Research Article

On Sum Degree-Based Topological Indices of Some Connected Graphs

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In mathematical chemistry, molecular structure of any chemical substance can be expressed by a numeric number or polynomial or sequence of number which represents the whole graph is called topological index. An important branch of graph theory is the chemical graph theory. As a consequence of their worldwide uses, chemical networks have inspired researchers since their development. Determination of the expressions for topological indices of different derived graphs of graphs is a new and interesting problem in graph theory. In this article, some graphs which are derived from honeycomb structure are studied and obtained their exact results for sum degree-based indices. Additionally, a comparison is shown graphically among all the indices.

1. Introduction and Preliminary Results

Atoms are connected through covalent bonds in molecular frames. Atoms are called vertices and covalent bonds as edges in graph theory. Information sciences, Mathematics, and Chemistry are combined in Cheminformatics. This is a new research field that attracts the attention of researchers. Dominating David Derived networks are under consideration in the present research article which are further extracted from honeycomb structures. Honeycomb structures taken from bee honeycombs had observed a lot of applications in different fields including chemical engineering, transportation, mechanical engineering, architecture, nanofabrication, and recently bioinformatics. The greatest challenge in this research direction is to comprehend the unique characteristics of honeycomb structures, relying on their structures, scales, and the materials used to allow the minimization of the materials to get minimum weight and maximum strength. Nowadays, there is a substantial use of honeycomb networks in computer graphics, processing of images, chemistry as the portrayal of benzenoid hydrocarbons, and cellular phone base stations by using hexagon arrangements with repetitive building of honeycomb structures [1].

A topological index (TI), sometimes also known as a graph-theoretic index, is a numerical invariant of a chemical graph [2]. There are many types of TI’s, but most popular and authentic TI’s are distance-based, degree-based, and neighbourhood degree-based indices. These indices contain a lot of information within themselves.

The method of drawing Dominating David Derived networks (dimension t) is as follows:

Step 1: consider a honeycomb network HC(t) dimension t as shown in Figure 1(a).
Step 2: split each edge into two by embedding another vertex as shown in Figure 1(b).
Step 3: in each hexagon cell, connect the new vertices by an edge if they are at a distance of 4 units within the cell as shown in Figure 1(c).
Step 4: place vertices at new edge crossings as shown in Figure 1(d).
Step 5: remove initial vertices and edges of honeycomb network as shown in Figure 1(e).

Step 6: split each horizontal edge into two edges by inserting a new vertex. The resulting graph is called Dominating David Derived system $\text{DDD}(t)$ of measurement $(t)$ as shown in Figure 1(f) [1, 3].

The first type of Dominating David Derived network $D_1(t)$ can be obtained by connecting vertices of degree two by an edge, which are not in the boundary (see Figure 2).

The second type of Dominating David Derived network $D_2(t)$ can be obtained by subdividing once the new edge introduced in $D_1(t)$ (see Figure 3).

The third type of Dominating David Derived network $D_3(t)$ can be obtained from $D_1(t)$ by introducing parallel path of length 2 between the vertices of degree two which are not in the boundary. See Figure 4 for third type of Dominating David Derived network of dimension 2, $D_3(2)$. Moreover, isomorphic graph of Dominating David Derived network of dimension 2 can be seen in Figure 5.

In this article, $\Upsilon$ is considered a network with a $V(\Upsilon)$ vertex set and an edge set of $E(\Upsilon)$, and $d_r$ is the degree of vertex $r \in V(\Upsilon)$. Let $S_r(r)$ denote the sum of the degrees of all vertices adjacent to a vertex $r$. Graovac et al. [4] defined fifth M-Zagreb indices as polynomials for a molecular graph and these are characterized as follows.

Let $\Upsilon$ be a graph. Then,

$$M_1G_5(\Upsilon) = \sum_{rs \in E(\Upsilon)} (S_r(r) + S_s(s)), \quad (1)$$

$$M_2G_5(\Upsilon) = \sum_{rs \in E(\Upsilon)} (S_r(r) \times S_s(s)) \quad (2)$$

Kulli [5, 6] motivated by above indices described some new topological indices and named them as the fifth M-Zagreb indices of first and second type and fifth hyper-M-

Zagreb indices of first and second type of a graph $\Upsilon$. They are defined as

$$M_1^aG_5(\Upsilon) = \sum_{rs \in E(\Upsilon)} (S_r(r) \times S_s(s))^a, \quad (3)$$

$$M_2^aG_5(\Upsilon) = \sum_{rs \in E(\Upsilon)} (S_r(r) \times S_s(s))^a, \quad (4)$$

$$HM_1^2G_5(\Upsilon) = \sum_{rs \in E(\Upsilon)} (S_r(r) \times S_s(s))^2, \quad (5)$$

$$HM_2^2G_5(\Upsilon) = \sum_{rs \in E(\Upsilon)} (S_r(r) \times S_s(s))^2, \quad (6)$$

They also define a new version of Zagreb index which they call as third Zagreb index or fifth $M_3$-Zagreb [7].

Figure 1: Construction algorithm for Dominating David Derived network $\text{DDD}(2)$. (a) Step 1. (b) Step 2. (c) Step 3. (d) Step 4. (e) Step 5. (f) Step 6 [3].

Figure 2: First type of Dominating David Derived network $D_1(2)$ [3].
Figure 3: Second type of Dominating David Derived network $D_2(2)$ [3].

Figure 4: Third type of Dominating David Derived network $D_3(2)$ [3].

Figure 5: Isomorphic graph of DDD(2) [3].
\[ M_3 G_5 (Y) = \sum_{r \in E(Y)} |S_G (r) - S_G (s)|. \]  

(7)

2. Results

We have studied the new topological indices described by Kulli named as fifth M-Zagreb indices, fifth M-Zagreb polynomials, and \( M_2 \)-Zagreb index and give closed formulae of these indices for Dominating David Derived networks. In the following publications, the authors found out and described many other interesting characterizations for topological indices for different networks. For example, Ali et al. [8] studied degree-based topological indices and computed some new topological indices [9] for various networks. Bača et al. [10] and Baig et al. [11] found some topological indices of carbon nanotube network and polyoxide networks, respectively. Baig et al. [12] and Imran [13] studied about topological polynomials of certain nanostructures and mesh-derived networks, respectively. Eliasi et al. [14] and Liu et al. [15] computed multiplicative indices of first Zagreb and carbon graphite \( t \)-levels, respectively. Furthermore, Gao et al. [16] computed multiple ABC and GA index; Imran et al. [17] studied topological properties of Dominating David Derived network; Imran et al. [18] investigated topological properties of diamond-like networks. To know more about topological indices of various graph families, see [19, 20]. For the basic notations and definitions, see [21, 22].

2.1. Results for First Type of Dominating David Derived Networks. In this section, we calculate degree-based topological indices of the dimension \( t \) for first type of Dominating David Derived networks. In the coming theorems, we compute M-Zagreb indices and polynomials.

Theorem 1. Let \( Y_1 \equiv D_1 (t) \) be the first type of Dominating David Derived network, then the first and second fifth M-Zagreb indices are equal to

\[
\begin{align*}
M_1 G_5(Y_1) &= 1002 - 2714t + 2178t^2, \\
M_2 G_5(Y_1) &= 8065 - 20581t + 14697t^2.
\end{align*}
\]

(8)

Proof. The outcome can be obtained by using the edge partition in Table 1. By using equation (1),

By doing some calculations, we get

\[
\begin{align*}
M_1 G_5(Y_1) &= (6 + 6)|E_1 (Y_1 (t))| + (6 + 6)|E_2 (Y_1 (t))| + (6 + 6)|E_3 (Y_1 (t))| \\
&\quad + (6 + 6)|E_4 (Y_1 (t))| + (7 + 9)|E_5 (Y_1 (t))| + (7 + 12)|E_6 (Y_1 (t))| \\
&\quad + (8 + 11)|E_7 (Y_1 (t))| + (8 + 13)|E_8 (Y_1 (t))| + (9 + 9)|E_9 (Y_1 (t))| + (9 + 14) \\
&\quad \cdot |E_{10} (Y_1 (t))| + (11 + 11)|E_{11} (Y_1 (t))| + (11 + 12)|E_{12} (Y_1 (t))| + (11 + 13)|E_{13} (Y_1 (t))| + (11 + 14)|E_{14} (Y_1 (t))| \\
&\quad + (11 + 16)|E_{15} (Y_1 (t))| + (12 + 14)|E_{16} (Y_1 (t))| \\
&\quad + (13 + 14)|E_{17} (Y_1 (t))| + (13 + 16)|E_{18} (Y_1 (t))| + (14 + 14)|E_{19} (Y_1 (t))| + (14 + 16)|E_{20} (Y_1 (t)),
\end{align*}
\]

\[
M_2 G_5(Y_1) = (6 + 6)(4t) + (6 + 6)|E_1 (Y_1 (t))| + (6 + 11)(4t) + (6 + 12)(4t) + (6 + 14)(4t) + (7 + 9)(4t) + (7 + 12) \\
&\quad \cdot (4t - 4) + (8 + 11)(12t - 8) + (8 + 13)(4t - 4) + (9 + 9)(2t - 2) + (9 + 14) \\
&\quad \cdot (4t - 4) + (11 + 11)(9t^2 - 7t + 3) + (11 + 12)(4t - 4) + (11 + 13)(4t - 4) + (11 + 14)(36t^2 - 68t + 32) \\
&\quad + (11 + 16)(4t - 4) + (12 + 14)(4t - 4) + (13 + 14)(4t - 4) + (13 + 16)(4t - 4) \\
&\quad + (14 + 14)(4t - 4) + (14 + 16)(36t^2 - 76t + 40).
\]

Thus, from equation (2),

\[
M_1 G_5(Y_1) = 1002 - 2714t + 2178t^2.
\]

(10)
Let \( \nu \) be the first type of DDD network \( \Upsilon \) for the sum of the degrees of each end vertex of each edge.

Consider the first type of DDD network \( \Upsilon \) based on the sum of the degrees of each end vertex of each edge.

\[
\begin{align*}
\nu \equiv M_2 G_5 (\Upsilon) &= \frac{1}{2} \left[ (11 \times 13) |E_{13} (\Upsilon (t))| + (11 \times 14) |E_{14} (\Upsilon (t))| + (11 \times 16) |E_{16} (\Upsilon (t))| + (12 \times 14) |E_{14} (\Upsilon (t))| + (13 \times 14) |E_{17} (\Upsilon (t))| + (13 \times 16) |E_{18} (\Upsilon (t))| + (14 \times 14) |E_{19} (\Upsilon (t))| + (14 \times 16) |E_{20} (\Upsilon (t))|, \\
&= (6 \times 6) (4t) + (6 \times 11) (4t) + (6 \times 12) (4) + (6 \times 14) (4t - 4) + (7 \times 9) (4t - 4) + (7 \times 12) (4t - 4) + (8 \times 11) (12t - 8) + (8 \times 13) (4t - 4) + (9 \times 9) (2t - 2) + (9 \times 14) (4t - 4) + (11 \times 11) (9t^2 - 7t + 3) + (11 \times 12) (4t - 4) + (11 \times 13) (4t - 4) + (11 \times 14) (36t^2 - 68t + 32) + (11 \times 16) (4t - 4) + (12 \times 14) (4t - 4) + (13 \times 14) (4t - 4) + (13 \times 16) (4t - 4) + (14 \times 14) (4t - 4) + (14 \times 16) (36t^2 - 76t + 40). \\
\end{align*}
\]

By doing some calculations, we get
\[
\nu := M_2 G_5 (\Upsilon) = 8065 - 20581t + 14697t^2. 
\]

\( \blacksquare \)

**Theorem 2.** Consider the first type of DDD network \( \Upsilon \) for \( t \in \mathbb{N} \). Then the first and second general fifth \( M \)-Zagreb indices are equal to

\[
\begin{align*}
\nu &\equiv M_2 G_5 (\Upsilon) = \sum_{r \in E(\Upsilon)} \left( S_G (r) + S_G (s) \right)^a,
\end{align*}
\]

By using edge partitions in Table 1, we get

\[
\begin{align*}
M_2 G_5 (\Upsilon) &= (6 \times 6) |E_1 (\Upsilon (t))| + (6 \times 11) |E_2 (\Upsilon (t))| + (6 \times 12) |E_3 (\Upsilon (t))| + (6 \times 14) |E_4 (\Upsilon (t))| + (7 \times 9) |E_5 (\Upsilon (t))| + (7 \times 12) |E_6 (\Upsilon (t))| + (8 \times 11) |E_7 (\Upsilon (t))| + (8 \times 13) |E_8 (\Upsilon (t))| + (9 \times 9) |E_9 (\Upsilon (t))| + (9 \times 14) + (11 \times 11) |E_{10} (\Upsilon (t))| + (11 \times 12) |E_{11} (\Upsilon (t))| + (11 \times 13) |E_{12} (\Upsilon (t))| + (11 \times 14) |E_{13} (\Upsilon (t))| + (11 \times 16) |E_{14} (\Upsilon (t))|,
\end{align*}
\]
By doing some calculations, we have

\[\begin{align*}
M_5^G(Y) &= \left[\begin{array}{c}
-2^{16}t^{10} \cdot 4^{18}t^{13} - 4^{17}t^{18} - 4^{17}t^{14} - 4^{16}t^{13} - 2^{16}t^{13} - 2^{14}13 - 12 \times 19 - 4 \times 21 - 3 + 22 - 32 + 25 - 8 \times 27 - 4 \times 29 + 4t
\end{array}\right] \\
M_5^G(Y) &= \left[\begin{array}{c}
-2^{11}t^{10} \cdot 4^{18}t^{13} - 4^{17}t^{18} - 4^{17}t^{14} - 4^{16}t^{13} - 2^{16}t^{13} - 2^{14}13 - 12 \times 19 - 4 \times 21 - 3 + 22 - 32 + 25 - 8 \times 27 - 4 \times 29 + 4t
\end{array}\right]
\end{align*}\]

(16)

From equation (4), we have

\[M_5^G(Y) = \sum_{r \in E(Y)} (S_G (r) + S_G (s))^a,\] (17)

By using edge partitions in Table 1, we get

\[M_5^G(Y) = (6 \times 6)^2[E_1 (Y_1 (t))] + (6 \times 11)^2[E_2 (Y_1 (t))] + (6 \times 12)^2[E_3 (Y_1 (t))] + (6 \times 14)^2[E_4 (Y_1 (t))] \]

+ (7 \times 9)^2[E_5 (Y_1 (t))] + (7 \times 12)^2[E_6 (Y_1 (t))] + (8 \times 11)^2[E_7 (Y_1 (t))] + (8 \times 13)^2[E_8 (Y_1 (t))] + (9 \times 9)^2

\[E_9 (Y_1 (t)) + (9 \times 14)^2[E_{10} (Y_1 (t))] + (11 \times 11)^2[E_{11} (Y_1 (t))] + (11 \times 12)^2[E_{12} (Y_1 (t))]

+ (11 \times 13)^2[E_{13} (Y_1 (t))] + (11 \times 14)^2[E_{14} (Y_1 (t))] + (11 \times 16)^2[E_{15} (Y_1 (t))] + (12 \times 14)^2[E_{16} (Y_1 (t))]

+ (13 \times 14)^2[E_{17} (Y_1 (t))] + (13 \times 16)^2[E_{18} (Y_1 (t))] + (14 \times 14)^2[E_{19} (Y_1 (t))] + (14 \times 16)^2[E_{20} (Y_1 (t))],

\[M_5^G(Y) = (6 \times 6)^2(4t) + (6 \times 11)^2(4t) + (6 \times 12)^2(4t) + (6 \times 14)^2(4t) + (7 \times 9)^2(4t) + (7 \times 12)^2(4t) + (7 \times 12)^2(4t) \]

+ (8 \times 11)^2(12r - 8) + (8 \times 13)^2(4t - 4) + (9 \times 9)^2(2t - 2) + (9 \times 14)^2(4t - 4)

+ (9 \times 14)^2(9t^2 - 7t + 3) + (11 \times 14)^2(4t - 4) + (11 \times 14)^2(36t^2 - 68t + 32) + (11 \times 16)^2(4t - 4)

+ (12 \times 14)^2(4t - 4) + (13 \times 14)^2(4t - 4) + (13 \times 16)^2(4t - 4) + (14 \times 14)^2(4t - 4) + (14 \times 16)^2(36t^2 - 76t + 40).

(18)

By doing some calculations, we have

\[M_5^G(Y) = \left[\begin{array}{c}
-2^{16}t^{10} \cdot 4^{18}t^{13} - 4^{17}t^{18} - 4^{17}t^{14} - 4^{16}t^{13} - 2^{16}t^{13} - 2^{14}13 - 12 \times 19 - 4 \times 21 - 3 + 22 - 32 + 25 - 8 \times 27 - 4 \times 29 + 4t
\end{array}\right]

\[M_5^G(Y) = \left[\begin{array}{c}
-2^{11}t^{10} \cdot 4^{18}t^{13} - 4^{17}t^{18} - 4^{17}t^{14} - 4^{16}t^{13} - 2^{16}t^{13} - 2^{14}13 - 12 \times 19 - 4 \times 21 - 3 + 22 - 32 + 25 - 8 \times 27 - 4 \times 29 + 4t
\end{array}\right]

(19)
Theorem 3. Consider the first type of DDD network $Y_1 \equiv D_1 (t)$ for $t \in \mathbb{N}$. Then, the first and second hyper fifth M-Zagreb indices are equal to

\begin{align*}
HM^1_{G_5}(Y_1) &= 8 \left( 4005 - 10286t + 7407t^2 \right), \\
HM^2_{G_5}(Y_1) &= (1868329 - 4442989t + 27918811t^2).
\end{align*}

Proof. Let $Y_1$ be the first type of DDD network. Table 1 shows such an edge partition of $D_1(t)$. Thus, from equation (5), it follows that

\begin{equation}
HM^1_{G_5}(Y) = \sum_{rs \in E(Y)} (S_G(r) + S_G(s))^2. 
\end{equation}

By using edge partitions in Table 1, we get

\begin{align*}
HM^2_{G_5}(Y_1) &= (6 + 6)^2|E_1(Y_1(t))| + (6 + 11)^2|E_2(Y_1(t))| + (6 + 12)^2|E_3(Y_1(t))| \\
&\quad + (6 + 14)^2|E_4(Y_1(t))| + (6 + 12)^2|E_5(Y_1(t))| \\
&\quad + (6 + 11)^2|E_6(Y_1(t))| + (6 + 13)^2|E_7(Y_1(t))| \\
&\quad + (6 + 9)^2|E_8(Y_1(t))| + (6 + 9)^2|E_9(Y_1(t))| \\
&\quad + (6 + 11)^2|E_{10}(Y_1(t))| + (6 + 12)^2|E_{11}(Y_1(t))| \\
&\quad + (6 + 13)^2|E_{12}(Y_1(t))| + (6 + 14)^2|E_{13}(Y_1(t))| \\
&\quad + (6 + 9)^2|E_{14}(Y_1(t))| + (6 + 9)^2|E_{15}(Y_1(t))| \\
&\quad + (6 + 11)^2|E_{16}(Y_1(t))| + (6 + 12)^2|E_{17}(Y_1(t))| \\
&\quad + (6 + 10)^2|E_{18}(Y_1(t))| + (6 + 11)^2|E_{19}(Y_1(t))| \\
&\quad + (6 + 12)^2|E_{20}(Y_1(t))|,
\end{align*}

By doing some calculations, we have

\begin{equation}
HM^1_{G_5}(Y_1) = 8 \left( 4005 - 10286t + 7407t^2 \right). 
\end{equation}

From equation (6), we have

\begin{align*}
HM^2_{G_5}(Y_1) &= (6 + 6)^2|E_1(Y_1(t))| + (6 + 11)^2|E_2(Y_1(t))| + (6 + 12)^2|E_3(Y_1(t))| \\
&\quad + (6 + 14)^2|E_4(Y_1(t))| + (6 + 12)^2|E_5(Y_1(t))| \\
&\quad + (6 + 11)^2|E_6(Y_1(t))| + (6 + 13)^2|E_7(Y_1(t))| \\
&\quad + (6 + 9)^2|E_8(Y_1(t))| + (6 + 9)^2|E_9(Y_1(t))| \\
&\quad + (6 + 11)^2|E_{10}(Y_1(t))| + (6 + 12)^2|E_{11}(Y_1(t))| \\
&\quad + (6 + 13)^2|E_{12}(Y_1(t))| + (6 + 14)^2|E_{13}(Y_1(t))| \\
&\quad + (6 + 9)^2|E_{14}(Y_1(t))| + (6 + 9)^2|E_{15}(Y_1(t))| \\
&\quad + (6 + 11)^2|E_{16}(Y_1(t))| + (6 + 12)^2|E_{17}(Y_1(t))| \\
&\quad + (6 + 10)^2|E_{18}(Y_1(t))| + (6 + 11)^2|E_{19}(Y_1(t))| \\
&\quad + (6 + 12)^2|E_{20}(Y_1(t))|.
\end{align*}
By doing some calculations, we have
\[ \Rightarrow M_3G_5 (Y_1) = (1868329 - 4442989t + 2791881t^2). \] (26)

**Theorem 4.** Consider the first type of DDD network \( Y_1 \equiv D_1(t) \) for \( t \in \mathbb{N} \). Then, the third \( M \)-Zagreb index is equal to
\[ M_3G_5 (Y_1) = 4\left(7 - 37t + 45t^2\right). \] (27)

\[ M_3G_5 (Y_1) = 6\cdot E_1(Y_1(t)) + 6\cdot E_2(Y_1(t)) + 6\cdot E_3(Y_1(t)) + 6\cdot E_4(Y_1(t)) + 6\cdot E_5(Y_1(t)) + 6\cdot E_6(Y_1(t)) + 6\cdot E_7(Y_1(t)) + 6\cdot E_8(Y_1(t)) + 6\cdot E_9(Y_1(t)) + 6\cdot E_{10}(Y_1(t)) + 6\cdot E_{11}(Y_1(t)) + 6\cdot E_{12}(Y_1(t)) + 6\cdot E_{13}(Y_1(t)) + 6\cdot E_{14}(Y_1(t)) + 6\cdot E_{15}(Y_1(t)) + 6\cdot E_{16}(Y_1(t)) + 6\cdot E_{17}(Y_1(t)) + 6\cdot E_{18}(Y_1(t)) + 6\cdot E_{19}(Y_1(t)) + 6\cdot E_{20}(Y_1(t)) + 6\cdot E_{21}(Y_1(t)) + 6\cdot E_{22}(Y_1(t)). \] (29)

By doing some calculations, we have
\[ \Rightarrow M_3G_5 (Y_1) = 4\left(7 - 37t + 45t^2\right). \] (30)

2.2. Results for Second Type of Dominating David Derived Network. Now, we are calculating fifth \( M \)-Zagreb topological indices of the \( Y_2 \equiv D_2(t) \), where \( t \in \mathbb{N} \) for second type of Dominating David Derived network.

**Theorem 5.** Let \( Y_2 \equiv D_2(t) \) be the second type of Dominating David Derived network, then the first and second fifth \( M \)-Zagreb indices are equal to
\[ M_2G_2 (Y) = \sum_{r \in E(Y)} (S_G(r) \times S_G(s)), \]
\[ M_2G_5 (Y_2) = (6 \times 6)\cdot E_1(Y_2(t)) + (6 \times 2)\cdot E_2(Y_2(t)) + (6 \times 10)\cdot E_3(Y_2(t)) + (6 \times 11)\cdot E_4(Y_2(t)) + (6 \times 12)\cdot E_5(Y_2(t)) + (6 \times 14)\cdot E_6(Y_2(t)) + (6 \times 7)\cdot E_7(Y_2(t)) + (6 \times 12)\cdot E_8(Y_2(t)) + (6 \times 11)\cdot E_9(Y_2(t)) + (6 \times 12)\cdot E_{10}(Y_2(t)) + (6 \times 14)\cdot E_{11}(Y_2(t)) + (6 \times 12)\cdot E_{12}(Y_2(t)) + (6 \times 11)\cdot E_{13}(Y_2(t)) + (6 \times 14)\cdot E_{14}(Y_2(t)) + (6 \times 12)\cdot E_{15}(Y_2(t)) + (6 \times 11)\cdot E_{16}(Y_2(t)) + (6 \times 12)\cdot E_{17}(Y_2(t)) + (6 \times 12)\cdot E_{18}(Y_2(t)) + (6 \times 14)\cdot E_{19}(Y_2(t)) + (6 \times 12)\cdot E_{20}(Y_2(t)) + (6 \times 14)\cdot E_{21}(Y_2(t)) + (6 \times 12)\cdot E_{22}(Y_2(t)). \]

\[ M_1G_5 (Y_2) = 8\left(129 - 349t + 279t^2\right), \]
\[ M_2G_5 (Y_2) = 24\left(321 - 823t + 591t^2\right). \] (31)

**Proof.** The outcome can be obtained by using the edge partition in Table 2. By using equation (1), the first \( M \)-Zagreb index is equal to
\[ \Rightarrow M_2G_2 (Y) = 8\left(129 - 349t + 279t^2\right). \] (32)

**Proof.** Let \( Y_1 \) be the first type of DDD network. Table 1 shows such an edge partition of \( D_1(t) \). Thus, from equation (7), it follows that
\[ M_3G_5 (Y) = \sum_{r \in E(Y)} \left| S_G(r) \times S_G(s) \right|. \] (28)

By using edge partitions in Table 1, we get
\[ M_3G_5 (Y) = 4\left(7 - 37t + 45t^2\right). \] (29)

**Proof.** Let \( Y_1 \) be the first type of DDD network. Table 1 shows such an edge partition of \( D_1(t) \). Thus, from equation (7), it follows that
\[ M_3G_5 (Y) = \sum_{r \in E(Y)} \left| S_G(r) \times S_G(s) \right|. \] (28)

By using edge partitions in Table 1, we get
\[ M_3G_5 (Y) = 4\left(7 - 37t + 45t^2\right). \] (29)
Table 2: Edge partition of second type of Dominating David Derived network \((D_2(t))\) based on sum of degrees of end vertices of each edge.

<table>
<thead>
<tr>
<th>((S_s, S_r)) where (rs \in E(Y_2))</th>
<th>Number of edges</th>
<th>((S_s, S_r)) where (rs \in E(Y_2))</th>
<th>Number of edges</th>
</tr>
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<tbody>
<tr>
<td>((6, 6))</td>
<td>(4t)</td>
<td>((10, 11))</td>
<td>(8t - 4)</td>
</tr>
<tr>
<td>((6, 8))</td>
<td>(4t - 4)</td>
<td>((10, 13))</td>
<td>(4t - 4)</td>
</tr>
<tr>
<td>((6, 10))</td>
<td>(18t^2 - 30r + 14)</td>
<td>((10, 14))</td>
<td>(36t^2 - 72t + 36)</td>
</tr>
<tr>
<td>((6, 11))</td>
<td>(4t)</td>
<td>((11, 12))</td>
<td>(4)</td>
</tr>
<tr>
<td>((6, 12))</td>
<td>(4)</td>
<td>((11, 14))</td>
<td>(4t - 4)</td>
</tr>
<tr>
<td>((6, 14))</td>
<td>(4t - 4)</td>
<td>((11, 16))</td>
<td>(4t - 4)</td>
</tr>
<tr>
<td>((7, 8))</td>
<td>(4t - 4)</td>
<td>((12, 14))</td>
<td>(4t - 4)</td>
</tr>
<tr>
<td>((7, 12))</td>
<td>(4t - 4)</td>
<td>((13, 14))</td>
<td>(4t - 4)</td>
</tr>
<tr>
<td>((8, 11))</td>
<td>(12t - 8)</td>
<td>((13, 16))</td>
<td>(4t - 4)</td>
</tr>
<tr>
<td>((8, 13))</td>
<td>(4t - 4)</td>
<td>((14, 14))</td>
<td>(4t - 4)</td>
</tr>
<tr>
<td>((8, 14))</td>
<td>(4t - 4)</td>
<td>((14, 16))</td>
<td>(36t^2 - 76t + 40)</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
= (6 \times 6)(4t) &+ (6 \times 8)(4t - 4) + (6 \times 10)(18t^2 - 30t + 14) + (6 \times 11)(4t) \\ &+ (6 \times 12)(4) + (6 \times 14)(4t - 4) + (7 \times 8)(4t - 4) + (7 \times 12)(4t - 4) + (8 \times 11)(12t - 8) + (8 \times 13) \\
\cdot (4t - 4) &+ (4t - 4) + (10 \times 11)(8t - 4) + (10 \times 13)(4t - 4) + (10 \times 14)(36t^2 - 72t + 36) \\
&+ (11 \times 12)(4) + (11 \times 14)(4t - 4) + (11 \times 16)(4t - 4) + (12 \times 14)(4t - 4) + \\
\cdot (13 \times 14)(4t - 4) &+ (13 \times 16)(4t - 4) + (14 \times 14)(4t - 4) + (14 \times 16)(36t^2 - 76t + 40).
\end{align*}
\]

By doing some calculations, we get

\[
\Rightarrow M_2 G_5(Y_2) = 24(321 - 823t + 59t^2).
\]
Theorem 6. Consider the second type of DDD network \( Y_2 \equiv D_2(t) \) for \( t \in \mathbb{N} \). Then the first and second general fifth \( M \)-Zagreb indices are equal to

\[
M_1^a G_5(Y) = \sum_{rst \in E(Y)} (S_G(r) + S_G(s))^a.
\]

By using edge partitions in Table 2, we get

\[
M_1^a G_5(Y) = \sum_{rst \in E(Y)} (S_G(r) + S_G(s))^a.
\]

By using edge partitions in Table 2, we get

\[
M_2^a G_5(Y) = \sum_{rst \in E(Y)} (S_G(r) + S_G(s))^a.
\]
Consider the second type of DDD network $\Upsilon_2$.

**Theorem 7.**

By doing some calculations, we have

\[
M^5_2G_5(\Upsilon_2) = \left[ 2^{2+3a}g^a + 4^{1+a}a^{33a} + 2^{3+2a}a_{21a}(t - 1) + 9 \times 4^{1+a}a^{35a}(t - 1)^2 + t4^{1+a}a^{9a} + t2^{2+3a}a^{33a} + \\
48^a(4t - 4) + 56^a(4t - 4) + 104^a(4t - 4) + 112^a(4t - 4) + 130^a(4t - 4) + 154^a \\
(4t - 4) + 168^a(4t - 4) + 176^a(4t - 4) + 182^a(4t - 4) + 196^a(4t - 4) + 208^a(4t - 4) \\
110^a(8t - 4) + 88^a(12t - 8) + 60^a(18t^2 - 30 + 14) + 224^a(36t^2 - 76t + 40) \right].
\]

**(42)**

**Proof.** Let $\Upsilon_2$ be the second type of DDD network. Table 2 shows such an edge partition of $D_2(t)$. Thus, from equation (5), it follows that

\[
HM^5_2G_5(\Upsilon_2) = \sum_{r \in E(\Upsilon)} (S_G(r) + S_G(s))^2.
\]

**(44)**

By using edge partitions in Table 2, we get

\[
M^5_2G_5(\Upsilon_2) = (6 + 6)^2[E_1(\Upsilon_2(t))] + (6 + 8)^2[E_2(\Upsilon_2(t))] + (6 + 10)^2[E_3(\Upsilon_2(t))] + (6 + 11)^2[E_4(\Upsilon_2(t))] \\
+ (6 + 12)^2[E_5(\Upsilon_2(t))] + (6 + 14)^2[E_6(\Upsilon_2(t))] + (7 + 8)^2[E_7(\Upsilon_2(t))] + (7 + 12)^2[E_8(\Upsilon_2(t))] \\
+ (8 + 11)^2[E_9(\Upsilon_2(t))] + (8 + 13)^2[E_{10}(\Upsilon_2(t))] + (8 + 14)^2[E_{11}(\Upsilon_2(t))] \\
+ (10 + 11)^2[E_{12}(\Upsilon_2(t))] + (10 + 13)^2[E_{13}(\Upsilon_2(t))] + (10 + 14)^2[E_{14}(\Upsilon_2(t))] \\
+ (11 + 12)^2[E_{15}(\Upsilon_2(t))] + (11 + 14)^2[E_{16}(\Upsilon_2(t))] + (11 + 16)^2 \\
\cdot |E_{17}(\Upsilon_2(t))| + (12 + 14)^2[E_{18}(\Upsilon_2(t))] + (13 + 14)^2[E_{19}(\Upsilon_2(t))] + (13 + 16)^2[E_{20}(\Upsilon_2(t))] \\
+ (14 + 14)^2[E_{21}(\Upsilon_2(t))] + (14 + 16)^2[E_{22}(\Upsilon_2(t))].
\]

**(45)**
By doing some calculations, we have
\[ \Rightarrow HM^2_G(Y_2) = 8(3875 - 9985t + 7218t^2). \] (46)

By using edge partitions in Table 2, we get
\[ HM^2_G(Y) = \sum_{r \leq s \in E(Y)} (S_G(r) + S_G(s))^2. \] (47)

**Theorem 8.** Consider the second type of DDD network \( Y_2 \equiv D_2(t) \) for \( t \in \mathbb{N} \). Then the third \( M \)-Zagreb index is equal to
\[ M_3G_5(Y_2) = 8(13 - 40t + 36t^2). \] (50)
By doing some calculations, we have
\[ \Rightarrow M_2G_5(Y_2) = 8\left(13 - 40t + 36t^2\right). \] (53)

2.3. Results for Third Type of Dominating David Derived Network. In this section, we calculate degree-based topological indices of the dimension \( t \) for third type of Dominating David Derived Networks. In the coming theorems, we compute M-Zagreb indices and polynomials.

\[ M_1G_5(Y) = \sum_{rs \in E(Y)} (S_G(r) - S_G(s)), \]
\[ M_1G_5(Y_3) = (6 + 6)E_5(Y_3(t)) + (6 + 12)E_2(Y_3(t)) + (6 + 14)E_3(Y_3(t)) + (8 + 10)|E_4(Y_3(t))| + (8 + 12)|E_5(Y_3(t))| + (10 + 16)|E_6(Y_3(t))| + (12 + 12)|E_8(Y_3(t))| + (12 + 14)|E_9(Y_3(t))| + (12 + 16)|E_{10}(Y_3(t))| + (14 + 14)|E_{11}(Y_3(t))| \]
\[ + (14 + 16)|E_{12}(Y_3(t))| + (16 + 16)|E_{13}(Y_3(t))| \]
\[ = (6 + 6)(4t) + (6 + 12)(4t + 4) + (6 + 14)(4t - 4) + (8 + 10)(12t - 12) + (8 + 12)(36t^2 - 44t + 16) + (8 + 14)(4t - 4) + (10 + 16)(4t - 4) + (12 + 12)(8t) + (12 + 14)(8t - 8) + (12 + 16)(36t^2 - 60t + 16) + (14 + 14)(4t - 4) + (14 + 16)(4t + 4) + (16 + 16)(36t^2 - 76t + 40). \]

\[ \Rightarrow M_1G_5(Y_3) = 8\left(179 - 469t + 360t^2\right). \] (56)

Thus, from equation (2),

\[ M_2G_5(Y_3) = 16(720 - 1763t + 1224t^2). \] (58)

\[ M_1G_5(Y_3) = 2^{16}\left[5\times 2^{4t} - 2 \times 9^t + 3 \times 10^t - 11t - 3 \times 15^t + 3 \times 14^t + 15^t + \left\{4 \times 4 + 6 \times 9 - 10^{1+2+3+4} + 11^t + 3 \times 15^t - 14^{1+2+3} + 15^t - 19 \times 16^t\right\} + 9t \times 10^{1+2+3+4} + 16^t\right]. \]
\[ M_2G_5(Y_3) = 2^{48} + 4^{4t} + 8^t + 16^t + 256^t + 36^t + 60^t + 16^t + 96^t + 44^t + 16^t. \] (59)
Proof. Let $Y_3$ be the first type of DDD network. Table 3 shows such an edge partition of $D_1(t)$. Thus, from equation (3), it follows that

$$M^a_{Y_3} = \sum_{rs \in E(Y_3)} (S_G(r) - S_G(s))^a. \tag{60}$$

By using edge partitions in Table 3, we get

$$M^a_{Y_3} = (6 + 6)^a |E_1(Y_3(t))| + (6 + 12)^a |E_3(Y_3(t))| + (6 + 14)^a |E_5(Y_3(t))| + (8 + 10)^a |E_7(Y_3(t))|$$

$$+ (8 + 12)^a |E_8(Y_3(t))| + (8 + 16)^a |E_9(Y_3(t))| + (10 + 12)^a |E_10(Y_3(t))| + (12 + 12)^a |E_12(Y_3(t))|$$

$$+ (12 + 14)^a |E_11(Y_3(t))| + (14 + 14)^a |E_13(Y_3(t))|,$$

$$= (6 + 6)^a (4t) + (6 + 12)^a (4t + 4) + (6 + 14)^a (4t - 4) + (8 + 10)^a (12t - 12)$$

$$+ (8 + 12)^a (36t^2 - 44t + 16) + (8 + 14)^a (4t - 4) + (10 + 12)^a (8t) + (12 + 12)^a (8t - 8)$$

$$+ (12 + 14)^a (36t^2 - 60t + 16) + (14 + 14)^a (4t - 4) + (14 + 16)^a (4t + 4) + (16 + 16)^a (36t^2 - 76t + 40). \tag{61}$$

By doing some calculations, we have

$$\Rightarrow M^a_{Y_3} = 2^{2a} \left( \left[ 5 \times 2^{1+4a} - 2 \times 9^a + 3 \times 10^a - 11^a - 3 \times 13^a + 3 \times 14^a + 15^a \right] + t \left[ 2^{1+4a} 3^a + 6^a + 4 \times 9^a - 10^{1+4a} + 11^a + 3 \times 13^a - 14^{1+4a} + 15^a - 19 \times 16^a \right] + 9t^2 \left[ 10^a + 14^a + 16^a \right] \right). \tag{62}$$

From equation (4), we have

$$M^a_{Y_3} = \sum_{rs \in E(Y_3)} (S_G(r) - S_G(s))^a. \tag{63}$$

By using edge partitions in Table 3, we get

$$M^a_{Y_3} = (6 + 6)^a |E_1(Y_3(t))| + (6 + 12)^a |E_3(Y_3(t))| + (6 + 14)^a |E_5(Y_3(t))| + (8 + 10)^a |E_7(Y_3(t))|$$

$$+ (8 + 12)^a |E_8(Y_3(t))| + (8 + 16)^a |E_9(Y_3(t))| + (10 + 12)^a |E_10(Y_3(t))| + (12 + 12)^a |E_12(Y_3(t))|$$

$$+ (12 + 14)^a |E_11(Y_3(t))| + (14 + 14)^a |E_13(Y_3(t))|,$$

$$= (6 + 6)^a (4t) + (6 + 12)^a (4t + 4) + (6 + 14)^a (4t - 4) + (8 + 10)^a (12t - 12) + (8 + 12)^a$$

$$\times \left[ 36t^2 - 44t + 16 \right] + (8 + 14)^a (4t - 4) + (10 + 16)^a (4t - 4) + (12 + 12)^a (8t) + (12 + 14)^a (8t - 8)$$

$$+ (12 + 16)^a (36t^2 - 60t + 16) + (14 + 14)^a (4t - 4) + (14 + 16)^a (4t + 4) + (16 + 16)^a (36t^2 - 76t + 40). \tag{64}$$
By doing some calculations, we have

\[
\Rightarrow M^2_2G_5(Y_3) = \begin{bmatrix}
2^{3+4t}9^a t + 4^{1+4t}9^a t + 84^a (4t - 4) + 112^a (4t - 4) + 160^a (4t - 4) + 196^a (4t - 4) \\
+72^a (4t + 4) + 224^a (4t + 4) + 168^a (8t - 8) + 80^a (12t - 12) + 256^a (36t^2 - 76t + 40) \\
+192^a (36t^2 - 60t + 16) + 96^a (36t^2 - 44t + 16)
\end{bmatrix}
\]  

(65)

**Theorem 11.** Consider the third type of DDD network \( Y_3 \equiv D_3(t) \) for \( t \in \mathbb{N} \). Then the first and second hyper fifth M-Zagreb indices are equal to

\[
HM^3_1G_5(Y_3) = 48(961 - 2369t + 1866t^2),
\]
\[
HM^2_2G_5(Y_3) = 64(45986 - 102633t + 62784t^2).
\]

(66)

**Proof.** Let \( Y_3 \) be the third type of DDD network. Table 3 shows such an edge partition of \( D_3(t) \). Thus, from equation (5), it follows that

\[
HM^3_1G_5(Y) = \sum_{rs \in E(Y)} (S_G(r) - S_G(s))^2.
\]

(67)

By using edge partitions in Table 3, we get

\[
M^3_2G_5(Y_3) = (6 \times 6)^2 |E_1(Y_3(t))| + (6 + 12)^2 |E_2(Y_3(t))| + (6 + 14)^2 |E_3(Y_3(t))| + (8 + 10)^2 |E_4(Y_3(t))| \\
+ (8 + 12)^2 |E_5(Y_3(t))| + (8 + 14)^2 |E_6(Y_3(t))| + (10 + 16)^2 |E_7(Y_3(t))| + (12 + 12)^2 |E_8(Y_3(t))| \\
+ (12 + 14)^2 |E_9(Y_3(t))| + (12 + 16)^2 |E_{10}(Y_3(t))| + (14 + 14)^2 |E_{11}(Y_3(t))| \\
+ (14 + 16)^2 |E_{12}(Y_3(t))| + (16 + 16)^2 |E_{13}(Y_3(t))|,
\]

(68)

By doing some calculations, we have

\[
\Rightarrow HM^3_1G_5(Y_3) = 48(961 - 2369t + 1866t^2).
\]

(69)

From equation (6), we have

\[
M^3_2G_5(Y_3) = (6 \times 6)^2 |E_1(Y_3(t))| + (6 \times 12)^2 |E_2(Y_3(t))| + (6 \times 14)^2 |E_3(Y_3(t))| + (8 \times 10)^2 |E_4(Y_3(t))| \\
+ (8 \times 12)^2 |E_5(Y_3(t))| + (8 \times 14)^2 |E_6(Y_3(t))| + (10 \times 16)^2 |E_7(Y_3(t))| + (12 \times 12)^2 |E_8(Y_3(t))| \\
+ (12 \times 14)^2 |E_9(Y_3(t))| + (12 \times 16)^2 |E_{10}(Y_3(t))| + (14 \times 14)^2 |E_{11}(Y_3(t))| \\
+ (14 \times 16)^2 |E_{12}(Y_3(t))| + (16 \times 16)^2 |E_{13}(Y_3(t))|,
\]

(70)

By using edge partitions in Table 3, we get

\[
HM^3_1G_5(Y) = \sum_{rs \in E(Y)} (S_G(r) - S_G(s))^2.
\]

By doing some calculations, we have

\[
\Rightarrow HM^3_1G_5(Y_3) = 48(961 - 2369t + 1866t^2).
\]

(71)
By doing some calculations, we have

\[
\Rightarrow HM^2_G(Y_3) = 64\left(45986 - 102633t + 62784t^2\right). 
\]  \hspace{1cm} (72)

\textbf{Theorem 12.} Consider the third type of DDD network \(Y_3 \cong D_3(t)\) for \(t \in \mathbb{N}\). Then the third M-Zagreb index is equal to

\[
M_3G(Y_3) = 8\left(5 - 33t + 36t^2\right). 
\]  \hspace{1cm} (73)

\textbf{Proof.} Let \(Y_3\) be the third type of DDD network. Table 3 shows such an edge partition of \(D_3(t)\). Thus from equation (7) it follows that...
Figure 12: Comparison between $M_1 G_3(Y_3)$ and $M_2 G_3(Y_3)$.

Figure 13: Comparison between $HM_1^2 G_3(Y_3)$ and $HM_2^2 G_3(Y_3)$.

Figure 14: Graphical detail of $M_3 G_3(Y_3)$. 
\[ M_3G_5(Y) = \sum_{r,s \in E(Y)} |S_G(r) - S_G(s)|. \]  

By using edge partitions in Table 3, we get

\[ M_3G_5(Y_3) = 6 - 6\|E_1(Y_3(t))\| + 6 - 12\|E_2(Y_3(t))\| + 6 - 14\|E_3(Y_3(t))\| + 8 - 10\|E_4(Y_3(t))\| 
\]
\[ + [12 - 14\|E_5(Y_3(t))\| + 12 - 16\|E_6(Y_3(t))\| + 10 - 16\|E_7(Y_3(t))\| + [12 - 12\|E_8(Y_3(t))\| + 10 - 16\|E_9(Y_3(t))\| + 16 - 16\|E_{11}(Y_3(t))\|] + 16. \]

By doing some calculations, we have

\[ \Rightarrow M_3G_5(Y_3) = 8(5 - 33t + 36t^2). \]

3. Comparison of Indices

This section contains the comparison of indices and graphical details of different types of DDD networks as given in Figures 6–14.

4. Conclusion

In this manuscript, we computed sum of degree-based indices for some derived graphs of honeycomb structure. We also computed certain sum of degree-based polynomials such as fifth M-Zagreb, fifth hyper M-Zagreb, generalized fifth M-Zagreb indices for all types of Dominating David Derived networks and we also provide comparison of indices in form of graphs, all the graphs are increasing, increase with the value of \( t \). These facts may be useful for people working in computer science and chemistry who encounter honeycomb networks. These results can also play a vital part in the determination of the significance of honeycomb derived networks. As other topological indices, determining the representations of derived graphs like these, is an open question for many other topological indices.

Honeycomb structure is present in different products of sports, aerospace, woodworking and loudspeaker technology due to its physical and chemical properties. Today, mathematicians are working on derived structures of honeycomb to enhance its physical and chemical properties. In future, for more stable structures we will be able to derive some new graphs of honeycomb structure and will find their physical and chemical properties via topological indices.

Data Availability

The data that used for this research paper are included in this article.

Conflicts of Interest

The authors declare no conflicts of interest.

References


