Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 2771494, 18 pages
https://doi.org/10.1155/2022/2771494

*Research Article*

# Mathematical Modelling of $M^X/G(a, b)/1$ Bulk Service Queue Model with Two Vacations and Setup Time in Ceramic Technology

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Received 1 August 2022; Accepted 9 September 2022; Published 27 October 2022

Academic Editor: Marcello Pellicciari



The orthodox way of tile printing results in mulish production. To tame this, digital inkjet printing technology was imposed. In this article, a mathematical prototype is designed exclusively for the tile printing wing. The various plants coming under printing are correlated to the proposed prototype. The article throws light on the functioning of the prototype. The prototype turns up subservient to bulk service queue models with two types of vacations. The prototype is resolved under the supplementary variable technique. The prototype with peculiar conditions is proved to be the existing model. Anticipated line distance, awaited active duration, awaited standby time, and expected idle period were computed. Thus, the total average cost is acquired for diverse vacation values. The main intention of the pattern is to downside the total average cost.

## 1. Introduction

As we all know, queues are a typical everyday encounter. When resources are guarded, there queues form. Having queues makes financial sense in reality. One of the major issues within the examination of any activity system is the investigation of delay. Delay is a more unpretentious concept. Queuing theory deals with consequences which include queuing (or holding up). Typical instances might be banks, supermarkets, computers, public transport, etc. In general, all queuing frameworks can be broken down into individual subsystems such as arrival process, service mechanism, queue characteristics, which are briefly discussed in [1].

Real-life systems barely ever reach a consistent state. Instead of this, a simple queuing formula can allow us a few understandings of how a system might carry on very rapidly. One factor that's of note is traffic intensity equals (entry rate)/(departure rate) where entry-rated notes total of entries every time and departure rate represent the tally of departures per time. Traffic intensity may be a measure of the blockage of the system. In case it is close to zero, there's exceptionally small lining in common, as the traffic intensity escalate (to approach 1 or indeed more than 1) the amount of queue upsurge. Before the implementation of digital printing over tiles, the manufacturers faced difficulties for design setting, because if the same design is asked it is tedious to bring back the same design accurately and definitely, there will be few moderations. The main advantage of using digital technology is one can change the colours as they need and even fine-tune designs and patterns if requested by customers. One more advantage is the designs can be stored for future use. A mathematical model is proposed for the digital printing technology. Here, the model considered is the bulk service queue model where the arrival of tiles and printing (service) is done in batches. Before printing the tiles, it has to be washed and cleaned which is mentioned as setup work. Apart from printing designs, the server is utilized for two more secondary jobs which are referred to as type 1 and type 2 jobs.

## 2. Literature Survey

Arumuganathan and Jeyakumar [2] investigated $M^x/G(a,b)/1$ the queuing framework for $N$-policy vacations, as well as setup and closedown times. The server does a closedown job and takes a vacation in case the waiting line length is fewer to "$a$." The attendant goes on another vacation after the service is completed. In case the queue distance is smaller than $N$, setup work begins, followed by service to "$b$" customers. In addition, the authors discussed a few performance metrics. Sasikala and Kandaiyan [3] devised a bulk queuing system that included setup time, balking, multiple vacations, and close down. Arumuganathan and Malliga [4] looked at a bulk queue system with setup time and several vacations. $G$-Queues, along with server breakdown and reservice, have been considered by Ayyappan and Supraja [5]. Karpagam [6] has considered a queue system which has more application in industries. Ali and Reza[7] have considered a queue model with $k$ sequential heterogeneous service steps and vacations.

Arumuganathan and Ramaswami [8] investigated a bulk queuing system along with a fast and moderate service rate, as well as multiple vacations. In addition, the same researchers looked at a state-dependent arrival, which means that while the server is busy, the rate of arrival would progressively grow, but when it is on vacation, the arrival rate will gradually fall. Ghosh et al. [9] have analysed a finite-buffer bulk service queue system where arrival service is correlated. The $GI^x/M/C/N$ queue system was analysed by Laxmi and Gupta [10]. Singh et al. [11] have analysed a bulk service queue system with a modified bulk service rule. Nithya and Haridass [12] have devised a system that divides bulk service into two stages. Niranjan et al. [13] investigated an actual issue including a retrial model with a threshold, nondisruptive service when the server failed and avail various vacations. Singh et al. [14] have analysed the operation of a washing machine under queueing system and its performance measures. In addition to setup time, Krishnareddy et al. [15] examined a batch queuing framework with $N$-policy.

Jeyakumar along with Senthilnathan [16] have also studied a queuing system including setup time, closedown time, and various vacations, and it does cause any. Renovation of the service station was also explored by the authors. Kavitha et al. [17] have developed a queue model which is used for mobile wallet system by applying blockchain. Goswami and Mohan [18] determine an explicit form of sojourn-time distribution and evaluate the distribution function for any specific time. Haridass and Arumuganathan [19] have considered a mass service queue system with two types of vacations. The server performs three types of jobs namely service, two types of secondary jobs type1 and type 2, respectively.

As an extension to the above-mentioned model in this article, a mathematical model is developed by introducing setup time. In this model, the service is given in bulk, and setup work is carried out after the first batch gets service. In the proposed model, the server performs four types of jobs namely setup work, service, and two types of secondary jobs type I vacation and type II vacation.

## 3. Model Description

The motivation for this mathematical model is materialized from the tile manufacturing and design sector. The traditional way of designing tile has multifold drawbacks such as long setup times, patterns repeat frequently, difficulty in colour management in case of repeated orders, and inflexible production planning, and only flat tiles can be decorated. To sweep off these difficulties, digital inkjet printing technology was imposed on tile designing. The mathematical model is created purely for designing part of the manufacturing unit. After the tiles are manufactured, the tiles are grated (if required), washed with cleaning agents, and dried. The tiles are moved to the printing area, and the dimensions of tiles are ensured. Meanwhile, the designs for printing were kept ready. We refer to the above process as setup work in our model. Tiles have to coat with base coat before printing the designs and have to dry for the next process, and this is termed as service, and it is done for $n$ number of tiles at once. Hence, the suitable queuing model for the process is bulk service. The server is idle until the printed tiles are dried. To utilize the idle time, the server will perform two types of work: cleaning the dried tiles and applying the top coat which makes the tiles look glossy. And the second job is final grinding and side coating. In this model, the former work is mentioned as type 1 vacation, and the later work is quoted as type 2 vacation. The major goal of this model is to lower manufacturing costs. Table 1 is a classification of the procedure described as follows.

The method described above is developed as a $M^x/G(a,b)/1$ bulk framework having two different types of various vacations and setup time. Consumers arrive in bulk in this scenario, and the server starts the service after "$a$" customers arrive. At the service terminus epoch, if the total clients are under "$a$," type 1 vacation is available. If the queue volume is more than that of "$a$," service will begin for the first $b$ customers. Finally, after the type 1 vacation completion epoch, the server checks the number of customers; if the number is less than "$a$," it continues type 1 vacation. Alternatively, if the system has a capacity greater than or equal to "$N$," setup work is required. Alternatively, if the majority of customers have a capacity more than or equal to "$N$," setup work begins. The server starts type 2 vacations if the total number of consumers is more than "a" but less than "$N$." If the server detects less than "$N$" clients during type 2 vacation completion stage, it will either continue type 2 vacation or begin setup work. After the completion of setup work, service is provided to a set of clients with the smallest

Table 1: Work flow.

| S. no | Mathematical term | Work |
|---|---|---|
| 1 | Setup work | Grating of tiles if required<br>Washing the tiles with a cleaning agent and drying<br>Bringing it to the printing area, ensuring the dimensions of tiles. Selecting designs for printing and arranging the designs |
| 2 | Service | Applying base coat on the tiles and loading the media. Printing the media on the tiles and drying for few hrs. |
| 3 | Type I vacation | Cleaning the previous batch tile for top coating. Applying top coating. |
| 4 | Type II vacation | Final grinding and side coating. |

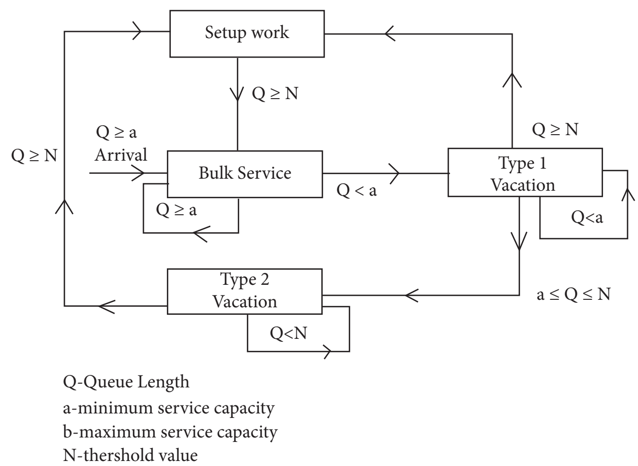


Figure 1: Bulk service with setup time and two types of vacation.

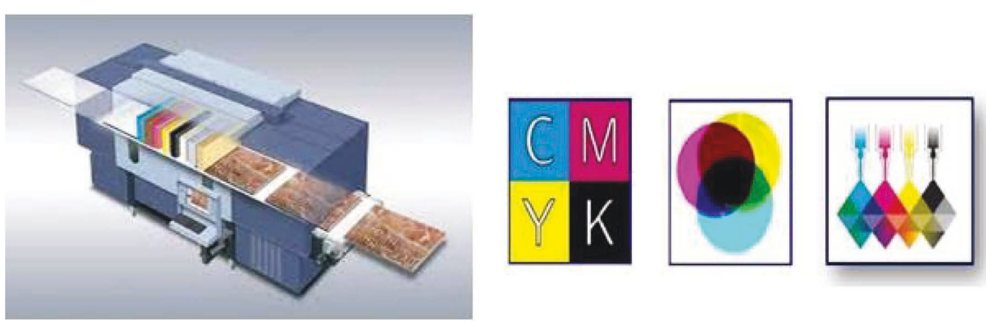


Figure 2: Tile printing machine and colour variants.

size "$a$" and the largest size "$b$." Figure 1 depicts the service flow diagrammatically.

The process of tile printing through inkjet printer is shown in Figure 2.

## 4. Notation

The arrival's group size random variable is denoted as $X$. Let $X(z)$ be the probability generating function of $X$. The chance that "$k$" customers appear in a group is symbolized as $g_k$. $\lambda$ be a sign of the arrival rate. (Hereafter throughout the article, $\varepsilon$ represents arrival rate). $Q_n(t)$, $S_n(t)$ indicate the count of customers in the line and customers availing service. Also,

$$U(t) = \begin{cases} 0 & \text{server is availing vacation,} \\ 1 & \text{server is busy in service,} \\ 2 & \text{server is in setup work.} \end{cases} \tag{1}$$

Let $h^1(t) = k, h^2(t) = k$ represents the service provider is on $g^{\text{th}}$ vacation of type 1 and type 2, respectively. Service time, setup time, secondary job time of type 1 and type 2 have a cumulative distribution function as $P(.), C(.), G(.), A(.)$. The Laplace-Stieltjes transform of $P, C, G$ and $A$ be $\widetilde{P}(\theta)$, $\widetilde{C}(\theta)$, $\widetilde{G}(\theta)$, $\widetilde{A}(\theta)$. Also, $a(x)$, $p(x)$, $c(x)$, $g(x)$ be probability density functions of setup time, service time, type 1vacation, type 2 vacation. $P^0(t)$, $C^0(t)$, $G^0(t)$, $A^0(t)$ designate the enduring period of service of the batch in service and type 1's, type 2's excess vacation time and remaining setup time randomly. The left over service time, setup time, and vacation time of the service provider are considered as supplementary variables. The steady-state difference differential equations for the system are addressed as follows.

The probabilities of each state are

$$
\begin{aligned}
P_{ki}(x,t)\mathrm{dt} &= P\left[S_n(t) = k,\ Q_n(t) = i,\ x \le P^0(t) \le x + \mathrm{dt},\ U(t) = 1\right],\\
C_{jn}(x,t)\mathrm{dt} &= P\left[Q_n(t) = n,\ x \le C^0(t) \le x + \mathrm{dt},\ U(t) = 0, h^1(t) = j,\ j \ge 1,\ n > 0\right],\\
G_{jn}(x,t)\mathrm{dt} &= P\left[Q_n(t) = n,\ x \le G^0(t) \le x + \mathrm{dt},\ U(t) = 0, h^2(t) = j,\ j \ge 1, n > 0\right],\\
A_n(x,t)\mathrm{dt} &= P\left[Q_n(t) = n,\ x \le A_0(t) \le x + \mathrm{dt},\ U(t) = 2,\ n \ge 0\right].
\end{aligned}
\tag{2}
$$

The SVT (supplementary variable technique) is utilized to construct, and the subsequent equations for the queueing process are as follows:

$$
\begin{aligned}
&P_{i0}(x-\Delta t,\ t+\Delta t) = (1-\varepsilon\Delta t)P_{i0}(x,t) + \sum_{m=a}^{b} P_{mi}(0,t)p(x)\Delta t,\ a \le i \le b,\\
&P_{ij}(x-\Delta t, t+\Delta t) = P_{ij}(x,t)(1-\varepsilon\Delta t) + \sum_{c=1}^{j} P_{ij-c}(x,t)\varepsilon g_c\Delta t,\ j \ge 1, a \le i \le b-1,\\
&P_{bj}(x-\Delta t, t+\Delta t) = P_{bj}(x,t)(1-\varepsilon\Delta t) + \sum_{m=a}^{b} P_{mb+j}(0,t)p(x)\Delta t + \sum_{h=1}^{j} P_{bj-h}(x,t)\varepsilon g_h\Delta t,\ 1 \le j \le N-(1+b),\\
&P_{bj}(x-\Delta t, t+\Delta t) = P_{bj}(x,t)(1-\varepsilon\Delta t) + \sum_{m=a}^{b} P_{mb+j}(0,t)p(x)\Delta t + \sum_{h=1}^{j} P_{bj-h}(x,t)\varepsilon g_h\Delta t\ +\ A_{b+j}(0)p(x),\ j \ge N-b,\\
&C_{10}(x-\Delta t, t+\Delta t) = C_{10}(x,t)(1-\varepsilon\Delta t) + \sum_{g=a}^{b} P_{g0}(0,t)c(x)\Delta t,\\
&C_{1n}(x-\Delta t, t+\Delta t) = C_{1n}(x,t)(1-\varepsilon\Delta t) + \sum_{m=a}^{b} c(x)P_{mn}(0,t)\Delta t + \sum_{k=1}^{n} C_{1n-k}(x,t), n \ge a-1,\\
&C_{jn}(x-\Delta t, t+\Delta t) = C_{jn}(x,t)(1-\varepsilon\Delta t) + \sum_{k=1}^{n} C_{jn-k}(x,t)\varepsilon g_k\Delta t,\ j \ge 1,\ n \ge a,\\
&C_{j0}(x-\Delta t, t+\Delta t) = C_{j0}(x,t)(1-\varepsilon\Delta t) + C_{j-1,0}(0,t)c(x)\Delta t,\ j \ge 2,\\
&C_{jn}(x-\Delta t, t+\Delta t) = C_{jn}(x,t)(1-\varepsilon\Delta t) + \sum_{k=1}^{n} C_{jn-k}(x,t)\varepsilon g_k\Delta t + C_{j-1n}C(0,t)c(x)\Delta t,\ a-1 \ge n,\ j \ge 2,\\
&G_{1n}(x-\Delta t, t+\Delta t) = (1-\varepsilon\Delta t)G_{1n}(x,t) + \sum_{k=1}^{\infty} C_{kn}(0,t)\Delta t g(x) + \sum_{k=1}^{n} G_{1n-k}(x,t)\varepsilon g_k\Delta t, N-1 \ge n \ge a,\\
&G_{jn}(x-\Delta t, t+\Delta t) = G_{jn}(x,t)(1-\varepsilon\Delta t) + \sum_{k=1}^{n} G_{j-1,n}(x,t)\varepsilon g_k\Delta t + G_{j-1,n}(0,t)g(x)\Delta t,\ j \ge 2,\ a \le n \le N-1,\\
&G_{jn}(x-\Delta t, t+\Delta t) = (1-\varepsilon\Delta t)G_{jn}(x,t) + \sum_{k=1}^{n} G_{j,n-k}(x,t)\varepsilon g_k\Delta t,\ n \ge N-1,\ j \ge 1,\\
&A_n(x-\Delta t, t+\Delta t) = A_n(x,t)(1-\varepsilon\Delta t) + \sum_{h=1}^{n} A_{n-h}(x,t)\varepsilon g_h\Delta t + \sum_{f=1}^{\infty} C_{fn}(0,t)a(x)\Delta t\\
&\qquad + \sum_{d=1}^{\infty} G_{dn}(0,t)a(x)\Delta t,\ n \ge N.
\end{aligned}
\tag{3}
$$

## 5. Queue Size Distribution

By treating the remaining service time, remaining setup time, and remaining vacation time of the service provider as supplementary variables, the steady state difference differential equations for the system are addressed as

$$-\frac{\mathrm{d}}{\mathrm{dx}}P_{i0}(x) = -\varepsilon P_{i0}(x) + \sum_{m=a}^{b} P_{mi}(0)p(x),\ a \le i \le b, \tag{4}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{dx}}P_{\mathrm{ij}}(x) &= -\varepsilon P_{ij}(x)\\ &\quad+\sum_{c=1}^{j}\varepsilon P_{\mathrm{ij}-c}(x)g_c,\ j \ge 1, a \le i \le b-1,\end{aligned} \tag{5}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}P_{bj}(x) &= -\varepsilon P_{bj}(x) + \sum_{k=1}^{j} P_{bj-k}(x)\varepsilon g_k\\ &\quad+\sum_{m=a}^{b} P_{mb+j}(0)p(x),\ 1 \le j \le N-(1+b),\end{aligned} \tag{6}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{dx}}P_{bj}(x) &= -\varepsilon P_{bj}(x) + \sum_{m=a}^{b} P_{mb+j}(0)s(x)\\ &\quad+\sum_{h=1}^{j} P_{bj-h}(x)\varepsilon g_h + A_{b+j}(0)p(x),\ j \ge N-b,\end{aligned} \tag{7}$$

$$-\frac{\mathrm{d}}{\mathrm{dx}}C_{10}(x) = -\varepsilon C_{10}(x) + \sum_{g=a}^{b} P_{g0}(0)c(x), \tag{8}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}C_{1n}(x) &= -\varepsilon C_{1n}(x) + \sum_{m=a}^{b} c(x)P_{mn}(0)\\ &\quad-\sum_{k=1}^{n} C_{1n-k}(x)\varepsilon g_k,\ a-1 \ge n,\end{aligned} \tag{9}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}C_{jn}(x) = -\varepsilon C_{jn}(x) + \sum_{k=1}^{n} C_{jn-k}(x)\varepsilon g_k,\quad j \ge 1,\ n \ge a, \tag{10}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}C_{j0}(x) = -\varepsilon C_{j0}(x) + C_{j-10}(0)c(x),\ j \ge 2, \tag{11}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}C_{jn}(x) &= -\varepsilon C_{jn}(x) + \sum_{k=1}^{n} C_{jn-k}(x)\varepsilon g_k\\ &\quad+C_{j-1,n}(0)g(x),\ j \ge 2, a-1 \ge n,\end{aligned} \tag{12}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}G_{1n}(x) &= -\varepsilon G_{1n}(x) + \sum_{k=1}^{\infty} C_{kn}(0)g(x)\\ &\quad+\sum_{k=1}^{n} G_{1n-k}(x)\varepsilon g_k,\ a \le n \le N-1,\end{aligned} \tag{13}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}G_{jn}(x) &= -\varepsilon G_{jn}(x) + \sum_{k=1}^{n} G_{j,n-k}(x)\varepsilon g_k\\ &\quad+G_{j,n-k}(0)c(x),\quad j \ge 2,\ a \le n \le N-1,\end{aligned} \tag{14}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}G_{kn}(x) &= -\varepsilon G_{kn}(x)\\ &\quad+\sum_{u=1}^{n} G_{k,n-u}(x)\varepsilon g_u,\quad n \ge N-1,\ k > 0,\end{aligned} \tag{15}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}A_n(x) &= -\varepsilon A_n(x) + \sum_{h=1}^{n} A_{n-h}(x)\varepsilon g_h + \sum_{f=1}^{\infty} C_{fn}(0)a(x)\\ &\quad+\sum_{d=1}^{\infty} G_{dn}(0)a(x),\ n \ge N.\end{aligned} \tag{16}$$

Applying the Laplace-Stieltjes transform to two sides of equations (4) to (16), we obtain

$$
\begin{aligned}
&\theta\widetilde{P}_{i0}(\theta)-P_{i0}(0)=\varepsilon\widetilde{P}_{i0}(\theta)-\sum_{m=a}^{b}P_{mi}(0)\widetilde{P}(\theta)\quad a\le i\le b,\\
&\theta\widetilde{P}_{ij}(\theta)-P_{ij}(0)=\varepsilon\widetilde{P}_{ij}(\theta)-\lambda\sum_{c=1}^{j}\widetilde{P}_{ij-c}(\theta)g_c,\quad a\le i\le b-1,\ j\ge 1,\\
&\theta\widetilde{P}_{bj}(\theta)-P_{bj}(0)=\varepsilon\widetilde{P}_{bj}(\theta)-\sum_{k=a}^{b}P_{k,b+j}(0)\widetilde{P}(\theta)-\sum_{m=1}^{j}\widetilde{P}_{b,j-m}(\theta)\varepsilon g_m,\quad 1\le j\le N-(1+b),\\
&\theta\widetilde{P}_{bj}(\theta)-P_{bj}(0)=\varepsilon\widetilde{P}_{bj}(\theta)-\sum_{m=a}^{b}P_{m,b+j}(0)\widetilde{P}(\theta)-\sum_{h=1}^{j}\widetilde{P}_{b,j-h}(\theta)\varepsilon g_h-A_{b+j}(0)\widetilde{P}(\theta),\quad j\ge N-b,\\
&\theta\widetilde{C}_{10}(\theta)-C_{10}(0)=\varepsilon\widetilde{C}_{10}(\theta)-\sum_{g=a}^{b}P_{g,0}(0)\widetilde{C}(\theta),\\
&\theta\widetilde{C}_{1n}(\theta)-C_{1n}(0)=\varepsilon\widetilde{C}_{1n}(\theta)-\widetilde{C}(\theta)\sum_{m=a}^{b}P_{m,n}(0)+\sum_{k=1}^{n}\widetilde{C}_{1n-k}(\theta)\varepsilon g_k,\quad a-1\ge n,\\
&\theta\widetilde{C}_{jn}(\theta)-C_{jn}(0)=\varepsilon\widetilde{C}_{jn}(\theta)-\sum_{k=1}^{n}\widetilde{C}_{jn-k}(\theta)\varepsilon g_k,\quad j\ge 1,\ n\ge a,\\
&\theta\widetilde{C}_{j0}(\theta)-C_{j0}(0)=\varepsilon\widetilde{C}_{j0}(\theta)-C_{j-1,0}(0)\widetilde{C}(\theta),\quad j\ge 2,\\
&\theta\widetilde{C}_{jn}(\theta)-C_{jn}(0)=\widetilde{\varepsilon C}_{jn}(\theta)-\sum_{k=1}^{n}\widetilde{C}_{jn-k}(\theta)\varepsilon g_k-C_{j-1n}(0)\widetilde{C}(\theta),\quad j\ge 2,\ 0\le n\le a-1,\\
&\theta\widetilde{G}_{1n}(\theta)-G_{1n}(0)=\varepsilon\widetilde{G}_{1n}(\theta)-\sum_{k=1}^{\infty}C_{kn}(0)\widetilde{G}(\theta)-\sum_{k=1}^{n}G_{1n-k}(\theta)\varepsilon g_k,\quad a\le n\le N-1,\\
&\theta\widetilde{G}_{jn}(\theta)-G_{jn}(0)=\varepsilon\widetilde{G}_{jn}(\theta)-\sum_{k=1}^{n}\widetilde{G}_{j,n-k}(\theta)\varepsilon g_k-G_{j-1,n}(0)\widetilde{G}(\theta),\quad j\ge 2,\ a\le n\le N-1,\\
&\theta\widetilde{G}_{jn}(\theta)-G_{jn}(0)=\varepsilon\widetilde{G}_{jn}(\theta)-\sum_{k=1}^{n}\widetilde{G}_{j,n-k}(\theta)\varepsilon g_k,\quad n\ge N-1,\ j\ge 1,\\
&\theta\widetilde{A}_{n}(\theta)-A_{n}(0)=\varepsilon\widetilde{A}_{n}(\theta)-\sum_{k=1}^{n}\widetilde{A}_{n-k}(\theta)\varepsilon g_k-\sum_{l=1}^{\infty}C_{\ln}(0)\widetilde{A}(\theta)-\sum_{l=1}^{\infty}G_{\ln}(0)\widetilde{A}(\theta),\quad n\ge N.
\end{aligned}
\tag{17}
$$

Let

$$
\begin{aligned}
&\widetilde{P}_i(z,\theta) = \sum_{j=0}^{\infty} \widetilde{P}_{ij}(\theta) z^j \\
&\widetilde{P}_i(z,0) = \sum_{j=0}^{\infty} P_{ij}(0) z^j \quad a \le i \le b \\
&\widetilde{C}_j(z,\theta) = \sum_{n=0}^{\infty} \widetilde{C}_{jn}(\theta) z^n \\
&C_j(z,0) = \sum_{n=0}^{\infty} C_{jn}(0) z^n \quad j \ge 1 \\
&\widetilde{G}_j(z,\theta) = \sum_{n=a}^{\infty} \widetilde{G}_{jn}(\theta) z^n \\
&G_j(z,0) = \sum_{n=a}^{\infty} G_{jn}(0) z^n \quad j \ge 1 \\
&\widetilde{A}(z,\theta) = \sum_{n=N}^{\infty} \widetilde{A}(\theta) z^n \\
&A(z,0) = \sum_{n=N}^{\infty} A_n(0) z^n.
\end{aligned} \tag{18}
$$

Multiplying the above equation by $z^0$ and by $z^n$ and aggregating and utilizing (18), we got

$$(\theta - \varepsilon + \varepsilon X(z))\widetilde{C}_1(z,\theta) = C_1(z,0) - \widetilde{C}(\theta) \sum_{k=0}^{a-1} \sum_{c=a}^{b} P_{ck}(0) z^k, \tag{19}$$

$$(\theta - \varepsilon + \varepsilon X(z))\widetilde{C}_j(z,\theta) = C_j(z,0) - \widetilde{C}(\theta) \sum_{d=0}^{a-1} C_{j-1,d}(0) z^d \ (j \ge 2), \tag{20}$$

$$(\theta - \varepsilon + \varepsilon X(z))\widetilde{G}_1(z,\theta) = G_1(z,0) - \widetilde{G}(\theta) \sum_{n=a}^{N-1} \sum_{k=1}^{\infty} C_{kn}(0) z^n \ (2 \ge j), \tag{21}$$

$$(\theta - \varepsilon + \varepsilon X(z))\widetilde{G}_j(z,\theta) = G_j(z,0) - \widetilde{G}(\theta) \sum_{n=a}^{\infty} G_{j-1,n}(0) z^n, \tag{22}$$

$$(\theta - \varepsilon + \varepsilon X(z))\widetilde{P}_i(z,\theta) = P_i(z,0) - \widetilde{P}(\theta) \sum_{m=a}^{b} P_{mi}(0) (a \le i \le b-1), \tag{23}$$

$$\widetilde{P}_b(z,\theta) z^b (\theta - \varepsilon + \varepsilon X(z)) = z^b P_b(z,0) - \widetilde{P}(\theta) \sum_{h=a}^{b} \left[ P_h(z,0) - \sum_{j=0}^{b-1} P_{hj}(0) z^j \right] + A(z,0), \tag{24}$$

$$
\begin{aligned}
(\theta - \varepsilon + \varepsilon X(z))\widetilde{A}(z,\theta) = A(z,0) &- \widetilde{A}(\theta) \left\{ \sum_{c=1}^{\infty} \left[ C_c(z,0) - \sum_{n=0}^{N-1} C_{cn}(0) z^n \right] \right\} \\
&- \widetilde{A}(\theta) \left\{ \sum_{d=1}^{\infty} \left[ G_d(z,0) - \sum_{n=0}^{N-1} G_{dn}(0) z^n \right] \right\}.
\end{aligned} \tag{25}
$$

Replacing $(\theta = \varepsilon + \varepsilon X(z))$ in equations (19) to (25), we get

$$C_1(z,0) = \widetilde{C}(\varepsilon - \varepsilon X(z)) \sum_{k=0}^{a-1} \sum_{f=a}^{b} P_{fk}(0) z^k, \tag{26}$$

$$
\begin{aligned}
&C_j(z,0) = \widetilde{C}(\varepsilon - \varepsilon X(z)) \sum_{d=0}^{a-1} C_{j-1,d}(0) z^d \ (2 \le j), \\
&G_1(z,0) = \sum_{n=a}^{N-1} \sum_{k=1}^{\infty} C_{kn}(0) z^n \widetilde{G}(\varepsilon - \varepsilon X(z)), \\
&G_j(z,0) = \sum_{n=a}^{N-1} G_{j-1,n}(0) z^n \widetilde{G}(\varepsilon - \varepsilon X(z)), \quad j \ge 2,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
A(z,0) = \widetilde{A}(\varepsilon - \varepsilon X(z)) \Bigg\{ &\sum_{f=1}^{\infty} \left[ C_f(z,0) - \sum_{n=0}^{N-1} C_{fn}(0) z^n \right] \\
&+ \sum_{h=1}^{\infty} \left[ G_h(z,0) - \sum_{j=0}^{N-1} G_{hj}(0) z^j \right] \Bigg\},
\end{aligned} \tag{28}
$$

$$P_i(z,0) = \widetilde{P}(\varepsilon - \varepsilon X(z)) \sum_{g=a}^{b} P_{gi}(0) \ (a \le i \le b-1), \tag{29}$$

$$z^b P_b(z,0) = \widetilde{P}(\varepsilon - \varepsilon X(z)) \sum_{m=a}^{b} \left[ P_m(z,0) - \sum_{j=0}^{b-1} P_{mj}(0) z^j \right] + A(z,0). \tag{30}$$

Substituting the above values in their corresponding equation, we have

$$\begin{aligned}
\widetilde{C}_1(z,\theta) &= \frac{\widetilde{C}(\varepsilon-\varepsilon X(z))-\widetilde{C}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{mn}(0)z^n,\\
\widetilde{C}_j^1(z,\theta) &= \frac{\widetilde{C}(\varepsilon-\varepsilon X(z))-\widetilde{C}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{n=0}^{a-1}C_{j-1,n}(0)z^n\ (j\geq 2),\\
\widetilde{G}_1(z,\theta) &= \frac{\widetilde{G}(\varepsilon-\varepsilon X(z))-\widetilde{G}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{n=a}^{N-1}\sum_{k=1}^{\infty}C_{kn}(0)z^n,\\
\widetilde{G}_j(z,\theta) &= \frac{\widetilde{G}(\varepsilon-\varepsilon X(z))-\widetilde{G}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{n=a}^{N-1}G_{j-1,n}(0)z^n,\ j\geq 2,\\
\widetilde{A}(z,\theta) &= \frac{\widetilde{A}(\varepsilon-\varepsilon X(z))-\widetilde{A}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\left\{\sum_{l=1}^{\infty}\left[C_1(z,0)-\sum_{n=0}^{N-1}C_{\ln}(0)z^n\right]+\sum_{l=1}^{\infty}\left[G_l(z,0)-\sum_{n=0}^{N-1}G_{\ln}(0)z^n\right]\right\},\\
\widetilde{P}_i(z,\theta) &= \frac{\widetilde{P}(\varepsilon-\varepsilon X(z))-\widetilde{P}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{m=a}^{b}P_{mi}(0).
\end{aligned} \tag{31}$$

Consider (30)

$$\begin{aligned}
z^bP_b(z,0) &= \widetilde{P}(\varepsilon-\varepsilon X(z))\left[\sum_{h=a}^{b}[P_h(z,0)]+P_b(z,0)-\sum_{m=a}^{b}\sum_{k=0}^{b-1}P_{mk}(0)z^k+A(z,0)\right],\\
\left(z^b-\widetilde{P}(\varepsilon-\varepsilon X(z))\right)P_b(z,0) &= \widetilde{P}(\varepsilon-\varepsilon X(z))\left[\sum_{m=a}^{b}[P_m(z,0)]-\sum_{k=a}^{b}\sum_{j=0}^{b-1}P_{kj}(0)z^j+A(z,0)\right].
\end{aligned} \tag{32}$$

Substituting $P_b(z,0)$ value in $P_b(z,\theta)$ equation, we obtain

$$\begin{aligned}
z^b(\theta-\varepsilon+\varepsilon X(z))\widetilde{P}_b(z,\theta) &= P_b(z,0)\left(z^b-\widetilde{P}(\theta)\right)\\
&\quad-\widetilde{P}(\theta)\left\{\sum_{k=a}^{b-1}\left[\widetilde{P}(\varepsilon-\varepsilon X(z))\sum_{j=a}^{b}P_{kj}(0)-\sum_{j=0}^{b-1}P_{kj}(0)z^j\right]+A(z,0)\right\}.
\end{aligned} \tag{33}$$

Let

$$\begin{aligned}
\sum_{g=a}^{b}P_{gi}(0) &= P_i,\\
\sum_{f=1}^{\infty}C_{fi}(0) &= C_i,\\
\sum_{h=1}^{\infty}G_{hi}(0) &= G_i^2.
\end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned}
\widetilde{C}_1(z,\theta) &= \sum_{n=0}^{a-1} P_n z^n \frac{\widetilde{C}(\varepsilon-\varepsilon X(z))-\widetilde{C}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))},\\
\widetilde{C}_j(z,\theta) &= \frac{\widetilde{C}(\varepsilon-\varepsilon X(z))-\widetilde{C}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{n=0}^{a-1} C_{j-1,n}(0)z^n,\\
\widetilde{G}_1(z,\theta) &= \frac{\widetilde{G}(\varepsilon-\varepsilon X(z))-\widetilde{G}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{n=a}^{N-1} C_n z^n,\\
\widetilde{G}_j(z,\theta) &= \frac{\widetilde{G}(\varepsilon-\varepsilon X(z))-\widetilde{G}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\sum_{n=a}^{N-1} G_{j-1}(0)z^n, \quad j\ge 2,\\
\widetilde{A}(z,\theta) &= \frac{\widetilde{A}(\varepsilon-\varepsilon X(z))-\widetilde{A}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}\left\{\sum_{l=1}^{\infty}\left[C_1(z,0)-\sum_{n=0}^{N-1} C_j z^n\right]-\sum_{l=1}^{\infty}\left[\sum_{h=0}^{N-1} G_h z^h - G_l(z,0)\right]\right\},\\
\widetilde{P}_j(z,\theta) &= \left(P_j\right)\frac{\widetilde{P}(\varepsilon-\varepsilon X(z))-\widetilde{P}(\theta)}{(\theta-\varepsilon+\varepsilon X(z))}.
\end{aligned} \tag{35}$$

From (28) and substituting the values, we get

$$\begin{aligned}
A(z,0) = \widetilde{A}((\varepsilon-\varepsilon X(z)))\Bigg\{&\widetilde{C}(\varepsilon-\varepsilon X(z))\sum_{n=0}^{a-1} P_n z^n + \widetilde{C}(\varepsilon-\varepsilon X(z))\sum_{n=0}^{-(1-a)} C_n z^n - \sum_{j=0}^{N-1} C_j z^j + \widetilde{V}_2(\varepsilon-\varepsilon X(z))\\
&\sum_{n=a}^{N-1} C_n z^n + \widetilde{G}(\varepsilon-\varepsilon X(z))\sum_{n=a}^{N-1} G_n z^n - \sum_{j=0}^{N-1} G_j z^j\Bigg\}.
\end{aligned} \tag{36}$$

Substituting the above value in $P_b(z,\theta)$, we get

$$P_b(z,\theta) = [k(z)]\frac{\widetilde{P}(\varepsilon-\varepsilon X(z))-\widetilde{P}(\theta)}{\left(z^b-\widetilde{P}(\varepsilon-\varepsilon X(z))\right)(\theta-\varepsilon+\varepsilon X(z))}, \tag{37}$$

where

$$k(z) = \widetilde{P}(\varepsilon-\varepsilon X(z))\sum_{h=a}^{b-1} P_h - \sum_{f=0}^{b-1} P_f z^f + \widetilde{A}(\varepsilon-\varepsilon X(z))\left\{\begin{array}{c}\widetilde{C}(\varepsilon-\varepsilon X(z))\sum_{n=0}^{a-1}(P_n+C_n)z^n\\ +\widetilde{G}(\varepsilon-\varepsilon X(z))\sum_{n=a}^{N-1}(C_n+G_n)z^n - \sum_{n=0}^{N-1}(C_n+G_n)z^n\end{array}\right\}. \tag{38}$$

If $P(z)$ is the probability-generating function of queue size at any time period, then

$$P(z) = \widetilde{A}(z,0) + \sum_{k=a}^{b-1}\widetilde{P}_k(z,0) + \widetilde{P}_b(z,0) + \sum_{f=1}^{\infty}\widetilde{C}_f(z,0) + \sum_{v=1}^{\infty}\widetilde{G_v}(z,0). \tag{39}$$

On substituting the values, we get

$$P(z)=\frac{\left\{\begin{array}{c}\sum_{i=a}^{b-1}P_i\left(z^b-z^j\right)(\widetilde{P}(\varepsilon-\varepsilon X(z))-1)+\sum_{n=0}^{a-1}p_nz^n\left(z^b-1\right)(\widetilde{A}(\varepsilon-\varepsilon X(z))\widetilde{C}(\varepsilon-\varepsilon X(z))-1)\\ +\sum_{n=a}^{N-1}(C_n+G_n)\widetilde{A}(\varepsilon-\varepsilon X(z))(\widetilde{G}(\varepsilon-\varepsilon X(z))-1)z^n\left(z^b-1\right)\\ +\sum_{n=0}^{a-1}C_nz^n\left(z^b-1\right)(\widetilde{C}(\varepsilon-\varepsilon X(z))-1)\widetilde{A}(\varepsilon-\varepsilon X(z))\end{array}\right\}}{\left(-\varepsilon+\varepsilon X(z)\left(z^b-\widetilde{P}(\varepsilon-\varepsilon X(z))\right)\right)}. \tag{40}$$

$P(z)$ need to fulfil $P(1)=1$. To qualify this, the L'Hospital rule is applied, and the formulation is compared with 1. For the existence of a steady state, the criteria to be convinced is $\rho<1$ where $\rho=\varepsilon E(x)E(s)/b$ and $E(p)$ is the expected service time.

## 6. Computational Aspects

Equation (40) has $N+b$ unknowns. The three accompanying theorems have been shown to represent $C_i$ and $G_i$ in the form of $p_i$, and as a result, the numerator has $b$ constant. Equation (40) now shows a probability-generating function containing only "$b$" unknowns. Using Rouché's theorem, we can prove that there are $b-1$ zero inside and one zero on the region $|z|=1$. The above-said equations are solved using MATLAB software.

**Theorem 1.** *The constant $C_n$ in $P(z)$ are communicated in terms of $p_n$ as $C_f=\sum_{i=0}^{f}b_{f-i}p_i$ where $b_n=1/1-\xi_0[\xi_n+\sum_{j=1}^{n}b_{n-j}\xi_i]$, $n=0,1,\ldots a-1$ and $\xi_s$ is the chance that "s" clients arrive during a vacation session.*

*Proof.* We know that equation (26) demonstrates

$$\begin{aligned}C_1(z,0)&=\widetilde{C}(\varepsilon-\varepsilon X(z))\sum_{n=0}^{a-1}\sum_{m=a}^{b}p_{mn}(0)z^n=\left(\sum_{n=0}^{\infty}\xi_nz^n\right)\cdot\left(\sum_{n=0}^{a-1}p_nz^n\right)\\ \sum_{j=1}^{\infty}C_j(z,0)&=C_1(z,0)+\sum_{j=2}^{\infty}C_j(z,0)\\ &=\left(\sum_{n=0}^{\infty}\xi_nz^n\right)\cdot\left(\sum_{n=0}^{a-1}p_nz^n\right)+\sum_{j=2}^{\infty}\widetilde{C}(\varepsilon-\varepsilon X(z))\sum_{n=0}^{a-1}C_{j-1,n}(0)z^n.\end{aligned} \tag{41}$$

From

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}C_{jn}(0)z^n&=\left(\sum_{n=0}^{\infty}\xi_nz^n\right)\cdot\left(\sum_{n=0}^{a-1}p_nz^n\right)+\left(\sum_{n=0}^{\infty}\xi_nz^n\right)\cdot\left(\sum_{n=0}^{a-1}C_nz^n\right)\\ \left(\sum_{n=0}^{\infty}C_nz^n\right)&=\left(\sum_{n=0}^{\infty}\xi_nz^n\right)\cdot\left(\sum_{n=0}^{a-1}(p_n+C_n)z^n\right)\\ \left(\sum_{n=0}^{a-1}C_nz^n\right)+\left(\sum_{n=a}^{\infty}C_nz^n\right)&=\left(\sum_{n=0}^{a-1}\left[\sum_{h=0}^{n}\xi_h(p_{n-h}+C_{n-h})z^n\right]\right)\cdot+\left(\sum_{n=a}^{\infty}\left[\sum_{j=0}^{a-1}\xi_{n-j}\left(p_j+C_j\right)z^n\right]\right).\end{aligned} \tag{42}$$

We get

$$\begin{aligned}
&C_n = \sum_{j=0}^{n} \xi_j \left(p_{n-j} + C_{n-j}\right),\\
&C_n = \sum_{j=0}^{n} \xi_j p_{n-j} + \sum_{j=0}^{n} \xi_j C_{n-j},\\
&C_n = \sum_{j=0}^{n} \xi_j p_{n-j} + \sum_{j=1}^{n} \xi_j C_{n-j} + \xi_0 C_n,\\
&\therefore C_n = \frac{1}{1-\xi_0}\left[\sum_{j=0}^{n} \xi_j p_{n-j} + \sum_{j=1}^{n} \xi_j C_{n-j}\right],\\
&\text{coefficient of } p_n \text{in } C_n = \frac{1}{1-\xi_0}\left(\xi_0\right) = b_0,\\
&\text{coefficient of } p_{n-1} \text{in } C_n = \frac{1}{1-\xi_0}\left(\xi_1 + \xi_1\left(\frac{1}{1-\xi_0}\right)\xi_0\right) = b_1,\\
&\text{coefficient of } p_{n-2} \text{in } C_n = \frac{1}{1-\xi_0}\left(\xi_2 + \xi_1\left(\frac{1}{1-\xi_0}\right)\left(\xi_1 + \xi_1\xi_0\left(\frac{1}{1-\xi_0}\right)\right) + \left(\xi_2\xi_0\left(\frac{1}{1-\xi_0}\right)\right)\right) = b_2\\
&\text{coefficient of } p_{n-n} \text{in } C_n = \frac{1}{1-\xi_0}\left(\xi_n + \left(\xi_n b_0 + \xi_{n-1} b_1 + \cdots \xi_1 b_{n-1}\right)\right) \text{ where } b_k = \frac{1}{1-\xi_0}\left[\xi_k + \sum_{j=1}^{n} b_{k-j}\xi_i\right],\\
&\therefore C_n = \sum_{j=0}^{n} b_{n-j} p_j \text{ where } b_k = \frac{1}{1-\xi_0}\left[\xi_k + \sum_{j=1}^{n} b_{k-j}\xi_i\right], \quad k = 0, 1, \ldots a-1,
\end{aligned} \tag{43}$$

where

$$b_k = \frac{1}{1-\xi_0}\left[\xi_k + \sum_{j=1}^{n} b_{k-j}\xi_i\right], \quad k = 0, 1, \ldots a-1. \tag{44}$$

□

**Theorem 2.** *The constant $C_h$ in P(z) are pointed in terms of $p_h$ as $C_h = \sum_{k=0}^{a-1}[\xi_{h-k} + \sum_{j=0}^{a-1-k} b_j \xi_{h-k-j}]p_k$ where $b_j = 1/1-\xi_0[\xi_j + \sum_{f=1}^{j} b_{j-f}\xi_f]$, $n = a, a+1, \ldots N-1$ where $\xi_c$ represent the chance that "c" clients enter the system during type 1secondary job.*

*Proof.* On both sides, the coefficients of $z^h, h = a, a+1, \ldots N-1$ are equated

$$\begin{aligned}
&C_h = \sum_{j=0}^{a-1} \xi_{h-j}\left(p_j + c_j\right)\\
&\Rightarrow C_h = \sum_{j=0}^{a-1} \xi_{h-j} p_j + \sum_{j=0}^{a-1} \xi_{h-j} C_j.
\end{aligned} \tag{45}$$

Substituting $C_s, s = 0 \text{to} a-1$ from Theorem 1, we get

$$\begin{aligned}
&C_h = \sum_{s=0}^{a-1} \xi_{n-s} p_s + \sum_{s=0}^{a-1} \xi_{h-s}\left(b_k p_{s-k}\right)\\
&C_h = \sum_{s=0}^{a-1} \xi_{h-s} p_s + \sum_{f=0}^{a-1}\left[\sum_{s=0}^{a-1-f} b_s \xi_{h-f-s}\right] p_f\\
&C_h = \sum_{s=0}^{a-1}\left[\xi_{h-s} + \sum_{f=0}^{a-1-s} b_s \xi_{h-s-f}\right] p_s.
\end{aligned} \tag{46}$$

□

**Theorem 3.** *Let $\phi_i = \sum_{j=a}^{n} C_j \eta_{n-j}$ where $G_k$ present in P (z) can be conveyed with reference to $p_k$ as $G_h = 1/1-\delta_0[\phi_i + \sum_{i=a}^{h-1} G_i \eta_{h-i}]$, $h = a, a+1, a+2, \ldots N-1$, and also $G_a = \phi a/1-\eta_0$ where $\eta_f$ is the chance that "f" customers enter the system during type two job.*

*Proof.* Consider

$$
\begin{aligned}
G_1(z,0) &= \sum_{g=a}^{N-1}\sum_{k=1}^{\infty} C_{kg}(0)z^g\widetilde{G}(\varepsilon-\varepsilon X(z)) = \left(\sum_{g=0}^{\infty}\eta_g z^g\right).\left(\sum_{g=a}^{N-1} C_g z^g\right)\\
\sum_{h=1}^{\infty} G_h(z,0) &= G_1(z,0)+\sum_{h=2}^{\infty} G_h(z,0)\\
&= \left(\sum_{h=0}^{\infty}\eta_h z^h\right).\left(\sum_{h=a}^{N-1} C_h z^h\right)+\sum_{j=2}^{\infty}\widetilde{G}(\varepsilon-\varepsilon X(z))\sum_{h=a}^{N-1} G_{j-1}(0)z^h\\
\sum_{j=1}^{\infty}\sum_{n=a}^{\infty} G_{jn}(0)z^n &= \left(\sum_{n=0}^{\infty}\eta_n z^n\right).\left(\sum_{n=a}^{N-1} C_n z^n\right)+\left(\sum_{n=0}^{\infty}\eta_n z^n\right).\left(\sum_{n=a}^{N-1} G_n z^n\right)\\
\left(\sum_{h=a}^{\infty} G_h z^h\right) &= \left(\sum_{h=0}^{\infty}\eta_h z^h\right).\left(\sum_{h=a}^{N-1}\left(C_h+G_{\backslash h}\right)z^h\right)\\
&= \left(\sum_{h=0}^{\infty}\eta_h z^h\right).\left(\sum_{h=a}^{N-1}(w_h)z^h\right),\text{ where } w_h = C_h+G_h\\
&= \sum_{n=a}^{N-1}\left(\sum_{i=a}^{n}\eta_{n-i}z^n w_i\right)z^n+\sum_{n=N}^{\infty}\left(\sum_{i=a}^{N-1}\eta_{n-i}z^n w_i\right).
\end{aligned}
\tag{47}
$$

Equating the coefficients of $z^k$, $k = a, a+1, \ldots N-1$

$$
\begin{aligned}
\therefore G_n &= \frac{1}{1-\delta_0}\left[\sum_{i=a}^{n} C_i\eta_{n-i}+\sum_{i=a}^{n-1} G_i\eta_{n-i}\right]\\
G_n &= \frac{1}{1-\delta_0}\left[\phi_i+\sum_{i=a}^{n-1} G_i\eta_{n-i}\right]\text{where } \phi_i = \sum_{i=a}^{n} C_i\eta_{n-i},\quad a,\ a+1,\ a+2,\ \ldots N-1.
\end{aligned}
\tag{48}
$$

Solving the above equation for $G_n$, we get $G_a = \phi a/1-\eta_0$, $G_n = 1/1-\delta_0[\phi_i +\sum_{i=a}^{n-1} G_i\eta_n -i]$,$\phi_i = \sum_{i=a}^{n} C_i\eta_{n-i}, a, 1+a, 2+a, \ldots N-1$. □

## 7. Particular Case

*Case 1.* In the event that there is only one secondary job (i.e., $\widetilde{G}(\varepsilon-\varepsilon X(z)) = \widetilde{C}(\varepsilon-\varepsilon X(z))$) and setup time is $A(\varepsilon-\varepsilon X(z)) = 1$, the equation (40) reduces to

$$
P(z) = \frac{\left\{\sum_{i=a}^{b-1} P_i\left(z^b-z^j\right)(\widetilde{P}-1)+\left(z^b-1\right)\left[\sum_{n=0}^{a-1} p_n z^n+\sum_{n=0}^{N-1} q_n^1 z^n\right](\widetilde{C}-1)\right\}}{\left(z^b-\widetilde{P}(\varepsilon-\varepsilon X(z))\right)(-\varepsilon+\varepsilon X(z))}. \tag{49}
$$

This result coincides with the distribution discussed in [15].

*Case 2.* If there is no setup time, that is, $A(\varepsilon-\varepsilon X(z)) = 1$, we get

$$
P(z) = \frac{\left\{\begin{array}{l}\sum_{i=a}^{b-1} P_i\left(z^b-z^j\right)(\widetilde{P}(\varepsilon-\varepsilon X(z))-1)+\sum_{n=0}^{a-1} p_n z^n\left(z^b-1\right)(\widetilde{C}(\varepsilon-\varepsilon X(z))-1)\\ +\sum_{n=a}^{N-1}(C_n+G_n)z^n\left(z^b-1\right)(\widetilde{G}(\varepsilon-\varepsilon X(z))-1)+\sum_{n=0}^{a-1} C_n z^n\left(z^b-1\right)(\widetilde{C}-1)\end{array}\right\}}{\left(z^b-\widetilde{P}\right)(-\varepsilon+\varepsilon X(z))}, \tag{50}
$$

is the distribution of bulk queuing system with vacation. The outcome coincides with the result stated in [6].

## 8. Performance Measures

A few performance measures are accomplished in this part, which are applicable in building up the cost model and furthermore to correlate the framework with several features of secondary jobs.

*8.1. Awaited Queue Length (EQL).* The awaited line length $E(Q)$ is obtained as

$$E(Q) = \underset{z\longrightarrow 1}{Lt}\ \frac{\mathrm{d}P(z)}{\mathrm{d}z}, \tag{51}$$

i.e.,

$$E(Q) = \frac{1}{4\lambda^2 E(X)^2 (b-f_1)^2}\left[\begin{array}{c}\sum_{i=a}^{b-1} P_i\left[T_2(b-f_1) + 2T_1(b-i)f_1\right] + \sum_{n=0}^{a-1} P_n\left[T_3(b-f_1) + 2b(f_0+f_2)T_1\right] \\ +\sum_{n=0}^{a-1} q_n^1\left[T_4(b-f_1) + 2b(f_2)T_1\right] + \sum_{n=a}^{N-1}\left(q_n^1+q_n^2\right)\left[T_5(b-f_1) + 2b(f_5)T_1\right]\end{array}\right], \tag{52}$$

where

$$\begin{aligned}
f_1 &= \varepsilon E(x)E(P),\\
f_2 &= \varepsilon E^2(x)E\left(P^2\right),\\
f_3 &= \varepsilon E(x)E(C),\\
f_4 &= \varepsilon E^2(x)E\left(C^2\right),\\
f_5 &= \varepsilon E(x)E(G),\\
f_6 &= \varepsilon E^2(x)E\left(G^2\right),\\
f_7 &= \varepsilon X''(1)E(C),\\
f_8 &= \varepsilon X''(1)E(G),\\
f_9 &= \varepsilon X''(1)E(P),\\
f_{10} &= \varepsilon E(x)E(A),\\
f_{11} &= \varepsilon E(x)^3 E\left(P^3\right),\\
f_{12} &= \varepsilon X''(1)E(A),\\
T_1 &= -3\varepsilon b(b-1)E(x) + b\varepsilon E(x)f_9 + 3\varepsilon^2 E(x)f_2 - 3\varepsilon bX''(1),\\
T_2 &= \left(3f_1\left(b^2-b-i^2+i\right) + 3f_9(b-i) + 3\varepsilon f_2(b-i)\right)2\varepsilon E(x),\\
T_3 &= 2\varepsilon E(x)(b-f_1)\left(\begin{array}{c}3b\varepsilon f_4 + 6b\left(f_{10} * f_3\right) + 3b\varepsilon f_{13} + 2b\left(f_{10}+f_3\right)\\ +b\left(f_{12}+f_7\right) + \left(f_{10}+f_3\right)(b+n) * (b+n-1) - \left(n^2-n\right)\end{array}\right),\\
T_4 &= 3b\left(f_3 * f_{10}\right) + 3b\varepsilon f_4 - 3bf_7 + 3 * \left((b+n)(b+n-1) - \left(n^2-n\right)\right)f_3,\\
T_5 &= 2\varepsilon E(x)(b-f_1) * \left[3 * \left((b+n) * (b+n-1) - \left(n^2-n\right)\right) * f_5\right] - 3 * bf_8 + 3b\varepsilon f_6 + 3b\left(f_5 * f_{10}\right).
\end{aligned} \tag{53}$$

*8.2. Awaited Length of Idle Period (EIP).* Let us denote the recreation period by $I$. The anticipated leisure period is $E(I) = E(I1) + E(I2) + E(G)$.

Let $U1$ denote a random variable. $U1$ takes the value 0, if at least "$a$" clients are spotted at the wind-up of the first vacation of type 1 secondary job, and $U1$ takes the value 1, if smaller than "$a$" clients are found at the end of the 1st vacation of type one secondary job.

$$E(I_1) = E\left(\frac{I_1}{U_1} = 0\right)P(U_1 = 1) + E\left(\frac{I_1}{U_1} = 0\right)P(U_1 = 0)$$
$$= [E(C) + E(I_1)]P(U_1 = 1) + E(C)P(U_1 = 0),$$
$$E(I_1) = \frac{E(C)}{P(U_1 = 0)}. \tag{54}$$

$U2 = 1$ if it does not find as many as "$N$" clients after the first type 2 vacations, $U2 = 0$, in case the server notices at least "$N$" clients after the first type 2 vacations and where $U2$ is a random variable.

$$E(I_2) = E\left(\frac{I_1}{U_1} = 0\right)P(U_2 = 0) + E\left(\frac{I_1}{U_1} = 1\right)P(U_2 = 1)$$
$$= E(G)P(U_2 = 0) + [E(G) + E(I_2)]P(U_2 = 1),$$
$$E(I_2) = \frac{E(G)}{P(U_2 = 0)},$$
$$\therefore E(I) = \frac{E(C)}{P(U_1 = 0)} + \frac{E(G)}{P(U_2 = 0)} + E(A),$$
$$\therefore E(I) = \frac{E(C)}{1 - \sum_{h=0}^{a-1}\sum_{k=0}^{h}\beta_k P_{h-k}} + \frac{E(G)}{1 - \sum_{h=a}^{N-1}\sum_{k=a}^{h} C_k \delta_{h-k}} + E(A). \tag{55}$$

*8.3. Anticipated Span of Busy Time (EBP).* Consider $B$ as a variable of the busy session. We characterise $D$ after the first service the attendant finds not exactly "$a$" clients then $J = 0$. $J = 1$, if in any event "$a$" customers are found by the server following the initial service.

Now,

$$E(B) = P(J = 1)E\left(\frac{D}{J = 1}\right) + P(J = 0)E\left(\frac{D}{J} = 0\right)$$
$$= [E(S) + E(D)]P(J = 1) + E(S)P(J = 0) \tag{56}$$
$$\therefore E(D) = \frac{E(S)}{P(J = 0)} = \frac{E(S)}{\sum_{J}^{a-1} P_i}.$$

*8.4. Expected Halting Period in the Line (EWT).* The average halting period of clients is acquired as $E(w) = E(Q)/\lambda E$.

## 9. Cost Model

It is important to detect the complete expense involved in the production for the management. This segment deals with the cost pattern. To acquire the total average cost, the following are assumed. Cs1, Cs2 signify the start-up cost and setup cost; Ch, C0 stand for retaining cost per tile and working expense per time. Compensation charges per unit period on account of the secondary job of type 1 and type 2 are addressed as Cr1, Cr2.

Total average cost (TAC) = Foundation cost/cycle + Setup cost/cycle + Retaining amount cost $*$ $\rho$) − benefit owing to type 1 vacation for each cycle benefit owing to type 2 vacation for each cycle

$$\begin{aligned}\text{TAC} &= C_{s1}.\frac{1}{E(Tc)} + C_{s2}.E(A)\frac{1}{E(Tc)} + C_h E(Q) \\ &+ C_0\rho - \frac{1}{E(Tc)}\left(C_{r1}.\frac{E(C)}{P(U_1 = 0)}\right) \\ &+ -\frac{1}{E(Tc)}\left(C_{r2}.\frac{E(G)}{P(U_2 = 0)}\right),\end{aligned} \tag{57}$$

where $E(Lc)$ symbolize the length of the cycle.

## 10. Numerical Example

The tile manufacturing management can utilize the result illustrated as follows. An example is illustrated in this segment for various lengths of secondary jobs, and cost analysis is examined.

Let entry of tiles follows a geometric distribution with mean 2.

Printing designs on tiles are K-Erlang where $k = 2$ (service time distribution).

Minimum batch service volume is $a$ with $a = 2$.

Maximum batch service volume is $b$ with $b = 4$.

Top coating and drying follow exponential with parameter (first type of secondary job that is type 1 vacation) is $\eta_1 = 8$.

Final grinding and side coating are exponential with parameter is $\eta_2 = 10$ (second type of secondary job that is type 2 vacation).

Threshold value is $N = 6$.

Operating expense for each tile is approximately 5.25 rupees.

Retaining charge for each customer was 1.00 rupees.

Establishment cost is Rs. 3.00.

Amount for preworks such as grinding, washing, shifting, dimension verification, and selecting designs is set as Rs. 0.50.

Amount per unit time for cleaning and applying top coat (type one vacation) is assumed as 2.00 rupees.

Amount per unit time for final grinding and side coating is assumed as Rs. 4.00.

Table 2: Evaluation of performance measures for $a = 2$, $b = 4$, $N = 6$, $\eta 1 = 8$, $\eta 2 = 10$, $r = 7$.

| $\varepsilon$ | $\mu$ | P | $P$ (0) | $P$ (1) | $P$ (2) | $P$ (3) | EQL | EBP | EIP | EWT | TAC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 2 | 0.25 | 0.0697 | 0.0955 | 0.3395 | 0.3463 | 0.057 | 6.0519 | 0.4257 | 0.057 | 1.7148 |
| 0.6 | 2 | 0.3 | 0.0729 | 0.104 | 0.3344 | 0.3499 | 0.1588 | 5.6545 | 0.4281 | 0.1323 | 2.1005 |
| 0.7 | 2 | 0.35 | 0.0738 | 0.1092 | 0.3256 | 0.3494 | 0.2862 | 5.4665 | 0.4291 | 0.2044 | 2.5016 |
| 0.8 | 2 | 0.4 | 0.0728 | 0.1114 | 0.3133 | 0.3448 | 0.4443 | 5.4293 | 0.4291 | 0.2777 | 2.9246 |
| 1 | 2 | 0.5 | 0.0665 | 0.1082 | 0.28 | 0.3233 | 0.8834 | 5.7242 | 0.4263 | 0.4417 | 3.8718 |
| 0.6 | 3 | 0.2 | 0.0921 | 0.1215 | 0.4848 | 0.4903 | 0.0216 | 3.1212 | 0.4374 | 0.018 | 1.6916 |
| 0.7 | 3 | 0.2333 | 0.0979 | 0.133 | 0.4815 | 0.4963 | 0.0781 | 2.8869 | 0.4414 | 0.0558 | 1.9627 |
| 0.8 | 3 | 0.2667 | 0.1018 | 0.1423 | 0.4759 | 0.4998 | 0.1434 | 2.7316 | 0.4444 | 0.0896 | 2.2326 |
| 1 | 3 | 0.3333 | 0.1049 | 0.1543 | 0.4583 | 0.4993 | 0.3042 | 2.5725 | 0.4474 | 0.1521 | 2.7761 |
| 0.7 | 4 | 0.175 | 0.1243 | 0.079 | 0.3681 | 0.9786 | 0.0799 | 2.366 | 0.439 | 0.0571 | 1.7623 |
| 0.8 | 4 | 0.2 | 0.1185 | 0.1571 | 0.6155 | 0.6323 | 0.0659 | 1.8141 | 0.4534 | 0.0412 | 2.0708 |
| 1 | 4 | 0.25 | 0.128 | 0.1773 | 0.6034 | 0.6389 | 0.1627 | 1.6373 | 0.4608 | 0.0813 | 2.498 |

Table 3: Comparison of performance measures when $T1 < T2$ and $T1 = T2$, $a = 2$, $b = 4$, $N = 6$, $r = 7$.

| $\varepsilon$ | $\mu$ | $\rho$ | EQL | | EBP | | EIP | | EWT | | TAC | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $T1 < T2$ | $T1 = T2$ | $T1 < T2$ | $T1 = T2$ | $T1 < T2$ | $T1 = T2$ | $T1 < T2$ | $T1 = T2$ | $T1 < T2$ | $T1 = T2$ |
| 0.5 | 2 | 0.25 | 0.057 | 0.0633 | 6.0519 | 6.0657 | 0.4257 | 0.4256 | 0.057 | 0.0633 | 1.7148 | 1.7204 |
| 0.6 | 2 | 0.3 | 0.1588 | 0.1659 | 5.6545 | 5.6694 | 0.4281 | 0.4279 | 0.1323 | 0.1383 | 2.1005 | 2.1068 |
| 0.7 | 2 | 0.35 | 0.2862 | 0.294 | 5.4665 | 5.4825 | 0.4291 | 0.4289 | 0.2044 | 0.21 | 2.5016 | 2.5085 |
| 0.8 | 2 | 0.4 | 0.4443 | 0.4526 | 5.4293 | 5.4467 | 0.4291 | 0.4289 | 0.2777 | 0.2829 | 2.9246 | 2.9319 |
| 1 | 2 | 0.5 | 0.8834 | 0.8926 | 5.7242 | 5.7452 | 0.4263 | 0.4261 | 0.4417 | 0.4463 | 3.8718 | 3.8798 |
| 0.6 | 3 | 0.2 | 0.0216 | 0.0295 | 3.1212 | 3.1301 | 0.4374 | 0.4372 | 0.018 | 0.0246 | 1.6916 | 1.6982 |

Table 4: Comparison of performance measures when $T1 < T2$ and $T1 > T2$, $a = 2$, $b = 4$, $N = 6$, $r = 7$.

| $\varepsilon$ | $\mu$ | $\rho$ | EQL | | EBP | | EIP | | EWT | | TAC | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $T1 < T2$ | $T1 > T2$ | $T1 < T2$ | $T1 > T2$ | $T1 < T2$ | $T1 > T2$ | $T1 < T2$ | $T1 > T2$ | $T1 < T2$ | $T1 > T2$ |
| 0.7 | 3 | 0.233 | 0.0781 | 0.0718 | 2.8869 | 2.8593 | 0.4414 | 0.3823 | 0.0558 | 0.0513 | 1.9627 | 2.0263 |
| 0.8 | 3 | 0.2667 | 0.1434 | 0.1366 | 2.7316 | 2.7033 | 0.4444 | 0.3848 | 0.0896 | 0.0854 | 2.2326 | 2.3023 |
| 1 | 3 | 0.3333 | 0.3042 | 0.2967 | 2.5725 | 2.5425 | 0.4474 | 0.3874 | 0.1521 | 0.1484 | 2.7761 | 2.8511 |
| 0.7 | 4 | 0.175 | 0.0799 | 0.0178 | 2.366 | 1.9359 | 0.439 | 0.388 | 0.0571 | 0.0127 | 1.7623 | 1.9501 |
| 0.8 | 4 | 0.2 | 0.0659 | 0.0589 | 1.8141 | 1.7951 | 0.4534 | 0.3922 | 0.0412 | 0.0368 | 2.0708 | 2.181 |
| 1 | 4 | 0.25 | 0.1627 | 0.1545 | 1.6373 | 1.6178 | 0.4608 | 0.3985 | 0.0813 | 0.0772 | 2.498 | 2.6222 |

## 11. Cost Analysis

In Table 2, for diverse arrival rates and service rates, steady-state conditions are verified, and the state probabilities are calculated. Performance measures are found. The total average cost is received numerically w.r.t variant secondary jobs.

The following conclusions can be drawn from Table 2.

(i) As the service rate improves, the called queue distance, peak time, and holding time decrease, while the predicted spare period increases. As a result, while the service rate is at its highest, the server spends more time idle.

(ii) It is also witnessed that when the service rate is maximum, the overall average cost is minimum. With an increment in the arrival rate, the sum of the mean price magnifies proportionally.

(iii) Anticipated line distance, awaited idle period, and expected halting period rise as there is an increment in the arrival rate. The expected busy period declines as there is a growth in the arrival rate.

From Table 3, the followings are ascertained:

(i) If the settings for both secondary jobs (type 1 and type 2) are the same, the subsequent busy time, line length, as well as halting time rise, but the expected idle period decreases.

(ii) When analysing costs, if the parameters for both secondary jobs are equal, the production costs will rise.

The following conclusions can be drawn from Table 4:

(i) When the parameter of type 1 secondary jobs is bigger than type 2 secondary jobs, performance measurements such as anticipated busy time, queue length, idle session, as well as halt period decrease.

(ii) When the cost of type 1 vacation exceeds the cost of type 2 vacation, the production cost rises.

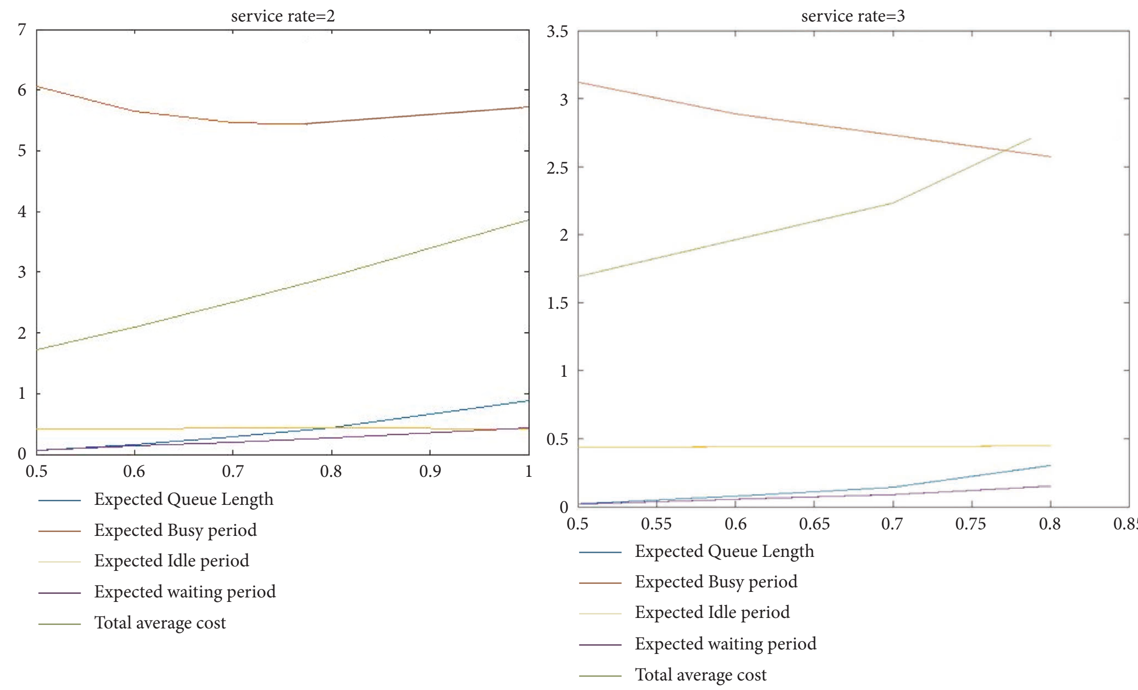


Figure 3: $L_q$, $L_w$, $L_b$, $L_i$, TAC for various service rates.

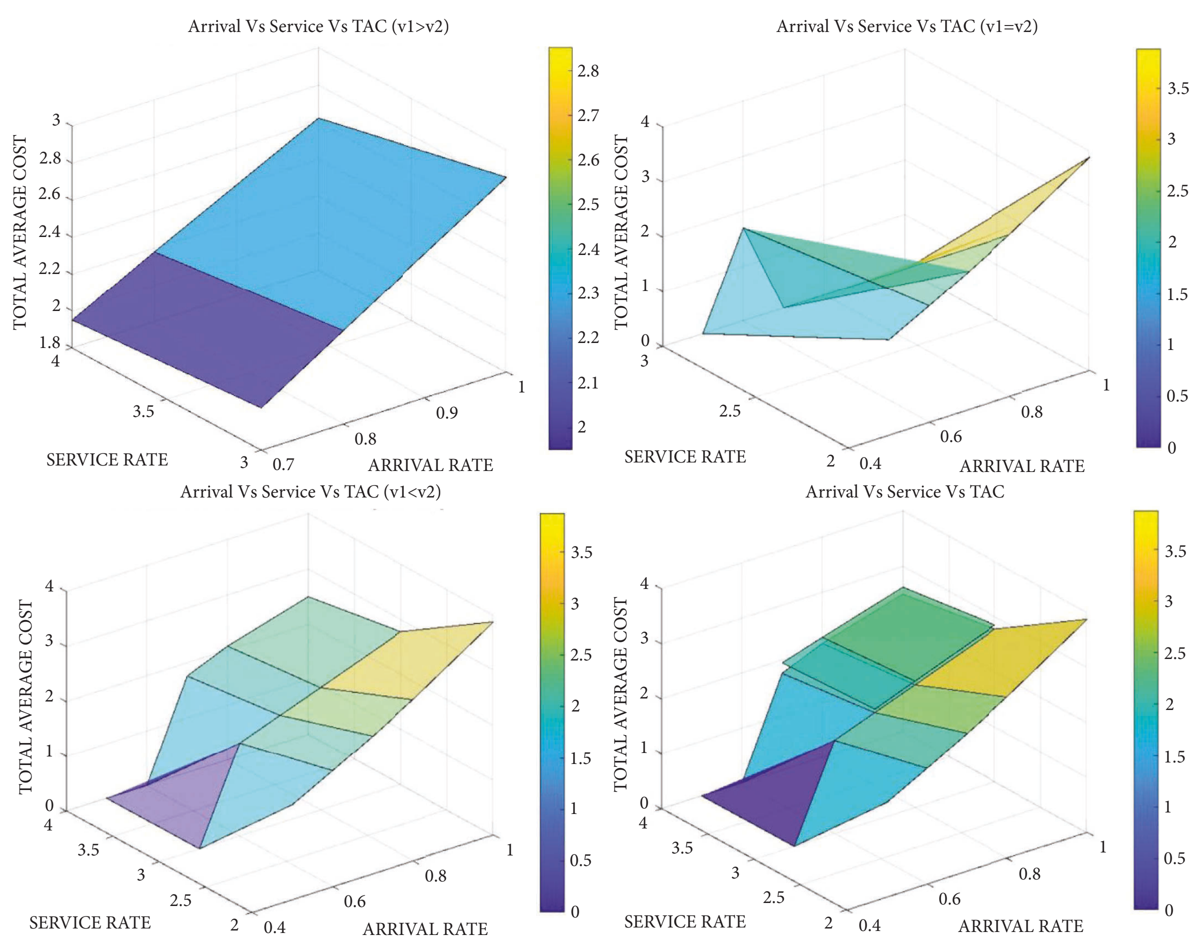


Figure 4: Arrival vs service vs TAC.

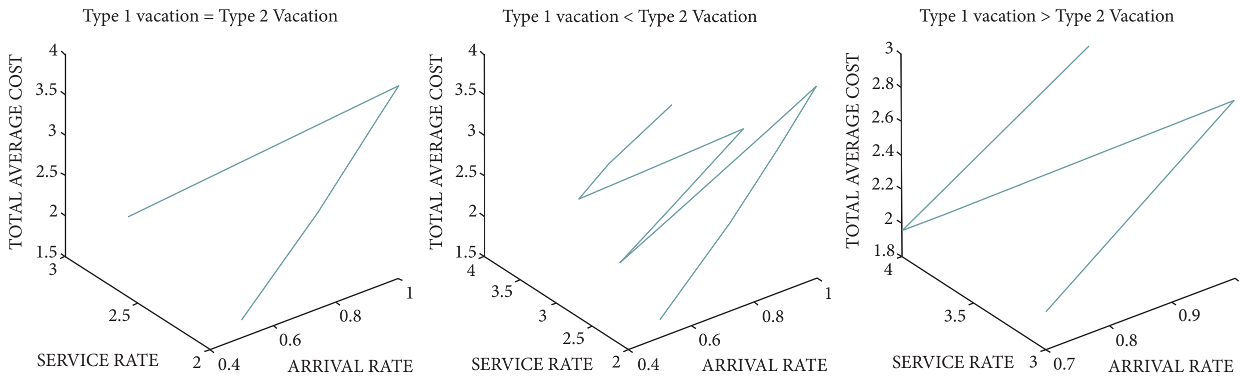


Figure 5: Arrival vs service vs TAC.

Figure 3 demonstrates the expected line length $L_q$, awaited busy time $L_b$, anticipated halt period $L_w$, predicted idle session $L_i$, and the total average cost (TAC), for service rates 2 and 3.

Figure 4 serves as a relationship between arrival rate, service rate, and total average cost. First part shows when the parameter of type1 vacation is greater than type 2 vacation. The second part represents both type1 and type 2 vacations are given the same parameter. The third part depicts when the parameter of type 2 vacation is big compared to type 1 vacation. The last part gives all the three parts in one graph.

Figure 5 depicts arrival rate vs service rate vs TAC. The first part explains when type 1 and type 2 vacations are equal, whereas the second part describes that the total average cost fluctuates when type 2 vacation is greater. Finally, the third part shows that when type 2 vacation is least compared to type 1, there are smooth fluctuations.

As a result, the mathematical model and its numerical findings confirm that the server is being used to its full potential while also lowering the overall production cost. It is also suggested that the parameters of type 1 vacations should always be smaller than those of type 2.

## 12. Conclusion

The proposed mathematical model is compared to other mathematical models, and it is concluded that the current model is effective. The model appears to be successful based on the numerical findings obtained. The model can be used by management to optimize production. The model could be expanded in the future to include a repair or reservice approach. The same model can be used for fog computing technology by assigning data transferring as service, and other works such as storing, allocating space, and verifying can be categorized as secondary jobs [20].

## Data Availability

The data used to support the findings of this study are included within the article.

## Disclosure

This study was performed.

## Conflicts of Interest

The authors declare no conflicts of interest.

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