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Research Article

New Results of Fixed-Point Theorems in Complete Metric Spaces

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In this paper, fixed point theorems of multiple values of self-mappings have been investigated in Banach space. In addition, a new modification of the same mentioned study has been considered in a metric space with at most three valued self-mappings. Moreover, we have extended and improved some existent results in a complete metric space. Finally, we present a slight improvement on some well-known lemmas that are related to nonexpansive mappings and nonincreasing distances mappings.

1. Introduction

In mathematics and related fields, the fixed point theory plays a significant role in several fields, see [1–4]. In this recent work, we continue studying some existent results by many researchers on fixed point theorems in Banach and metric spaces. In 2014, a general fixed point theorem for multivalued mappings that is not always a contraction was proposed by Abdul Latif[2]. In addition, he derived some modern studies on contraction mappings. In 2020, Singh and Anthony [5] studied and extended the concept of more than one fixed point into mappings on a cone Sb-metric space. Moreover, they provided some robust examples to validate their results.

The structure of our proposed paper is organized as follows:

In Section 2, we present our first generalization of results by Ray [6], where we provide some well-known results consisting of multivalued mappings in a complete metric space (X, d). Moreover, we present the same study for the results provided by Taskovic [7].

In section 3, we provide some results about the fixed point theorem of three mappings f_1, f_2, and f_3 in a complete metric space (X, d). Furthermore, a brief discussion of few common fixed point theorems is going to be presented. Also, we will give a study of the three mappings f_1, f_2, and f_3 that satisfy the following equation:

$$[d(f_1x, f_2y)]^2 \leq \phi[d(f_3x, f_1x)d(f_3y, f_2y), d(f_3x, f_2y)d(f_3y, f_1x), d(f_3x, f_1x)d(f_3x, f_2y), d(f_3y, f_1x)d(f_3y, f_2y)]. \tag{1}$$

where $x, y \in X$; $f_1f_3 = f_3f_1$, $f_2f_3 = f_3f_2$, $f_1(x) \subset f_3(x)$; $f_2(x) \subset f_3(x)$; and ϕ are continuous function that satisfy the following condition:

$$\phi(s, s, a_1s, a_2s) < s \text{ for any } s > 0 \text{ and } a_i \in \{0, 1, 2\} \text{ such that } a_1 + a_2 = 2. \tag{2}$$

More studies have been done and considered by many authors regarding theorems that satisfy different functional inequalities and in terms of fixed point theorems in different spaces as well as several applications such as solving nonlinear and differential equations, see [8], [3, 9, 10], and [11–15].

In Section 4, we mainly extend the results that have been obtained by Kirk in [16] and in [1]. In fact, we work on modifying the given lemmas that are related to multivalued mappings in order to extend the results for the sake of generalization.

2. Fixed Point in Metric Space

Jleli and Bessem in [17] introduced a novel definition of generalized metric spaces, where they worked on extending several existing fixed-point findings along with the Banach contraction fundamental theorem [7]. In addition, Taskovic presented and proved a fixed-point theorem on a metric space (X, d) along with a mapping T, which is not necessarily continuous and satisfy a condition of the type,

$$ad(Tx, Ty) + b\, d(x, Tx) + c\, d(y, Ty) - \min\{d(x, Ty), (y, Tx)\} \leq q\, d(x, y), \tag{3}$$

for all $x, y \in X$.

In the following theorem, we present the first result for two mappings f_1 and f_2 that generalizes the above-mentioned result.

Theorem 1. *Suppose that $f_!$ and f_2 are a couple self-functionals of a complete metric space (X, d) with the following condition:*

$$ad(f_!x, f_2y) + bd(x, f_!x) + cd(y, f_2y) - \min\{d(x, f_2y), d(y, f_1x)\} \leq q\, d(x, y), \tag{4}$$

for all $x, y \in X$ such that the numbers $a, b, c \geq 0$ and $q > 0$ with $a > q + 1$ and $a + c > 0$. Then f_1 and f_2 will contain a unique common fixed point.

Now, we introduce the below generalized theorem of Ray [6] for a couple of self-mapping on a metric space.

Theorem 2. *Suppose that f_1 is a continuous functional in a complete metric space (X, d) and suppose that $f_2 \subset f_1$ is another mapping in the same space where the two mappings commute with each other, and f_2 satisfies the following condition:*

$$\begin{aligned} d(f_2(x), f_2(y)) \leq{}& \alpha(d(f_1(x), f_1(y)))d(f_1(x)f_1(y)) \\ &+ \beta(d(f_1(x), f_1(y))[d(f_1(x)f_2(x))d(f_1(y), f_2(y))]\gamma(d(f_1(x), f_1(y))[d(f_1(x)g(y)) \\ &+ d(f_1(y), f_2(x))], \end{aligned} \tag{5}$$

for each $f_1(x) \neq f_1(y)$, where α, β, and γ are monotonically decreasing functions from $(0, \infty)$ into $(0, 1)$ with $\alpha(t) + 2\beta(s) + 2\gamma(s) < 1$ and $s \in (0, \infty)$. Then, the two mappings f_1 and f_2 have only one common fixed point in X.

The proof of the above theorem is not going to be presented as it goes in a similar way as that in [18]. Therefore, the proof is omitted.

Remark 1. If f_1 happens to be an identity mapping, we get Theorem 2 of Ray [6] without the diagram of f_2 being closed. Further, if $\beta = \gamma = 0$, we have the result of Rokotch [19].

Another generalization of Theorem 2 of Ray goes as follows:

Theorem 3. *Suppose that f_1 and f_2 are two self-functionals in a complete metric space (X, d), where the two mappings satisfy the following condition:*

$$\begin{aligned} d(f_1x, f_2y) \leq{}& \alpha(d(x, y))d(x, y) + \beta(d(x, y)) \\ &[d(x, f_1x) + d(y, f_2y)] \\ &+ \gamma(d(x, y))[d(x, f_2y) + d(y, f_1x)], \end{aligned} \tag{6}$$

for every $x \neq y \in X$, such that α, β, and γ are monotonically decreasing functions from $(0, \infty)$ into $[0,1)$ with $\alpha(s) + 2\beta(s) + 2\gamma(s) < 1$, $s \in (0, \infty)$ Then the two mappings f_1 and f_2 have only one common fixed point in X.

3. Consequences

Let us assume that ϕ represent an upper semicontinuous function from the space R^{+4} to the space R^{+}, where this function is monotonically increasing in every variable and for each number $s > 0$ the following condition is satisfied:

$$\phi(s, s, a_1 s \cdot a_2 s) < s, \tag{7}$$

where $a_i \in \{0, 1, 2\}$ such that $a_1 + a_2 = 2$.

In the following, we present the main theorem of this work.

Theorem 4. *Suppose that f_1, f_2, and f_3 are three self-functionals of a complete metric space (X, d), which satisfy the following conditions:*

$$\begin{aligned} &f_1 f_3 = f_3 f_1,\ f_2 f_3 = f_3 f_2,\ f_1(x) \subset f_3(x) \text{ and } f_2(x) \subset f_3(x) \\ &[d(f_1 x, f_2 y)]^2 \leq \phi[d(f_3 x, f_1 y) d(f_3 y, f_2 y), d(f_3 x, f_2 y) d(f_3 y, f_1 x), \\ &\qquad d(f_3 x, f_1 x) d(f_3 x, f_2 y), d(f_3 y, f_1 x) d(f_3 y, f_2 y)], \end{aligned} \tag{8}$$

for all $x, y \in X$, where ϕ is defined in (7), then f_1 and f_2 have only one common fixed point.

for all $x, y \in X$. Further, let f_3 be continuous, then f_1, f_2, and f_3 have only one common fixed point in X. The proof of this theorem goes in a similar way to the one by Devi Prasad [9]. Therefore, we omit the proof.

The following result is a corollary to the previous theorem by taking $f_3 = I$, where I represents the identity map.

Corollary 1. *Let f_1 and f_2 be a couple of mappings from a complete metric space (X, d) into itself, and they satisfy the following condition:*

$$\begin{aligned} [d(f_1 x, f_2 y)]^2 \leq \phi[&d(x, f_1 x) d(y, f_2 y), d(x, f_2 y) d(x, f_2 y), \\ &d(y, f_1 x) d(x, f_2 y), d(y, f_1 x) d(y, f_2 y)], \end{aligned} \tag{9}$$

Now, we present the following three theorems, where the proof of each one follows from Theorem 4.

Theorem 5. *Suppose that f_1, f_2, and f_3 are three self-functionals of a complete metric space (X, d) sin such a way that,*

$f_1 f_3 = f_3 f_1$, $f_2 f_3 = f_3 f_2$, $f_1(x) \subset f_3^r(x)$, and $f_2(x) \subset f_3^r(x)$, then there is a non-negative and nonzero integers p, q, and r satisfying the following condition:

$$d[(f_1^p x, f_2^q y)]^2 = \phi \begin{bmatrix} d(f_3^r x, f_1^p x) d(f_3^r y, f_2^q y), d(f_3^p x, f_2^q y) d(f_3^p y, f_1^p x), \\ d(f_3^r x, f_1^p x) d(f_3^r x, f_2^q y), d(f_3^r y, f_1^p x) d(f_3^r y, f_2^q y) \end{bmatrix}, \tag{10}$$

for all $x, y \in X$, and if f_3^r is continuous, then f_1, f_2, and f_3 have only one common fixed point.

Theorem 6. *Suppose that $\{f_{1n}\}$, $\{f_{2n}\}$, and $\{f_{3n}\}$ are sequences of self-functionals of a complete metric space (X, d) such that $\{f_{1n}\}$, $\{f_{2n}\}$, and $\{f_{3n}\}$ converge uniformly to self-mappings f_1, f_2, and f_3 on X with f_3 continuous. Suppose that for each $n \geq 1$, x_n is a common fixed point of f_{1n} and f_{3n}, and y_n is a common fixed point of f_{2n} and f_{3n}. Furthermore, let f_1, f_2, and f_3 satisfy conditions (8) and (7). If x_0 is the common fixed point of f_1, f_2, and f_3, and $\sup d(x_n x_0) < \infty$ and $\sup d(y_n, x_0) < \infty$. Then, $x_n \longrightarrow x_0$ and $y_n \longrightarrow y_0$.*

Theorem 7. *Let $\{f_{1n}\}$, $\{f_{2n}\}$, and $\{f_{3n}\}$ be sequences of self-mappings of a complete metric space (X, d) such that f_{3n} is continuous for each n and f_{1n}, f_{2n}, and f_{3n} satisfy conditions (8) and (7) for each $n \geq 1$. If f_1, f_2, and f_3 are uniform limits of $\{f_{1n}\}$, $\{f_{2n}\}$, and $\{f_{3n}\}$ respectively, then, f_1, f_2, and f_3 also satisfy conditions (8) and (7). Also, let $\{x_n\}$ be a sequence of the only one common fixed point of f_{1n}, f_{2n}, and f_{3n} converges to the only one common fixed point x_0 of f_1, f_2, and f_3, whenever $\sup d(x_n, x_0) < \infty$.*

4. Main Results

Suppose that (Ω, d) is a metric space and let Q be a class of countably compact subsets of Ω, and suppose that this class is stable under intersections, normal, and contains all the closed balls in Ω. Also, suppose that $\{\Lambda_i, (i \in N_0)\}$ are subsets of Ω. Moreover, we define ζ as

$\zeta(\Lambda) = \begin{cases} r(\Lambda)/\delta(\Lambda) & \text{if } \delta(\Lambda) > 0 \\ 1 & \text{if } \delta(\Lambda) = 0 \end{cases}$ such that $\delta(\Lambda) = \sup \{d(\lambda_1, \lambda_2): \lambda_1, \lambda_2 \in \Lambda\} r_u(\Lambda) = \sup \{d(\lambda_1, \lambda_2): \lambda_2 \in \Lambda\}$ $(\lambda_1 \in \Lambda) r(\Lambda) = \inf \{r_{\lambda_1}(\Lambda): \lambda_1 \in \Lambda\}$

We shall take this assumption in order to initiate our proposed theorems.

Theorem 8. *If $T: \Omega \longrightarrow F(x)$ be a mapping such that,*

$$T(x) \cap \Omega \neq \phi \quad \forall x \in \Omega, \tag{11}$$

and for all $x \in \Omega$.

$$T(x) \cap Q = \{T(x) \cap \Lambda: \Lambda \in Q\}, \tag{12}$$

is a compact class of which each nonempty member is a compact subset of X. For any $G \in Q$ satisfying $T(\xi) \cap G \neq \phi$, for all $\xi \in G$,

$$H(T(x) \cap G, T(y) \cap G) \leq d(x, y), \forall x, y \in G. \tag{13}$$

Then, T has a fixed point in Ω.

Proof. A slight modification on the main lemma by kirk [1], we show that for each $\varepsilon > 0$, there exists a nonempty set $\Omega(\varepsilon) \in Q$ such that,

$$T(x) \cap \Omega(\varepsilon) \neq \phi, \tag{14}$$

for all $x \in \Omega(\varepsilon)$ and for which $\delta(\Omega(\varepsilon)) \leq (\zeta(\Omega) + \varepsilon)\delta(\Omega)$. For this, we take $\Omega(\varepsilon) = \Omega$ if $\delta(\Omega) = 0$. Otherwise, construct $\Omega(\varepsilon)$ as follows:

Now, let$r = (\zeta(\Omega) + \varepsilon)\delta(\Omega)$. By the definition of ζ, the set $C = \{z \in \Omega: \Omega \subset B(\alpha, r)\}$ is nonempty, where $B(z, p)$ represents the closed ball with the center α and the radius r. Let

$$F = \{\Lambda \in Q; C \subset \Lambda, T(x) \cap \Lambda \neq \phi, \forall x \in \Lambda\}, \tag{15}$$

and let $\tau\{\Lambda_i\}_{i\in_{N_0}}$ be a decreasing chain in F. Let $\Lambda_0 = \cap \Lambda_I$, then $\Lambda_0 \in Q$ and $C \subset \Lambda_0$. Furthermore, since $\tau\{\Lambda_i\}_{i\in_{N_0}}$ is decreasing, it follows that for each $x \in \Lambda_0$, the family $\{T(x) \cap \Lambda_i\}_{i\in N_0}$ has a finite intersection property. So, by hypothesis (13),

$$T(x) \cap \left(\cap_{i\in N_0} \Lambda_i\right) \neq \phi. \tag{16}$$

That is $T(x) \cap \Lambda_0 \neq \phi$, thus every decreasing chain in F has a lower bound. Therefore, by Zorn's lemma, F has a minimal element L. Let $A = C \cup T(L)$, where

$$T(\mathrm{L}) = \bigcup_{x\in\Lambda} (T(x) \cap L). \tag{17}$$

Then, $T(L) \subset L$. So, $cov(A) = \cap \{\Lambda \in Q; A \subset \Lambda\} \subset L$. Also, for $x \in cov(A), T(x) \cap L \neq \phi$ and $T(x) \cap L \subset T(L)$. So, $T(x) \cap A \neq \phi$. Hence, $T(x) \cap cov(A) \neq \phi$. Thus, $cov(A) \in F$.

Since L is a minimal member of F and $cov(A) \subset L$, it follows that $cov(A) = L$. Let

$$\Omega(\varepsilon) = \{x \in L; L \subset B(x, r)\}. \tag{18}$$

Then, $\Omega(\varepsilon) \neq \phi$. Since $\Omega(\varepsilon) \subset C$, and let $x \in \Omega(\varepsilon)$. Then, $T(x) \cap L \neq \phi$. Take $x' \in T(x) \cap L$. Let $0 = L \cap B(x', r)$. Next take $\eta \in 0$, then $\eta \in L$ and by (13),

$$H(T(x) \cap L, T(\eta) \cap L) \leq d(x, \eta) \leq r (\because x \in \Omega(\varepsilon), \eta \in L). \tag{19}$$

Since $T(x) \cap L$ and $T(\eta) \cap L$ are nonempty compact sets and $x' \in T(x) \cap L$, there exists $\eta' \in T(\eta) \cap L$, such that

$$d(x', \eta') \leq H(T(x) \cap L, T(\eta) \cap L) \leq r. \tag{20}$$

So, $\eta' \in 0$, hence, $T(\eta) \cap 0 \neq \phi$. Thus, $0 \in F$ and $0 \subset L$. Since L is a minimal element in F, it follows that $0 = L$. So $d(x', y') \leq r \forall y \in L$, i.e., $B(x', r) \subset L\cdot$ So, $x' \in \Omega(\varepsilon)$, which implies $T(x) \cap \Omega(\varepsilon) \neq \phi$. Now, let $m = \{\Lambda\} \in Q;\ \Lambda \neq \phi, T(x) \cap \Lambda \neq \phi,\ \forall x \in (\Lambda\}$ and for each $\Lambda \in m$, let

$$\delta_0(\Lambda) = \inf\{\delta(F); F \in m, F \subset \Lambda\}. \tag{21}$$

The rest of the proof is omitted as it goes in a similar way as the one by Kirk in [1]. □

Remark 2. The necessity of condition (14) of the theorem has been studied by Samanta in [20, 21].

In the following, we mention two theorems by Kirk in [1] and Samanta in [20] as we consider them corollaries of Theorem 4.

Corollary 2 (see [1]). *Suppose that (Ω, d) is a metric space, which is bounded and nonempty and let Q be a countably compact class of subsets in Ω, which is normal and stable under random intersections. In addition, suppose that we have closed balls of Ω that are contained in Q. Then, there exists a fixed point in each nonexpansive mapping T of Ω into itself.*

Corollary 3 (see [20]). *Let S be bounded and a closed convex subset in a reflexive Banach space X, and if $\Psi: S \longrightarrow 2^x$ is a mapping such that,*

$$\Psi(x) \cap S \neq \phi, \quad \forall x \in S, \tag{22}$$

for any closed convex subset G of S satisfying $l(\xi) \cap G \neq \phi$, $\forall \xi \in G$

$$H(\psi(x) \cap G, \psi(y) \cap G) \leq \|x - y\| \text{whenever } x, y (\neq x) \in G. \tag{23}$$

Then, l has a fixed point.

5. Conclusions

We have presented in this paper generalizations of various fixed-point theorems in a metric space and in a Banach space. A generalization with multiple values of self-mappings in those mentioned spaces have been intensively studied. We saw that mapping scheme of the fixed-point theorem is vast no doubt. Therefore, the existent result by Kirk was improved by extending the proof and working on the fixed-point theorem of multiself-mappings.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have read and agreed to the published version of the manuscript.

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