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*Research Article*

# $(\alpha, \beta)$-Normal Operators in Several Variables

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We consider an extension of the concept of $(\alpha, \beta)$-normal operators in single variable operator to tuples of operators, similar to those extensions of the concepts of normality to joint normality, hyponormality to joint hyponormality, and quasi-hyponormality to joint quasi-hyponormality.

## 1. Introduction

Through this paper, we denote Hilbert space over the field of complex numbers $\mathbb{C}$ by $(\mathscr{H}, \langle .|.\rangle)$. We write $\mathscr{B}(\mathscr{H})$ for the set of all bounded linear operators on $\mathscr{H}$. For an operator $T \in \mathscr{B}(\mathscr{H})$, the adjoint, the kernel, and the range of $T$ are denoted by $T^*$, $\mathscr{N}(T)$, and $\mathscr{R}(T)$, respectively. For $T, S \in \mathscr{B}(\mathscr{H})$, $T$ is said to be positive if $\langle Tx \mid x\rangle \geq 0$ for all $x \in \mathscr{H}$ and $T \geq S$ if and only if $T - S \geq 0$. We let $[T, S]: = TS - ST$.

The class of normal operators on Hilbert spaces plays a critical role in operator theory. This class of operators has been generalized by many authors to non-normal operators.

Let $(\alpha, \beta) \in \mathbb{R}^2$ such that $0 \leq \alpha \leq 1 \leq \beta$. An operator $T \in \mathscr{B}(\mathscr{H})$ is called

(1) Normal if $[T^*, T] = 0$ $(\Leftrightarrow \|Tx\| = \|T^*x\|, \forall x \in \mathscr{H})$ [1].

(2) Hyponormal if $[T^*, T] \geq 0$ $(\Leftrightarrow \|Tx\| \geq \|T^*x\|, \forall x \in \mathscr{H})$ [2, 3].

(3) $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$ [4] if

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T, \quad (1)$$

or equivalently

$$\alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\|, \quad (2)$$

for all $x \in \mathscr{H}$. It should be noted that for $\alpha = 1 = \beta$, $T$ is a normal operator. For $\alpha = 1$, we observe that $T^*$ is hyponormal and for $\beta = 1$, we obtain that $T$ is hyponormal. Interested readers can find more details on $(\alpha, \beta)$-normal operators in [4–9]. The concept of $p$-$(\alpha, \beta)$-normal operators was introduced by Senthilkumar and Shanthi [10]. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be $p$-$(\alpha, \beta)$-normal operator for $0 < p \leq 1$ if

$$\alpha^2 (T^*T)^p \leq (TT^*)^p \leq \beta^2 (T^*T)^p, \quad 0 \leq \alpha \leq 1 \leq \beta. \quad (3)$$

When $p = 1$, this coincides with $(\alpha, \beta)$-normal operators.

The study of tuples of commuting operators has received great attention in recent years by many authors, and the interested reader can refer to [11–20] for complete details.

Let $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$ be a $m$-tuple of operators. $T$ is said to be jointly normal if $T_iT_j = T_jT_i$ and every $T_i$ is a normal operator for all $i, j = 1, \ldots, m$ (see [21]).

Given an $m$-tuple $T: = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$, let $[T^*, T] \in \mathscr{B}(\mathscr{H} \oplus \ldots \oplus \mathscr{H})$ denote the self-commutator of $T$, defined by $[T^*, T]_{i,j}: = [T_j^*, T_i]$ for all $i, j \in \{1, \ldots, m\}$. In [22], the author has introduced the concept of joint hyponormality of operators. An $m$-tuple $T = (T_1, \ldots, T_m)$ of operators is called joint hyponormal, if the operator matrix

$$[T^*,T]=\left([T_j^*,T_i]\right)_{i,j=1}^m=\begin{pmatrix}[T_1^*,T_1] & [T_2^*,T_1] & \cdots & [T_m^*,T_1]\\ ,[T_1^*,T_2] & [T_2^*,T_2] & \cdots & [T_m^*,T_2]\\ \vdots & \vdots & \vdots & \vdots\\ [T_1^*,T_n] & [T_2^*,T_n] & \cdots & [T_m^*,T_m]\end{pmatrix}, \tag{4}$$

is positive on the direct sum of $m$ copies of $\mathcal{H}$ (i.e., $\oplus_m\mathcal{H}:=\mathcal{H}\oplus\ldots\oplus\mathcal{H}$ ) or equivalently if $\sum_{1\le i,j\le m}\langle[T_i^*,T_j]u_i|u_j\rangle\ge 0$, for each finite collection $u_1,\ldots,u_m$ of $\mathcal{H}$ (see [22]).

Recently, the first named author in [23] has introduced the concept of joint $m$-quasi-hyponormal tuple of operators as follows. An $m$-tuple of operators $T=(T_1,\cdots,T_m)\in\mathcal{B}(\mathcal{H})^m$ is said to be joint $m$-quasi-hyponormal if and only if $T$ satisfies

$$\sum_{1\le i,j\le m}\langle T_j^*\left[T_j^*,T_i\right]T_iu_j|u_i\rangle\ge 0, \tag{5}$$

for $u_1\cdots,u_m\in\mathcal{H}$. Or equivalently, is joint $m$-quasi-hyponormal tuple if the operator matrix

$$T^*\left[T^*,T\right]T=\left(T_j^*\left[T_j^*,T_i\right]T_i\right)_{i,j=1}^m=\begin{pmatrix}T_1^*[T_1^*,T_1]T_1 & T_2^*[T_2^*,T_1]T_1 & \cdots & T_m^*[T_m^*,T_1]T_1\\ T_1^*[T_1^*,T_2]T_2 & T_2^*[T_2^*,T_2]T_2 & \cdots & T_m^*[T_m^*,T_2]T_2\\ \vdots & \vdots & \vdots & \vdots\\ T_1^*[T_1^*,T_m]T_m & T_2^*[TT_2^*,T_m]T_m & \cdots & T_m^*[T_m^*,T_m]T_m\end{pmatrix}, \tag{6}$$

is positive on the space $\oplus_{k=1}^m\mathcal{H}$.

For $T\in\mathcal{B}(\mathcal{H})$, the numerical radius $\omega(T)$ of an operator $T\in\mathcal{B}(\mathcal{H})$ is defined by

$$\omega(T)=\sup\{|\langle Tx|x\rangle|:\ \|x\|=1\}. \tag{7}$$

These concepts were later generalized by Dekker [24] to joint numerical radius of $T=(T_1,\ldots,T_m)\in\mathcal{B}(\mathcal{H})^m$ as follows.

The joint numerical radius of $T\in\mathcal{B}(\mathcal{H})^m$ is defined [25] by

$$\omega(T)=\sup\left\{\left(\sum_{1\le k\le m}\left|\langle T_kx|x\rangle\right|^2\right)^{1/2},x\in\mathcal{H}:\ \|x\|=1\right\}. \tag{8}$$

Now, for a given $m$-tuple of operators $T=(T_1,\ldots,T_m)$, we consider the joint operator norm of $T$ defined by

$$\|T\|=\sup\left\{\left(\sum_{1\le k\le m}\left\|T_kx\right\|\right)^{1/2},x\in\mathcal{H}:\ \|x\|=1\right\}. \tag{9}$$

Note that $\|T\|$ was introduced by Cho and Takaguchi in [26].

It was proved in [6] that if $T\in\mathcal{B}(\mathcal{H})$ is an $(\alpha,\beta)$-normal operator and $\lambda\in\mathbb{C}$, then

$$\alpha\|T\|^2\le\omega\left(T^2\right)+\frac{2\beta\left\|T-\lambda T^*\right\|^2}{(1+|\lambda|\alpha)^2}, \tag{10}$$

and in [7],

$$\left(1+\alpha^2\right)\|T\|^2\le\frac{1}{2}\left\|T-T^*\right\|^2+\omega\left(T^2\right). \tag{11}$$

The study of classes of operators and related topics is one of the hottest areas in operator theory in Hilbert spaces. In this work, we are going to consider an extension of the concept of $(\alpha,\beta)$-normal operators in single variable to tuples of operators, similar to those extensions of the concepts of normality to joint normality, hyponormality to joint hyponormality, and quasi-hyponormality to joint quasi-hyponormality (for more details, see [21–23]).

The outline of the paper is as follows. Section 2 is devoted to the study of our new class of multioperators. We give several remarks and examples, which try to clarify the context of the concept of joint $(\alpha,\beta)$-normal tuples. We show that the product of a joint $(\alpha,\beta)$-normal tuple by an $m$-tuple of operators satisfying suitable conditions is joint $(\alpha,\beta)$-normal tuple. In Section 3, some inequalities involving joint operator norms and joint numerical radius for joint $(\alpha,\beta)$-normal tuples are proved under suitable conditions.

## 2. Joint $(\alpha,\beta)$-Normal Operators

This section is devoted to the study of the class of jointly $(\alpha,\beta)$-normal operators acting on complex Hilbert spaces. We show some basic results related to this class of operators.

Let $T=(T_1,\ldots,T_m)\in\mathcal{B}(\mathcal{H})^m$ and set

$$[T]=\begin{pmatrix}T_1^*T_1 & T_2^*T_1 & \cdots & T_m^*T_1\\ T_1^*T_2 & T_2^*T_2 & \cdots & T_m^*T_2\\ \vdots & \vdots & \vdots & \vdots\\ T_1^*T_m & T_2^*T_m & \cdots & T_m^*T_m\end{pmatrix}\text{and } [T^*]=\begin{pmatrix}T_1T_1^* & T_1T_2^* & \cdots & T_mT_1^*\\ T_1T_2^* & T_2T_2^* & \cdots & T_1T^*\\ \vdots & \vdots & \vdots & \vdots\\ T_mT_1^* & T_mT_2^* & \cdots & T_mT_m^*\end{pmatrix}. \tag{12}$$

*Definition 1.* Let $T=(T_1,\ldots,T_m)\in\mathcal{B}(\mathcal{H})^m$ be an $m$-tuple of operators and let $0\le\alpha\le 1\le\beta$. We say that $\mathbf{T}$ is a joint $(\alpha,\beta)$-normal tuple of operators if the operator matrix $[T]:=(T_i^*T_j)_{1\le i,j\le m}$ satisfies

$$\alpha^2[T] \leq [T^*] \leq \beta^2[T] \quad \left(\text{or} \quad \beta^2[T] \geq [T^*] \geq \alpha^2[T]\right), \tag{13}$$

on the direct sum of $m$-copies of $\mathscr{H}$ or equivalently

$$\begin{cases} \beta^2\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right) - \sum_{1\leq i,j\leq m} \langle T_i^* x_i | T_j^* x_j\rangle \geq 0, \\ \sum_{1\leq i,j\leq m} \langle T_i^* x_i | T_j^* x_j\rangle - \alpha^2\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right) \geq 0, \\ \text{for} \quad x_1,\cdots,x_m \in \mathscr{H}. \end{cases} \tag{14}$$

*Remark 1.* If $m = 1$, (14) coincides with (1).

*Remark 2.* Let $T = (T_1,\ldots,T_m) \in \mathscr{B}(\mathscr{H})^m$ be a joint normal tuple of operators; then, $T$ is a joint $(1,1)$-normal tuple.

**Proposition 1.** *Let $T = (T_1,\ldots,T_m) \in \mathscr{B}(\mathscr{H})^m$ such that $T_iT_j^* = 0$ for $i \neq j$, $1 \leq i, j \leq m$. Then, $T$ is joint $(\alpha,\beta)$-normal if and only if*

$$\alpha^2\left(\sum_{1\leq k\leq m} \|T_k x_k\|^2\right) \leq \left(\sum_{1\leq k\leq m} \|T_k^* x_k\|^2\right) \leq \beta^2\left(\sum_{1\leq k\leq m} \|T_k x_k\|^2\right), \tag{15}$$

*for $x_1,\ldots,x_m$ in $\mathscr{H}$.*

*Proof.* Under the assumptions that $T_iT_j^* = 0$ for $i \neq j$ and $1\leq i, j\leq m$, it follows that for each finite collections $x_1,\ldots,x_m \in \mathscr{H}$,

$$\begin{gathered} \beta^2\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right) - \sum_{1\leq i,j;\leq m} \langle T_i^* x_i | T_j^* x_j\rangle \geq 0, \\ \beta^2\left(\sum_{1\leq k\leq m} \|T_k x_k\|^2\right) - \left(\sum_{1\leq k\leq m} \|T_k^* x_k\|^2\right) \geq 0, \end{gathered} \tag{16}$$

and

$$\begin{gathered} \sum_{1\leq i,j\leq m} \langle T_i^* x_i | T_j^* x_j\rangle - \alpha^2\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right) \geq 0, \\ \left(\sum_{1\leq k\leq m} \|T_k^* x_k\|^2\right) - \alpha^2\left(\sum_{1\leq k\leq m} \|T_k x_k\|^2\right) \geq 0. \end{gathered} \tag{17}$$

For $z \in \mathbb{C}$, let Re(z) denote the real part of $z$. ◻

**Theorem 1.** *Let $T = (T_1,\ldots,T_m) \in \mathscr{B}(\mathscr{H})^m$ be an $m$-tuple of operators. If $T$ is a joint $(\alpha,\beta)$-normal operator, then*

$$\begin{aligned} \alpha^2 \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right) &\leq \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_i^* x_i | T_j^* x_j\rangle\right) \\ &\leq \beta^2 \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right), \end{aligned} \tag{18}$$

*for $x_1,\ldots,x_m \in \mathscr{H}$.*

*Proof.* Since $T$ is a joint $(\alpha,\beta)$-normal operator, it follows from (14)that

$$\begin{cases} \beta^2 \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right) - \operatorname{Re}\left(\sum_{1\leq i,j\leq;m} \langle T_i^* x_i | T_j^* x_j\rangle\right) \geq 0, \\ \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_i^* x_i | T_j^* x_j\rangle\right) - \alpha^2 \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle\right) \geq 0, \\ \text{for} \quad x_1,\ldots,x_m \in \mathscr{H}. \end{cases} \tag{19}$$

We deduce from the above inequalities that

$$\begin{cases} \beta^2 \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_j x_i \mid T_i x_j\rangle\right) - \operatorname{Re}\left(\sum_{1\leq i,j\leq;m} \langle T_i^* x_i \mid T_j^* x_j\rangle\right) \geq 0, \\ \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_i^* x_i \mid T_j^* x_j\rangle\right) - \alpha^2 \operatorname{Re}\left(\sum_{1\leq i,j\leq m} \langle T_j x_i \mid T_i x_j\rangle\right) \geq 0, \\ \textit{for } x_1,\ldots,x_m \in \mathscr{H}, \end{cases} \tag{20}$$

from which (18) follows.

***Question.*** Assume that $T$ satisfies (18); is it then true that $T$ is a joint $(\alpha,\beta)$-normal operator? ◻

*Remark 3.* If $T = (T_1,\ldots,T_m) \in \mathscr{B}(\mathscr{H})^m$ is joint $(0,1)$-normal, then $T$ is joint hyponormal. In fact, taking $\alpha = 0$ and $\beta = 1$ in (14), we get

$$\left(\sum_{1\leq i,j\leq m} \langle T_i^* T_j x_i | x_j\rangle - \sum_{1\leq i,j\leq m} \langle T_j T_i^* x_i | x_j\rangle\right) \geq 0, \tag{21}$$

for $x_1,\ldots,x_m \in \mathscr{H}$, or equivalently

$$\sum_{1\leq i,j\leq m} \langle (T_i^* T_j - T_j T_i^*) x_i | x_j\rangle = \sum_{1\leq i,j\leq m} \langle [T_i^*, T_j] x_i | x_j\rangle \geq 0, \tag{22}$$

which yields $[T^*, T] \geq 0$ on $\oplus_m \mathscr{H}$. Therefore, $T$ is joint hyponormal.

*Remark 4.* Let $T = (T_1,\ldots,T_m) \in \mathscr{B}(\mathscr{H})^m$ be a commuting tuple of operators. If $T$ is joint $(1,1)$-normal, then $T$ is a joint normal operator.

In fact, since $T$ is joint $(1,1)$-normal, it follows from (14) that

$$\sum_{1\leq i,j\leq m} \langle T_j x_i | T_i x_j\rangle - \sum_{1\leq i,j\leq m} \langle T_i^* x_i | T_j^* x_j\rangle \geq 0, \tag{23}$$

and

$$\sum_{1\le i,j\le m}\langle T_i^* x_i | T_j^* x_j\rangle - \sum_{1\le i,j\le m}\langle T_j x_i | T_i x_j\rangle \ge 0. \quad (24)$$

We deduce that

$$\sum_{1\le i,j\le m}\langle T_i^* x_i | T_j^* x_j\rangle = \sum_{1\le i,j\le m}\langle T_j x_i | T_i x_j\rangle. \quad (25)$$

This implies that

$$\sum_{1\le i,j\le m}\langle T_j T_i^* x | x\rangle = \sum_{1\le i,j\le m}\langle T_i^* T_j x ; | x\rangle, \quad (26)$$

for each $x \in \mathscr{H}$. In particular,

$$\langle\!\left(T_j T_i^* - T_i^* T_j\right) x | x\rangle = 0, \forall\, x \in \mathscr{H} \text{ and } \forall\, i, j = 1, \ldots, m. \quad (27)$$

Consequently, $T_i T_i^* = T_i^* T_i$ for $i = 1, \ldots, m$. So, $T$ is joint normal.

*Example 1.* Consider $T = (T_1, T_2)$ where $T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $T_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. A simple calculation shows that $T$ is joint $(\alpha, \beta)$-normal for all $\alpha, \beta$: $0 \le \alpha \le 1 \le \beta$.

The following example shows that there exists a $m$-tuple of operators $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$ such that each $T_j$ is an $(\alpha, \beta)$-normal operator for all $j = 1, \ldots, m$, but $T = (T_1, \ldots, T_m)$ is not joint $(\alpha, \beta)$-normal.

*Example 2.* Let $\mathscr{H} = \mathbb{C}^2$ and $\{e_i\}_{i=1}^2$ be the standard orthonormal basis of $\mathbb{C}^2$. Consider $T = (T_1, T_2) \in \mathscr{B}(\mathscr{H})^2$ where $T_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. It was observed that $T_1$ and $T_2$ are $(\alpha, \beta)$-normal with $\alpha = \sqrt{3 - \sqrt{5}/2}$ and $\beta = \sqrt{3 + \sqrt{5}/2}$ (see [4, 8]). However, a short calculation shows that

$$\sum_{1\le i,j\le 2}\langle T_i^* e_i | T_j^* e_j\rangle - \frac{3+\sqrt{5}}{2}\left(\sum_{1\le i,j\le 2}\langle T_i e_j | T_j e_i\rangle\right) \ge 0. \quad (28)$$

Consequently, $T$ is not joint $(\sqrt{3 - \sqrt{5}/2}, \sqrt{3 - \sqrt{5}/2})$-normal.

*Example 3.* Let $T \in \mathscr{B}(\mathscr{H})$ be $(\alpha, \beta)$-normal. Then, $T = (T, \ldots, T) \in \mathscr{B}(\mathscr{H})^m$ is joint $(\alpha, \beta)$-normal. Indeed, Since $T$ is $(\alpha, \beta)$-normal, it follows that

$$\alpha^2 \langle Tx | Tx\rangle \le \langle T^* x | T^* x\rangle \le \beta^2 \langle Tx | Tx\rangle, \forall\, x \in \mathscr{H}. \quad (29)$$

Let $x_1, x_2, \ldots, x_m \in \mathscr{H}$. Using these inequalities for $x = \sum_{1\le i\le m} x_i$, we can write

$$\beta^2 \langle T\left(\sum_{1\le i\le m} x_i\right) | T\left(\sum_{1\le i\le m} x_i\right)\rangle - \langle T^*\left(\sum_{1\le i\le mm} x_i\right) | T^*\left(\sum_{1\le i\le m} x_i\right)\rangle \ge 0, \quad (30)$$

and

$$\langle T^*\left(\sum_{1\le i\le m} x_i\right) | T^*\left(\sum_{1\le i\le m} x_i\right)\rangle - \alpha^2 \langle T\left(\sum_{1\le i\le m} x_i\right) | T\left(\sum_{1\le i\le m} x_i\right)\rangle \ge 0. \quad (31)$$

Moreover, one has

$$\beta^2\left(\sum_{1\le i,j\le m}\langle Tx_i | Tx_j\rangle\right) - \left(\sum_{1\le i,j\le m}\langle T^* x_i | T^* x_j\rangle\right) \ge 0, \quad (32)$$

and

$$\left(\sum_{1\le i,j\le m}\langle T^* x_i | T^* x_j\rangle\right) - \alpha^2\left(\sum_{1\le i,j\le m}\langle Tx_i | Tx_j\rangle\right) \ge 0, \quad (33)$$

for $x_1, \ldots, x_m \in \mathscr{H}$. This means that $T$ is joint $(\alpha, \beta)$-normal.

**Proposition 2.** *Let $T \in \mathscr{B}(\mathscr{H})$ and consider $T = (T, \ldots, T) \in \mathscr{B}(\mathscr{H})^m$. Then, $T$ is joint $(\alpha, \beta)$-normal if and only if $T$ is $(\alpha, \beta)$-normal.*

*Proof.* Assume that $T$ is $(\alpha, \beta)$-normal. In view of Example 3, we know that $T$ is joint $(\alpha, \beta)$-normal. Conversely, assume that $T$ is joint $(\alpha, \beta)$-normal. From Definition 1, it follows that

$$\beta^2\left(\sum_{1\le i,j\le m}\langle Tx_i | Tx_j\rangle\right) - \sum_{1\le i,j\le m}\langle T^* x_i | T^* x_j\rangle \ge 0, \quad (34)$$

and

$$\sum_{1\le i,j\le m}\langle T^* x_i | T^* x_j\rangle - \alpha^2\left(\sum_{1\le i,j\le m}\langle Tx_i | Tx_j\rangle\right) \ge 0, \quad (35)$$

for each collection for $(x_i)_{1\le i\le m} \in \mathscr{H}$. In particular, for $x = x_1 = \cdots = x_m$, we get

$$\beta^2 \langle Tx | Tx\rangle - \langle T^* x | T^* x\rangle \ge 0, \quad (36)$$

and

$$\langle T^* x | T^* x\rangle - \alpha^2 \langle Tx | Tx\rangle \ge 0, \quad (37)$$

or equivalently,

$$\alpha^2 \langle Tx | Tx\rangle \le \langle T^* x | T^* x\rangle \le \beta^2 \langle Tx | Tx\rangle, \quad (38)$$

for all $x \in \mathscr{H}$. So, $T$ is $(\alpha, \beta)$-normal.

In the general case, we have the following theorem. □

**Theorem 2.** *Let $T = (T_1, \ldots, T_m)$ be $m$-tuple of operators on $\mathscr{H}$ and let $0 \le \alpha \le 1 \le \beta$. Assume that*

$$\begin{cases} \beta^2 T_i^* T_j - T_j T_i^* = 0 \ \ for\, i \neq j, \\ T_j T_i^* - \alpha^2 T_i^* T_j = 0 \ \ for\, i \neq j. \end{cases} \tag{39}$$

Then, $T = (T_1, \ldots, T_m)$ is joint $(\alpha, \beta)$ normal tuple if and only if each $T_i$ is $(\alpha, \beta)$ normal for $i = 1, \ldots, m$.

*Proof.* We have that $T = (T_1, \ldots, T_m)$ is joint $(\alpha, \beta)$ normal tuple if and only if

$$\begin{cases} \beta^2 [T] - [T^*] \geq 0, \\ [T^*] - \alpha^2 [T] \geq 0. \end{cases} \tag{40}$$

However, by using (37), we get

$$\begin{aligned} \begin{cases} \beta^2 [T] - [T^*] \geq 0, \\ [T^*] - \alpha^2 [T] \geq 0. \end{cases} &\Leftrightarrow \begin{cases} \beta^2 \left( \sum_{1 \leq k \leq m} \|T_k x\|^2 \right) - \left( \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right) \geq 0, \ \ \forall\, x \in \mathscr{H} \\ \left( \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right) - \alpha^2 \left( \sum_{1 \leq k \leq m} \|\|T_k x\|\|^2 \right) \geq 0, \ \ \forall\, x \in \mathscr{H} \end{cases} \\ &\Leftrightarrow \begin{cases} \sum_{1 \leq k \leq m} \left( \beta^2 \|T_k x\|^2 - \|T_k^* x\|^2 \right) \geq 0, \ \ \forall\, x \in \mathscr{H}. \\ \sum_{1 \leq k \leq m} \left( \|T_k^* x\|^2 - \alpha^2 \|T_k x\|^2 \right) \geq 0 \ \ \forall\, x \in \mathscr{H}. \end{cases} \\ &\Leftrightarrow \begin{cases} \left( \beta^2 \|T_k x\|^2 - \|T_k^* x\|^2 \right) \geq 0, \ \ \forall\, x \in \mathscr{H} \\ \left( \|T_k^* x\|^2 - \alpha^2 \|T_k x\|^2 \right) \geq 0, \ \ \forall\, x \in \mathscr{H} \end{cases} \\ &\Leftrightarrow \ \ T_k \ is\ (\alpha, \beta) - normal\ for\ k = 1, \ldots, m. \end{aligned} \tag{41}$$

□

**Proposition 3.** *Let $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$ and let $(\alpha, \beta) \in \mathbb{R}^2$ such that $0 < \alpha \leq 1 \leq \beta$. If $T_i T_j^* = 0$ for all $i \neq j$, then $T$ is joint $(\alpha, \beta)$-normal if and only if $T^* = (T_1^*, \ldots, T_m^*)$ is joint $(1/\beta, 1/\alpha)$-normal.*

*Proof.* Assume that $T$ is joint $(\alpha, \beta)$-normal and prove that $T^*$ is joint $(1/\beta, 1/\alpha)$-normal. To do so, we have the following in view of (15):

$$\begin{aligned} \alpha^2 \left( \sum_{1 \leq k \leq m} \|T_k x\|^2 \right) &\leq \left( \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right) \\ &\leq \beta^2 \left( \sum_{1 \leq k \leq m} \|T_k x\|^2 \right), \forall\, x \in \mathscr{H}. \end{aligned} \tag{42}$$

from which it follows that

$$\begin{aligned} \frac{1}{\beta^2} \left( \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right) &\leq \left( \sum_{1 \leq k \leq m} \|T_k x\|^2 \right) \\ &\leq \left( \frac{1}{\alpha^2} \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right), \forall\, x \in \mathscr{H}. \end{aligned} \tag{43}$$

Therefore, $T^*$ is joint $(1/\beta, 1/\alpha)$-normal tuple. Conversely, if $T^*$ is joint $(1/\beta, 1/\alpha)$-normal, we get

$$\begin{aligned} \frac{1}{\beta^2} \left( \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right) &\leq \left( \sum_{1 \leq k \leq m} \|T_k x\|^2 \right) \\ &\leq \left( \frac{1}{\alpha^2} \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right), \forall\, x \in \mathscr{H}. \end{aligned} \tag{44}$$

This means that

$$\begin{aligned} \alpha^2 \left( \sum_{1 \leq k \leq m} \|T_k x\|^2 \right) &\leq \left( \sum_{1 \leq k \leq m} \|T_k^* x\|^2 \right) \\ &\leq \beta^2 \left( \sum_{1 \leq k \leq m} \|T_k x\|^2 \right), \forall\, x \in \mathscr{H}. \end{aligned} \tag{45}$$

Hence, $T$ is a joint $(\alpha, \beta)$-normal tuple by (13).

The following corollary is an immediate consequence of Proposition 3. □

**Corollary 1.** *Let $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$. Let $(\alpha, \beta) \in \mathbb{R}^2$ such that $0 < \alpha \leq 1 \leq \beta$ and $\alpha\beta = 1$. If $T_i T_j^* = 0$ for $i \neq j$, then $T$ is joint $(\alpha, \beta)$-normal if and only if $T^*$ is joint $(\alpha, \beta)$-normal.*

**Proposition 4.** *Let $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$ be joint $(\alpha, \beta)$-normal. The following statements hold.*

*(1) $\mu T: = (\mu_1 T_1, \ldots, \mu_m T_m)$ is joint $(\alpha, \beta)$-normal for all $\mu: = (\mu_1, \ldots, \mu_m) \in \mathbb{C}^m$.*

*(2) If $V \in \mathscr{B}(\mathscr{H})$ is an isometry, then $VTV^* = (VT_1V^*, \ldots, VT_mV^*)$ is joint $(\alpha,\beta)$-normal.*

*Proof.* (1) Since **T** is joint $(\alpha,\beta)$-normal, it follows for each collection $x_1, \ldots, x_m \in \mathscr{H}$ that

$$\begin{aligned} &= \beta^2\left(\sum_{1\le i,j\le m} \langle \mu_j T_j x_i | \mu_i T_i x_j\rangle\right) \\ &\quad - \left(\sum_{1\le i,j\le m} \langle (\mu_i T_i)^* x_i | (\mu_j T_j)^* x_j\rangle\right) \\ &= \beta^2\left(\sum_{1\le i,j\le m} \langle T_j \overline{\mu_i} x_i | T_i \overline{\mu_j} x_j\rangle\right) \\ &\quad - \left(\sum_{1\le i,j\le m} \langle T_i^* \overline{\mu_i} x_i | T_j^* \overline{\mu_j} x_j\rangle\right) \\ &= \beta^2\left(\sum_{1\le i,j\le m} \langle T_j u_i | T_i u_j\rangle\right) \\ &\quad - \left(\sum_{1\le i,j\le m} \langle T_i^* u_i; | T_j^* u_j\rangle\right) \quad (u_i = \overline{\mu_i} x_i) \\ &\ge 0. \end{aligned} \tag{46}$$

On the other hand, for same reason, we have

$$\begin{aligned} &= \beta^2\left(\sum_{1\le i,j\le m} \langle \mu_j T_j x_i | \mu_i T_i x_j\rangle\right) \\ &\quad - \left(\sum_{1\le i,j\le m} \langle (\mu_i T_i)^* x_i | (\mu_j T_j)^* x_j\rangle\right) \\ &= \beta^2\left(\sum_{1\le i,j\le m} \langle T_j \overline{\mu_i} x_i | T_i \overline{\mu_j} x_j\rangle\right) \\ &\quad - \left(\sum_{1\le i,j\le m} \langle T_i^* \overline{\mu_i} x_i | T_j^* \overline{\mu_j} x_j\rangle\right) \\ &= \beta^2\left(\sum_{1\le i,j\le m} \langle T_j u_i | T_i u_j\rangle\right) \\ &\quad - \left(\sum_{1\le i,j\le m} \langle T_i^* u_i; | T_j^* u_j\rangle\right) \quad (u_i = \overline{\mu_i} x_i) \\ &\ge 0. \end{aligned} \tag{47}$$

Consequently,

$$\alpha^2 [\mu T] \le [\mu T]^* \le \beta^2 [\mu T]. \tag{48}$$

Thus, $\mu T$ is joint $(\alpha,\beta)$-normal as required. (2) Under the assumption that $V$ is an isometry, we have for each collection $x_1, \ldots, x_m \in \mathscr{H}$,

$$\begin{aligned} &\sum_{1\le i,j\le m} \langle (VT_jV^*)^* x_i | (VT_iV^*)^* x_j\rangle \\ &\quad - \alpha^2\left(\sum_{1\le i,j\le m} \langle VT_iV^* x_i | VT_jV^* x_j\rangle\right) \\ &= \sum_{1\le i,j\le m} \langle T_i^* V^* x_i | T_j^* V^* x_j\rangle \\ &\quad - \alpha^2\left(\sum_{1\le i,j\le m} \langle T_jV^* x_i | T_iV^* x_j\rangle\right) \\ &= \sum_{1\le i,j\le m} \langle T_i^* y_i | T_j^* y_j\rangle \\ &\quad - \alpha^2\left(\sum_{1\le i,j\le m} \langle T_j y_i | T_i y_j\rangle\right) \quad (y_i = V^* x_i) \\ &\ge 0. \end{aligned} \tag{49}$$

On the other hand,

$$\begin{aligned} &\beta^2\left(\sum_{1\le i,j\le m} \langle VT_jV^* x_i | VT_iV^* x_j\rangle\right) \\ &\quad - \sum_{1\le i,j\le m} \langle (VT_iV^*)^* x_i | (VT_jV^*)^* x_j\rangle \\ &= \beta^2\left(\sum_{1\le i,j\le m} \langle T_jV^* x_i | T_iV^* x_j\rangle\right) \\ &\quad - \sum_{1\le i,j\le m} \langle T_i^* V^* x_i | T_j^* V^* x_j\rangle \\ &= \beta^2\left(\sum_{1\le i,j\le m} \langle T_j y_i | T_i y_j\rangle\right) - \sum_{1\le i,j\le m} \langle T_i^* y_i | T_j^* y_j\rangle \\ &\ge 0. \end{aligned} \tag{50}$$

More precisely, we have shown that $[VTV^*]$ satisfies

$$\alpha^2 [VTV^*] \le [VTV^*]^* \le \beta^2 [VTV^*]. \tag{51}$$

It is obvious that if $T \in \mathscr{B}(\mathscr{H})$ is $(\alpha,\beta)$-normal, then $\mathscr{N}(T) = \mathscr{N}(T^*)$. The following lemma extended this result to the multivariable case. □

**Lemma 1.** *Let $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$ be joint $(\alpha,\beta)$-normal. Then,*

$$\bigcap_{1\le k\le m} \mathscr{N}(T_k) = \bigcap_{1\le k\le m} \mathscr{N}(T_k^*). \tag{52}$$

*Proof.* Since $T$ is joint $(\alpha,\beta)$-normal, it follows that

$$\beta^2\left(\sum_{1\le i,j\le m} \langle T_j x_i | T_i x_j\rangle\right) - \left(\sum_{1\le i,j\le m} \langle T_i^* x_i | T_j^* x_j\rangle\right) \ge 0, \tag{53}$$

and

$$\left(\sum_{1\le i,j\le m}\langle T_i^* x_i | T_j^* x_j\rangle\right) - \alpha^2\left(\sum_{1\le i,j\le m}\langle T_j x_i | T_i x_j\rangle\right) \ge 0, \quad (54)$$

for $x_1 \cdots, x_m \in \mathcal{H}$. In particular, for $x \in \mathcal{H}$, we get

$$\beta^2\left(\sum_{1\le i,j\le m}\langle T_j x | T_i x\rangle\right) - \left(\sum_{1\le i,j\le m}\langle T_i^* x | T_j^* x\rangle\right) \ge 0, \quad (55)$$

and

$$\left(\sum_{1\le i,j\le m}\langle T_i^* x | T_j^* x\rangle\right) - \alpha^2\left(\sum_{1\le i,j\le m}\langle T_j x | T_i x\rangle\right) \ge 0. \quad (56)$$

If we take $x \in \cap_{1\le k\le m}\mathcal{N}(T_k)$, one can see that

$$\sum_{1\le i,j\le m}\langle T_j^* x | T_i^* x\rangle = 0. \quad (57)$$

This implies that $T_i T_j^* x = 0$; $\forall i, j = 1, \dots, m$, and we deduce that $T_j^* x = 0$ for all $j = 1, \dots, m$. So, $x \in \cap_{1\le k\le m}\mathcal{N}(T_k)$. Conversely, let $x \in \cap_{1\le k\le m}\mathcal{N}(T_k)$. From the above inequalities, we infer immediately that

$$\sum_{1\le i,j\le m}\langle T_j x | T_i x\rangle = 0. \quad (58)$$

Hence, we always have $T_i^* T_j x = 0$; $\forall i, j = 1, \dots, m$, and we deduce that $T_j x = 0$ for all $j = 1, \dots, m$. This yields $x \in \cap_{1\le k\le m}\mathcal{N}(T_k)$. □

**Proposition 5.** *Let $T = (T_1, \dots, T_m) \in \mathcal{B}(\mathcal{H})^m$ and let $S = (S_1, \dots, S_m) \in \mathcal{B}(\mathcal{H})^m$ such that $S_j^* T_i = 0$ for $i, j = 1, \dots, m$. If $T$ and $S$ are joint $(\alpha, \beta)$-normal, then so is $T + S = (T_1 + S_1, \dots, T_m + S_m)$.*

*Proof.* Let $x_1, \dots, x_m \in \mathcal{H}$. Under the assertions that $S_j^* T_i = 0$ for $i, j = 1, \dots, m$, we get

$$\begin{aligned}\langle (T_j + S_j)x_i | (T_i + S_i)x_j\rangle &= \langle (T_j x_i) | T_i x_j\rangle \\ &\quad + \langle S_j x_i | S_i x_j\rangle \text{ for } i, j = 1, \dots, m,\end{aligned} \quad (59)$$

and

$$\begin{aligned}\langle (T_i + S_i)^* x_i | (T_j + S_j)^* x_j\rangle &= \langle (T_i^* x_i) | T_j^* x_j\rangle \\ &\quad + \langle S_i^* x_i | S_j^* x_j\rangle \text{ for } i, j = 1, \dots, m.\end{aligned} \quad (60)$$

Since **T** and **S** are $(\alpha, \beta)$-normal, it follows that

$$\begin{aligned} &= \sum_{1\le i,j\le m}\left(\langle (T_i + S_i)^* x_i | (T_j + S_j)^* x_j\rangle\right) - \alpha^2 \sum_{1\le i,j\le m}\left(\langle (T_j + S_j)x_i | (T_i + S_i)x_j\rangle\right) \\ &\quad \underbrace{\sum_{1\le i,j\le m}\langle (T_i^* x_i | T_j^* x_j\rangle - \alpha^2 \sum_{1\le i,j\le m}\langle T_j x_i | T_i x_j\rangle}_{\ge 0} + \underbrace{\sum_{1\le i,j\le m}\langle S_i^* x_i | S_j^* x_j\rangle - \alpha^2 \sum_{1\le i,j\le m}\langle S_j x_i | S_i x_j\rangle}_{\ge 0} \\ &\ge 0, \end{aligned} \quad (61)$$

and

$$\begin{aligned} &\beta^2\left(\sum_{1\le i,j\le m}\left(\langle (T_j + S_j)x_i | (T_i + S_i)x_j\rangle\right)\right) - \left(\sum_{1\le i,j\le m}\left(\langle (T_i + S_i)^* x_i | (T_j + S_j)^* x_j\rangle\right)\right) \\ &= \underbrace{\beta^2\left(\sum_{1\le i,j\le m}\langle (T_i x_j) | T_j x_i\rangle\right) - \left(\sum_{1\le i,j\le m}\langle T_i^* x_i | T_j^* x_j\rangle\right)}_{\ge 0} + \underbrace{\beta^2\left(\sum_{1\le i,j\le m}\langle S_j x_i | S_i x_j\rangle\right) - \left(\sum_{1\le i,j\le m}\langle S_i^* x_i | S_j^* x_j\rangle\right)}_{\ge 0} \\ &\ge 0. \end{aligned} \quad (62)$$

From the previous calculations, we see that $T + S$ is a joint $(\alpha, \beta)$-normal operator.

In [26, Theorem 4], the authors have proved that if $T \in \mathcal{B}(\mathcal{H})$ is an $(\alpha, \beta)$-normal operator and $S \in \mathcal{B}(\mathcal{H})$ is an $(\alpha, \beta)$-normal operator such that $T^* S = S T^*$, then $TS$ and $ST$ are $(\alpha\alpha', \beta\beta')$-normal operators. The following example proves that even if $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$ are joint $(\alpha, \beta)$-normal operators, their product $TS = (T_1 S_1, \dots, T_m S_m)$ is not in general a joint $(\alpha, \beta)$-normal operator. □

*Example 4*

(1) Consider $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ which are $(\alpha, \beta)$-normal and their product $TS = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ is $(\alpha, \beta)$-normal. In view of Proposition 2, it follows that $T = (T, \ldots, T)$, $S = (S, \ldots, S)$, and $TS = (TS, \ldots, TS)$ are joint $(\alpha, \beta)$-normal operators.

(2) Consider $T_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $S_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ which are $(\alpha, \beta)$-normal, whereas their product $T_1 S_1 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$ is not $(\alpha, \beta)$-normal. Hence, by applying Proposition 2, we observe that $T_1 = (T_1, \ldots, T_1)$ and $S_1 = (S_1, \ldots, S_1)$ are joint $(\alpha, \beta)$-normal operators. However, $T_1 S_1 = (T_1 S_1, \ldots, T_1 S_1)$ is not a joint $(\alpha, \beta)$-normal operator.

In the following theorems, we show that the product of a joint $(\alpha, \beta)$-normal tuple by an $m$-tuple of operators satisfying suitable conditions is joint $(\alpha, \beta)$-normal tuple.

**Theorem 3.** *Let $T = (T_1, \ldots, T_m) \in \mathcal{B}(\mathcal{H})^m$ be joint $(\alpha, \beta)$-normal and let $A = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$ be joint normal operators such that*

$$T_j A_i = A_i T_j \quad \text{for all } j,\ i = 1, \ldots, m. \tag{63}$$

Then, $TA = (T_1 A_1, \ldots, T_m A_m)$ is a joint $(\alpha, \beta)$-normal operator.

*Proof.* Since $A$ is joint normal and $A_i T_j = T_j A_i$ for all $i, j = 1, \ldots, m$, it follows from Fuglede theorem [1] that

$$A_i^* A_j = A_j A_i^* \text{ an d } A_i^* T_j = T_j A_i^* \ \forall i, j = 1, \ldots, m. \tag{64}$$

By elementary calculations based on the above arguments, we get for all $i, j = 1, \ldots, m$ and for $x_1, \ldots, x_m \in \mathcal{H}$,

$$\langle T_j A_j x_i | T_i A_i x_j \rangle = \begin{matrix} \langle T_j A_j x_i | T_i A_i x_j \rangle \\ \langle A_j T_j x_i | A_i T_i x_j \rangle \\ \langle A_i^* A_j T_j x_i | T_i x_j \rangle \\ \langle T_j A_i^* x_i | T_i A_j^* x_j \rangle \end{matrix}. \tag{65}$$

We have

$$\begin{aligned} &\sum_{1 \le i,j \le m} \langle (T_i A_i)^* x_i | (T_j A_j)^* x_j \rangle - \alpha^2 \left( \sum_{1 \le i,j \le m} \langle T_j A_j x_i | T_i A_i x_j \rangle \right) \\ &= \left( \sum_{1 \le i,j \le m} \langle T_i^* A_i^* x_i | T_j^* A_j^* x_j \rangle \right) - \alpha^2 \left( \sum_{1 \le i,j \le m} \langle T_j A_i^* x_i | T_i A_j^* x_j \rangle \right) \\ &= \sum_{1 \le i,j \le m} \langle T_j^* z_i | T_i^* z_j \rangle - \alpha^2 \left( \sum_{1 \le i,j \le m} \langle T_j z_i | T_i z_j \rangle \right) \quad (z_i = A_i^* x_i) \\ &\ge 0, \end{aligned} \tag{66}$$

and

$$\begin{aligned} &\beta^2 \left( \sum_{1 \le i,j \le m} \langle T_j A_j x_i | T_i A_i x_j \rangle \right) - \sum_{1 \le i,j \le m} \langle (T_i A_i)^* x_i | (T_j A_j)^* x_j \rangle \\ &= \beta^2 \left( \sum_{1 \le i,j \le m} \langle T_j A_i^* x_i | T_i A_j^* x_j \rangle \right) - \left( \sum_{1 \le i,j \le m} \langle T_i^* A_i^* x_i | T_j^* A_j^* x_j \rangle \right) \\ &= \beta^2 \left( \sum_{1 \le i,j \le m} \langle T_j z_i | T_i z_j \rangle \right) - \sum_{1 \le i,j \le m} \langle T_j^* z_i | T_i^* z_j \rangle \quad (z_i = A_i^* x_i) \\ &\ge 0. \end{aligned} \tag{67}$$

So, $TA$ satisfies (14). Thus, the proof is complete. □

**Corollary 2.** *Let $T = (T_1, \ldots, T_m) \in \mathcal{B}(\mathcal{H})^m$ be joint $(\alpha, \beta)$-normal and let $A = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$ be a commuting tuple of operators such that each $A_i$ is self-adjoint for $i = 1, \ldots, m$. If $T_j A_i = A_i T_j$ for all $i, j = 1, \ldots, m$, then $TA = (T_1 A_1, \ldots, T_m A_m)$ is joint $(\alpha, \beta)$-normal.*

*Proof.* This is an immediate consequence of Theorem 3. □

**Theorem 4.** *Let $T = (T_1, \ldots, T_m) \in \mathcal{B}(\mathcal{H})^m$ be a joint $(\alpha, \beta)$-normal tuple and $S = (S_1, \ldots, S_m) \in \mathcal{B}(\mathcal{H})^m$ such that $S_l^* S_k = S_k S_l^*$ and $S_l^* T_k = T_k S_l^*$ for $(k, l) \in \{1, \ldots, m\}^2$; then, $ST: = (S_1 T_1, \ldots, S_m T_m)$ is a joint $(\alpha, \beta)$-normal tuple.*

*Proof.* Let $x_1, \ldots, x_m \in \mathcal{H}$; under the assumptions

$$S_l^* S_k = S_k S_l^* \text{ and } T_l S_k^* = S_k^* T_l, \ \forall (k, l) \in \{1, \ldots, m\}^2 \tag{68}$$

and the fact that $T$ is jointly $(\alpha, \beta)$-normal, it follows that

$$\begin{aligned} &\beta^2 \left( \sum_{1 \le i,j \le m} \langle S_j T_j x_i | S_i T_i x_j \rangle \right) - \left( \sum_{1 \le i,j \le m} \langle (S_i T_i)^* x_i | (S_j T_j)^* x_j \rangle \right) \\ &= \beta^2 \left( \sum_{1 \le i,j \le m} \langle T_j S_i^* x_i | T_i S_j^* x_j \rangle \right) - \left( \sum_{1 \le i,j \le m} \langle T_i^* S_i^* x_i | T_j^* S_j^* x_j \rangle \right) \\ &\ge \beta^2 \left( \sum_{1 \le i,j \le m} \langle T_j y_i | T_i y_j \rangle \right) - \left( \sum_{1 \le i,j \le m} \langle T_i^* y_i | T_j^* y_j \rangle \right) \quad (y_i = S_i^* x_i) \\ &\ge 0. \end{aligned} \tag{69}$$

Similarly,

$$\begin{aligned} &\left( \sum_{1 \le i,j \le m} \langle (S_i T_i)^* x_i | (S_j T_j)^* x_j \rangle \right) - \alpha^2 \left( \sum_{1 \le i,j \le m} \langle (S_j T_j) x_i | (S_i T_i) x_j \rangle \right) \\ &= \left( \sum_{1 \le i,j \le m} \langle T_i^* S_i^* x_i | T_j^* S_j^* x_j \rangle \right) - \alpha^2 \left( \sum_{1 \le i,j \le m} \langle T_j S_i^* x_i | T_i S_j^* x_j \rangle \right) \\ &\ge \beta^2 \left( \sum_{1 \le i,j \le m} \langle T_i^* y_i | T_j^* y_j \rangle \right) - \alpha^2 \left( \sum_{1 \le i,j \le m} \langle T_j y_i | T_i y_j \rangle \right) \quad (y_i = S_i^* x_i) \\ &\ge 0. \end{aligned} \tag{70}$$

□

## 3. Some Inequalities for Joint $(\alpha, \beta)$-Normal Operators

Finding some inequalities involving norms and numerical radius of Hilbert space operators is a topic which has been considered by many mathematicians in recent years. Recently, a generalization of the numerical radius, known as joint numerical radius, has been introduced and studied and several inequalities containing numerical radius are extended.

**Proposition 6.** *Let $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$ such that $T_iT_j = T_iT_j^* = 0$ for $i \neq j, 1 \leq i, j \leq m$. If $T$ is joint $(\alpha, \beta)$-normal tuple, then*

$$\omega(T)^2 \leq \frac{1}{2}\left(\beta\|T\| + \omega\left(T^2\right)\right), \tag{71}$$

*where $T^2 := (T_1^2, \ldots, T_m^2)$.*

*Proof.* By taking into account the well-known inequality (see [6]),

$$\left|\langle u|w_0\rangle\langle w_0|v\rangle\right| \leq \frac{1}{2}\left(\|u\|\|v\| + |\langle u|v\rangle|\right), \tag{72}$$

for any $u, v, w_0 \in \mathscr{H}$ with $\|w_0\| = 1$. Let $x \in \mathscr{H}$ with $\|x\| = 1$ and set $u = \sum_{1\leq j\leq m} T_j x$ and $v = \sum_{1\leq j\leq m} T_j x$. We get

$$\begin{aligned} &\left|\langle \sum_{1\leq j\leq m} T_j x|x\rangle\langle x|\sum_{1\leq j\leq m} T_j^* x\rangle\right| \\ &\leq \frac{1}{2}\left(\sum_{1\leq j\leq m} T_j x\right\|\sum_{1\leq j\leq m} T_j^* x\| + \left|\langle \sum_{1\leq j\leq m} T_j x|\sum_{1\leq j\leq m} T_j^* x\rangle\right|\Bigg) \\ &= \frac{1}{2}\left(\left(\sum_{1\leq j\leq m} T_j x\|^2\right)^{1/2}\left(\sum_{1\leq j\leq m} T_j^* x\|^2\right)^{1/2} + \left|\sum_{1\leq j\leq m}\langle T_j^2 x|x\rangle\right|\right) \\ &\leq \frac{1}{2}\left(\left(\sum_{1\leq j\leq m} T_j x\|^2\right)^{1/2}\left(\beta^2\sum_{1\leq j\leq m} T_j x\|^2\right)^{1/2} + \sum_{1\leq j\leq m}\left|\langle T_j^2 x|x\rangle\right|\right) \\ &\leq \frac{1}{2}\left(\beta\left(\sum_{1\leq j\leq m} T_j x\|^2\right) + \sum_{1\leq j\leq m}\left|\langle T_j^2 x|x\rangle\right|\right), \end{aligned} \tag{73}$$

from which it follows that

$$\begin{aligned} &\left|\langle \sum_{1\leq j\leq m} T_j x|x\rangle^2\right| \\ &\leq \frac{1}{2}\left(\beta\left(\sum_{1\leq j\leq m} T_j x\|^2\right) + \sum_{1\leq j\leq m}\left|\langle T_j^2 x|x\rangle\right|\right), \end{aligned} \tag{74}$$

so that

$$\begin{aligned} &\sup_{\|x\|=1}\left|\langle \sum_{1\leq j\leq m} T_j x \mid x\rangle^2\right| \\ &\leq \frac{1}{2}\left(\beta\sup_{\|x\|=1}\left(\|\sum_{1\leq j\leq m} T_j x\|^2\right)\right. \\ &\quad\left. + \sup_{\|x\|=1}\left(\sum_{1\leq j\leq m}\left|\langle T_j^2 x \mid x\rangle\right|\right)\right), \end{aligned} \tag{75}$$

$$\omega(T)^2 \leq \frac{1}{2}\left(\beta\|T\| + \omega\left(T^2\right)\right).$$

□

**Theorem 5.** *Let $T = (T_1, \ldots, T_m) \in \mathscr{B}(\mathscr{H})^m$ be a joint $(\alpha, \beta)$-normal tuple. If $T_iT_j = T_iT_j^* = 0$ for $i \neq j, 1 \leq i, j \leq m$, then*

$$\alpha\|T\|^2 \leq \omega\left(T^2\right) + \frac{2\beta}{(1+\alpha\lambda)^2}\left\|T - \lambda T^*\right\|^2, \quad \lambda \in \mathbb{C}. \tag{76}$$

*Proof.* Taking into account the following inequality ([6]):

$$\|u\|\|v\| \leq |\langle u|v\rangle| + \frac{\|u\|\|v\|\|u-v\|^2}{(\|u\| + \|v\|)^2}, \quad u, v \in \mathscr{H} \quad u, v \neq 0, \tag{77}$$

for $x_1, \ldots, x_m \in \mathscr{H}$, set $u = \sum_{1\leq k\leq m} T_k x_k$ and $v = \lambda\sum_{1\leq k\leq m} T_k^* x_k$.

Since $T_iT_j = 0$ for $i \neq j$, we get

$$\begin{aligned} \left\|\sum_{1\leq k\leq m} T_k x_k\right\|\left\|\lambda\sum_{1\leq k\leq m} T_k^* x_k\right\| &= \left(\left\|\sum_{1\leq k\leq m} T_k x_k\right\|^2\right)^{1/2}|\lambda|\left(\left\|\sum_{1\leq k\leq m} T_k^* x_k\right\|^2\right)^{1/2} \\ &= \left(\sum_{1\leq k\leq m}\left\|T_k x_k\right\|^2\right)^{1/2}|\lambda|\left(\sum_{1\leq k\leq m}\left\|T_k^* x_k\right\|^2\right)^{1/2} \\ &\geq \alpha|\lambda|\left(\sum_{1\leq k\leq m}\left\|T_k x_k\right\|^2\right) \quad (by\,(2.2)). \end{aligned} \tag{78}$$

Similarly, we get from (13),

$$\left\|\sum_{1\le k\le m} T_k x_k\right\| \left\|\lambda \sum_{1\le k\le m} T_k^* x_k\right\| \le |\lambda|\beta\left(\sum_{1\le k\le m} \|T_k x_k\|^2\right). \tag{79}$$

Moreover,

$$\left|\left\langle \sum_{1\le k\le m} T_k x_k \lambda \sum_{1\le k\le m} T_k^* x_k\right\rangle\right| = |\lambda| \left|\sum_{1\le k\le m} \langle T_k^2 x | x\rangle\right| \le |\lambda| \sum_{1\le k\le m} \left|\langle T_k^2 x | x\rangle\right|, \tag{80}$$

and

$$\begin{aligned}\left(\left\|\sum_{1\le k\le m} T_k x_k\right\| + \left\|\lambda \sum_{1\le k\le m} T_k^* x_k\right\|\right)^2 &= \left(\left(\left\|\sum_{1\le k\le m} T_k x_k\right\|^2\right)^{1/2} + |\lambda|\left(\left\|\sum_{1\le k\le m} T_k^* x_k\right\|^2\right)^{1/2}\right)^2 \\ &= \left(\left(\sum_{1\le k\le m} \|T_k x_k\|^2\right)^{1/2} + |\lambda|\left(\sum_{1\le k\le m} \|T_k^* x_k\|^2\right)^{1/2}\right)^2 \\ &\ge (1+|\lambda|\alpha)^2\left(\sum_{1\le k\le m} \|T_k x_k\|^2\right) \quad (\text{by } (2.2)).\end{aligned} \tag{81}$$

From the above calculations, we get

$$\begin{aligned}\alpha|\lambda|\left(\sum_{1\le k\le m} \|T_k x_k\|^2\right) &\le |\lambda| \sum_{1\le k\le m} \left|\langle T_k^2 x_k | x_k\rangle\right| \\ &+ \frac{2|\lambda|\beta}{(1+|\lambda|\alpha)^2}\left\|\sum_{1\le k\le m} (T_k - \lambda T_k^*) x_k\right\|.\end{aligned} \tag{82}$$

For $x = x_1 = \cdots = x_m$, we get

$$\begin{aligned}\alpha|\lambda|\left(\sum_{1\le k\le m} \|T_k x\|^2\right) &\le |\lambda| \sum_{1\le k\le m} \left|\langle T_k^2 x | x\rangle\right| \\ &+ \frac{2|\lambda|\beta}{(1+|\lambda|\alpha)^2} \sum_{1\le k\le m} (T_k - \lambda T_k^*) x\|.\end{aligned} \tag{83}$$

Taking the supremum in (83) over $x \in \mathscr{H}$, $\|x\| = 1$, we get the desired inequality (76). □

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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