Research Article
Boundary Integral Technique of 2nd Order Partial Differential Equation by Using Radial Basis

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The solution of second order partial differential equation, with continuous change in coefficients by the formation of integral equation and then using radial basis function approximation (RBSA), has been developed in this paper. Use of boundary element method (BEM), which gives the solution of heat or mass diffusion in non-homogenous medium with function varying smoothly in space is also the part of this article. Discretization of boundary of integral domain instead of entire domain of the problem concerned is also the distinction of the recent work. The numerical solution of some problems with known value of the variable has also been included at the end.

1. Introduction


\[
\begin{align*}
\frac{\partial}{\partial x_1} (h_{11} \frac{\partial \hat{w}}{\partial x_1}) + \frac{\partial}{\partial x_1} (h_{12} \frac{\partial \hat{w}}{\partial x_2}) &= 0, \\
\frac{\partial}{\partial x_2} (h_{21} \frac{\partial \hat{w}}{\partial x_1}) + \frac{\partial}{\partial x_2} (h_{22} \frac{\partial \hat{w}}{\partial x_2}) &= 0
\end{align*}
\]

(1)

Equation (1) is valid for two-dimensional steady state flow in anisotropic medium or diffusion of mass in anisotropic medium, where \( \hat{w} \) = temperature or concentration, \( h_{ij} \) = diffusion or conduction coefficient and also \( h_{ij} = h_{ji} \).

Furthermore,

\[
\begin{align*}
h_{11} \Delta_1 \Delta_1 + h_{12} \Delta_1 \Delta_2 &> 0 \\
h_{11} \Delta_1 \Delta_2 + h_{12} \Delta_2 \Delta_2 &> 0
\end{align*}
\]

Such that \( \Delta_1^2 + \Delta_2^2 \neq 0 \). (2)

Equation (2) presents the constraints for the integral base scheme associated with equation (1). Reutskiy [3], Al-Jawary and Wrobel [4], Rangelov et al. [5], and Ferreira [6] studied the behavior of graded material in nonhomogeneous media.

Efficiency in computation and accuracy in treatment makes the numerical method based on integral equation more advantageous for the treatment of such B.V.Ps.

To solve the integral equations derived from such numerical techniques, trial functions play a vital role. Trial functions are of many types such as polynomials, trigonometric functions, and radial basis functions. Radial basis functions as conical and multiquadric radial basis functions have been found useful recently by a number of researchers Lin et al. [7]. Use of radial basis functions in modern era as Lin and Reutskiy [8] has revolutionized the process of research in a number of fields.

Clements [9] and Ooi et al. [10] established the solution of equation (1) with constant value of \( h_{ij} \) but when \( h_{ij} \) is
continuously changing, then solution of equation (1) is really a challenging case.

Suitable fundamental solution of equation (1) for varying coefficient $h_{ij}$ is difficult though not possible. If fundamental solution is used to form integral equation for the special case when $h_{ij} = \tau_{ij}g(x_1, x_2)$. When $g(x_1, x_2)$ is uniformly changing function and $\tau_{ij}$ is constant, then the resulting formulation is not only boundary integral but domain integral containing $\tilde{w}$ as integrand Brebbia and Nardini [11] used dual reciprocity method to find approximate solution in terms of boundary integrals. Ang [12] and Tanaka et al. [13] purposed dual reciprocity method for $h_{ij} = \tau_{ij}g(x_1, x_2)$.

This paper enhanced the use of boundary element method (BEM) for steady state diffusion equation by adding source term and taking $h_{ij}$ as smoothly varying function. No restriction is imposed on $h_{11}, h_{12},$ and $h_{22}$ as in the work of Ang [14], Dineva et al. [15], and Rangogni [16]. Condition of $h_{11}, h_{12},$ and $h_{22}$ is to satisfy the definiteness condition (2) in solution domain. This paper also employs radial basis trial function to approximate $\tilde{w}(x_1, x_2)$ to convert the given diffusion equation to elliptic diffusion equation as Ang et al. [17], Fahmy [18] explain that integral formulation here does not involve any domain integral. At the end, specific problems are solved for unknown by converting the problem into a set of algebraic equations.

2. Steps for Solution

Steady state anisotropic diffusion equation is

$$\frac{\partial}{\partial x_1} \left( h_{11} \frac{\partial \tilde{w}}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( h_{12} \frac{\partial \tilde{w}}{\partial x_2} \right) + g_1(x_1, x_2) \tilde{w} + g_2(x_1, x_2) = 0$$

where $g_1(x_1, x_2)$ is the source term.

$$\tilde{w}(x_1, x_2) = \zeta(x_1, x_2)\text{ for } (x_1, x_2) \in \mathcal{C}_1$$

$$h_{11}(x_1, x_2) n_1(x_1, x_2) \frac{\partial \tilde{w}}{\partial x_1} + h_{12}(x_1, x_2) n_2(x_1, x_2) \frac{\partial \tilde{w}}{\partial x_2}$$

$$= \zeta(x_1, x_2)\text{ for } (x_1, x_2) \in \mathcal{C}_2$$

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2, n_i = \text{unit normal vector to } \mathcal{C}$$

3. Reformation

We rewrite the system of equation (3) as

$$\left[ h_{11}^{(0)} \frac{\partial^2 \tilde{w}}{\partial x_1^2} + \frac{\partial}{\partial x_1} \left( h_{11}^{(0)} \frac{\partial \tilde{w}}{\partial x_1} \right) + h_{12}^{(0)} \frac{\partial^2 \tilde{w}}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_2} \left( h_{12}^{(0)} \frac{\partial \tilde{w}}{\partial x_2} \right) \right]$$

$$+ \left[ h_{12}^{(0)} \frac{\partial \tilde{w}}{\partial x_1} + h_{22}^{(0)} \frac{\partial^2 \tilde{w}}{\partial x_2^2} + \frac{\partial}{\partial x_1} \left( h_{12}^{(0)} \frac{\partial \tilde{w}}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( h_{22}^{(0)} \frac{\partial \tilde{w}}{\partial x_2} \right) \right]$$

$$= -g_1(x_1, x_2) \tilde{w} - g_2(x_1, x_2).$$

(5)

Here, $h_{ij}^{(1)} = h_{ij} - h_{ij}^{(0)}$ are functions of $x_1$ and $x_2$ and varies smoothly, and $h_{ij}^{(0)}$ are constant terms.

Let the substitution be

$$\tilde{w}(x_1, x_2) = v_1(x_1, x_2) + v_2(x_1, x_2),$$

where $v_1$ is related to $\tilde{w}$ by

$$h_{11}^{(0)} \frac{\partial^2 v_1}{\partial x_1^2} + h_{12}^{(0)} \frac{\partial^2 v_1}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_2} \left( h_{12}^{(0)} \frac{\partial \tilde{w}}{\partial x_2} \right)$$

$$+ h_{22}^{(0)} \frac{\partial^2 v_2}{\partial x_2^2} + h_{12}^{(0)} \frac{\partial^2 v_1}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_2} \left( h_{12}^{(0)} \frac{\partial \tilde{w}}{\partial x_2} \right)$$

$$= -g_1(x_1, x_2) \tilde{w} - g_2(x_1, x_2).$$

(7)

$v_2$ is related to $\tilde{w}$ by

$$h_{11}^{(0)} \frac{\partial^2 v_2}{\partial x_1^2} + h_{12}^{(0)} \frac{\partial^2 v_2}{\partial x_1 \partial x_2} + h_{22}^{(0)} \frac{\partial^2 v_2}{\partial x_2^2} + h_{12}^{(0)} \frac{\partial v_2}{\partial x_2} = 0.$$  

(8)

It is obvious that above equations satisfy equation (3). Discretize equation (8) into linear algebraic equations. Solve resulting algebraic equations using B.Cs.

4. Trial Function Substitution

Trial functions such as radial basis function (RBF) are used to approximate unknown and are also used to discretize the domain and numerically solve partial differential equations by considering following approximation:

$$\sum_{j=1}^{N} \left( h_{ij}^{(0)} \frac{\partial \eta_1}{\partial x_j} + h_{ij}^{(1)} \frac{\partial \tilde{w}}{\partial x_j} \right) \equiv \sum_{j=1}^{N} a_i^{(0)}(r)(x_1, x_2).$$

(9)

Here, $a_i^{(0)}$ are constant and $(r)(x_1, x_2)$ are radial basis functions centered at $(\eta_1^{(0)}, \eta_2^{(0)})$.

Using equation (9) in equation (7),
We consider the following substitution:

\[ \alpha'^{(r)} \frac{\partial}{\partial x_1} (r') (x_1, x_2) + \alpha'^{(r)} \frac{\partial}{\partial x_2} (r') (x_1, x_2) \]

\[ = -g_1 (x_1, x_2) \hat{w} - g_2 (x_1, x_2), \tag{10} \]

where \( \tilde{N} \) is the number of interpolation points.

We consider the following substitution:

\[ \hat{w}(x_1, x_2) = \beta_1^{(1)} (x_1, x_2) + \beta_1^{(2)} (x_1, x_2) + \beta_1^{(3)} (x_1, x_2) \]

\[ + \cdots + \mu_1^{(m)} (x_1, x_2), \quad \text{for} \quad \mu_1^{(m)} (x_1, x_2) = d_1^{(1)} (x_1, x_2) \]

\[ + d_1^{(2)} (x_1, x_2) + \cdots + d_1^{(\tilde{N})} (x_1, x_2), \]

where \( \alpha \) is the number of interpolation points.

We rearrange this system of equations for constants \( \beta^{(r)} \) and \( d^{(r)} \) which results in

\[ \hat{w}(x_1, x_2) = \sum_{r'=1}^{\tilde{N}} \sum_{m=1}^{N} \Psi (r^{(m)})(r^{(r)}) \psi^{(m)} (x_1, x_2) \]

\[ v(x_1, x_2) = \sum_{r'=1}^{\tilde{N}} \sum_{m=1}^{N} \Psi (r^{(m)})(r^{(r)}) \nu^{(m)} (x_1, x_2) \] \tag{11}

Equation (9) becomes with the use of equation (12):

\[ \alpha^{(1)} (x_1, x_2) \psi^{(m)} (x_1, x_2) + \alpha^{(2)} (x_1, x_2) \nu^{(m)} (x_1, x_2) + \cdots + \alpha^{(\tilde{N})} (x_1, x_2) \psi^{(m)} (x_1, x_2) \]

\[ = \sum_{m=1}^{\tilde{N}} \Psi (r^{(m)})(r^{(r)}) \psi^{(m)} (x_1, x_2) \]

where \( m = 1, 2, \ldots, \tilde{N} \)

From the above equation,

\[ \Psi^{(r^{(m)})(r^{(r)})} \psi^{(m)} (x_1, x_2) \]

\[ = \mu^{(m)} (x_1, x_2) \]

\[ \nu^{(m)} (x_1, x_2) \]

From equation (14),

\[ \alpha^{(i)} = \sum_{m=1}^{\tilde{N}} \left( t_i^{(r^{(m)})} \psi^{(m)} + O_i \psi^{(m)} \right) \] \tag{16}

with

\[ t_i^{(r^{(m)})} = \sum_{n=1}^{N} g_i^{(r^{(m)})} \phi^{(r^{(m)})} \]

\[ O_i = \sum_{n=1}^{N} \Psi_i^{(r^{(m)})} \phi^{(r^{(m)})} \] \tag{17}

Using equation (16) into equation (10),

\[ g_1^{(\mu^{(m)})} \psi^{(m)} + \cdots + g_1^{(\mu^{(m)})} \psi^{(m)} = -g_2^{(\mu^{(m)})} \]

\[ \text{for} \quad \mu^{(m)} = 1, 2, \ldots, \tilde{N}, \]

where \( g_i^{(\mu^{(m)})} = g_i^{(\mu^{(m)})} \eta_1^{(\mu^{(m)})} \eta_2^{(\mu^{(m)})} \)

Here, the relation for \( \psi^{(m)}(x_1, x_2) \) and \( \nu^{(m)}(x_1, x_2) \) are given as

\[ \hat{w}^{(m)}(x_1, x_2) = \hat{w}^{(\mu^{(m)})} (x_1, x_2) \]

\[ v^{(m)}(x_1, x_2) = \nu^{(m)} (x_1, x_2) \] \tag{13}

\[ \sum_{m=1}^{\tilde{N}} \psi^{(r^{(m)})}(x_1, x_2) \eta_1^{(m)} \eta_2^{(m)} = \begin{cases} 1, & \text{when } r' = S', \\ 0, & \text{when } r' \neq S', \end{cases} \]

We collocate at \( (x_1, x_2) = (\eta_1^{(m)}, \eta_2^{(m)}) \) where \( m' = 1, 2, 3, \ldots, \tilde{N} \).

Equation (15) becomes with the use of equation (12):

\[ \alpha^{(1)} (x_1, x_2) + \alpha^{(2)} (x_1, x_2) + \cdots + \alpha^{(\tilde{N})} (x_1, x_2) \]

\[ = \sum_{m=1}^{\tilde{N}} \mu^{(m)} (x_1, x_2) \psi^{(m)} (x_1, x_2) \]

\[ \sum_{m=1}^{\tilde{N}} \psi^{(r^{(m)})}(x_1, x_2) \eta_1^{(m)} \eta_2^{(m)} = \begin{cases} 1, & \text{when } r' = S', \\ 0, & \text{when } r' \neq S', \end{cases} \]

We remember that \( \psi^{(m)}(x_1, x_2) \) and \( \nu^{(m)}(x_1, x_2) \) is known as radial basis trial function for partial differential equation (7).

Multiquadric radial basis function is

\[ (r_1) (x_1, x_2) = \left[ 1 + \left( x_1 - \eta_1^{(r)} \right)^2 + \left( x_2 - \eta_2^{(r)} \right)^2 \right]^{1/2} \] \tag{20}

We consider the following trial function found in the work of Zhang et al. [19].

\[ (r_1) (x_1, x_2) = \left[ 1 + \left( x_1 - \eta_1^{(r)} \right)^2 + \left( x_2 - \eta_2^{(r)} \right)^2 \right]^{3/2} \] \tag{21}
5. Formation of Boundary Integral Equation

Partial differential equation given by equation (8) can be converted into boundary integral equation as

\[\delta(\eta_1, \eta_2)\frac{\partial}{\partial x_j}(\omega(x_1, x_2)) ds(x_1, x_2),\]

where

\[\delta(\eta_1, \eta_2) = \begin{cases} 1, & \text{when } (\eta_1, \eta_2) \text{ lies in interior of domain } D, \\ \frac{1}{2}, & \text{when } (\eta_1, \eta_2) \text{ lies on smooth part of } \hat{C}. \end{cases}\]

By putting equation (6) into equation (22),

\[\delta(\eta_1, \eta_2)(\hat{\omega}(\eta_1, \eta_2) - v_1(\eta_1, \eta_2)) = \int_C \left( Y(x_1, x_2, \eta_1, \eta_2)(\hat{\omega}(x_1, x_2) - v_1(x_1, x_2)) \right) ds(x_1, x_2).\]  

(24)

where

\[\hat{\omega}(\eta_1, \eta_2) = \sum_{m=1}^{M} \left( w^{m_1} - v_1^{m_1} \right) C^{m_1} Y(x_1, x_2, \eta_1^{m_1}, \eta_2^{m_1}) ds(x_1, x_2)\]

and

\[\sum_{m=1}^{M} \left( P^{m_2} - t_1^{m_2} \right) C^{m_2} \phi(x_1, x_2, \eta_1^{m_2}, \eta_2^{m_2}) ds(x_1, x_2).\]

(28)

For \(n' = M + i\), where \(1 \leq i \leq N\)

It is the boundary integral approximation of partial differential equation (8).

6. Mathematical Methodology

Boundary conditions given in equation (4) can be expressed in terms of algebraic equations as

\[P^{(m_2)} + \sum_{g=1}^{M+N} y^{(m_2g)} \hat{\omega}^{(g)} = \sum_{i=1}^{2} \sum_{j=1}^{2} h_{ij}(x_1, x_2) n_i(x_1, x_2) \frac{\partial \hat{\omega}}{\partial x_j} \text{ is restricted to } C^{x_{ij}}.\]

The coefficient \(y^{(m_2g)}\) is given as

\[\hat{\omega}^{(g)} = \sum_{i=1}^{2} \sum_{j=1}^{2} h_{ij}(x_1, x_2) n_i(x_1, x_2) \frac{\partial \hat{\omega}}{\partial x_j} \text{ is restricted to } C^{x_{ij}}.\]

(30)
To solve the problem above, we apply the following B. Cs:

\[
y^{(m,p)} = \sum_{i=1}^{2} \sum_{j=1}^{2} h_{ij}^{(1)} (x_{i1}, x_{i2}) \phi_{ij}^{(m,p)} \int \frac{\partial}{\partial x_j} \left( r(x_1, x_2) \right) dx_1 dx_2 = \left( \eta_{11}^{(m)}, \eta_{12}^{(m)} \right).
\]  

(31)

To obtain M-set of linear algebraic equations by using equation (12) and second part of equation (25) as

\[
j^{(m,p)} - \sum_{g=1}^{M+N} z^{(m,p)} v^{(g)} = 0, m, p = 1, 2, ..., M,
\]  

(32)

with

\[
z^{(m,p)} = \sum_{i=1}^{2} \sum_{j=1}^{2} h_{ij}^{(0)} (x_{i1}, x_{i2}) \phi_{ij}^{(m,p)} \int \frac{\partial}{\partial x_j} \left( \eta(x_1, x_2) \right) dx_1 dx_2
\]

\[
\left( \eta_{11}, \eta_{12} \right).
\]  

(33)

So, for the solution of boundary value problem, we solve system of 4M + 2N linear algebraic equations given by in equations (18), (28), (29), and (32).

7. Numerical Application of Boundary Integral Technique (B.I.T)

Algorithm for the solution of specific problems by B.I.T using trial basis function introduced in equation (21) has been used in this numerical computation.

PROBLEM#01: Considering the following specific values of \( h_{ij} \) and \( g_i \):

\[
\frac{1}{2} h_{11} = h_{22} = h_{12} = h_{21} = \left( x_1^2 - 2x_1 x_2 + 2 \right)^2, g_1 = g_2 = 0.
\]  

(34)

Here, the domain of the problem is \( 0 < x_i < 1 \) (i = 1, 2).

To solve the problem above, we apply the following B. Cs:

\[
\tilde{w}(x_i, 0) = \frac{1 + x_i}{x_i^2 + 2}
\]

\[
\tilde{w}(x_i, 1) = \frac{x_i}{x_i^2 - 2x_i + 2}
\]

when \( x_1, x_2 \in (0, 1) \).

\[
\tilde{w}(0, x_2) = \frac{1 - x_2}{2}
\]

\[
\tilde{w}(1, x_2) = \frac{2 - x_2}{3 - 2x_2}
\]

The analytic solution of the above problem is

\[
\tilde{w}(x_1, x_2) = \frac{1 + x_1 - x_2}{x_i^2 - 2x_i x_2 + 2}.
\]  

(35)

\( \tilde{w}^{(0)} \) denotes the average value of \( h_{ij} \) on the interior nodes, and its value is

\[
\tilde{w}(x_1, x_2) = \frac{1 + x_1 - x_2}{x_i^2 - 2x_i x_2 + 2}.
\]  

(36)

Analytical and numerical values of \( \tilde{w} \) are compared graphically.

Figure 1 shows comparison of number of boundary elements, and interior collocation points has been drawn graphically. It is worth observing here that the accuracy of solution has improved with more element discretization of boundary curve. Value of \( \tilde{w} \) at some specific points like (10, 4) and (40, 19) is considered to compare with analytic solution of \( \tilde{w} \) at selected interior points. \( \tilde{w} \) used in equation (12) is differentiated w.r.t \( x_i \) (i = 1, 2) in order to derive partial derivatives of first order for \( \tilde{w} \).

PROBLEM#02: Now considering the following values of \( h_{ij} \) and \( g_i \):

\[
\begin{align*}
\frac{1}{2} h_{11} &= e^{x_1} + 1, \\
\frac{1}{2} h_{22} &= e^{-x_1}, \\
\frac{1}{2} h_{12} &= h_{21} = 0 \\
g_1 &= -e^{2x_1} x_2 - 6e^{-x_1} - 2e^{-x_2} - 4, g_2 = 1
\end{align*}
\]  

(37)

Here, the domain of the problem is \( 0 < x_i < 1 \) (i = 1, 2).

To solve the problem above, we apply the following B. Cs:

\[
\begin{align*}
\sum_{j=1}^{2} h_{ij} \frac{\partial \tilde{w}}{\partial x_j} |_{x_j=0} &= -e^{-2x_1} \\
\sum_{j=1}^{2} h_{ij} \frac{\partial \tilde{w}}{\partial x_j} |_{x_j=1} &= -e^{-2(x_1+x_2)} \\
\tilde{w}(0, x_2) &= 0, x_2 \in (0, 1), \\
\tilde{w}(1, x_2) &= e^{-2x_2} \\
\end{align*}
\]  

(38)

Here, again \( \tilde{w}^{(0)} \) denotes the average value of \( h_{ij} \) on the interior nodes, and its value is same as equation (37).

The analytic solution of concerned problem is

\[
\tilde{w}(x_1, x_2) = e^{-2x_1-x_2}.
\]  

(39)

Analytical and numerical values of \( \tilde{w} \) are compared graphically by taking \( x_2 = 0.50 \) and varying \( x_1 \) such that \( x_1 = 0.1, 0.2, ..., 0.9 \).

In the graph above, analytical solution of \( \tilde{w} \) at fixed value of \( x_2 = 0.50 \) is compared with approximated value of \( \tilde{w} \) by taking different values of \( x_1 \). The graph predicts that analytical solution agrees well with the numerical value.
8. Conclusion

Numerical technique used in the article requires only the boundary to be discretized for the solution of 2-D steady state mass diffusion or heat conduction using trial function approximation. Specialty is that it does not include collocation points only but also interior points distributed in a mannered way. The accuracy and validity of the method are verified by applying to a problem with known solutions. The solution obtained numerically agrees well with the known results.

It is also noted here that in this paper, the boundary integral equation, obtained and used in this method, is discretized using elements with constant value and this makes the error as minimum as desired. Also, the reduction in error is observed by increasing the number of boundary elements and related interior collocation points. The selected method based on trial function and boundary integral approximation provides effective and reliable alternatives to all the existing mathematical techniques for the solution of heat and mass conduction in the anisotropic medium. The possibility of further improvement in the work to solve problems related to anisotropic media is also the part of this paper as was performed earlier by Fahmy [20], Marin and Lesnic [21], Baron [22] and Aksoy and Senocak [23], and Dobroskok and Linkov [24].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


