Research Article

On an Explicit Characterization of Spherical Curves in Dual Lorentzian 3-Space $D^3_1$

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Received 30 March 2022; Accepted 18 May 2022; Published 25 June 2022

Academic Editor: Francesco Tornabene

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In this study, we developed a mathematical framework for studying spherical and null curves in dual Lorentzian 3-space $D^3_1$. Considering the causal character of the dual curve, sufficient and necessary conditions for spacelike curves to be dual Lorentzian spherical were given. Also, new sufficient and necessary conditions for non-null curves to be hyperbolic dual spherical were presented. Then, an insight on curves with null-principal normal is reported.

1. Introduction

Nowadays, with the evolution of the relativity theory, geometers have extended some subjects’ geometrical studies of Riemannian manifolds to that of Lorentzian manifold. For example, in the Euclidean 3-space $E^3$, a curve for which the tangent vector has a constant angle with a fixed direction is named a helix (distinguished by $\tau/\kappa$ is a nonconstant linear function of the arc length parameter), spherical curves (distinguished by $(\sigma \rho')^2 + \rho^2 = \text{constant}$, with $\kappa \rho = 1$, $\sigma \tau = 1$), and rectifying curves (distinguished by $\tau/\kappa$ is a nonconstant linear function of the arc length parameter), where $\tau$ and $\kappa$ are the torsion and curvature of the curve, respectively (see [1–5]). Due to its connection with physical sciences in Minkowski space, the helices, spherical, normal, and rectifying curves have been studied widely in [6–13].

Dual numbers were originally introduced to the subject of geometrical studies and later exploited to deal with the spatial kinematics and applied mechanics [14–17]. E. Study used it as an instrument for his studies on the differential line geometry. He dedicated specific attention to the representation of directed lines by dual unit vectors and considered the mapping that is recognized by his name. The set of dual points on the dual unit sphere in dual 3-space $D^3_1$ is in one-to-one correspondence with the set of all directed lines in Euclidean 3-space $E^3$. Specific details on the needful fundamental concepts concerning with the dual elements and the one-to-one correspondence between the ruled surface and the one-parameter dual spherical motion can be found in [14–17]. If we take the Minkowski 3-space $E^3_1$ instead of $E^3$, the E. Study map can be stated as follows: The set of dual points on the hyperbolic (Lorentzian) dual unit sphere $H^3_1$ (resp., $S^3_1$) in the Lorentzian dual 3-space $D^3_1$ is in one-to-one correspondence with the set of all timelike (resp., spacelike) directed lines in Minkowski 3-space $E^3_1$.

In this work, we give some new characterizations of the spherical and null curves in $D^3_1$. Then, several known properties of the helices, spherical, normal, and rectifying curves at the Euclidean 3-space $E^3$ are extended to the dual Lorentzian 3-space $D^3_1$. In addition, new sufficient and necessary conditions for non-null curves to be hyperbolic dual spherical were presented. Also, the case of null dual curves to be spherical in dual Lorentzian 3-space was examined. Therefore, this work gives a connection with the classical surface theory since a differentiable curve on the hyperbolic (resp., Lorentzian) dual unit sphere $H^3_1$ (resp., $S^3_1$) corresponds to a timelike (resp., spacelike or timelike) ruled surface in $E^3_1$ [18–21].
2. Preliminaries

It is started with some basic concepts on the theory of dual numbers, dual Lorentzian vectors and E. Study map (see [18–21]). If and refer to real numbers, the combination is named a dual number. At this point, is the dual number subject to the rules \( e \neq 0, e^2 = 0, e \cdot 1 = 1 \cdot e = e \). The set of dual numbers, \( \mathbb{D} \), presents the commutative ring with the numbers \( ea^* \) as divisors of zero, not a field. No number \( ea^* \) has inverse in algebra. However, the algebra of dual numbers has similar lows as the complex numbers. Therefore, this set

\[
\mathbb{D}^3 = \{ \mathbf{a} = a + ea^* = (a_1, a_2, a_3) \},
\]

(1)
in addition to the Lorentzian scalar product

\[
\langle \mathbf{a}, \mathbf{b} \rangle = a_1b_1 + a_2b_2 - a_3b_3,
\]

(2)
resulted in what is named dual Lorentzian 3-space \( \mathbb{D}^3_1 \). This yields

\[
\begin{align*}
\mathbf{j}_1 \times \mathbf{j}_2 &= -\mathbf{j}_3, \\
\mathbf{j}_1 \cdot \mathbf{j}_2 &= \mathbf{j}_3, \\
\mathbf{j}_1 \cdot \mathbf{j}_3 &= \mathbf{j}_2,
\end{align*}
\]

(3)
where \( \mathbf{j}_1, \mathbf{j}_2, \) and \( \mathbf{j}_3 \) are the dual base at the origin point \( \mathbf{O} = (0, 0, 0) \) of the dual Lorentzian 3-space \( \mathbb{D}^3_1 \). Then, this point \( \mathbf{a} = (a_1, a_2, a_3) \) has dual coordinates \( \mathbf{\alpha} = (a_1^*, a_2^*, a_3^*) \in \mathbb{D}^3_1 \). If \( a \neq 0 \), the norm \( ||\mathbf{\alpha}|| \) of \( \mathbf{\alpha} = a + ea^* \) is defined by

\[
||\mathbf{\alpha}|| = \sqrt{\langle \mathbf{\alpha}, \mathbf{\alpha} \rangle} = ||a|| \left( 1 + e \frac{\langle a, a^* \rangle}{||a||^2} \right).
\]

(4)
Then, the vector \( \mathbf{\alpha} \) is named the spacelike (resp., timelike) dual unit vector in case \( \langle \mathbf{\alpha}, \mathbf{\alpha} \rangle = 1 \) (resp., \( \langle \mathbf{\alpha}, \mathbf{\alpha} \rangle = -1 \)). It is clear that

\[
\langle \mathbf{\alpha}, \mathbf{\alpha} \rangle = \pm 1 \Leftrightarrow ||\mathbf{\alpha}||^2 = \pm 1, \langle a, a^* \rangle = 0.
\]

(5)
The six components \( a_1, a_1^* \) \( i = 1, 2, 3 \) of \( a \), and \( a_1^* \) are named the normed Plücker coordinates of the line.

Assume \( \mathbf{\bar{m}}(\bar{m}_1, \bar{m}_2, \bar{m}_3) \) is the fixed dual point in \( \mathbb{D}^3_1 \). Therefore,

\[
S^2_1(\mathbf{\bar{m}}) = \{ \mathbf{\alpha} \in \mathbb{D}^3_1 | (\bar{m}_1 - \bar{m}_1) + (\bar{m}_2 - \bar{m}_2) - (\bar{m}_3 - \bar{m}_3)^2 = 1 \},
\]

(6)
defines the Lorentzian dual unit sphere with the center \( \mathbf{\bar{m}} \) and

\[
H^2_1(\mathbf{\bar{m}}) = \{ \mathbf{\alpha} \in \mathbb{D}^3_1 | (\bar{m}_1 - \bar{m}_1)^2 + (\bar{m}_2 - \bar{m}_2)^2 - (\bar{m}_3 - \bar{m}_3)^2 = -1 \},
\]

(7)
defines the hyperbolic dual unit sphere that has the center \( \mathbf{\bar{m}} \). Then, the hyperbolic and Lorentzian (de Sitter space) dual unit spheres that has the center \( \mathbf{O} \) are

\[
H^2_1 = \{ \mathbf{\alpha} \in \mathbb{D}^3_1 | a_1^2 + a_2^2 - a_3^2 = -1 \},
\]

(8)
and

\[
S^2_1 = \{ \mathbf{\alpha} \in \mathbb{D}^3_1 | a_1^2 + a_2^2 - a_3^2 = 1 \},
\]

(9)
respectively. As a result of this, the following map (E. Study map) is given as follows: The dual unit spheres are shaped as a pair of conjugate hyperboloids. The combined asymptotic cone presents the set of null (lightlike) lines, the ring-shaped hyperboloid represents the set of spacelike lines, the oval-shaped hyperboloid forms the set of timelike lines, and opposite points of each hyperboloid present the pair of opposite vectors on the line (see Figure 1).

Assume \( \alpha(t) = (a_1^*(t), a_2^*(t), a_3^*(t)) \) and \( \alpha^*(t) = (a_1^*(t), a_2^*(t), a_3^*(t)) \) are two real smooth differentiable curves at Minkowski 3-space \( \mathbb{E}^3_1 \). Therefore, the differentiable curve

\[
\mathbf{\bar{a}}: t \in \mathbb{R} \rightarrow \mathbb{D}^3_1
\]

(10)
represents the curve in the dual Lorentzian 3-space \( \mathbb{D}^3_1 \) and is named the dual space curve. The pseudo dual arc length of \( \mathbf{\bar{a}}(t) \) from \( t_0 \) to \( t \) is defined as

\[
\mathbf{\bar{a}}: t \in \mathbb{R} \rightarrow \mathbb{D}^3_1
\]

(11)
where \( t \) defines the unit tangent vector of \( \alpha(t) \). From here, the pseudo dual arc length \( \mathbf{\bar{a}}(t) \) will be used as the parameter rather than \( t \in \mathbb{R} \). Therefore, \( \mathbf{\bar{a}}(\mathbf{\bar{s}}) \) is named the dual arc-length parameter curve. Also, we shall often not write the dual parameter \( \mathbf{\bar{s}} \) explicitly in our formulae.

Let \( \mathbf{\bar{f}}(\mathbf{\bar{s}}), \mathbf{\bar{n}}(\mathbf{\bar{s}}), \mathbf{\bar{b}}(\mathbf{\bar{s}}) \) be the moving dual Serret–Frenet frame along \( \mathbf{\bar{a}}(\mathbf{\bar{s}}) \), such that \( \mathbf{\bar{f}}, \mathbf{\bar{n}}, \) and \( \mathbf{\bar{b}} \) are the unit tangent, the principal normal, and the binormal vector, respectively. The dual arc-length derivative of the dual Serret–Frenet frame is

\[
\begin{pmatrix}
\mathbf{\bar{t}}' \\
\mathbf{\bar{n}}' \\
\mathbf{\bar{b}}'
\end{pmatrix} =
\begin{pmatrix}
0 & \mathbf{\bar{r}}(\mathbf{\bar{s}}) & 0 \\
-\kappa_2 \mathbf{\bar{r}}(\mathbf{\bar{s}}) & 0 & \mathbf{\bar{t}}(\mathbf{\bar{s}}) \\
0 & -\kappa_1 \mathbf{\bar{t}}(\mathbf{\bar{s}}) & 0
\end{pmatrix} \times
\begin{pmatrix}
\mathbf{\bar{f}} \\
\mathbf{\bar{n}} \\
\mathbf{\bar{b}}
\end{pmatrix},
\]

(12)
where \( \kappa_1 = \langle \mathbf{\bar{f}}, \mathbf{\bar{f}} \rangle = \pm 1, \kappa_2 = \langle \mathbf{\bar{n}}, \mathbf{\bar{n}} \rangle = \pm 1, \kappa_3 = \langle \mathbf{\bar{b}}, \mathbf{\bar{b}} \rangle = \pm 1, \kappa_1 \kappa_2 \kappa_3 = -1, \) and \( \mathbf{\bar{r}} = \kappa + e \kappa' \) are nowhere pure dual curvature, and \( \mathbf{\bar{t}} = \tau + e \tau' \) is nowhere pure dual torsion. Here "prime" indicates the derivative respecting to the pseudo dual parameter. Furthermore, in case \( \mathbf{\bar{a}}(\mathbf{\bar{s}}) \) is the unit speed spacelike dual curve with the null principal normal \( \mathbf{\bar{n}}(\mathbf{\bar{s}}) \), we have the following equation:

\[
\begin{pmatrix}
\mathbf{\bar{f}}' \\
\mathbf{\bar{n}}' \\
\mathbf{\bar{b}}'
\end{pmatrix} =
\begin{pmatrix}
0 & \mathbf{\bar{r}} & 0 \\
-\kappa & 0 & \mathbf{\bar{t}} \\
\kappa' & -\kappa & 0
\end{pmatrix} \times
\begin{pmatrix}
\mathbf{\bar{f}} \\
\mathbf{\bar{n}} \\
\mathbf{\bar{b}}
\end{pmatrix},
\]

(13)
where
functions of the following equation:

\[ \alpha \cdot \kappa \] and

\[ \alpha \cdot \tau \]

respect to \( \tilde{s} \), with equation (12), we obtain the following equation:

\[-\tilde{a}' = (\tilde{a}'_1 + \tilde{a}'_2 + \tilde{a}'_3) \tilde{r} + (\tilde{a}'_1 + \tilde{a}'_2 + \tilde{a}'_3) \tilde{n} + (\tilde{a}'_1 + \tilde{a}'_2 + \tilde{a}'_3) \tilde{b}. \]

(18)

From equation (18), we have the following equation:

\[
\begin{align*}
\tilde{a}'_1 - e_1 e_2 \tilde{a}'_2 - \tilde{a}'_3 &= -1, \\
\tilde{a}'_1 + \tilde{a}'_2 - e_2 e_3 \tilde{a}'_3 &= 0, \\
\tilde{a}'_1 + \tilde{a}'_2 + \tilde{a}'_3 &= 0.
\end{align*}
\]

(19)

If the unit speed non-null dual curve with a spacelike or a timelike rectifying dual plane lies at the hyperbolic or Lorentzian dual unit sphere, then \( \tilde{a}, \tilde{a} > 0 \) for \( e_0 = -1 \) or \( e_0 = 1 \), respectively. So, considering equation (17), it gives us the following equation:

\[ e_1 \tilde{a}'_1 + e_2 \tilde{a}'_2 + e_3 \tilde{a}'_3 = e_0. \]

(20)

Differentiating equation (20), with using equation (19), we obtain the following equation:

\[ \tilde{a}'_1 = 0, \tilde{a}'_2 - e_1 e_2 \tilde{a}'_3 = 0, \tilde{a}'_3 - e_2 e_3 (\frac{1}{\tilde{\kappa}})' = 0, \]

(21)

where \( \tilde{\kappa} \) and \( \tilde{\tau} \) are nowhere pure duals. If we substitute equation (21) in equation (17), we have the following equation:

\[-\tilde{a}' = \left( \frac{e_1 e_2}{\tilde{\kappa}} \right) \tilde{n} + \left( \frac{e_1 e_3}{\tilde{\kappa}} \right)' \tilde{b}. \]

(22)

Thus, from the fact that \( \langle \tilde{a}, \tilde{a} \rangle = e_0 (\pm 1) \), we obtain the following equation:

\[ e_1 \left( \frac{1}{\tilde{\kappa}} \right)' + e_3 \left[ \frac{1}{\tilde{\kappa} - \tilde{\tau}} \right]' = e_0, \]

(23)

Therefore, the following corollaries will be given:

**Corollary 1.** There is no unit speed timelike dual curve which lies on \( \mathbb{H}^2_1 \).

**Proof.** Suppose \( \tilde{a}(\tilde{s}) \) is the unit speed timelike dual curve which lies at \( \mathbb{H}^2_1 \). Therefore, we have the following equation:

\[ \left( \frac{1}{\tilde{\kappa}} \right)' + \left[ \frac{1}{\tilde{\kappa} - \tilde{\tau}} \right]' = -1, \]

(24)

and this is a contradiction. \( \square \)

**Corollary 2.** Suppose \( \tilde{a}(\tilde{s}) \) is the unit speed timelike dual curve lies on \( \mathbb{S}^2_1 \), with \( \tilde{\kappa} \) and \( \tilde{\tau} \) are nowhere pure duals for each \( \tilde{s} \in \tilde{\mathbb{I}} \subset \mathbb{D} \), then

\[ \left( \frac{1}{\tilde{\kappa}} \right)' + \left[ \frac{1}{\tilde{\kappa} - \tilde{\tau}} \right]' = 1. \]

(25)

**Theorem 1.** A unit speed spacelike dual curve \( \tilde{a}(\tilde{s}) \) with a spacelike principal normal lies at the hyperbolic dual sphere with imaginary dual radius \( \tilde{r} \) if \( \tilde{r} \in \mathbb{D}^+ \) iff

\[ \tilde{r} \in \mathbb{D}^+ \]
\[
\left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 > \left( \frac{1}{\hat{\tau}} \right)^2 \quad \text{and} \quad \tilde{\tau} = \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right] = 0,
\]

where \( \hat{\tau} \) and \( \tilde{\tau} \) are nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \).

**Proof.** Assume that \( \hat{a} (\hat{s}) \) is the unit speed spacelike dual curve with the spacelike principal normal lying on the hyperbolic dual sphere with imaginary dual radius \( \hat{r} \) for \( \hat{r} \in D^+ \). Then, if \( \hat{\tau} \) and \( \tilde{\tau} = \hat{\tau} + \epsilon \hat{r} \) nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \), we have the following equation:

\[
\left( \frac{1}{\hat{\tau}} \right)^2 - \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 = -\hat{\tau}^2.
\]

It follows that

\[
\left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 > \left( \frac{1}{\hat{\tau}} \right)^2.
\]

This inequality requires that \( (1/\hat{\tau})' \neq 0 \); i.e., it is not constant. Furthermore, the differentiation of relation (27) equals to

\[
2 \frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)}' - 2 \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]' = 0,
\]

and consequently

\[
\tilde{\tau} = \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]' = 0.
\]

Conversely, assume that the unit speed spacelike dual curve \( \hat{a} = \hat{a} (\hat{s}) \) with a spacelike principal normal satisfies the relations

\[
\left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 > \left( \frac{1}{\hat{\tau}} \right)^2 \quad \text{and} \quad \tilde{\tau} = \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]' = 0,
\]

such that \( \hat{\tau} \) and \( \tilde{\tau} \) are nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \). From the inequality, we know that \( (1/\hat{\tau})' \neq 0 \), and also, by the fact that \( \hat{\tau} \) and \( \tilde{\tau} \) are nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \), a simple calculation gives us the following equation:

\[
2 \frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)}' - 2 \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]' = 0,
\]

which is the differential of the equation

\[
\left( \frac{1}{\hat{\tau}} \right)^2 - \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 = \tilde{\tau} < 0,
\]

so that we may take \( \sqrt{\tilde{\tau}} = \sqrt{-\tilde{\tau}} \). This implies that \( \hat{a} = \hat{a} (\hat{s}) \) lies at the hyperbolic sphere of with imaginary dual radius \( \hat{r} \) for \( \hat{r} \in D^+ \).

In a similar form, the following corollaries are given. \( \square \)

**Corollary 3.** Suppose \( \hat{a} (\hat{s}) \) is the unit speed spacelike dual curve with a timelike principal normal. If \( \hat{a} (\hat{s}) \) lies at the hyperbolic dual sphere with imaginary dual radius \( \hat{r} \) for \( \hat{r} \in D^+ \), then

\[
\left( \frac{1}{\hat{\tau}} \right)^2 - \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 = -\hat{\tau},
\]

such that \( \hat{\tau} \) and \( \tilde{\tau} \) are nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \). This relation is satisfied iff

\[
\left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 < \left( \frac{1}{\hat{\tau}} \right)^2 \quad \text{and} \quad \tilde{\tau} = \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]' = 0.
\]

The Lorentzian spherical spacelike curves with spacelike principal normal were studied by Pekmen and Pasali [13]. The following corollary is the dual version of the corollary given in [13].

**Corollary 4.** Let \( \hat{a} = \hat{a} (\hat{s}) \) be the unit speed spacelike dual curve with the timelike principal normal. If \( \hat{a} \) lies on the dual Lorentzian sphere of radius \( \hat{r} \) for \( r \in D^+ \), then

\[
-\left( \frac{1}{\hat{\tau}} \right)^2 + \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 = \hat{r},
\]

such that \( \hat{\tau} \) and \( \tilde{\tau} \) are nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \). This relation exists iff

\[
\left[\frac{1}{\hat{\tau}} \left( \frac{1}{\hat{\tau}} \right) \right]^2 > \left( \frac{1}{\hat{\tau}} \right)^2 \quad \text{and} \quad \tilde{\tau} = \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]' = 0.
\]

The Lorentzian spherical spacelike curves with timelike principal normal were studied by Petrovic–Torgasev and Sucurovic. The following corollary is the dual version of the given in Theorems 2 and 3 of [14].

**Corollary 5.** Let \( \hat{a} = \hat{a} (\hat{s}) \) be the unit speed spacelike dual curve with the spacelike principal normal. If \( \hat{a} \) lies at the Lorentzian dual sphere of radius \( \hat{r} \) for \( \hat{r} \in D^+ \), then \( \hat{s} \in \hat{T} \subseteq D \); then

\[
\left( \frac{1}{\hat{\tau}} \right)^2 - \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 = \tilde{\tau}^2,
\]

such that \( \hat{\tau} \) and \( \tilde{\tau} \) are nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \). This relation exists iff

\[
\tilde{\tau} = \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]' \quad \text{and} \quad \left[\frac{1}{\hat{\tau} \left( \frac{1}{\hat{\tau}} \right)} \right]^2 < \left( \frac{1}{\hat{\tau}} \right)^2.
\]

Afterwards, some necessary and sufficient conditions of \( \hat{a} = \hat{a} (\hat{s}) \) lie at \( H^2 \) (or \( S_1^2 \)) for \( \hat{\tau} \) and \( \tilde{\tau} \) are nowhere pure duals for all \( \hat{s} \in \hat{T} \subseteq D \). Here, if we make a parameter change like

\[
\hat{\psi} (\hat{s}) = \int_0^{\tilde{\tau}} \hat{\tau} \hat{d}s,
\]

we obtain the following theorem:

**Theorem 2.** A non-null unit speed dual curve \( \hat{a} = \hat{a} (\hat{s}) \) with timelike or spacelike rectifying plane lies on a Lorentzian dual sphere with radius \( \sqrt{\epsilon_3 \hat{\tau}^2 + \epsilon_1 \hat{\tau}^2} \) (or a hyperbolic dual sphere with imaginary dual radius \( \sqrt{\epsilon_3 \hat{\tau}^2 + \epsilon_1 \hat{\tau}^2} \)) iff there exists a C^1-dual function \( \tilde{f} \), such that

\[
\tilde{f} - \frac{d\hat{\psi}}{d\hat{\psi}} = 0, \quad \frac{d\tilde{\tau}}{d\hat{\psi}} - \epsilon_1 \hat{\tau} = 0, \quad \hat{\psi} (\hat{s}) = \int_0^{\tilde{\tau}} \hat{\tau} \hat{d}s, \quad \text{with} \quad \hat{\tau} = 1,
\]

where \( \tilde{\tau} \) is nowhere pure dual.

**Proof.** Assume that the unit speed non-null dual curve \( \hat{a} = \hat{a} (\hat{s}) \) with spacelike or timelike rectifying dual plane lies on a
Lorentzian dual sphere with radius $\sqrt{\epsilon_2 p^2 + \epsilon_3 f^2}$ (or a hyperbolic dual sphere with imaginary dual radius $\sqrt{\epsilon_2 p^2 - \epsilon_3 f^2}$). So, for $\tilde{\pi} = 1$, by equations (22) and (23), we can introduce $C^1$ dual functions $\tilde{\rho} = \tilde{\rho}(\tilde{s})$ and $\tilde{f} = \tilde{f}(\tilde{s})$ defined by

$$\tilde{\rho} = -\epsilon_1(\tilde{a}, \tilde{b}),$$

and a vector dual function $\tilde{\eta}(\tilde{s})$ is defined by $\tilde{\eta} = \tilde{a} - \tilde{\beta}$, then

$$\langle \tilde{\eta}, \tilde{\eta} \rangle = \epsilon_2 p^2 + \epsilon_3 f^2. \tag{44}$$

Differentiating equations (43) and (44) with respect $\tilde{s}$, and making use of equation (8), we obtain the following equation:

$$\tilde{\rho}' = 0 \text{ and } \frac{d}{d\tilde{s}} \langle \tilde{\eta}, \tilde{\eta} \rangle = 0. \tag{45}$$

These show that the dual vectors $\tilde{\rho}$ and $\tilde{\eta}$ are constant dual vectors; therefore, the non-null dual curve $\tilde{a}(\tilde{s})$ lies on a Lorentzian dual sphere with radius $\sqrt{\epsilon_2 p^2 + \epsilon_3 f^2}$ (or a hyperbolic dual sphere with imaginary dual radius $\sqrt{\epsilon_2 p^2 - \epsilon_3 f^2}$).

According to Theorem 2, we have the following theorems.

**Theorem 3.** The unit speed timelike dual curve $\tilde{a} = \tilde{a}(\tilde{s})$ lies on a Lorentzian dual sphere of radius $\sqrt{\tilde{a}^2 + \tilde{b}^2}$ iff there are dual constants $\tilde{a}, \tilde{b} \in \mathbb{D}$; that is,

$$\tilde{\rho}(\tilde{s}) = \tilde{a} \cos \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s} + \tilde{b} \sin \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s}, \tag{46}$$

holds for every $\tilde{s} \in T \subseteq \mathbb{D}$.

**Proof.** First assume that $\tilde{a} = \tilde{a}(\tilde{s})$ lies on a Lorentzian dual sphere of radius $\sqrt{\tilde{a}^2 + \tilde{b}^2}$; that is, it satisfies the conditions in Theorem 3. Thus, there is a differential dual function $\tilde{f}(\tilde{s})$ such that

$$\tilde{f} - \frac{d\tilde{\rho}}{d\tilde{\psi}} = 0 \text{ and } \frac{d\tilde{f}}{d\tilde{\psi}} + \tilde{\rho} = 0. \tag{47}$$

Furthermore, the $C^1$ dual functions $\tilde{h}(\tilde{s})$ and $\tilde{g}(\tilde{s})$ are defined by

$$\tilde{g}(\tilde{s}) = \rho \cos \tilde{\psi} - \tilde{f} \sin \tilde{\psi} \text{ and } h(\tilde{s}) = \rho \sin \tilde{\psi} + \tilde{f} \cos \tilde{\psi}. \tag{48}$$

Differentiation of the dual functions $\tilde{h}(\tilde{s})$ and $\tilde{g}(\tilde{s})$ with respect to $\tilde{s}$ easily gives $\tilde{h}' = \tilde{g}' = 0$. Also, we get $\sqrt{\tilde{h}^2 + \tilde{g}^2} = \sqrt{\rho^2 + f^2}$, which is the radius of Lorentzian dual sphere. Hence, $\sqrt{\tilde{a}^2 + \tilde{b}^2} = \sqrt{\rho^2 + f^2}$; that is, $\tilde{g} = \tilde{a}$ and $\tilde{h} = \tilde{b}$ are dual constants, so the above relation becomes

$$\tilde{a} = \rho \cos \tilde{\psi} - \tilde{f} \sin \tilde{\psi} \text{ and } \tilde{b} = \rho \sin \tilde{\psi} + \tilde{f} \cos \tilde{\psi}. \tag{49}$$

Multiplying the first previous equations with $\cos \tilde{\psi}$, multiplying the second by $\sin \tilde{\psi}$, and then adding them together resulted in $\rho = \tilde{a} \cos \tilde{\psi} + \tilde{b} \sin \tilde{\psi}$. Thus, considering equation (27) gives equation (46).

On the other hand, suppose $\tilde{a}$ and $\tilde{b}$ are the dual constants; that is, equation (46) holds for all $\tilde{s} \in T \subseteq \mathbb{D}$. Then, $\tilde{\kappa}$ is nowhere pure dual, such that $\tilde{\kappa} = ||\tilde{\kappa}||$. The differentiation with respect to $\tilde{\psi}$ of the above relation (46) gives

$$\frac{d\tilde{\rho}}{d\tilde{\psi}} = -\tilde{a} \sin \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s} + \tilde{b} \cos \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s}. \tag{50}$$

Further, the differentiable function $\tilde{f}(\tilde{s})$ is defined by

$$\tilde{f} = -\tilde{a} \sin \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s} + \tilde{b} \cos \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s}. \tag{51}$$

Then, the above two relations give $\tilde{f} = \frac{d\tilde{\rho}}{d\tilde{\psi}}$. By differentiation with respect to $\tilde{\psi}$ of relation (51) and using relation (46), we obtain $\frac{d\tilde{f}}{d\tilde{\psi}} = -\tilde{\rho}$. Therefore, by Theorem 2, we obtain that this curve lies on the Lorentzian dual sphere with radius $\sqrt{\rho^2 + f^2} = \sqrt{\tilde{a}^2 + \tilde{b}^2}$.

**Theorem 4.** The unit speed spacelike dual curve $\tilde{a} = \tilde{a}(\tilde{s})$ with timelike or spacelike rectifying dual plane lies at the Lorentzian dual sphere of radius $\sqrt{\epsilon_2 \tilde{a}^2 + \epsilon_3 \tilde{b}^2}$ iff there are duals $\tilde{a}, \tilde{b} \in \mathbb{D}$ such that

$$\tilde{\rho} = \epsilon_1 \tilde{a} \cosh \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s} + \epsilon_3 \tilde{b} \sinh \int_0^{\tilde{s}} \tilde{\tau}d\tilde{s} \text{ and } \epsilon_2 \tilde{a}^2 + \epsilon_3 \tilde{b}^2 > 0, \tag{52}$$

holds for all $\tilde{s} \in T \subseteq \mathbb{D}$.

**Proof.** Assume that $a = a(s)$ is the unit speed spacelike dual curve with timelike or spacelike rectifying dual plane lies at the Lorentzian dual sphere of radius $\sqrt{\epsilon_2 a^2 + \epsilon_3 b^2}$.

Also, by Theorem 2, for a Lorentzian dual sphere of radius $\sqrt{\epsilon_2 \tilde{a}^2 + \epsilon_3 \tilde{b}^2}$, there exists a differential dual function $\tilde{f}(\tilde{s})$; that is,

$$\tilde{f} - \frac{d\tilde{\rho}}{d\tilde{\psi}} = 0 \text{ and } \frac{d\tilde{f}}{d\tilde{\psi}} - \tilde{\rho} = 0. \tag{53}$$

Let us define the $C^1$ dual functions $\tilde{h}(\tilde{s})$ and $\tilde{g}(\tilde{s})$ by...
\[ \ddot{g}(\bar{s}) = \epsilon_3 \ddot{p} \cosh \ddot{\psi} + \epsilon_3 \ddot{f} \sinh \ddot{\psi}, \quad \dddot{h}(\bar{s}) = \epsilon_3 \ddot{p} \sinh \dddot{\psi} + \epsilon_3 \ddot{f} \cosh \dddot{\psi}. \]  

(54)

Since \( \dddot{a} \) is a spacelike dual curve, that is, \( \epsilon_1 = 1 \), we know \( \epsilon_3 = -\epsilon_3 \). So, the last functions become
\[ \ddot{g}(\bar{s}) = \epsilon_3 \ddot{p} \cosh \dddot{\psi} - \epsilon_3 \ddot{f} \sinh \ddot{\psi}, \quad \dddot{h}(\bar{s}) = \epsilon_3 \ddot{p} \sinh \dddot{\psi} - \epsilon_3 \ddot{f} \cosh \dddot{\psi}. \]  

(55)

It is easily seen that \( \dddot{g}^2 - \dddot{h}^2 = \dddot{f}^2 - \ddot{p}^2 \), and this means that
\[ \sqrt{\epsilon_3 \dddot{g}^2 + \epsilon_3 \dddot{h}^2} = \sqrt{\epsilon_3 \ddot{p}^2 + \epsilon_3 \dddot{f}^2}. \]

In these regards, by our assumption we obtain
\[ \sqrt{\epsilon_3 \dddot{g}^2 + \epsilon_3 \dddot{h}^2} = \sqrt{\epsilon_3 \ddot{a}^2 + \epsilon_3 \ddot{b}^2}. \] Also, if we differentiate the dual functions \( \dddot{h}(\bar{s}) \) and \( \ddot{g}(\bar{s}) \) with respect to \( \bar{s} \), we obtain
\[ \dddot{g}'(\bar{s}) = \dddot{h}'(\bar{s}) = 0. \] Thus, \( \dddot{g}(\bar{s}) = \dddot{a} \) and \( \ddot{g}(\bar{s}) = \ddot{b} \) are dual constants, such that \( \sqrt{\epsilon_3 \dddot{a}^2 + \epsilon_3 \ddot{b}^2} > 0 \). Thus, the above relations become
\[ \dddot{a}(\bar{s}) = \epsilon_3 \ddot{p} \cosh \dddot{\psi} - \epsilon_3 \ddot{f} \sinh \ddot{\psi}, \quad \ddot{b}(\bar{s}) = \epsilon_3 \ddot{p} \sinh \dddot{\psi} - \epsilon_3 \ddot{f} \cosh \dddot{\psi}. \]  

(56)

Multiplying the first previous equations with \( \epsilon_3 \cosh \ddot{\psi} \), and multiplying the second by \( -\epsilon_3 \sinh \dddot{\psi} \), and then adding them together resulted in \( \dddot{\rho} = \epsilon_3 \ddot{a} \cosh \dddot{\psi} - \epsilon_3 \ddot{b} \sinh \ddot{\psi}. \) Consequently, by equation (40), we find
\[ \dddot{\rho} = \epsilon_3 \ddot{a} \cosh \int_0^{\bar{s}} \tau \ddot{\tau} ds - \epsilon_3 \ddot{b} \sinh \int_0^{\bar{s}} \tau \ddot{\tau} ds. \]  

(57)

Considering \( \epsilon_2 = -\epsilon_3 \) completes the proof of the necessary condition.

Conversely, let \( \dddot{a} \) and \( \ddot{b} \) be the real constants that are satisfying relation (52) for all \( \bar{s} \in \tilde{I} \subseteq \mathbb{D} \). Then, obviously \( \dddot{k} = \| \dddot{t} \| \) is nowhere pure dual for all \( \bar{s} \in \tilde{I} \subseteq \mathbb{D} \). The differentiation with respect to \( \dddot{\psi} \) of the above gives
\[ \frac{d \dddot{\rho}}{d \dddot{\psi}} = \epsilon_3 \ddot{a} \sinh \int_0^{\bar{s}} \tau \dddot{\tau} ds + \epsilon_3 \ddot{b} \cosh \int_0^{\bar{s}} \tau \dddot{\tau} ds. \]  

(58)

Also, the differentiable dual function \( \dddot{f}(\bar{s}) \) is defined as
\[ \dddot{f}(\bar{s}) = \epsilon_3 \ddot{a} \sinh \int_0^{\bar{s}} \tau \dddot{\tau} ds + \epsilon_3 \ddot{b} \cosh \int_0^{\bar{s}} \tau \dddot{\tau} ds. \]  

(59)

Then, the above two relations give \( \dddot{f} = \frac{d \dddot{\rho}}{d \dddot{\psi}} \). By differentiation with respect to \( \dddot{\psi} \) of relation (62) and using relation (52), we get \( \frac{d \dddot{f}}{d \dddot{\psi}} = \dddot{\rho} \). Therefore, by Theorem 2, this spacelike dual curve with a spacelike or the timelike rectifying dual plane lies on the Lorentzian dual sphere of radius \( \sqrt{\epsilon_3 \dddot{a}^2 + \epsilon_3 \ddot{b}^2} \).

The real parts of the special case of \( \epsilon_2 = -1 \) and \( \epsilon_3 = 1 \) in Theorem 4 correspond to Theorem 5 of [13] that can be seen in the following corollary.

**Corollary 6.** A unit speed spacelike dual curve with spacelike principal normal lies on the Lorentzian dual sphere of radius \( \sqrt{\dddot{a}^2 - \dddot{b}^2} \) iff there are dual constants \( \dddot{a}, \ddot{b} \in \mathbb{D} \) such that
\[ \dddot{\rho} = \dddot{a} \cosh \int_0^{\bar{s}} \tau \dddot{\tau} ds - \ddot{b} \sinh \int_0^{\bar{s}} \tau \dddot{\tau} ds \]  

(60)

hold for all \( \bar{s} \in \tilde{I} \subseteq \mathbb{D} \).

The real parts of the special case of \( \epsilon_2 = -1 \) and \( \epsilon_3 = 1 \) in Theorem 4 correspond to Theorem 5 of [14] that can be seen in the following corollary.

**Corollary 7.** A unit speed spacelike dual curve with timelike principal normal lies on the Lorentzian dual sphere of radius \( \sqrt{\dddot{b}^2 - \dddot{a}^2} \) iff there are constants \( \dddot{a} \) and \( \ddot{b} \in \mathbb{R} \) such that
\[ \dddot{\rho} = \dddot{b} \sinh \int_0^{\bar{s}} \tau \dddot{\tau} ds - \dddot{a} \cosh \int_0^{\bar{s}} \tau \dddot{\tau} ds \]  

(61)

hold for all \( \bar{s} \in \tilde{I} \subseteq \mathbb{D} \).

By interchanging the roles of hyperbolic dual sphere with Lorentzian dual sphere, we state the following theorems.

**Theorem 5.** The unit speed spacelike dual curve \( \dddot{a} = \dddot{a}(\bar{s}) \) with a spacelike or a timelike rectifying dual plane lies at the hyperbolic dual sphere of imaginary dual radius \( \sqrt{\dddot{b}^2 - \dddot{a}^2} \). By Theorem 2 for a hyperbolic dual sphere of imaginary dual radius \( \sqrt{\epsilon_3 \dddot{a}^2 + \epsilon_3 \dddot{b}^2} \), there is a differential dual function \( \dddot{f}(\bar{s}) \); that is,
\[ \dddot{f} - \frac{d \dddot{\rho}}{d \dddot{\psi}} = 0, \quad \frac{d \dddot{f}}{d \dddot{\psi}} \dddot{\rho} = 0. \]  

(63)

Let us define the \( C^1 \) dual functions \( \dddot{h}(\bar{s}) \) and \( \dddot{g}(\bar{s}) \) by
\[ \dddot{g}(\bar{s}) = \epsilon_3 \dddot{p} \cosh \dddot{\psi} - \epsilon_3 \dddot{f} \sinh \dddot{\psi}, \quad \dddot{h}(\bar{s}) = \epsilon_3 \dddot{p} \sinh \dddot{\psi} - \epsilon_3 \dddot{f} \cosh \dddot{\psi}. \]  

(64)

It is easily seen that \( \sqrt{\epsilon_3 \dddot{g}^2 + \epsilon_3 \dddot{h}^2} = \sqrt{\epsilon_3 \dddot{a}^2 + \epsilon_3 \dddot{b}^2} \). By our assumption, we get \( \sqrt{\epsilon_3 \dddot{g}^2 + \epsilon_3 \dddot{h}^2} = \sqrt{\dddot{a}^2 + \dddot{b}^2} \). Also, if we differentiate \( \dddot{h}(\bar{s}) \) and \( \dddot{g}(\bar{s}) \) with respect to \( \bar{s} \), we get
\[ \dddot{h}' = \dddot{g}' = 0. \] Thus, \( \dddot{a} = \dddot{a} \) and \( \dddot{b} = \dddot{b} \) are dual constants, such that \( \epsilon_3 \dddot{a}^2 + \epsilon_3 \dddot{b}^2 < 0 \). Thus, from the above relations, one can say
\[ \dddot{a} = \epsilon_3 \dddot{p} \cosh \dddot{\psi} - \epsilon_3 \dddot{f} \sinh \dddot{\psi}, \quad \dddot{b} = \epsilon_3 \dddot{p} \sinh \dddot{\psi} - \epsilon_3 \dddot{f} \cosh \dddot{\psi}. \]  

(65)

Multiplying the first previous equations with \( \epsilon_3 \cosh \dddot{\psi} \), and multiplying the second by \( -\epsilon_3 \sinh \dddot{\psi} \), and then adding...
them together resulted in \( \bar{p} = e_2 \bar{a} \cosh \bar{\psi} - e_2 \bar{b} \sinh \bar{\psi} \). Consequently, by equation (19), we find the following equation:

\[
\bar{p} = e_2 \bar{a} \cosh \int_0^s \tau \, ds - e_2 \bar{b} \sinh \int_0^s \tau \, ds.
\]  

(66)

Considering \( e_2 = -e_3 \) completes the proof of the necessary condition.

Conversely, let \( \bar{a} \) and \( \bar{b} \) be the dual constants that are satisfying the relation (60) for all \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \). Then, obviously \( \bar{k} = \| \bar{t}' \| \) is nowhere pure dual \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \). The differentiation with respect to \( \bar{\psi} \) of the above relation gives

\[
\frac{d \bar{p}}{d \bar{\psi}} = e_2 \bar{a} \sinh \int_0^s \tau \, ds + e_3 \bar{b} \cosh \int_0^s \tau \, ds.
\]  

(67)

Furthermore, the differentiable dual function \( \bar{f}(\bar{s}) \) is defined by

\[
\bar{f} = e_2 \bar{a} \sinh \int_0^s \tau \, ds + e_3 \bar{b} \cosh \int_0^s \tau \, ds.
\]  

(68)

Then, the above two relations give \( \bar{f} = \frac{d \bar{p}}{d \bar{\psi}} \). By differentiation with respect to \( \bar{\psi} \) of relation (68) and using relation (60), we get \( d \bar{f}/d \bar{\psi} = \bar{p} \). Therefore, by Theorem 2, this spacelike dual curve with a spacelike or a timelike rectifying dual plane lies at the hyperbolic dual sphere of imaginary dual radius \( \sqrt{e_2^2 \bar{a}^2 + e_3^2 \bar{b}^2} \).

**Corollary 8.** The unit speed spacelike dual curve with timelike principal normal lies at the hyperbolic dual sphere of imaginary dual radius \( \sqrt{\bar{a}^2 - \bar{b}^2} \) iff there are dual constants \( \bar{a} \) and \( \bar{b} \in \mathbb{D} \) such that

\[
\bar{p} = \bar{a} \cosh \int_0^s \tau \, ds - \bar{b} \sinh \int_0^s \tau \, ds \quad \text{and} \quad \bar{a}^2 - \bar{b}^2 < 0,
\]  

(69)

holds for all \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \).

**Corollary 9.** A unit speed spacelike dual curve with timelike principal normal lies on the Lorentzian dual sphere of dual radius \( \sqrt{\bar{b}^2 - \bar{a}^2} \) iff there are dual constants \( \bar{a} \) and \( \bar{b} \in \mathbb{D} \) such that

\[
\bar{p} = \bar{b} \sinh \int_0^s \tau \, ds - \bar{a} \cosh \int_0^s \tau \, ds \quad \text{and} \quad \bar{b}^2 - \bar{a}^2 < 0,
\]  

(70)

holds for all \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \).

3.1. Spacelike Dual Spherical Curves with Null Principal Normal. The geometric place of the curves with null principal normal can be given by the following theorem.

**Theorem 6.** Let \( \bar{\alpha} = \bar{\alpha}(\bar{s}) \) be the unit speed spacelike dual curve, with the null principal normal \( \bar{n} \in \mathbb{D} \). Therefore, \( \bar{\alpha} = \bar{\alpha}(\bar{s}) \) lies on \( \mathbb{H}^2_c \) (or \( \mathbb{S}^2_1 \)) iff \( \alpha(s) \) is the planar dual curve and

\[
-\alpha(s) = \frac{e_0}{2} \bar{n}(s) + \bar{b}(s),
\]  

(71)

is satisfied for all \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \).

**Proof.** Assume \( \bar{\alpha} = \bar{\alpha}(\bar{s}) \) is the dual curve satisfying the mentioned conditions which lies on \( \mathbb{H}^2_c \) (or \( \mathbb{S}^2_1 \)). As stated in the above section, we have the following equation:

\[
-\bar{a}(s) = \bar{a}_1 \bar{t} + \bar{a}_2 \bar{n} + \bar{a}_3 \bar{b},
\]  

(72)

where \( \bar{a}_1(\bar{s}), \bar{a}_2(\bar{s}), \) and \( \bar{a}_3(\bar{s}) \) are differentiable dual functions of \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \). Differentiating equation (72) with respect to \( \bar{s} \), and by the use of corresponding Serret–Frenet formulae, it is found that

\[
-\bar{t} = (\bar{a}'_1 - \bar{a}_3 \bar{k}) \bar{t} + (\bar{a}'_2 + \bar{a}_1 \bar{\kappa} + \bar{a}_2 \bar{\tau}) \bar{n} + (\bar{a}'_3 - \bar{a}_2 \bar{\kappa}) \bar{b},
\]  

(73)

and since in this case we have \( \bar{k} = 1 \), for all \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \), it follows that

\[
\begin{align*}
\bar{a}'_1 - \bar{a}_3 & = -1, \\
\bar{a}'_2 + \bar{a}_1 & + \bar{a}_2 \bar{\tau} = 0, \\
\bar{a}'_3 - \bar{a}_2 & \bar{\kappa} = 0.
\end{align*}
\]  

(74)

In view of equation (72), and from the fact that

\[
\langle \bar{a}, \bar{a} \rangle = e_0 = \begin{cases} -1, & \bar{a} \in \mathbb{H}^2_c \\ 1, & \bar{a} \in \mathbb{S}^2roll}}.}
\]  

(75)

we get the following equation:

\[
2 \bar{a}_3 \bar{a}_3 + \bar{a}_1^2 = e_0.
\]  

(76)

Differentiation of equation (76), with account of equation (74), we get the following equation:

\[
\bar{a}_1 = 0, \quad \bar{a}_2 = \frac{e_0}{2}, \quad \bar{a}_3 = 1, \quad \bar{\tau} = 0.
\]  

(77)

Consequently, \( \bar{\alpha} = \bar{\alpha}(\bar{s}) \) is the planar dual curve, and by replacing equation (77) in equation (72), we obtain

\[
-\bar{a} = \frac{e_0}{2} \bar{n} + \bar{b}.
\]  

(78)

Conversely, assume that \( \bar{\alpha} = \bar{\alpha}(\bar{s}) \) is the unit speed spacelike planar dual curve, with the null principal normal \( \bar{n} \), which satisfies equation (78) for all \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \). By differentiation of equation (78) with respect to \( \bar{s} \), it is found that

\[
-\bar{a}' = \frac{e_0}{2} \bar{n}' + \bar{b}'.
\]  

(79)

Since in this case we have \( \bar{k} = 1 \) and \( \bar{\tau} = 0 \) for all \( \bar{s} \in \bar{T} \subseteq \mathbb{D} \), the corresponding Serret–Frenet formulae read as

\[
\begin{pmatrix}
\bar{t} \\
\bar{n}' \end{pmatrix} = \begin{pmatrix}
0 & \bar{k} \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\bar{t} \\
\bar{n}
\end{pmatrix},
\]  

(80)

Combining this equation with equation (79) yields \( \bar{a}' = 0 \); i.e., the vector \( \bar{a} \) is constant dual vector. Thus, we find \( \| \bar{a} \| = \| e_0/2 \bar{n} + \bar{b} \|^2 = e_0 \), so that \( \bar{\alpha}(\bar{s}) \in \mathbb{H}^2_c \) (or \( \mathbb{S}^2_1 \)).
The Lorentzian spherical spacelike curves with null principal normal was studied by Petrovic–Torgasev and the following theorem was also proved by [14].

**Theorem 7.** Suppose \( \bar{a} = \bar{a}(\bar{s}) \) is the unit speed spacelike dual curve, with the null principal normal \( \bar{n} \), for all \( \bar{s} \in \mathbb{I} \subseteq \mathbb{D} \). \( \bar{a}(\bar{s}) \in S^1_1 \) iff there are dual constants \( \bar{a} \) and \( \bar{b} \in \mathbb{D} \) such that

\[
1 = \bar{a} \sinh \int_0^\bar{s} \bar{\tau} d\bar{s} - \bar{b} \cosh \int_0^\bar{s} \bar{\tau} d\bar{s}. \tag{81}
\]

**Proof.** Suppose \( \bar{a} = \bar{a}(\bar{s}) \) is the dual curve satisfying the mentioned conditions, which lies on \( S^1_1 \). Taking into consideration Theorem 6, we see that \( \bar{\tau} = 0 \), for all \( \bar{s} \in \mathbb{I} \subseteq \mathbb{D} \). Also consider the \( C^2 \) dual function \( \bar{\psi}(\bar{s}) \), the \( C^1 \) functions \( \bar{h}(\bar{s}) \), and \( \bar{g}(\bar{s}) \) by

\[
\bar{\psi}(\bar{s}) = \int_0^{\bar{s}} \bar{\tau} d\bar{s}, \quad \bar{g}(\bar{s}) = -\sinh \bar{\psi}, \quad \bar{h}(\bar{s}) = -\cosh \bar{\psi}. \tag{82}
\]

Since \( \bar{\tau} = 0 \), for all \( \bar{s} \in \mathbb{I} \subseteq \mathbb{D} \), we find that

\[
\bar{\psi}(\bar{s}) = \bar{c} \in \mathbb{D}, \quad \bar{g}(\bar{s}) = -\sinh \bar{c} = \bar{a}, \quad \bar{h}(\bar{s}) = -\cosh \bar{c} = \bar{b}. \tag{83}
\]

Therefore, there are dual constants \( \bar{a} \) and \( \bar{b} \in \mathbb{D} \); that is, the equation

\[
1 = \bar{a} \sinh \int_0^{\bar{s}} \bar{\tau} d\bar{s} \bar{\tau} - \bar{b} \cosh \int_0^{\bar{s}} \bar{\tau} d\bar{s}, \tag{84}
\]

holds for all \( \bar{s} \in \mathbb{I} \subseteq \mathbb{D} \).

Conversely, assume that here exist dual constants \( \bar{a} \) and \( \bar{b} \in \mathbb{D} \), such that the dual torsion \( \bar{\tau} \) of the dual curve \( \bar{a} = \bar{a}(\bar{s}) \) satisfies equation (84). By differentiation of equation (84) with respect to \( \bar{s} \), it is found that

\[
\bar{\tau} \left( \bar{a} \cosh \int_0^{\bar{s}} \bar{\tau} d\bar{s} - \bar{b} \sinh \int_0^{\bar{s}} \bar{\tau} d\bar{s} \right) = 0. \tag{85}
\]

Differentiating equation (85) with respect to \( \bar{s} \), it is found that

\[
\bar{\tau}' \left( \bar{a} \cosh \int_0^{\bar{s}} \bar{\tau} d\bar{s} - \bar{b} \sinh \int_0^{\bar{s}} \bar{\tau} d\bar{s} \right) + \bar{\tau}^2 \left( \bar{a} \sinh \int_0^{\bar{s}} \bar{\tau} d\bar{s} - \bar{b} \cosh \int_0^{\bar{s}} \bar{\tau} d\bar{s} \right) = 0. \tag{86}
\]

Applying equation (84) to equation (86), it gives directly:

\[
\bar{\tau}' \left( \bar{a} \cosh \int_0^{\bar{s}} \bar{\tau} d\bar{s} - \bar{b} \sinh \int_0^{\bar{s}} \bar{\tau} d\bar{s} \right) + \bar{\tau}^2 = 0. \tag{87}
\]

Multiplying the last equation with \( \bar{\tau} \), and using equation (74), we get \( \bar{\tau}^3 = 0 \), and consequently \( \bar{\tau} = 0 \), for every \( \bar{s} \in \mathbb{I} \subseteq \mathbb{D} \). Moreover, consider the dual curve defined by

\[
\bar{\beta}(\bar{s}) = \bar{a}(\bar{s}) + \frac{1}{2} \bar{\bar{n}}(\bar{s}) + \bar{\bar{b}}(\bar{s}). \tag{88}
\]

Since \( \bar{\tau} = 0 \), for all \( \bar{s} \in \mathbb{I} \subseteq \mathbb{D} \), it follows that \( \bar{\beta}' = 0 \), thereby \( \bar{\beta} \) is dual constant. Finally, we easily obtain \( \|\bar{\alpha}\|^2 = 1/(2\bar{n} + \bar{\bar{b}}) \|^2 = 1 \), so that \( \bar{\alpha} \in S^1_1 \).

By a similar procedure as in Theorem 7, we can give the following theorem:

**Theorem 8.** Suppose \( \bar{a} = \bar{a}(\bar{s}) \) is a unit speed spacelike dual curve, with the null principal normal \( \bar{n} \), for all \( \bar{s} \in \mathbb{I} \subseteq \mathbb{D} \). \( \bar{a} = \bar{a}(\bar{s}) \) lies on \( H^2_+ \) iff there are dual constants \( \bar{a} \) and \( \bar{b} \in \mathbb{D} \) such that

\[
1 = -\bar{a} \sinh \int_0^\bar{s} \bar{\tau} d\bar{s} + \bar{b} \cosh \int_0^\bar{s} \bar{\tau} d\bar{s}. \tag{89}
\]

### 3.2. Spherical Null Dual Curves

Now, let us give the characterization for spherical null dual curves with the following theorem.

**Theorem 9.** There is no null dual curves \( \bar{a}(\bar{s}) \) lying on \( \epsilon \in H^2_+ \) (or \( S^1_1 \)).

**Proof.** Suppose \( \bar{a} = \bar{a}(\bar{s}) \) is the null dual curve lying on \( H^2_+ \) (or \( S^1_1 \)). Then,

\[
-\bar{a} = \bar{a}_1 \bar{\tau} + \bar{a}_2 \bar{n} + \bar{a}_3 \bar{b}, \tag{90}
\]

where \( \bar{a}_1(s), \bar{a}_2(s), \) and \( \bar{a}_3(s) \) are differentiable functions of \( s \in \mathbb{I} \subseteq \mathbb{D} \). By a similar procedure as in Theorem 6, we have the following equation:

\[
\bar{a}_1' - \bar{a}_2 \bar{\tau} = 0, \quad \bar{a}_2' - \bar{a}_3 \bar{\tau} = 0, \quad \bar{a}_3' + \bar{a}_2 = 0. \tag{91}
\]

If \( \bar{a} = \bar{a}(\bar{s}) \) is an oriented line, i.e., \( \bar{\tau} = 0 \), then we have \( \bar{a}_1' = 0, \bar{a}_2' = 0, \bar{a}_3' + 1 = 0 \), and therefore

\[
\bar{a}_3 = c_3, \quad \bar{a}_2 = c_2, \quad \text{and} \quad \bar{a}_1 = -\bar{s} + c_1, \tag{92}
\]

where \( c_1, c_2, \) and \( c_3 \) are dual constants of integrations. Then,

\[
-\bar{a} = (-\bar{s} + c_1) \bar{\tau} + c_2 \bar{n} + c_3 \bar{b}. \tag{93}
\]

More specific, using the fact that \( \langle \bar{a}, \bar{\bar{a}} \rangle = 1 \) in equation (92) we obtain \( c_3^2 + 2c_2c_3 = 1 \), and consequently \( \bar{s} = (1 - c_2^2/2c_3) \) is a constant, which is a contradiction. On the other hand, if \( \bar{a} = \bar{a}(\bar{s}) \) is not an oriented line, i.e., \( \bar{\tau} = 1 \), we find that

\[
\bar{a}_1' - \bar{a}_2 = 0, \quad \bar{a}_2' - \bar{a}_3 = 0, \quad \bar{a}_1' + \bar{a}_2 = 0. \tag{94}
\]

Consequently, we have \( \bar{a}_2 = \bar{a}_3 = \bar{c} \exp \bar{s} \) and \( \bar{a}_1 = -\bar{s} + \bar{c} \exp \bar{s} \), where \( \bar{c} \in \mathbb{D} \). Then,

\[
-\bar{a} = -\bar{s} \bar{\beta} + \bar{\bar{c}} (\bar{\bar{\beta}} + \bar{\bar{\bar{b}}} \exp \bar{s}), \tag{95}
\]
which together with \( \langle \bar{a}, \bar{a} \rangle = \epsilon_0 \) gives \( \exp \bar{s} = 0 \). This contradiction completes the proof.

4. Conclusions

In the present paper, we dealt with spherical and null curves in dual Lorentzian 3-space \( \mathbb{D}^3 \). Then, without the precondition on the dual torsion is nowhere pure dual, a necessary and sufficient condition for a non-null curve to be dual Lorentzian spherical is presented. Also, new necessary and sufficient conditions for null curves to be hyperbolic dual spherical were reported. Hopefully, these results will be useful to physicists and those studying the general relativity theory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

The second author expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

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