(\in, \in \vee q)$–h-BFI$_R$ of M, respectively. Now, from our supposition, $(\chi_E \wedge \chi_F)^- \le (\chi_E \circ_h \chi_F)^-$. Also, $\chi_{E \cap F}^- = (\chi_E \wedge \chi_F)^- \le (\chi_E \circ_h \chi_F)^- = \chi_{\overline{EF}}^-$ which suggests that $E \cap F \subseteq \overline{EF}$. From Lemma 1, M is h-intrahemiregular. □

6. Comparative Study and Discussion

In [2], Shabir et al. used regular and intraregular semirings with the help of (α, β)–BFIs. We extended the work of [2] to hemirings with the help of (α, β)–h-BFIs. Semiring with zero and commutative addition is hemiring. We have characterized h-hemiregular and h-intrahemiregular hemirings by using (α, β)–h-BFIs. As h-ideal is a more restricted form of ideal, our extension is more applicable than the approach discussed in [2].

7. Conclusion

BFS is a dominant tool of mathematics to resolve the uncertainty in positive as well as a negative aspect of the data. In this paper, basic concepts, operations, and related properties with respect to (α, β)–h-BFIs are proposed. Generally, we have proved with an example that if a BFSS η of M is an $(\in, \in \vee q)$–h-BFI of M, then it satisfies three particular conditions, but the reverse may not hold. Also, we have studied the lower and upper parts of $(\in, \in \vee q)$–h-BFI of hemirings. We have characterized h-hemiregular and h-intrahemiregular hemirings by using (α, β)–h-BFIs.

The bipolar fuzzy set has membership degree ranges from −1 to 1, but we often face such critical situations in real life which cannot be handled by bipolar fuzzy sets due to their membership degree [−1,1]. To control these critical situations, Pythagorean fuzzy sets are more useful because of the sum of its membership degree and nonmembership degree which can be greater than 1.

In the future, we will extend this work for hyperstructures, LA-semigroups, near-rings, etc. We will utilize this proposed bipolar fuzzy model to enhance these studies such as fuzzy stochastic data envelopment analysis model, intuitionistic fuzzy linear regression model, bounded linear programs with trapezoidal fuzzy numbers, interval-valued trapezoidal fuzzy number, fuzzy arc waves based on artificial bee colony algorithm, and fuzzy efficiency measures in data envelopment analysis. In addition, we will study its real-life applications in medical science, computer science, management science, and many other fields.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Conceptualization was performed by S. B. and R. M. Methodology was prepared by Zu. Software was implemented by S. B. Validation was done by A. N. A. -K. Formal analysis was carried out by Zu. Investigation and data curation were done by R. M. Resources were obtained by A. N. A. -K. Original draft preparation was performed by Zu. Review and editing were conducted by S. B. and R. M. Visualization was done by R. M. Supervision was done by S. B. Project administration was performed by Zu. Funding acquisition was done by A. N. A. -K. All authors have read and agreed to the published version of the manuscript.

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