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Research Article

A Rule-Based Decision Support Method Combining Variable Precision Rough Set and Stochastic Multi-Objective Acceptability Analysis for Multi-Attribute Decision-Making

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In an open environment, the demands of users are diverse and dynamic because users can participate in product design from beginning to end. Owing to this, the disorderly and unplanned participation of users will greatly increase the complexity of multi-attribute decision-making (MADM) in the product design process. In order to ensure the smooth development of the open design process, the decision support model and method need to repeatedly provide decision makers (DMs) with necessary decision support information in a relatively short period of time, which can realize the evaluation of the scheme and improve the utilization efficiency of community resources. However, the process of eliciting preference information is complex, exhausting, inefficient, and time-consuming in existing methods, which will result in a poor decision-making. With the purpose of optimizing the eliciting process in MADM, a rule-based decision support method is proposed in this paper, where the process of eliciting preference information and decision-making are synchronized and guided by pre-extracted decision rules. The rules are deduced from comparison relations on attributes and their outcomes through the combination of variable precision rough set approach (VPRS) and stochastic multi-objective acceptability analysis (SMAA). With the concept of attribute reduction and approximation accuracy in rough set theory, the extracted rules could eliminate redundant attributes and assign the relative priority of preference information. Based on the extracted rules, the multi-attribute decision-making process could be carried out step by step in an orderly manner. In each step, DMs only need to provide partial preference information by non-quantitative statements according to extracted rules. Once the decision result is reliable enough, the eliciting and decision-making process can be terminated promptly. In order to validate the proposed approach, experiments of decision rule extraction are implemented, and the results show that the proposed approach is effective both in the weak rule extraction and the strong rule extraction.

1. Introduction

Multi-attribute decision-making (MADM), also referred to as multi-objective decision-making, is an important component of modern decision science [1]. It refers to the problem of evaluating and prioritizing a finite set of alternatives based on a collection of attributes. The theory and methods of MADM have been extensively applied to the fields of society, economy, management, military, engineering design, and so on. Generally, the process of solving MADM problems can be divided into two steps. One step is to collect decision information, thus determining attribute values and weights. The other step is to synthesize the collected decision information through a proper aggregation operator (e.g., the simple averaging operator and the ordered weighted averaging operator). The ranking of alternatives can be obtained according to their synthesized values.

The complexity of MADM is mainly due to the difficulty of assessing the relative importance of different attributes. In MADM problems, attributes are assigned different weight values to indicate their relative importance. Depending on the information provided, the methods of determining attribute weights are divided into subjective methods and objective methods [2]. In subjective methods, attribute weights are derived from value judgments about attributes given by decision makers (DMs), including eigenvector-based method [3–6], weighted averaging or least square method [7–10], AHP or ANP-based method [11, 12], and D-TOPSIS method [13]. And the objective methods determine attribute weights according to objective information (e.g., the information in the decision matrix), such as principal component analysis method [14], entropy method [15, 16], and multi-objective programming model-based method [17, 18]. Either way, the process of determining attribute weights is not easy. Especially in subjective methods, attribute weights reflect the group preference of DMs. Therefore, the first thing to address is to collect and aggregate DMs' preference information through proper means. In the study of most subjective methods, the process of eliciting preference information highly depends on frequent and deep involvement of DMs. For example, the Delphi method uses multiple rounds of surveys to collect and aggregate preference information through a series of data collection and analysis techniques interspersed with feedback [19]. In each round, DMs (experts) need to do a series of questionnaires focusing on problems, opportunities, solutions, or forecasts until an agreement is reached. In AHP and ANP-based methods, a complete pairwise comparison matrix is the prerequisite. Many research studies point out that it is hard to ensure the consistency of the pairwise comparison matrix, especially when the number of attributes is large [20, 21]. In addition, for various reasons, the comparison relation between attributes could not be given by DMs precisely and completely. Predictably, the process of eliciting preference information based on these methods would be very time-consuming and exhausting when the number of attributes and DMs is large. In order to the uncertainty and fuzziness of MADM, Ning et al. [22] proposed a novel probabilistic dual hesitant fuzzy-enabled MADM technique. Wang et al. [23] proposed a multi-objective particle swarm optimization-enabled approach to optimize park-level integrated energy systems with multi-attribute decision analysis frameworks. Li et al. [24] proposed a data-driven approach to solve the problem of personalized individual semantics under the background of MADM, and the experiment showed that the proposed approach can obtain personalized numerical scales of linguistic terms for a decision maker. In order to solve the problem of the AC transmission systems appropriate placement, Chinda et al. [25] proposed a novel MADM approach combined with a particle mobility honey bee algorithm, which can maximize efficiency of the energy planning. Rahman et al. [26] proposed a MADM approach combined with AND OR operations, which can effectively deal with the uncertainties of possibility neutrosophic hypersoft set. Zhang et al. [27] proposed a bisimulation-based generalized fuzzy variable precision rough set model to solve the problem of the decision-making, which can effectively tackle complicated problems including the attribute and the relational data. Ye et al. [28] proposed a fuzzy rough sets-enabled decision-making approach, which can effectively tackle the uncertainty and imprecision problem in MADM. Jiang et al. [29] proposed a rough sets-enabled risk decision-making approach, which not only consider the decision risk but also offer instructions to select the appropriate semantic explanation. Sarwar [30] proposed a rough D-TOPSIS method to manipulate the subjectiveness and vagueness of MADM, which unites the competency to analyze changeable and ambiguous information without additional assumptions. Fei et al. [31] proposed a rough D-TOPSIS method to solve the problem of the human resources selection, and an effectual method was applied by authors to describe the changeable data and information called D-numbers. Lin et al. [32] proposed a rough ELECTRE-II method to deal with the selection problem of the edge nodes, and an entropy measure is devised to measure the uncertainty degree. Akram et al. [33] proposed a rough ELECTRE-II method with the specific structure, through which the diverse opinions of decision experts can be well handled. Kuncova et al. [34] proposed a fuzzy rough PROMETHEE method which can evaluate the order of 14 regions of the Czech Republic in regard to economic indices. Liu et al. [35] proposed a fuzzy rough PROMETHEE method to accomplish the process for optimal alternative selection.

However, existing methods prefer to solve MADM problems through a "top-down" way, in which organizers collect preference information based on the essential characteristics of the problem. For example, in plant location selection (PLS) problem, alternatives are usually evaluated based on environmental index, economic index, social index, and geological index [36, 37]. However, in some cases, alternatives may be similar in some attributes. The relative importance of such attributes, which are called redundant attributes, have no effect on the final ranking result. The existing method cannot eliminate redundant attributes from the perspective of the attribute value. Therefore, it is necessary to clarify which attribute should be involved and which weight should be given priority.

In order to fill in the above research gaps, a rule-based decision support method for MADM (R-MADM) is proposed in this paper to help optimize the process of eliciting preference information in MADM. The contribution and motivation of this paper is summarized as follows:

(1) In R-MADM, the process of eliciting preference information and the process of decision-making are synchronized and guided by pre-extracted decision rules. In contrast to existing methods, R-MADM is a "bottom-up" method. The MADM problem could be solved through two stages based on the proposed method, namely, decision rule extraction and decision rule application.

(2) In the first stage, outcomes (ranking results) corresponding to different comparison relations are obtained using stochastic multi-objective acceptability analysis (SMAA). Decision rules are then extracted based on SMAA analysis results through the variable precision rough set (VPRS) approach.

(3) In the second stage, according to the decision rules, the preference eliciting process and the decision-making process could be carried out step by step. DMs only need to provide partial preference

information according to extracted rules in each step. Based on the SMAA analysis result, the measurement of the reliability of decision-making is given. Once the decision result is reliable enough, the process of eliciting preference information and the process of decision-making can be terminated. Therefore, the efficiency of MADM can be improved.

The remainder of this paper is organized as follows. Relevant theoretical foundations about VPRS and SMAA are introduced in Section 2. In Section 3, the proposed R-MADM method is illustrated in detail. A numerical example is used to illustrate the implementation of the proposed method in Section 4. Section 5 provides a conclusion and discusses the limitations of the proposed method as well as research directions for in-depth studies in the future.

2. Theoretical Foundations

2.1. Rough Set Theory. Rough set theory (RST) proposed by Pawlak provides a useful tool for extracting knowledge (decision rules) from inexact, uncertain, or vague information. RST has been widely used in the area of machine learning [38], knowledge management [39], expert system [40], decision support system [41], pattern recognition [42], yield prediction [43], and so on. The basic idea of RST is using available information to perform complete approximation (classification) of the given objects without any preliminary assumptions. In RST, the approximation of an object is related to its attribute information, and an information system is often expressed as a quadruple as

$$S = (U, A, V, f), \tag{1}$$

where U is referred to as the universe consisting of a finite set of objects, A is a non-empty and finite set of attributes, V is the union of attribute domains, i.e., $V = \cup V_a$ for any $a \in A$, and f is an information function which associates every object in U with a unique attribute value.

Generally, the information system (IS) in RST is represented in the form of a decision table (knowledge table), in which rows and columns correspond to objects and their attributes, respectively. The attribute set A is divided into the condition attribute set C and the decision attribute set D, i.e., $A = C \cup D$. The value of the decision attribute in D is related to the value of the condition attribute in C. The objects in U can be partitioned into a group of disjoint subsets based on the indiscernibility relation R_P, and it is defined as

$$R_P = \{(x, y) \in U \times U\colon\ f(x, a) = f(y, a) \forall a \in P, P \subseteq A\}. \tag{2}$$

Each indiscernibility relation partitions the universe U into a family of disjoint subsets U/R_P called equivalent classes, which is expressed as

$$U/R_P = \{[x]_P\colon\ x \in U\}. \tag{3}$$

Given $X \subseteq U$, X can be approximated by its lower approximation $\underline{R_P}(X)$ and upper approximation $\overline{R_P}(X)$. The lower approximation $\underline{R_P}(X)$ is the union of equivalent classes that belong to the subset X with certainty, i.e.,

$$\underline{R_P}(X) = \{x \in U\colon\ [x]_P \subseteq X\} = \cup \{[x]_P\colon\ [x]_P \subseteq X\}. \tag{4}$$

And the upper approximation $\overline{R_P}(X)$ is the union of equivalent classes that partly belong to the subset X, i.e.,

$$\overline{R_P}(X) = \{x \in U\colon\ [x]_P \cap X \neq \varphi\} = \cup \{[x]_P\colon\ [x]_P \cap X \neq \varphi\}. \tag{5}$$

The pair $(\underline{R_P}(X), \overline{R_P}(X))$ is called the rough set of X with respect to P. Based on the lower and upper approximation, the universe U could be partitioned into three regions.

(1) $\mathbf{POS}_P(X)$ is referred to as the positive region of X with respect to P, and it is the lower approximation of X, i.e., $\mathbf{POS}_P(X) = \underline{R_P}(X)$

(2) $\mathbf{BND}_P(X)$ is referred to as the boundary region of X with respect to P, and it is the subtraction of the upper approximation and the lower approximation of X, i.e., $\mathbf{BND}_P(X) = \overline{R_P}(X) - \underline{R_P}(X)$

(3) $\mathbf{NEG_P(X)}$ is referred to as the negative region of $\mathbf{X}$ with respect to P, and it is the subtraction of the universe and the upper approximation of $\mathbf{X}$, i.e., $\mathbf{NEG_P(X) = U - \overline{R_P}(X)}$

2.1.1. Variable Precision Rough Set Approach. The classical rough set approach only can be used to model fully correct classification problem. This is primarily due to the fact that the classical rough set approach is based on a deterministic approach which deliberately ignores the available probabilistic information in its formalism. The classification with a controlled misclassification error, which is referred to as partial classification, is outside the realm of this approach. This limitation severely reduces the applicability of RST to problems which are more probabilistic than deterministic in nature. To solve this problem, the variable-precision rough set (VPRS) approach is proposed by taking partial classification into account. In VPRS, the standard set inclusion relation used in the classical rough set approach is replaced by the majority inclusion relation. In the majority inclusion relation, the classification error of classifying objects of a set X into another set Y is measured by

$$\begin{aligned} c(X, Y) &= 1 - \frac{car\ d(X \cap Y)}{car\ d(X)}\ if\ car\ d(X) > 0, \\ c(X, Y) &= 0\ if\ car\ d(X) = 0, \end{aligned} \tag{6}$$

where $car\ d(X)$ denotes set cardinality. For a given admissible classification error $(0 \leq \beta \leq 0.5)$, the majority inclusion relation in VPRS can be expressed as

$$Y \supseteq^{\beta} X\ if\ an\ d\ only\ if\ c(X, Y) \leq \beta. \tag{7}$$

It can be observed that the standard set inclusion relation is a special case of the majority inclusion relation when $\beta = 0$. Based on the majority inclusion relation, Ziarko gives the definitions of β-lower and β-upper approximations of a set X with respect to $P \subseteq A$ in VPRS, which are defined as

$$\underline{R_P}^{\beta}(X) = \cup\{[x]_P: c([x]_P, X) \le \beta\},$$
$$\overline{R_P}^{\beta} = \cup\{[x]_P: c([x]_P, X) < 1-\beta\}. \tag{8}$$

In VPRS, the β-lower approximation of X constitutes its β-positive region $\mathbf{POS}_P^{\beta}(X)$, i.e.,

$$\mathbf{POS}_P^{\beta}(X) = \underline{R_P}^{\beta}(X). \tag{9}$$

And the β-boundary region is given by

$$\mathbf{BNR}_P^{\beta}(X) = \cup\{[x]_P: 1-\beta > c([x]_P, X) > \beta\}. \tag{10}$$

And the β -negative region of X is the complement of its β-upper approximation, i.e.,

$$\mathbf{NEG}_P^{\beta}(X) = \cup\{[x]_P: c([x]_P, X) \ge 1-\beta\}. \tag{11}$$

2.1.2. Dominance-Based Variable Precision Rough Set Approach. In both the classical rough set approach and VPRS approach, attribute information is assumed to be nominal, which cannot be arranged in any particular order. However, in some real-life applications, the information has obvious ordinal characteristic where better condition attribute values usually bring out better decision attribute values. The dominance-based variable precision rough set (DB-VPRS) approach improves the VPRS approach by replacing the indiscernibility relation with dominance relation, thus taking the preference-orders into consideration [44]. The dominance relation with respect to attribute p is represented as D_p, which is also denoted by $\succ_p$, and $x \succ_p y$ means that x dominates (is better than) y in terms of attribute p. For the benefit index, $x \succ_p y$ means that the attribute value of x is greater than the attribute value of y with respect to attribute p, i.e.,

$$x \succ_p y \Rightarrow f(x,p) \succ_p f(y,p) \Rightarrow f(x,p) > f(y,p). \tag{12}$$

And for the cost index, $x \succ_p y$ means that the attribute value of x is smaller than the attribute value of y with respect to attribute p, i.e.,

$$x \succ_p y \Rightarrow f(x,p) \succ_p f(y,p) \Rightarrow f(x,p) < f(y,p). \tag{13}$$

Based on the dominance relation, the dominance set of x is defined by

$$\begin{aligned} D_P^+(x) &= \{y \in U \mid y D_P x\} \\ &= \{y \in U \mid f(y,d) \succ_p f(x,d), \forall p \in P, P \subseteq A\}. \end{aligned} \tag{14}$$

And the dominated set of x is defined by

$$D_P^-(x) = \{y \in U \mid x D_P y\} = \left\{\begin{array}{c} y \in U \mid f(x,d) \succ_p f(y,d), \\ \forall p \in P, P \subseteq A \end{array}\right\} \tag{15}$$

Let $Cl_t, t \in \{1,2,\cdots,n\}$ be decision classes. Each decision class Cl_t is defined as $Cl_t = \{x \in U: f(x,d) = v_{d_t}, d \in D, D \subseteq A\}$, where v_{d_t} is the decision attribute value. It is assumed that the total order of decision attribute values is $v_{d_1} \prec_d v_{d_2} \prec_d \cdots \prec_d v_{d_n}$; and then, the order relation of decision classes can be written as $Cl_1 \prec Cl_2 \prec \cdots \prec Cl_n$. The upward and downward union of decision classes are defined as

$$\begin{aligned} &Cl_T^{\succcurlyeq} = \cup_{t \ge T} Cl_t \cdot Cl_T^{\preccurlyeq} = \cup_{t \le T} Cl_t \\ &t, T \in \{1,2,\cdots,n\}, \end{aligned} \tag{16}$$

where if $x \in Cl_T^{\succcurlyeq}$ then $f(x,d) \succ_d v_{d_T}$; and if $x \in Cl_T^{\preccurlyeq}$ then $f(x,d) \prec_d v_{d_T}$.

For a given confidence level $l \in (0.5, 1]$ in DB-VPRS, the lower and upper approximation of $Cl_T^{\succcurlyeq}$ and $Cl_T^{\preccurlyeq}$ are defined as

$$\underline{R}_P^l(Cl_t^{\succcurlyeq}) = \cup\left\{x \in Cl_t^{\succcurlyeq}: \frac{|D_P^+(x) \cap Cl_t^{\succcurlyeq}|}{|D_P^+(x)|} \ge l\right\}, \tag{17}$$

$$\begin{aligned} \overline{R_P}^l(Cl_t^{\succcurlyeq}) &= U - R_P^l(U - Cl_t^{\succcurlyeq}) = U - R_P^l(Cl_{t-1}^{\preccurlyeq}) \\ &= Cl_t^{\succcurlyeq} \cup \left\{ \cup \left\{ \begin{array}{c} x \in Cl_t^{\preccurlyeq}: \\ \dfrac{|D_P^-(x) \cap Cl_t^{\succcurlyeq}|}{|D_P^-(x)|} > 1 - l \end{array} \right\} \right\}, \end{aligned} \tag{18}$$

$$\underline{R}_P^l(Cl_t^{\preccurlyeq}) = \cup\left\{x \in Cl_t^{\preccurlyeq}: \frac{|D_P^-(x) \cap Cl_t^{\preccurlyeq}|}{|D_P^-(x)|} \ge l\right\}, \tag{19}$$

$$\begin{aligned} \overline{R_P}^l(Cl_t^{\preccurlyeq}) &= U - R_P^j(U - Cl_t^{\preccurlyeq}) = U - R_P^l(Cl_{t+1}^{\succcurlyeq}) \\ &= Cl_t^{\preccurlyeq} \cup \left\{ \cup \left\{ \begin{array}{c} x \in Cl_t^{\succcurlyeq}: \\ \dfrac{|D_P^+(x) \cap Cl_t^{\preccurlyeq}|}{|D_P^+(x)|} > 1 - l \end{array} \right\} \right\}. \end{aligned} \tag{20}$$

In DB-VPRS, the positive region of $Cl_T^{\succcurlyeq}$ with respect to $P \subseteq A$ is defined b.

Similarly, the positive region of $Cl_T^{\preccurlyeq}$ with respect to $P \subseteq A$ is defined as

$$\mathbf{POS}_P^l(Cl_t^{\succcurlyeq}) = \left\{ \begin{array}{c} x \in U: \\ \dfrac{|D_P^-(x) \cap Cl_t^{\succcurlyeq}|}{|D_P^-(x) \cap Cl_t^{\succcurlyeq}| + |D_P^+(x) \cap Cl_{t-1}^{\preccurlyeq}|} \ge l \end{array} \right\}. \tag{21}$$

Based on its duality characteristic, the negative regions of $Cl_T^{\succcurlyeq}$ and $Cl_T^{\preccurlyeq}$ are defined by

$$\begin{aligned} \mathbf{NEG}_P^l(Cl_t^{\succcurlyeq}) &= U - \mathbf{POS}_P^l(U - Cl_t^{\succcurlyeq}) \\ &= U - \mathbf{POS}_P^l(Cl_{t-1}^{\succcurlyeq}) \\ &= \left\{ \begin{array}{c} x \in U: \\ \dfrac{|D_P^-(x) \cap Cl_t^{\succcurlyeq}|}{|D_P^-(x) \cap Cl_t^{\succcurlyeq}| + |D_P^+(x) \cap Cl_{t-1}^{\preccurlyeq}|} \ge 1 - l \end{array} \right\}. \end{aligned} \tag{22}$$

The boundary regions of $Cl_T^{\geqslant}$ and $Cl_T^{\leqslant}$ can be obtained by

$$\mathbf{BNR}_P^l\left(Cl_t^{\geqslant}\right) = U - \mathbf{NEG}_P^l\left(Cl_t^{\geqslant}\right) - \mathbf{POS}_P^l\left(Cl_t^{\geqslant}\right). \quad (23)$$

2.1.3. Attribute Reduction and Decision Rule Extraction in VPRS. Attribute reduction for the analysis of important attributes and decision rule extraction for the approximation of the decision attribute by means of the condition attribute are major topics in RST.

(1) Attribute dependency and approximation quality.

For a given information system $S = (U, A, V, f)$, let $x, y \in U$, $P \subseteq C$, $B \in D$, $C \cup D = A$, and the equivalent classes $[x]_P = \{X_1, X_2, \cdots, X_m\}$ and $[y]_B = \{Y_1, Y_2, \cdots, Y_n\}$ are assumed to be known. Then, for a given confidence level μ, the dependency of B on P is measured by

$$0 \leq \gamma_P^\mu(B) = \sum_{i=1}^{n} \mathrm{card}\left(\underline{R_P}^{\mu}(Y_i)\right)/\mathrm{card}(U) \leq 1. \quad (24)$$

If $\gamma_P^\mu(B) = 1$, the partition $[y]_B$ is completely dependent on P; If $0 < \gamma_P^\mu(B) < 1$, $[y]_B$ is partly dependent on P; If $\gamma_P^\mu(B) = 0$, $[y]_B$ is completely independent on P.

(2) Relative importance of attribute.

According to the definition given by (28), the relative importance of an attribute $p \in P$ can be measured by

$$\mathrm{RI}(p, B) = \gamma_P^\mu(B) - \gamma_{P-\{p\}}^\mu(B). \quad (25)$$

The bigger the $\mathrm{RI}(p, B)$ is, the more important is the attribute. If $\mathrm{RI}(p, B) = 0$, the attribute p is regarded as a redundant attribute.

(3) μ-attribute reduct.

For a given confidence level μ, an μ-attribute reduct is the minimal subset $\mathbf{RED}^\mu(P, B)$ of condition attributes in P preserving the dependency level with the decision attribute B, $\mathbf{RED}^\mu(P, B)$ satisfies the following two criteria.

(a) $\gamma_P^\mu(B) = \gamma_{\mathbf{RED}^\mu(P,B)}^\mu(B)$;

(b) No attributes can be further eliminated from $\mathbf{RED}^\mu(P, B)$ without affecting the requirement (a).

There are usually more than one rough set reduct $\mathbf{RED}^\mu(P, B)$. The common part of available rough set reduct is called the core of rough set, i.e.,

$$\mathrm{Core}(P, B) = \cap\{\mathbf{RED}^\mu(P, B)\}. \quad (26)$$

(4) Decision rule extraction.

The approximation relation between condition attributes in P and the decision attribute B could be translated into a decision rule in the form of

$$\begin{aligned} &\wedge(p, v) \longrightarrow {}^{\mu}\vee(B, w), \\ &\textit{where } p \in P \subseteq C, B \in D, v \in V_p, w \in V_B. \end{aligned} \quad (27)$$

2.2. Stochastic Multi-Objective Acceptability Analysis. Stochastic multi-objective acceptability analysis (SMAA) is a multi-criteria decision support technique for decision-making problems with uncertainty [45, 46]. In SMAA, inaccurate or uncertain input data are represented in the form of probability distribution. It is usually assumed that all feasible combinations of criteria weights and criteria measurements are equally likely to be selected. In SMAA, alternatives are evaluated according to statistically descriptive measurements which are calculated by inverse analysis of uncertainty space.

It is supposed that a set of m alternatives with n attributes are under evaluation. The value of attribute j for alterative i is represented by g_{ij}. The weight of attribute j is represented by w_j. The function in (32) is usually used to calculate the utility value of an alternative, and it is expressed by a convex combination of the attribute weight vector $\mathbf{w} = (w_1, \cdots, w_n)$ and the attribute value vector $\mathbf{g}_i = (g_{i1}, \cdots, g_{im})$ in feasible weight space $\left(\mathbf{W} = \left\{\mathbf{w} \in \mathbb{R}^{\mathbf{n}}\colon w_j \geq 0 \wedge \sum w_j = 1\right\}\right)$. Here, attribute values are mapped into the range $[0, 1]$ by a partial utility function $u_*(b)$.

$$u_i(\mathbf{g}_i, \mathbf{w}) = \sum_{j=1}^{n} w_j u_*\left(g_{ij}\right), \mathbf{w} \in \mathbf{W}. \quad (28)$$

If attribute weights are available, a multi-attribute decision-making problem can be addressed by calculating utility values and choosing the alternative with the largest overall utility. However, for various reasons, it is often difficult to obtain precise measurements for some attributes. Thus, in SMAA, $\mathbf{g}_i$ is replaced with a stochastic variable γ_i. The joint probability density functions for attribute values and attribute weights are represented by $f_\gamma(\xi)$ and $f_{\mathbf{W}}(\mathbf{w})$, respectively. When attribute weights and values are uncertain, utility value distributions of alternatives can be obtained by computing all possible combinations of uncertain parameters, and it is usually realized through Monte Carlo simulation technique in SMAA.

The core idea of SMAA is to classify alternatives to those which should be taken into consideration and to those which should be eliminated. The standard SMAA method only considers the information about the alternative with the first rank lacking holistic evaluation on all alternatives. SMAA-2 extends the discussion on all ranking results and gives several descriptive measures: rank acceptability index and three best rank-type measures.

2.2.1. Rank Acceptability Index. In SMAA-2, the ranking result for alternative i could be calculated by

$$rank(i, \xi, \mathbf{w}) = 1 + \sum_{k \neq i} \rho\left(u_i(\xi_i, \mathbf{w}) \geq u_k(\xi_k, \mathbf{w})\right), \quad (29)$$

where $rank(i, \xi, \mathbf{w}) \in \{1, \cdots, m\}$, $\rho(true) = 1$, and $\rho(false) = 0$. For a given order r, its favorable ranking weights $\mathbf{W}_i^r$ is defined by

$$\mathbf{W}_i^r(\boldsymbol{\xi}) = \{\mathbf{w} \in \mathbf{W}\colon rank(i, \boldsymbol{\xi}, \mathbf{w}) = r\}. \quad (30)$$

Ranking acceptability index b_i^r describes the share of combinations of uncertain parameter which could make alternative i be ranked rth. It is computed as a multidimensional integral over criteria measurements and favorable weight space as

$$b_i^r = \int_{\xi \in \Upsilon} f_{\Upsilon}(\boldsymbol{\xi}) \int_{\mathbf{w} \in \mathbf{W}_i^r(\xi)} f_{\mathbf{W}}(\mathbf{w}) d\mathbf{w}\, d\boldsymbol{\xi}. \quad (31)$$

Rank acceptability index ranges from zero to one. Alternatives with zero-value ranking acceptability index are never ranked at the certain order, and alternatives with one-value ranking acceptability index could obtain the certain rank with any feasible weights combinations.

2.2.2. K-Best Rank Indices. K-best rank indices in SMAA-2 include k-best rank (k-br) acceptability $K\_b_i^k$, k-br central weight vector w_i^k, and k-br confidence factor p_i^k, which are defined as

$$\begin{aligned} K\_b_i^k &= \sum_{r=1}^{k} b_i^r, \\ \mathbf{w}_i^k &= \int_{\xi \in \Upsilon} f_{\Upsilon}(\boldsymbol{\xi}) \sum_{r=1}^{k} \int_{\mathbf{w} \in \mathbf{W}_i(\xi)} f_{\mathbf{W}}(\mathbf{w})\mathbf{w}\, d\mathbf{w}\, d\boldsymbol{\xi}, \\ p_i^k &= \int_{\xi \in \Upsilon:\ rank(i,\xi,\mathbf{w}) \le k} f_{\Upsilon}(\boldsymbol{\xi}) d\boldsymbol{\xi}. \end{aligned} \quad (32)$$

K-best rank indices extended the meaning of the descriptive measures in standard SMAA method by taking more alternatives into consideration. The k-br acceptability $K\_b_i^k$ is used as a performance indicator of the alternatives ranked in top k. It can help aggregate some alternatives with a good performance into a small group and to eliminate others. The k-br weight vector describes preference information of the k-best alternatives. And the k-br confidence factor gives the credibility for the alternatives to be judged into the k-best ones with k-br central weight vector.

3. A Rule-Based Decision Support Method for Multi-Attribute Decision-Making

In the majority of extant MADM methods, the problem solving process often contains two separate sub-processes: the process of eliciting preference information from DMs and the process of decision-making. Generally, the process of decision-making cannot start if the process of eliciting preference information is not complete. In addition, during the process of eliciting preference information, DMs usually cannot receive feedback of the consequence of their choices. This affects the efficiency and reliability of decision-making. In this paper, a rule-based decision support method is proposed to help optimize the problem solving process in MADM, which is referred to as R-MADM. The proposed method overcomes the defects mentioned above, in which the process of eliciting preference information and the process of decision-making are synchronized (see Figure 1). Once the decision result is reliable, the process of eliciting preference information and the decision-making process could be terminated promptly. In R-MADM, a MADM problem is solved through two stages, namely, decision rule extraction and decision rule application.

3.1. Decision Rule Extraction. As shown in Figure 1, the aim of the decision rule extraction stage is to provide rules for subsequent decision-making process. The decision rule extraction stage is implemented through two steps. The first step is to construct a preference information system (IS) which can reflect the relationship between the preference structure of DM and the corresponding alternative ranking result. The second step is to extract decision rules from the constructed IS using variable precision rough set (VPRS) approach.

3.1.1. Construct Preference Information. In this paper, the preference structure of decision maker (DM) is roughly represented by a combination of pairwise comparison relations on attributes called preference combination. If there are l kinds of pairwise comparison relations between two attributes, then for a given MADM problem consisting of m alternatives and n attributes, there will be a total of $l^{n(n-1)/2}$ possible combinations. Let PW_{ij} represent the pairwise comparison relation of attribute i and attribute j, then each preference combination can be represented by a tuple as $\cup\{PW_{ij}\}$ satisfying $\operatorname{card}(\cup\{PW_{ij}\}) = n(n-1)/2$. Here, three kinds of pairwise comparison relations are defined as follows:

(a) If $PW_{ij} = 1$, it means that attribute i is at least as important as attribute j, i.e., $w_i \ge w_j$, $i, j \in \{1, 2, \cdots, n\}, i \ne j$

(b) If $PW_{ij} = -1$, it means that attribute i is at most as important as attribute j, i.e, $w_i \le w_j$, $i, j \in \{1, 2, \cdots, n\}, i \ne j$

(c) If $PW_{ij} = 0$, it means that the pairwise comparison relation on attribute i and attribute j is not clear, which is expressed as $w_i \cdot w_j$.

In addition, the defined relations should satisfy the following properties.

$$\forall i, j \in \{1, 2, \cdots, n\}, i \ne j, \text{such that}\, PW_{ij} = PW_{ji}. \quad (33)$$

(2) The defined pairwise comparison relations are reflexive and transitive, i.e.,

$$\begin{gathered} \forall i, j, k \in \{1, 2, \cdots, n\}, i \ne j \ne k, \\ PW_{ij} = 1, PW_{jk} = 1, PW_{ik} = 1. \end{gathered} \quad (34)$$

Consequently, for a given MADM problem consisting of m alternatives and n attributes, there are at most $3^{n(n-1)/2}$ preference combinations. Take $n = 3$ as an example, in R-MADM, all preference combinations could be depicted by a multi-level decision tree (see Figure 2). Each branch is a possible preference combination consisting of pairwise comparison relations on attributes. Each preference combination is represented by $X_i, i \in \{1, 2, \cdots 3^{n(n-1)/2}\}$ at the end of the branch.

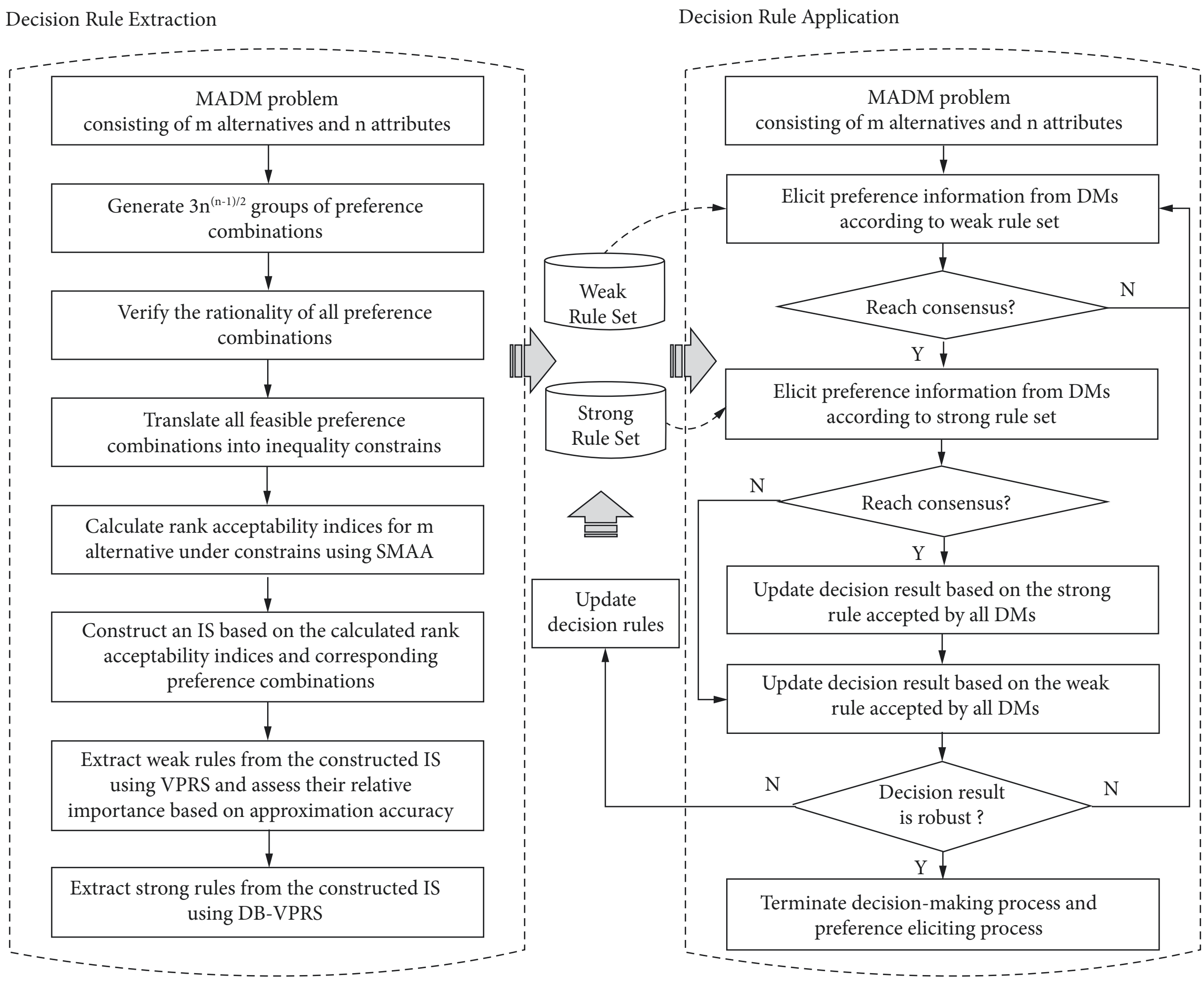

Figure 1: The rule-based decision-making process proposed by this paper.

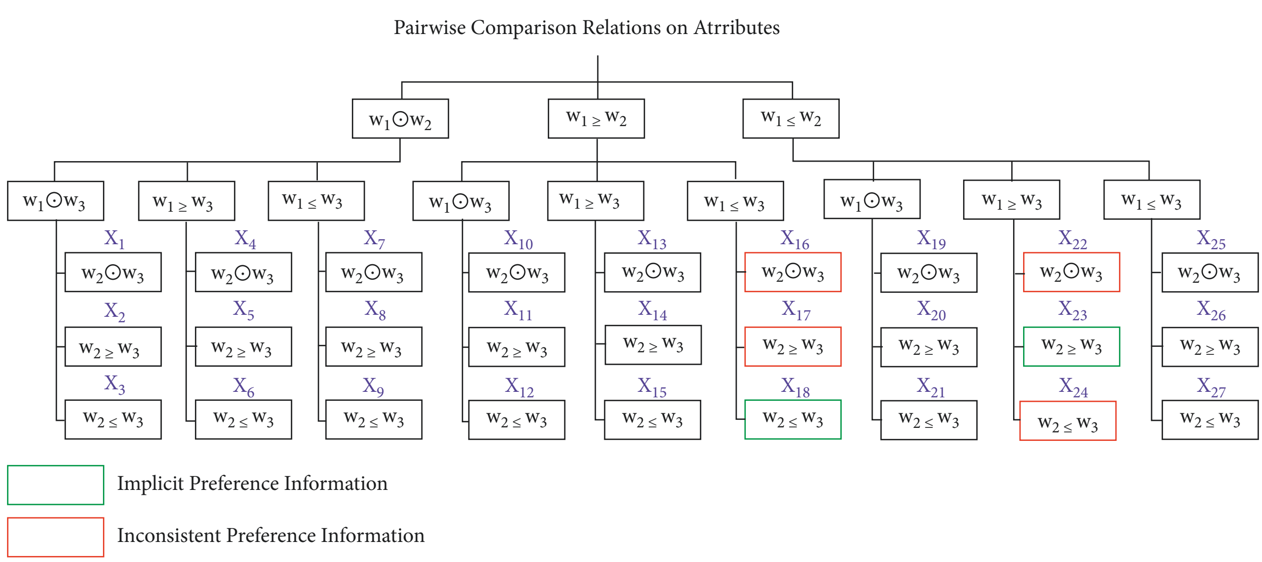

Figure 2: Preference combinations in the form of a decision tree (the number of attributes is three).

Nevertheless, not all combinations are rational in practice. Some preference combinations contain implicit preference information in which one pairwise comparison relations on attributes can be derived from others according to property (2). This kind of preference combination is marked with a green box in Figure 2. Some preference combinations contain inconsistent preference information in which pairwise comparison relations on attributes conflict with each other, and they are marked with red boxes in Figure 2. The preference combinations containing conflict preference information should be filtered.

In fact, each preference combination gives a group of no-conflicting inequality constrains of attribute weights. Under different constrain combinations, ranking results of alternatives can be obtained using stochastic multi-objective acceptability analysis (SMAA). Based on SMAA analysis results and corresponding preference combinations, an information system (IS) containing preference information is constructed, which is represented in the form of a decision table (see Table 1). In Table 1, pairwise comparison relations contained in preference combinations are condition attributes, while ranking results reflected by SMAA indices are decision attributes.

3.1.2. Extract Decision Rules from the Constructed Information System. Due to the stochastic nature of SMAA indices, the information system represented by Table 1 contains a great number of incomplete information. In this method, the variable precision rough set (VPRS) approach is used to extract decision rules from such incomplete information system.

Considering that, in MADM problem, both attribute weights and values are usually uncertain. For a given MADM problem consisting of m alternatives and n attributes, let U be the set of possible combinations of attribute weights and values, each combination can be represented by a tuple AW_i, and it expressed as

$$AW_i = \left\{ \begin{array}{c} A_1\left(c_{11}^i, c_{12}^i, \cdots, c_{1n}^i\right), \cdots, \\ A_m\left(c_{m1}^i, c_{m2}^i, \cdots, c_{mn}^i\right), \\ W\left(w_1^i, w_2^i, \cdots, w_n^i\right) \end{array} \right\}. \quad (35)$$

Here, $A_j(c_{j1}^i, c_{j2}^i, \cdots, c_{jn}^i)$ is a possible combination of n attribute values for alternative $j, j \in \{1, 2, \cdots, m\}$, $W(w_1^i, w_2^i, \cdots, w_n^i)$ is a possible combination of n attribute weights satisfying $\sum_{j=1}^n w_j^i = 1$. Each combination corresponds to a ranking result of m alternatives. Without loss of generality, the simple additive weighting method is used for reasoning the ranking result in this paper. In the proposed method, the ranking result is obtained by evaluating the overall utility values of alternatives. And, the overall utility value U_j for alternative j is a convex combination of $A_j(c_{j1}^i, c_{j2}^i, \cdots, c_{jn}^i)$ and $W(w_1^i, w_2^i, \cdots, w_n^i)$, and it is expressed as

$$U_j = \sum_{l=1}^n w_l^i \cdot c_{jl}^i. \quad (36)$$

(1) Weak Rule Extraction Based on the Combination of VPRS and SMAA. Let all possible combinations of attribute values and weights be the universe U, pairwise comparison relations contained in preference combinations be the condition attribute set C, and ranking results reflected by SMAA indices be the decision attribute set D. The indiscernibility relation with respect to condition attribute set $P \subseteq C$ is defined by

Table 1: An example of the decision table constructed based on SMAA analysis results and preference combinations (the number of attributes is three).

Preference combinations	Pairwise comparison relations			K-br acceptability		
	PW_{12}	PW_{13}	PW_{23}	$K\_b_1^k$	$K\_b_2^k$	$\cdots$
X_1	0	0	0	$\cdots$	$\cdots$	$\cdots$
X_2	0	0	1	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_{26}	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
X_{27}	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$$R_P = \left\{ \begin{array}{c} \left(AW_x, AW_y\right) \in U \times U: \\ PW_{ij}(AW_x) = PW_{ij}\left(AW_y\right), \\ \forall PW_{ij} \in P \subseteq C \end{array} \right\}. \quad (37)$$

The objects in U satisfying same pairwise comparison relations are indiscernible and can form an equivalent class with respect to $P \subseteq C$ as

$$U/R_P = \left\{ [AW]_P \text{: } AW \in U \right\}. \quad (38)$$

In this method, for a MADM problem consisting of m alternatives, there are m decision attributes defined. Each decision attribute $D_l \in D, l \in \{1, 2, \cdots, m\}$ describes whether alternative l could be ranked in top k according to the attribute values and weights given by $AW \in U$. Consequently, the indiscernibility relation with respect to a decision attribute $D_l \in D$ is defined as

$$R_{D_l} = \left\{ \begin{array}{c} \left(AW_x, AW_y\right) \in U \times U: \\ \sum_{l'=1, l' \neq l}^{m} \left(\mathscr{B}\left(\sum_{i=1}^n w_i^x \cdot c_{l'i}^x - \sum_{i=1}^n w_i^x \cdot c_{li}^x > 0 \right) \right) \\ < k \wedge \sum_{l'=1, l' \neq m}^{m} \left(\mathscr{B}\left(\sum_{i=1}^n w_i^y \cdot c_{l'i}^y - \sum_{i=1}^n v > 0 \right) \right) < k \end{array} \right\}, \quad (39)$$

where $\mathscr{B}(\cdot)$ is a Boolean variable, if the bracketed expression is true then $\mathscr{B}(\cdot) = 1$, otherwise $\mathscr{B}(\cdot) = 0$.

Let $X \subseteq U/R_P$ and $Y \subseteq U/R_{D_l}$, based on the majority relation in VPRS, the classification error of classifying objects in X into Y is measured by $c(X, Y) = 1 - card(X \cap Y)/car\ d(X)$. Here, $card(X)$ is the number of objects in U satisfying same pairwise comparison relations defined by X. It can be calculated by

$$card(X) = \sum_{b=1}^{B} \left(\prod_{j=1, i \neq j}^{n} \prod_{i=1, i \neq j}^{n} \left(\mathscr{B}\left(PW_{ij}(AW_b) = p_{ij}(X) \right) \right) \right), \quad (40)$$

where $B = card(U)$, $p_{ij} \in \{-1, 0, 1\}$ is the value of pairwise comparison relation between attribute i and attribute j given by X.

Similarly, $card(X\cap Y)$ is the number of objects in U which simultaneously satisfy two conditions: (1) They satisfy same pairwise comparison relations defined by X; (2) Alternative l could be ranked in top k according to the attribute values and weights given by $AW \in X$. It can be calculated by

$$card(X\cap Y)=\sum_{b=1}^{B}\left(\begin{array}{c}\left(\prod_{j=1,i\neq j}^{n}\prod_{i=1,i\neq j}^{n}\left(\mathscr{B}\left(PW_{ij}(AW_b)=p_{ij}(X)\right)\right)\right)\\ \cdot\mathscr{B}\left(\sum_{l'=1,l'\neq l}^{m}\left(\mathscr{B}\left(\sum_{i=1}^{n}w_i^b\cdot c_{l'i}^b-\sum_{i=1}^{n}w_i^b\cdot c_{li}^b>0\right)\right)<k\right)\end{array}\right). \tag{41}$$

The expression $card(X\cap Y)/card(X)$ has the same meaning with the k-best rank (k-br) acceptability $K\_b_i^k$ in SMAA, i.e., $c(X,Y)=1-b_l^k$. Consequently, for a given admissible classification error $(0\leq\beta\leq 0.5)$, the majority inclusion relation in VPRS can also be expressed as

$$Y\supseteq^{\beta}X\ if\ and\ only\ if\ 1-b_l^k\leq\beta. \tag{42}$$

In this way, the β-lower and β-upper approximation of the equivalent class $Y\subseteq U/R_{D_l}$ with respect to $P\subseteq C$ can be expressed as

$$\underline{R_P}^{\beta}(Y)=\cup\left\{X\colon 1-b_l^k\leq\beta\right\}, \tag{43}$$

$$\overline{R_P}^{\beta}(Y)=\cup\left\{X\colon 1-b_l^k\geq 1-\beta\right\} \tag{44}$$

According to (29), the relative importance of the pairwise relation for a decision attribute D_l can be measured by

$$\mathrm{RI}\left(PW_{ij},D_l\right)=\left|b_l^k(P)-b_l^k\left(P-PW_{ij}\right)\right|, \tag{45}$$

where $b_l^k(P)$ is the k-best rank acceptability under the constrain provided by P, $b_m^k(P-PW_{ij})$ is the k-best rank acceptability under the constrain provided by $P-PW_{ij}$. Then, the effect of pairwise comparison relations on ranking result is measured by

$$\mathrm{E}\left(PW_{ij}\right)=\sum_{l=1}^{m}b_l^k(\varphi)\cdot\mathrm{RI}\left(PW_{ij},D_l\right), \tag{46}$$

where $b_l^k(\varphi)$ is the k-best rank acceptability under no constrain.

Considering that b_l^k is usually obtained through the Monte Carlo simulation method, it often contains a computational error. Therefore, for a given threshold δ, the pairwise comparison relation PW_{ij} is regarded to be redundant if $\mathrm{RI}(PW_{ij},D_l)<\delta$.

In this method, whether or not an alternative could be ranked in top k is considered as a decision attribute. Therefore, for a decision class $Y\subseteq U/R_{D_l}$, there are two objects in it, i.e., $Y=\{Y_Y^l,Y_N^l\}$, Y_Y^l is the union of objects in U that could make alternative l to be ranked in top k; Y_N^l is the union of objects in U that could not make alternative l to be ranked in top k. According to the definition of positive region in VPRS given in section 2.1, preference combinations in the positive region of Y_Y^l are referred to as positive preference combinations. For a given confidence level $\gamma=1-\beta$, the positive preference combinations satisfy

$$X_j\longrightarrow b_l^k>\gamma\,\forall X_j\in\mathbf{POS}_P^{\beta}\left(Y_Y^l\right). \tag{47}$$

And preference combinations in the negative region of Y_Y^l are referred to as negative preference combinations which satisfy

$$X_j\longrightarrow b_l^k<1-\gamma\,\forall X_j\in\mathbf{NEG}_P^{\beta}\left(Y_Y^l\right). \tag{48}$$

In the proposed method, the intersection of positive preference combinations is referred to as a positive rule, which is expressed as

$$\mathbf{Rule}_{positive}\left(D_l\right)=X_j,X_j\in\mathbf{POS}_P^{\beta}\left(Y_Y^l\right). \tag{49}$$

And the intersection of negative preference combinations is referred to as a negative rule, which is expressed as

$$\mathbf{Rule}_{negative}\left(D_l\right)=X_j,X_j\in\mathbf{NEG}_P^{\beta}\left(Y_Y^l\right). \tag{50}$$

The positive rule $\mathbf{Rule}_{positive}(D_l)$ reflects the common characteristic of the preference structure which could ensure alternative l being ranked in top k. Similarly, the negative rule $\mathbf{Rule}_{negative}(D_l)$ gives the common characteristic of the preference structure which prevents alternative l from being ranked in top k. The ultimate rule is the aggregation of the positive rule $\mathbf{Rule}_{positive}(D_l)$ and the negative rule $\mathbf{Rule}_{negative}(D_l)$, which is expressed as

$$\mathbf{Rule}_{weak}\left(D_l\right)=\mathbf{Rule}_{positve}\left(D_l\right)\cup\overline{\mathbf{Rule}_{negative}\left(D_l\right)}, \tag{51}$$

$\mathbf{Rule}_{weak}(D_l)$is referred to as a weak rule of the decision attribute D_l. All weak rules for m decision attributes in D constitute the weak rule set in Figure 1, which can be represented as $\cup\left\{\mathbf{Rule}_{weak}(D_l)\right\}, l\in\{1,2,\cdots,m\}$. The advantages of eliciting preference information based on weak rules lie in the following two aspects: (1) With the concept of attribute reduction in VPRS, the process of eliciting preference information based on weak rules could avoid the evaluation on redundant attributes, thus reducing the burden of DMs. (2) The relative priority of preference information is given by weak rules. In this way, the preference information, which is sensitive to ranking result, could be obtained preferentially to ensure its reliability.

(2) Strong Rule Extraction Based on the Combination of DB-VPRS and SMAA. The rules extracted based on the above preference information system are weak rules in which pairwise comparison relations on attributes are roughly defined from a logical and qualitative point of view. In practice, the pairwise comparison relation on attributes can be further refined based on its significance level. The significance level reflects DMs' confidence in their evaluation of pairwise comparison relations on attributes. The significance level of pairwise comparison relations is essentially a kind of dominance relation. In the proposed method, to improve the reliability of the rule-based decision-making process, the extracted weak rules are refined using the combination of DB-VPRS and SMAA, which are defined as strong rules.

Similarly, let pairwise comparison relations considering dominance relation be condition attribute set C. The difference of attribute weights represented by $D(PW_{ij})$ is defined as the subtraction of two attribute weight values, i.e., $D(PW_{ij}) = |w_i - w_j|$. Based on the value of the difference of attribute weights, three kinds of significance levels are given as follows:

(a) If $D(PW_{ij}) \in [0, 0.05]$, then the significance level of the pairwise comparison relation $SIG(PW_{ij})$ between attribute i and attribute j is defined as low;

(b) If $\mathrm{D}(\mathrm{PW}_{\mathrm{ij}}) \in (0.05, 0.15]$, then the significance level of the pairwise comparison relation $\mathrm{SIG}(\mathrm{PW}_{\mathrm{ij}})$ between attribute i and attribute j is defined as middle;

(c) If $\mathrm{D}(\mathrm{PW}_{\mathrm{ij}}) \in (0.15, 1]$, then the significance level of the pairwise comparison relation $\mathrm{SIG}(\mathrm{PW}_{\mathrm{ij}})$ between attribute i and attribute j is defined as high.

It is assumed that, for any condition attribute $p \in P \subseteq C$, low $\prec_p$ middle $\prec_p$ high. Consequently, for two given objects $AW_x \in U$ and $AW_y \in U$, $SIG(PW_{ij}(AW_x)) \prec_p SIG(PW_{ij}(AW_y))$means that the pairwise comparison relation on attribute i and attribute j reflected by AW_y is more significant than AW_x. If $\forall p \in P \subseteq C$, $SIG(PW_{ij}(AW_x)) \prec_p SIG(PW_{ij}(AW_y))$ is true, AW_x is said to be dominated by AW_y with respect to P, i.e.,

$$\begin{aligned} &AW_y D_P AW_x \; SIG\big(PW_{ij}(AW_x)\big) \\ &\quad \prec_p SIG\big(PW_{ij}\big(AW_y\big)\big) \; for\,all\; p \in P \subseteq C. \end{aligned} \tag{52}$$

Consequently, the dominance and dominated union of AW_x with respect to P are represented by

$$D_P^+(AW_x) = \{AW_y \in U \colon AW_y D_P AW_x\}, \tag{53}$$

$$D_P^-(AW_x) = \{AW_y \in U \colon AW_x D_P AW_y\}. \tag{54}$$

By follow the definition in Section 2.1.2, let the ranking results of alternatives be decision attributes which is represented by $D^{\succ}$. Let $Cl_t^l, t = 1, 2, \cdots, m$ be decision classes with respect to a decision attribute $D_l^{\succ} \in D^{\succ}, l = 1, 2, \cdots, m$. The object in Cl_t^l could make the alternative l to be ranked in top k according to its attribute values and weights. The upward union of decision classes $Cl_t^{l\geqslant} = U_{s\leq t}\, Cl_s^l$ means that the alternative l could be ranked at least in top t. Similarly, the downward union of decision class $Cl_t^{l\leqslant} = U_{s\geq t}\, Cl_s^l$ means that the alternative l could be ranked at most in top t.

As discussed above, the approximation accuracy could be replaced by k-best rank (k-br) acceptability $K\_b_i^k$ in SMAA. Therefore, according to equations (18)–(21), the β-lower approximation and the β-upper approximation of $Cl_t^{l\geqslant}$ and $Cl_t^{l\leqslant}$ are represented by

$$\underline{R}_P^{\beta}\big(Cl_t^{l\geqslant}\big) = \left\{ AW \in Cl_t^{l\geqslant} \colon \frac{\left|D_P^+(AW_x) \cap Cl_t^{l\geqslant}\right|}{\left|D_P^+(AW_x)\right|} \geq \beta \right\}, \tag{55}$$

$$\underline{R}_P^{\beta}\big(Cl_t^{l\leqslant}\big) = \left\{ AW \in Cl_t^{l\geqslant} \colon \frac{\left|D_P^+(AW_x) \cap Cl_t^{l\geqslant}\right|}{\left|D_P^+(AW_x)\right|} \geq \beta \right\}, \tag{56}$$

$$\begin{aligned} \overline{R}_p^{\beta}\big(Cl_t^{l\geqslant}\big) &= Cl_t^{l\geqslant} \\ &\cdot \left\{ AW \in Cl_t^{l\leqslant} \colon \frac{\left|D_P^+(AW_x) \cap Cl_t^{l\geqslant}\right|}{\left|D_P^+(AW_x)\right|} > 1 - \beta \right\}, \end{aligned} \tag{57}$$

$$\begin{aligned} \overline{R}_P^{\beta}\big(Cl_t^{l\leqslant}\big) &= Cl_t^{l\leqslant} \\ &\cdot \left\{ AW \in Cl_t^{l\geqslant} \colon \frac{\left|D_P^+(AW_x) \cap Cl_t^{l\geqslant}\right|}{\left|D_P^+(AW_x)\right|} > 1 - \beta \right\}, \end{aligned} \tag{58}$$

In this way, the strong rules can be extracted from a preference information system containing dominance relation using a dominance-based variable precision rough set (DB-VPRS) approach, which are usually represented in the form of a series of if-then decision rules, and they are usually expressed as

$$if \wedge SIG\big(PW_{ij}\big) \succ_p low,\; then\, b_l^t \geq \beta. \tag{59}$$

3.2. Decision Rule Application. In the decision rule application stage, as shown in Figure 3, the process of eliciting preference information and the process of decision-making are guided by rules and carried out step by step.

First of all, for a given MADM problem, a weak rule set (WRS) and a strong rule set (SRS) could be obtained based on the attribute value of alternatives using the proposed decision rule extraction method. The rules in WRS could eliminate redundant attributes and give the relative priority of preference information. And the rules in SRS further improve the confidence of decision-making process by introducing the dominance relation. Then, in the first step, according to the rules in WRS, investigation questionnaires are designed in which DMs only need to evaluate the pairwise comparison relation of partial attributes. In this step, DMs are allowed to answer the questions in the designed questionnaire in an intuitive and linguistic way. In the second step, the designed questionnaires are sent to DMs to collect their preference information. In this step, DMs

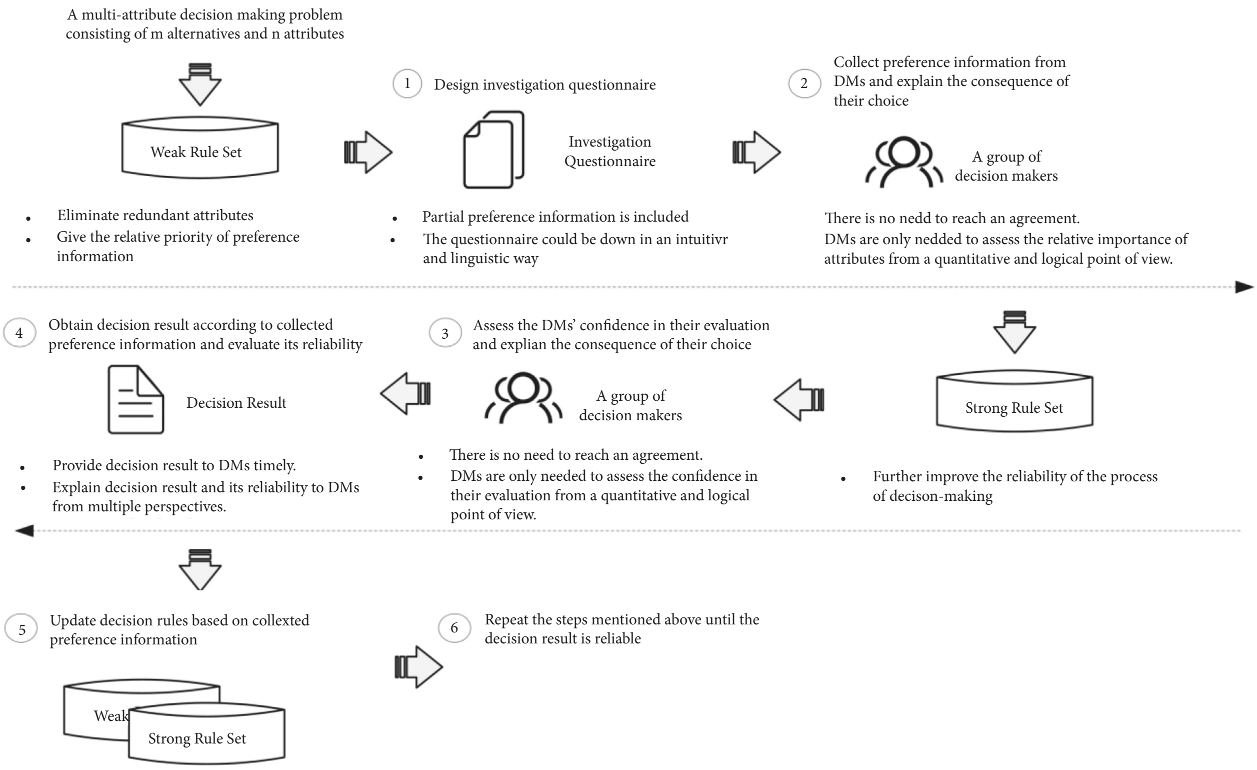

Figure 3: The decision rule application stage.

could express their preference in a way like that "attribute A is more important than attribute B," and they are informed the consequence of their choice. Moreover, even if DMs may not reach an agreement on the question, the following steps could still continue. When DMs agree on a pairwise comparison relation of two attributes, the significant level of this relation is further asked in the third step. According to rules in SRS, DMs are also informed the consequence of their choice in this step. In the fourth step, according to the collected preference information, a preliminary decision result can be obtained. Meanwhile, the reliability of the decision result is obtained in this step. In this paper, the reliability of the decision result is measured by the following equation (63).

$$R(DR_l) = \sum_{j=1}^{l}\left(K\_b_i^l(j)\right), \tag{60}$$

where $K\_b_i^l(j)$ is the jth element of a sequence in which the k-best rank (K-br) acceptability $K\_b_i^l$ is ordered from high to low. For a given threshold value ε, if $R(DR_l) > 100\%(l \cdot \varepsilon)$, the decision result is considered to be reliable. In this way, the decision result can be explained from multiple perspectives by adjusting the value of l and ε to adapt to different types of MADM problems.

If the decision result is reliable, the process of eliciting preference information and the process of decision-making could be terminated promptly. And if not, the rules in WRS and SRS should be updated based on collected preference information, and the steps above should be repeated again and again until the decision result is reliable.

Table 2: The normalized attribute values of twenty alternatives.

Alternative	Attribute 1	Attribute 2	Attribute 3
1	[0.00, 0.15]	[0.35, 0.46]	[0.42, 0.55]
2	[0.11, 0.25]	[0.15, 0.26]	[0.87, 1.00]
3	[0.15, 0.26]	[0.00, 0.15]	[0.73, 0.85]
⋮	⋮	⋮	⋮
18	[0.00, 0.18]	[0.85, 1.00]	[0.22, 0.35]
19	[0.00, 0.15]	[0.14, 0.26]	[0.85, 1.00]
20	[0.32, 0.45]	[0.22, 0.35]	[0.26, 0.45]

4. Illustrative Example

In this section, a MADM problem consisting of twenty alternatives is used to illustrate the application of the proposed method. In this example, each alternative is evaluated based on three attributes. Attribute values are given in Table 2 in the form of interval numbers. For simplicity, attribute values are normalized to the range of zero to one. It is supposed that all attributes are benefit indices and independent of each other.

It is necessary to carry out a preliminary screening on the twenty alternatives to reduce the complexity of the problem. According to equations (35) and (36), k-best rank (k-br) acceptability results for the twenty alternatives under no constrains are firstly obtained through Monte Carlo simulation. In order to ensure the accuracy of analysis results and reduce the computational complexity of the problem, descriptive sampling technique is used in Monte Carlo simulation. Each simulation is carried out for ten times with up to 50000 samples and less than 0.002 convergence tolerance.

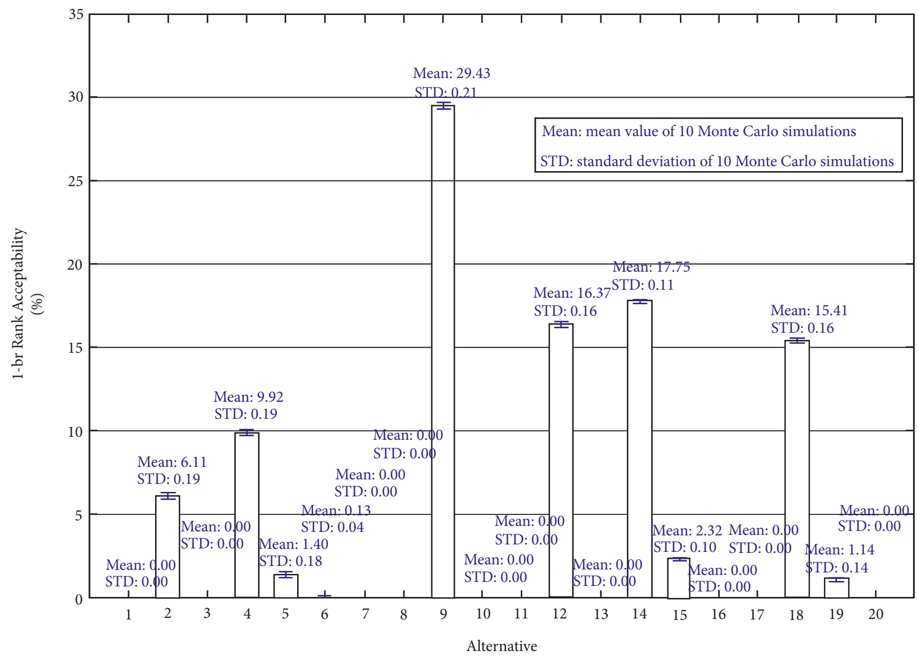

Figure 4: 1-br rank acceptability results of the twenty alternatives.

The mean value and standard deviation of 1-br and 3-br rank acceptability of the twenty alternatives in ten simulations are given in Figures 3 and4, respectively. It can be observed that thirteen alternatives including alternative 1, 3, 5, 6, 7, 8, 10, 11, 13, 16, 17, 19, and 20 have near-zero rank acceptability which will be eliminated in further analysis. In this way, the MADM problem consisting twenty alternatives is simplified to a problem with only seven alternatives.

4.1. Decision Rule Extraction. As discussed in Section 3.1, all feasible preference combinations are potential decision rules. In this example, 27 preference combinations are depicted in Figure 5 in the form of a three level decision tree, in which 24 combinations are feasible and each combination is represented by $X_i, i \in \{1, 2, \cdots, 27\}$. The preference combinations and their outcomes (ranking results) constitute a decision table (see Table 3), in which pairwise (Figure 6) comparison relations contained in preference combinations are condition attributes, while ranking results reflected by SMAA indices are decision attributes.

4.1.1. Weak Rule Extraction. According to equations (40) and (41), the universe U is partitioned by feasible preference combinations $\{X_{1-15}, X_{18-21}, X_{23-27}\}$ into 23 equivalent classes, i.e.,

$$U/C = \left\{[AW]_{X_i}, X_i \in \{X_{1-15}, X_{18-21}, X_{23-27}\}\right\}. \tag{61}$$

And each decision attribute $D_l, l \in \{2, 4, 9, 12, 14, 15, 18\}$ divides the universe U into two decision classes, i.e.,

$$U/D_l = \left\{Y^l_Y, Y^l_N\right\}, l \in \{2, 4, 9, 12, 14, 15, 18,\} \tag{62}$$

Let the confidence level $\beta = 0.8$. Then, the β-lower and β-upper approximations of the decision attribute D_l can be given by equations (46) and (47). Taking $l = 2$ as an example, it can be obtained that

$$\begin{gathered} Y^2_Y \supseteq^\beta [AW]_{X_9}; Y^2_Y \supseteq^\beta [AW]_{X_{18}}; \\ Y^2_Y \supseteq^\beta [AW]_{X_{21}}; Y^2_Y \supseteq^\beta [AW]_{X_{27}}, \\ \underline{R}^\beta_P\left(Y^2_Y\right) = \left\{[AW]_{X_9}, [AW]_{X_{18}}, [AW]_{X_{21}}, [AW]_{X_{27}}\right\}, \\ \overline{R}^\beta_X\left(Y^2_Y\right) = \left\{ \begin{array}{c} [AW]_{X_1}, [AW]_{X_3}, [AW]_{X_{6-10}}, [AW]_{X_{12}}, \\ [AW]_{X_{15}}, [AW]_{X_{18}}, [AW]_{X_{21}}, [AW]_{X_{25-27}} \end{array} \right\}. \end{gathered} \tag{63}$$

According to the definition in Section 2.1.1, it can be obtained that

$$\begin{gathered} POS^\beta_X\left(Y^2_Y\right) = \left\{[AW]_{X_9}, [AW]_{X_{18}}, [AW]_{X_{21}}, [AW]_{X_{27}}\right\}, \\ NEG^\beta_X\left(Y^2_Y\right) = \left\{ \begin{array}{c} [AW]_{X_2}, [AW]_{X_4}, [AW]_{X_5}, [AW]_{X_{11}}, \\ [AW]_{X_{13,14}}, [AW]_{X_{20}}, [AW]_{X_{23,24}} \end{array} \right\}, \\ BN^\beta_X\left(Y^2_Y\right) = \left\{ \begin{array}{c} [AW]_{X_1}, [AW]_{X_3}, [AW]_{X_{6-8}}, [AW]_{X_{10}}, \\ [AW]_{X_{12}}, [AW]_{X_{15}}, [AW]_{X_{25}}, [AW]_{X_{26}} \end{array} \right\}. \end{gathered} \tag{64}$$

Then, decision rules for the decision attributes D_2 are obtained by equations (52)–(54) as

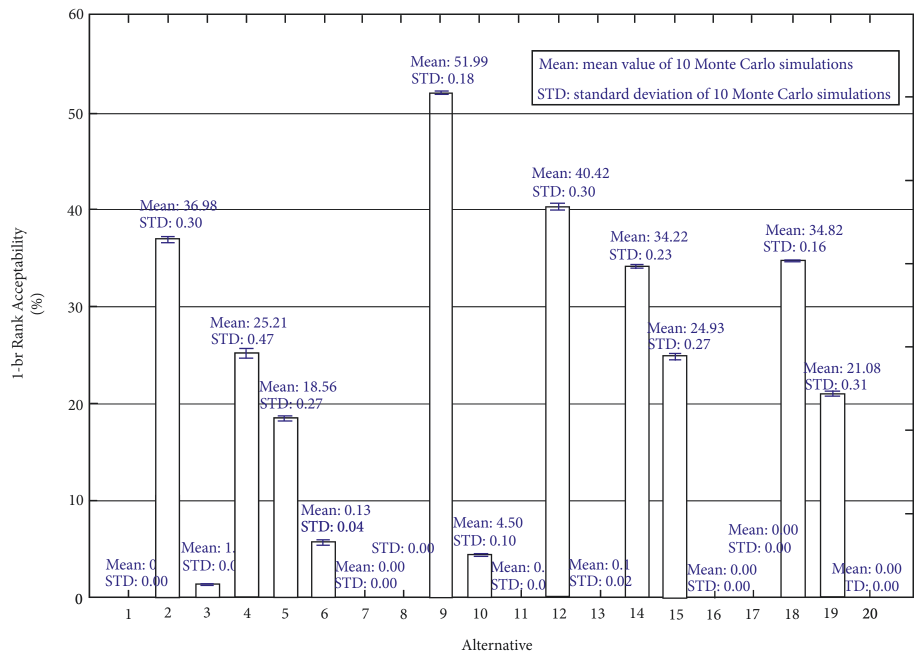

Figure 5: 3-br rank acceptability results of the twenty alternatives.

Table 3: A decision table consisting of SMAA analysis results and preference combinations for the proposed MADM problem.

Preference combinations	Pairwise comparison relations			K-br acceptability		
	PW_{12}	PW_{13}	PW_{23}	b_2^3	$\cdots$	b_{19}^3
X_1	0	0	0	39.60	$\cdots$	22.24
X_2	0	0	1	8.32	$\cdots$	2.84
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_{26}	−1	−1	1	25.15	$\cdots$	8.19
X_{27}	−1	−1	1	91.20	$\cdots$	22.79

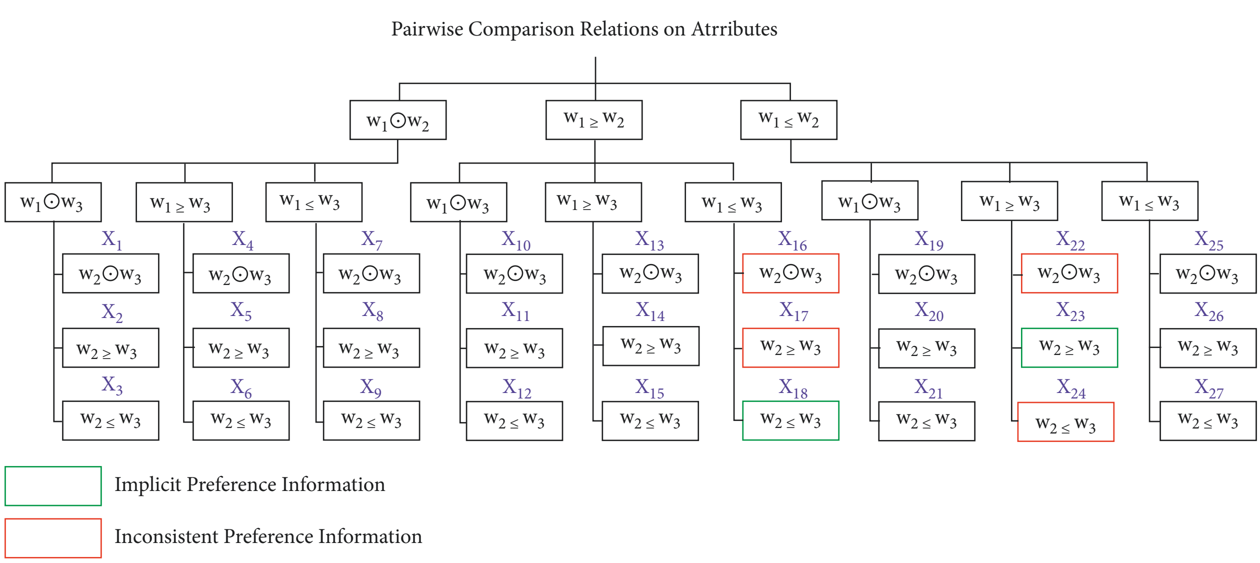

Figure 6: Preference combinations in the proposed example represented in the form of a decision tree.

$$\begin{aligned}
\mathbf{Rule}_{\text{positive}}(D_2) &= [AW]_{X_9} \cap [AW]_{X_{18}} \cap [AW]_{X_{21}} \cap [AW]_{X_{27}} \Rightarrow PW_{13} = -1, PW_{23} = -11,\\
\mathbf{Rule}_{\text{negative}}(D_2) &= [AW]_{X_2} \cap [AW]_{X_4} \cap [AW]_{X_5} \cap [AW]_{X_{11}} \cap [AW]_{X_{13}} \cap [AW]_{X_{14}} \cap [AW]_{X_{20}} \cap [AW]_{X_{23}} \cap [AW]_{X_{24}}\\
&\Rightarrow PW_{13} = 1,\\
\mathbf{Rule}_{\text{weak}}(D_2) &= \mathbf{Rule}_{\text{positve}}(D_2) \cup \overline{\mathbf{Rule}_{\text{negative}}(D_2)} \Rightarrow PW_{13} = -1, PW_{23} = -11.
\end{aligned} \tag{65}$$

Similarly, decision rules for other decision attributes can also be extracted from the information system like Table 3. All rules and their approximation accuracy are listed in Table 4.

The relative importance of pairwise comparison relations on attributes in weak rules are obtained by (48) and listed in Table 5. Let the threshold $\delta = 5\%$, it can be observed that the pairwise comparison on attribute 1 and attribute 2 is redundant for the rule X_{14}, which could be eliminated from the rule X_{14}. Based on (49), the comprehensive effects of pairwise comparison relations on ranking result are listed in the last row of Table 5. It can be observed that the pairwise comparison relation on attribute 1 and attribute 3 is more important than others. Based on the analysis result, when eliciting preference information from DMs, the priority of pairwise comparison relations is $PW_{13} \succ PW_{12} \succ PW_{23}$.

4.1.2. Strong Rule Extraction. The decision rules extracted based on the information provided by the decision table like Table 3 are weak rules, in which pairwise comparison relations on attributes are roughly defined. As discussed in the previous section, the weak rules could be refined by introducing dominance relations. The approximation accuracy of seven weak rules listed in Table 4 under different combinations of dominance relations are listed in Tables 6–11.

Take the weak rule X_2 as an example, as shown in Table 6, under different combinations of dominance relations, there are obvious differences on the k-br acceptability $K\_b_i^k$ of alternative 2.

Let the confidence level $\beta = 0.8$, based on equation (58), it can be obtained that

$$\begin{aligned}
&R_P^{\beta}\left(Cl_1^{2\geqslant}\right) = \varphi; R_P^{\beta}\left(Cl_2^{2\geqslant}\right) = \varphi\\
&R_P^{\beta}\left(Cl_3^{2\geqslant}\right) = \left\{S_3^2, S_5^2, S_6^2, S_8^2, S_9^2\right\} \Rightarrow\\
&SIG(PW_{13}) \succ_p Low \wedge SIG(PW_{23}) \succ_p High\\
&\text{or} SIG(PW_{13}) \succ_p Middle \wedge SIG(PW_{23}) \succ_p Middle\\
&R_P^{\beta}\left(Cl_4^{2\geqslant}\right) = \left\{S_2^2, S_3^2, S_5^2, S_6^2, S_7^2, S_8^2, S_9^2\right\}\\
&\quad \Rightarrow SIG(PW_{13}) \succ_p Low \wedge SIG(PW_{23})\\
&\quad \succ_p Middle\\
&R_P^{\beta}\left(Cl_5^{2\geqslant}\right) = \left\{S_1^2, S_2^2, S_3^2, S_5^2, S_6^2, S_7^2, S_8^2, S_9^2\right\}\\
&\quad \Rightarrow SIG(PW_{13}) \succ_p Low \wedge SIG(PW_{23}) \succ_p Low.
\end{aligned} \tag{66}$$

The obtained strong rules for different upward unions of decision classes are illustrated in Figure 7, in which different filled regions correspond to the strong rules for different upward unions of decision classes. When DMs' confidence in their evaluation of pairwise comparison relations on attributes fall within the filled region, the reliability of decision-making will be improved significantly. In addition, Table 6 could provide more decision support information to DMs. For example, even if DMs' confidence in their evaluation of pairwise comparison relations on attributes are both low, the alternative 2 still could be ranked in top 5 with at least 82.24% confidence level. Meanwhile, even if DMs' Tables 8–10confidence in their evaluation of pairwise comparison relations on attributes are both high, the probability of ranking alternative 2 in top 1 is at most 24.43%.

Similarly, based on the information provided in Tables 7–11, the refined strong rules for different weak rules are depicted in Figures 7–11 separately.

4.2. Decision Rule Extraction. In this section, a case is provided to illustrate Table 12 the process of decision-making based on the proposed method. It is assumed that the top three alternatives are of concern in this case.

First of all, according to the relative importance of pairwise comparison relations in weak rules given by Table 5, DMs are asked to evaluate the relative importance of attribute 1 and attribute 3, and they are informed of much decision support information. For example, the provided decision support information may consist of the following:

(i) In the absence of preference information, alternative 2, alternative 9, and alternative 12 are the most likely to be ranked in top three, and their possibility are 39.60%, 54.78%, and 50.17% separately.

(ii) If attribute 1 is more important than attribute 3, then the possibilities of eight alternatives being ranked in top three are at most 61.26%, 82.78%, 57.09%, 97.80%, 98.56%, 83.72%, 89.04%, and 27.33% according to Table 3.

(iii) If attribute 1 is less important than attribute 3, then the probabilities of eight alternatives being ranked in top three are at most 93.25%, 1.35%, 99.87%, 63.45%, 56.53%, 12.68%, 96.86%, and 62.48% according to Table 3.

It is assumed that DMs reach an agreement that attribute 1 is more important than attribute 3. Then, they are asked to give the confidence of their evaluation. And, it is assumed that the significant level of the pairwise comparison relation on attribute 1 and attribute 3 is high. Then according to strong rules in Figures 7–13, DMs are informed the following decision support information.

Table 4: Decision rules and their confidence of decision classes.

Decision attribute	Decision rules (β = 80%)									Approximation accuracy
	Rule$_{positive}$			**Rule**$_{negative}$			**Rule**$_{weak}$			
	PW_{12}	PW_{13}	PW_{23}	PW_{12}	PW_{13}	PW_{23}	PW_{12}	PW_{13}	PW_{23}	
D_2	×	−1	−1	×	1	×	×	−1	-1	91.20
D_4	1	1	×	×	×	×	1	1	×	78.63
D_9	×	−1	×	×	1	×	×	1	×	90.23
D_{12}	×	1	×	−1	×	−1	×	1	1	93.05
D_{14}	1	1	−1	−1	×	×	1	1	-1	98.55
D_{15}	−1	1	×	×	−1	×	−1	1	×	83.65
D_{18}	−1	×	×	1	×	×	−1	×	×	80.09

Table 5: The relative importance of pairwise comparison relations in weak rules.

D_l	RI(PW$_{12}$, D_1)	RI(PW$_{13}$, D_1)	RI(PW$_{23}$, D_1)
D_2	×	21.31%	22.71%
D_4	24.61%	26.40%	×
D_9	×	39.08%	×
D_{12}	×	9.62%	18.71%
D_{14}	0.01%	20.82%	22.32%
D_{15}	28.15%	51.05%	×
D_{15}	39.23%	×	×
E(PW$_{ij}$)	30.11%	65.55%	26.66%

Table 6: Approximation accuracy of the weak rule considering dominance relation.

Rule	Significant level		K-br acceptability (%)				
	$SIG(PW_{13})$	$SIG(PW_{23})$	$K\_b_2^1$	$K\_b_2^2$	$K\_b_2^3$	$K\_b_2^4$	$K\_b_2^5$
S_1^2	*Low*	*Low*	6.54	28.97	42.99	63.55	82.24
S_2^2	*Low*	*Middle*	8.33	36.18	63.82	82.72	92.28
S_3^2	*Low*	*High*	8.71	57.09	87.24	96.07	98.42
S_4^2	*Middle*	*Low*	2.33	36.65	62.29	77.97	90.04
S_5^2	*Middle*	*Middle*	11.00	55.23	81.56	88.02	94.44
S_6^2	*Middle*	*High*	18.63	66.47	91.94	98.64	99.75
S_7^2	*High*	*Low*	0.99	30.94	73.49	96.24	98.89
S_8^2	*High*	*Middle*	2.11	57.80	85.65	98.06	99.17
S_9^2	*High*	*High*	24.43	76.38	98.29	99.88	99.97

Table 7: Approximation accuracy of the weak rule considering dominance relation.

Rule	Significant level		K-br acceptability				
	$SIG(PW_{12})$	$SIG(PW_{13})$	b_4^1	b_4^2	b_4^3	b_4^4	b_4^5
S_1^4	*Low*	*Low*	0.85	4.27	9.40	17.09	28.21
S_2^4	*Low*	*Middle*	2.67	8.44	19.78	35.33	51.56
S_3^4	*Low*	*High*	2.75	24.44	62.73	82.27	93.47
S_4^4	*Middle*	*Low*	0	6.43	14.41	29.93	47.23
S_5^4	*Middle*	*Middle*	8.41	22.60	36.03	52.07	68.12
S_6^4	*Middle*	*High*	13.42	44.06	74.71	89.30	95.91
S_7^4	*High*	*Low*	0.05	7.77	30.85	63.65	82.58
S_8^4	*High*	*Middle*	2.06	41.25	66.63	82.63	92.40
S_9^4	*High*	*High*	45.18	86.47	95.12	98.84	99.75

(i) Based on their preference information, the alternative 2 could be ranked in top three with at least a probability of 73.49%, the alternative 4 could be ranked in top three with at least a probability of 62.73%, the alternative 9 could be ranked in top three with at least a probability of 74.57%, the alternative 12 could be ranked in top three with at least a probability of 36.31%, the alternative 14 could be ranked in top

Table 8: Approximation accuracy of the weak rule considering dominance relation.

Rule	Significant level	K-br acceptability				
	$SIG(PW_{13})$	b_9^1	b_9^2	b_9^3	b_9^4	b_9^5
S_1^9	*Low*	23.56	43.97	74.57	95.68	99.38
S_2^9	*Middle*	36.74	60.40	88.46	99.04	99.90
S_3^9	*High*	66.09	89.34	98.05	99.99	100.00

Table 9: Approximation accuracy of the weak rule considering dominance relation.

Rule	Significant level		K-br acceptability				
	$SIG(PW_{13})$	$SIG(PW_{23})$	b_{12}^1	b_{12}^2	b_{12}^3	b_{12}^4	b_{12}^5
S_1^{12}	*Low*	*Low*	31.49	56.17	74.04	85.53	92.34
S_2^{12}	*Low*	*Middle*	34.55	64.44	82.63	92.12	96.57
S_3^{12}	*Low*	*High*	16.19	89.54	97.05	99.33	99.89
S_4^{12}	*Middle*	*Low*	33.54	53.09	70.37	86.36	92.39
S_5^{12}	*Middle*	*Middle*	42.81	72.05	85.00	92.67	97.39
S_6^{12}	*Middle*	*High*	44.32	92.52	97.93	99.55	99.89
S_7^{12}	*High*	*Low*	8.66	18.71	36.31	87.74	98.06
S_8^{12}	*High*	*Middle*	20.23	36.80	85.59	97.57	99.04
S_9^{12}	*High*	*High*	61.25	85.43	98.01	98.10	99.91

Table 10: Approximation accuracy of the weak rule considering dominance relation.

Rule	Significant level		K-br acceptability				
	$SIG(PW_{13})$	$SIG(PW_{23})$	b_{14}^1	b_{14}^2	b_{14}^3	b_{14}^4	b_{14}^5
S_1^{14}	*Low*	*Low*	37.34	57.94	77.25	88.84	94.42
S_2^{14}	*Low*	*Middle*	53.74	78.91	90.93	97.28	99.49
S_3^{14}	*Low*	*High*	87.18	96.97	99.37	99.77	99.94
S_4^{14}	*Middle*	*Low*	47.40	76.41	87.66	94.16	97.84
S_5^{14}	*Middle*	*Middle*	71.89	88.33	94.86	97.11	99.00
S_6^{14}	*Middle*	*High*	93.67	98.78	99.69	99.90	99.97
S_7^{14}	*High*	*Low*	47.57	88.01	98.46	99.70	99.88
S_8^{14}	*High*	*Middle*	72.79	96.15	99.24	99.82	99.93
S_9^{14}	*High*	*High*	94.38	99.78	99.99	100.00	100.00

Table 11: Approximation accuracy of the weak rule considering dominance relation.

Rule	Significant level		K-br acceptability				
	$SIG(PW_{13})$	$SIG(PW_{23})$	b_{15}^1	b_{15}^2	b_{15}^3	b_{15}^4	b_{15}^5
S_1^{15}	*Low*	*Low*	7.63	24.10	50.60	70.28	85.14
S_2^{15}	*Low*	*Middle*	11.69	42.02	67.19	84.72	93.26
S_3^{15}	*Low*	*High*	11.85	65.11	84.66	95.86	98.60
S_4^{15}	*Middle*	*Low*	5.63	20.05	49.77	80.18	91.67
S_5^{15}	*Middle*	*Middle*	8.81	30.14	69.99	87.87	97.70
S_6^{15}	*Middle*	*High*	9.20	55.88	87.83	97.99	99.45
S_7^{15}	*High*	*Low*	0.85	5.26	63.43	97.29	99.66
S_8^{15}	*High*	*Middle*	1.01	6.00	80.99	98.56	99.64
S_9^{15}	*High*	*High*	1.91	17.18	96.83	99.79	99.98

three with at least a probability of 98.46%, and the alternative 15 could be ranked in top three with at least a probability of 63.43%.

According to (63), the reliability of decision result based on extant collected preference information is calculated as

$$R(DR_3) = 98.46\% + 74.57\% + 73.49\% = 246.52\%. \quad (67)$$

Let the threshold value $\varepsilon = 0.8$, then it can be obtained that

$$R(DR_3) = 246.52\% > 100\% \times 3 \times \varepsilon = 240\%. \quad (68)$$

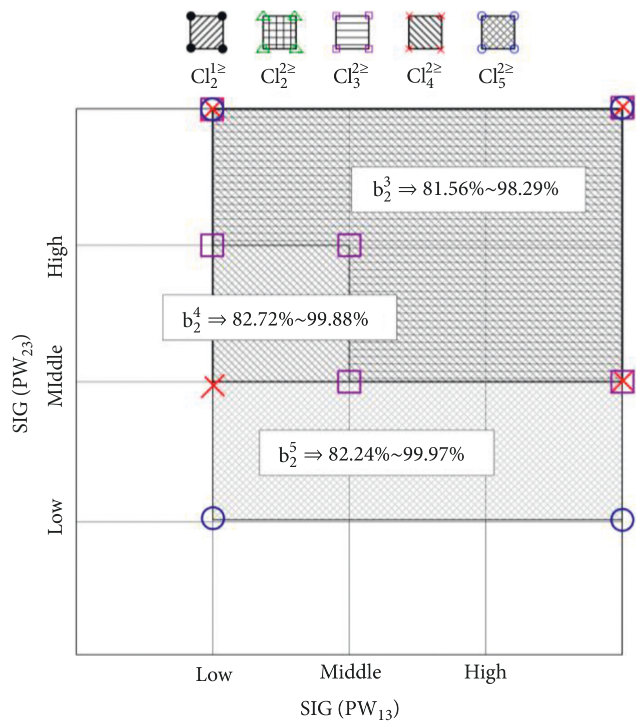

FIGURE 7: The strong rules of different upward unions of decision classes for alternative 2.

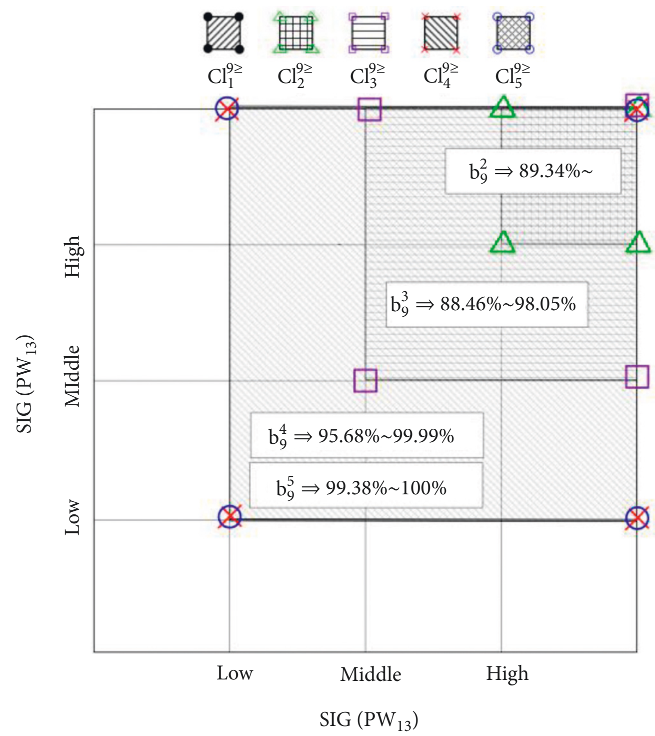

FIGURE 9: The strong rules of different upward unions of decision classes for alternative 9.

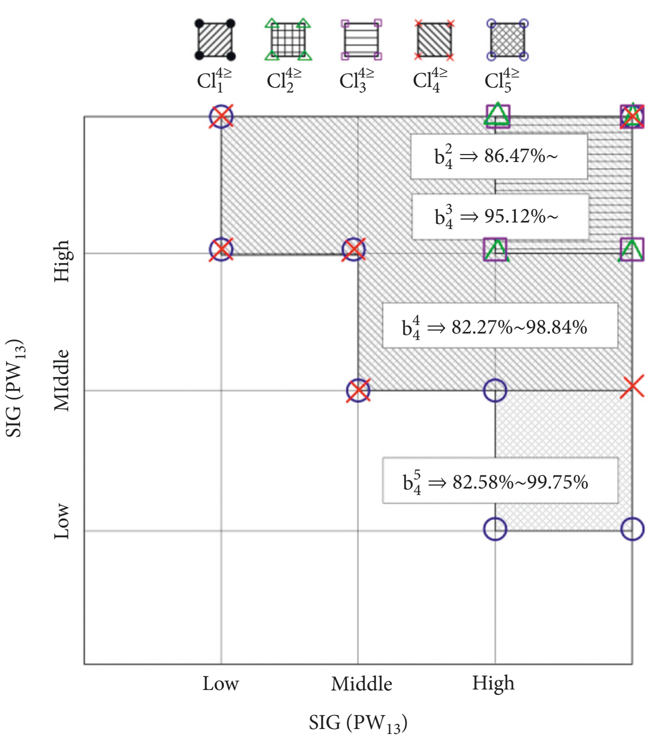

FIGURE 8: The strong rules of different upward unions of decision classes for alternative 4.

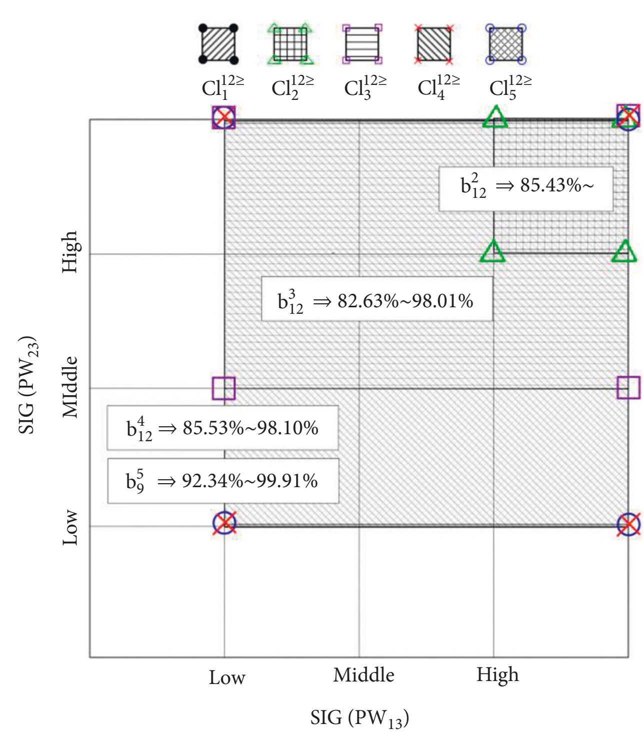

FIGURE 10: The strong rules of different upward unions of decision classes for alternative 12.

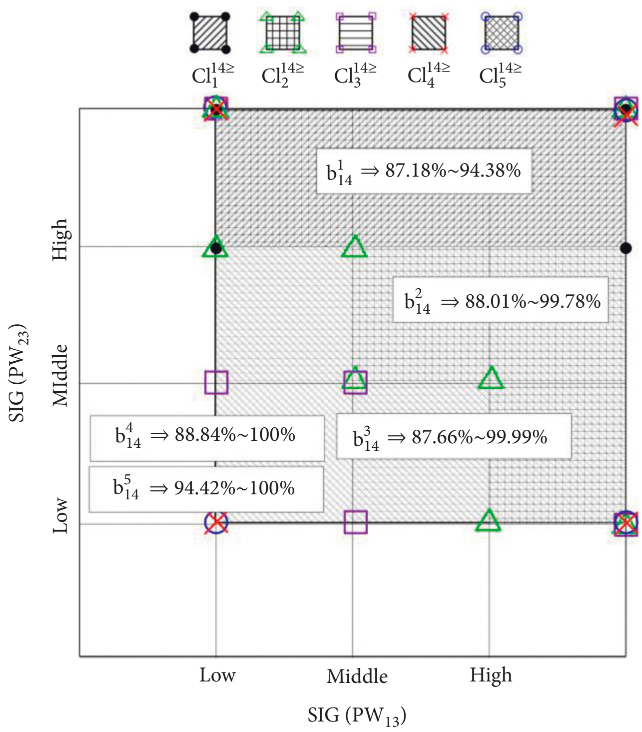

FIGURE 11: The strong rules of different upward unions of decision classes for alternative 14.

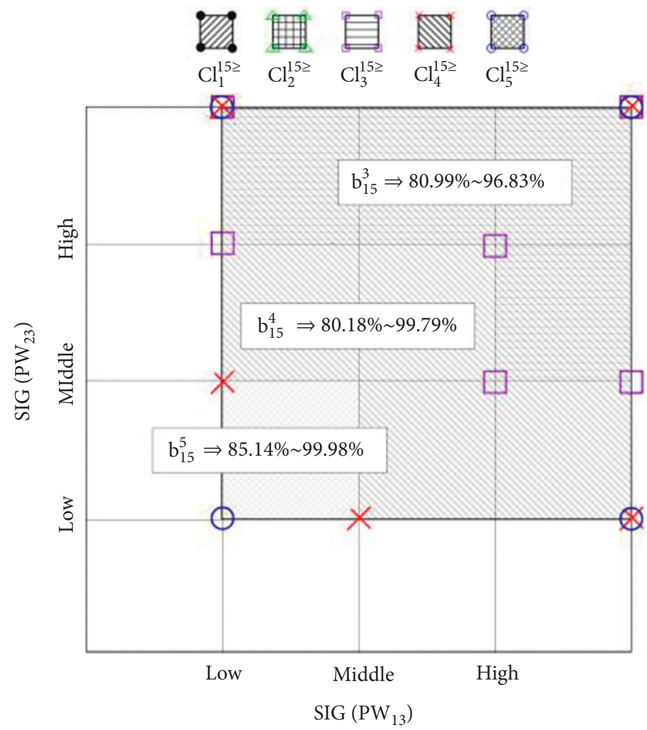

FIGURE 12: The strong rules of different upward unions of decision classes for alternative 15.

TABLE 12: Approximation accuracy of the weak rule considering dominance relation.

Rule	Significant level $SIG(PW_{12})$	K-br acceptability b_{15}^1	b_{15}^2	b_{15}^3	b_{15}^4	b_{15}^5
S_1^{18}	*Low*	3.15	13.92	30.51	49.88	81.39
S_2^{18}	*Middle*	7.68	28.97	51.34	71.87	93.91
S_3^{18}	*High*	41.46	71.94	81.28	97.90	99.72

It means that, with an 80% confidence level, the decision result is reliable. And then, the process of eliciting preference information and the process of decision-making is terminated. The decision result is that alternative 2, alternative 9, and alternative 14 are most likely to be selected.

The aim of the proposed method is to provide timely, various, and concise decision support information for DMs. According to the requirements of DMs, a unique and more robust decision result can be obtained by adjusting the confidence level and the number of alternatives considered in (63). In addition, to obtain a unique and more robust decision result, more subjective factors (e.g., the risk awareness of DMs) should7 be considered.

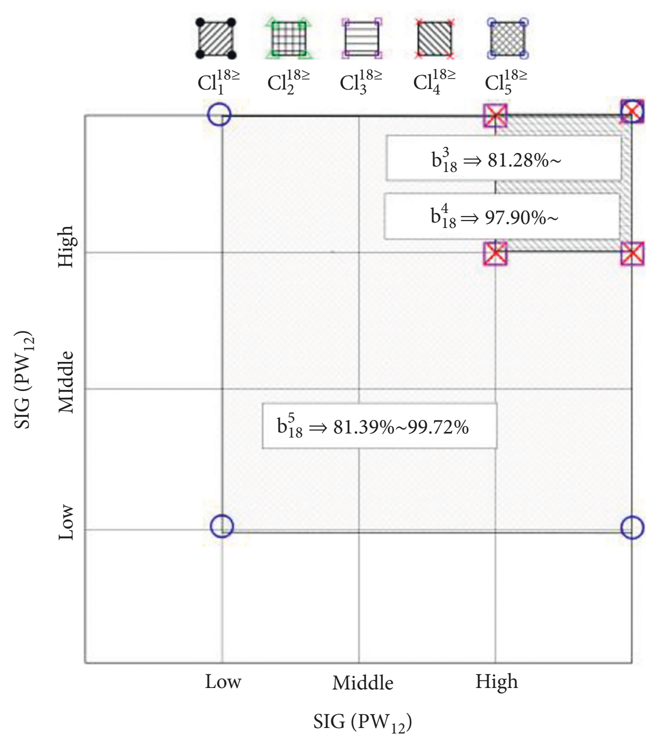

FIGURE 13: The strong rules of different upward unions of decision classes for alternative 18.

4.3. Comparative Experiments. In order to verify the excellent performance of the proposed algorithm, comparative experiments are implemented with AHP [47], ANP [48], D-TOSPIS [49], ELECTRE-II [50], and PROMETHEE [51]. Under the condition that the index evaluation information is inaccurate and the decision maker's preference information is incomplete, 20 potential design schemes including performance evaluation indexes of three products are analyzed and evaluated, which can be seen in Figure 14. Besides, the index accuracy is applied to describe the deviation

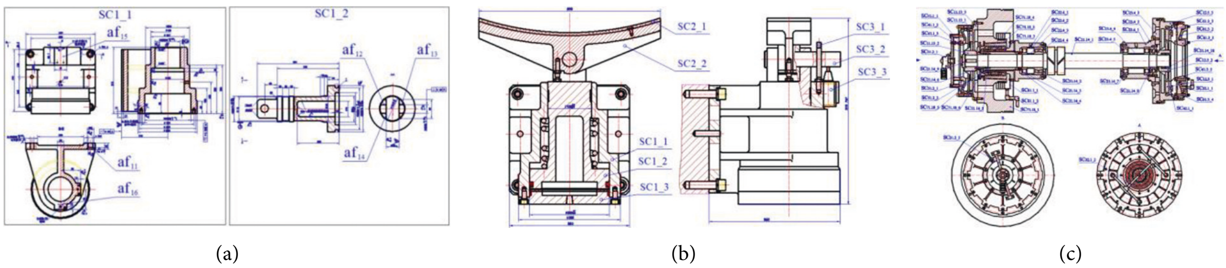

Figure 14: Real cases of three products. (a) Product no. 1. (b) Product no. 2. (c) Product no. 3.

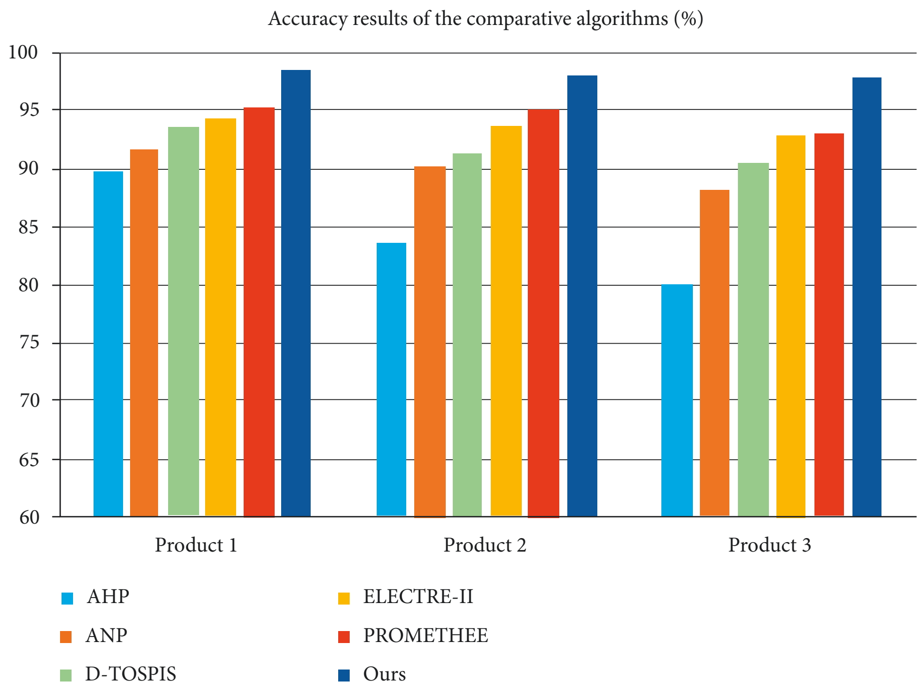

Figure 15: Accuracy results of the comparative algorithms.

Table 13: Accuracy results of the comparative algorithms in real cases of three products.

Product	Algorithm					
	AHP	ANP	D-TOSPIS	ELECTRE-II	PROMETHEE	Ours
No.1	89.78	91.53	93.52	94.15	95.22	**98.55**
No.2	83.65	90.16	91.28	93.78	94.95	**97.96**
No.3	80.09	88.15	90.52	92.77	93.01	**97.83**

between the relative importance of the extracted preference information and the relative importance of the real preference information. The accuracy results are shown in Figure 15, which indicates that the performance of the proposed algorithm is prior to the comparative algorithms.

Accuracy results of the comparative algorithms in real cases of three products are shown in Table 13 minutely. It is obvious that the accuracy of the proposed algorithm is higher than the other five comparative algorithms in the three real cases. Interestingly, along with the complexity of the cases increases, the accuracy of the six algorithms is decreased to a greater or lesser extent. However, the performance of the proposed algorithm is especially excellent in the complex case (e.g., product no.3).

5. Conclusion

In order to optimize the process of eliciting preference information from decision makers, a rule-based decision support method for multi-attribute decision-making is proposed in this paper. Based on the proposed method, it is unnecessary to ask decision makers to provide complete preference information. In this way, the decision burden of decision makers is reduced, and the efficiency of the multi-attribute decision-making is improved. The comparative experiments indicate that the performance of the proposed method is prior to the common methods (i.e., AHP, ANP, and TOSPIS). The significance of this paper is summarized as follows:

(1) The core of this method is extracting rules based on attribute values of alternatives and available preference information provided by decision makers through the combination of stochastic multi-objective acceptability analysis and a variable precision rough set approach. With the concept of attribute reduction and approximation accuracy, the extracted rules could eliminate redundant attributes and give the relative priority of preference information. According to the pre-extracted rules, the preference information that is sensitive to decision result will be collected preferentially.

(2) Based on the proposed method, the process of decision-making could be carried out step by step, and DMs are informed of much decision support information to help them make a choice. Moreover, the measurement of reliability of the decision result based on the k-best rank acceptability is given in this paper. In this way, according to the measurement, once the decision result is reliable enough, the preference eliciting and decision-making process can be terminated.

The limitation of this paper is that only the pairwise comparison relation on attributes is considered in the proposed method. Therefore, more types of relations between attributes will be studied in the future research. Moreover, considering that digital feedback information is often difficult to be understood for DMs, it is necessary to translate digital feedback information into semantic information, which is easier to be understood. The study on the transformation from digital feedback information to semantic information is also one of future research topics.

Data Availability

Raw data were generated at Navicat 15 for MySQL. Derived data supporting the findings of this study are available from the corre-sponding author on request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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