Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 3096866, 17 pages
https://doi.org/10.1155/2022/3096866

Research Article

Application of Malliavin Calculus in Mean-Variance Hedging Strategy

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Received 28 August 2022; Revised 21 November 2022; Accepted 23 November 2022; Published 5 December 2022

Academic Editor: Filippo Cacace

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This paper considers an approach of Malliavin calculus to obtain the hedging ratio for mean-variance hedging (MVH) strategy under the stochastic volatility model with pure jump Lévy process (SVJ). Specifically speaking, there exists a correspondence between the martingale representation theorem and the Clark-Ocone formula for a square integrable contingent claim. Therefore, we can replace the diffusion term coefficients with the functions containing Malliavin derivatives to get a closed-form representation for the MVH strategy. By fast Fourier transform (FFT) algorithm, some numerical examples are performed to analyze the sensitivity of MVH strategy concerning strike price and current time.

1. Introduction

Since the appearance of the Black-Scholes (B-S) model was proposed in 1973 [1], much research has been conducted on its application to the financial market. However, the different maturity dates and strike prices lead to the "smiles" and "incline" of volatility, which are inconsistent with the assumption that volatility is constant. Some classical models have been proposed to incorporate these phenomena in asset pricing, including two aspects. The first is presented by Heston [2], which assumes that volatility also follows a stochastic process, resulting in the volatility clustering effects can be captured. The second is proposed by Merton [3], which uses compound Poisson processes to describe the jump-diffusion phenomenon in the price process of the underlying asset. These models can be generally classified as Lévy processes for the above stochastic processes. Lévy processes have gained increasing importance in the option pricing literature due to the well-known flaws of the classical Black-Scholes geometric Brownian motion model [4]. They include the Brownian motion Poisson process and the compound Poisson process. Based on these characteristics, the model can well describe the leptokurtosis phenomenon and infinite small jump behaviours of asset return distribution which cannot be described by the Gaussian process. Therefore, it is useful and feasible to adopt Lévy process in option pricing research.

In recent years, many researchers have yet to be satisfied with the pricing performance of the existing models and have developed various pricing models. He and Zhu [5] study the two-factor Heston-CIR model in which the interest rate process obeyed the CIR process and derived closed-form series solutions to variance and volatility swaps. Diffusion processes with double exponential jump features were examined by Kou and Wang [6]. A pricing model with stochastic volatility and stochastic interest rates with pure jumps can be seen in [7], in which the jump components of asset price follow the compound Poisson process, and volatility is stochastic. Carr and Wu [8] find that the compound Poisson process cannot describe the many small jumps in financial assets. However, the infinite activity *Lévy* process allows the jump components to have infinite activity and admits nearly an arbitrary distribution. Using an infinite activity *Lévy* process instead of a compound Poisson process is natural. Feng et al. [9] present a generalized European option-pricing model with stochastic volatility and stochastic interest rates and pure jumps under Lévy processes, and alternatively, the FFT algorithm is used to obtain the solution of option price. Bakshi et al. [10] exhibit empirical results of all those models above for the S&P500 call option data by the square of sum of pricing errors optimization and find the SVJ model has the least bias.

Due to introducing a new random source, volatility, and pure jump, the market is usually incomplete such that the equivalent martingale measure is not unique. We cannot construct a portfolio to reduce intrinsic risk to zero in the incomplete market perfectly, so we focus on how to construct a hedging strategy to minimize risk. Gourieroux et al. [11] proposed the mean-variance hedging strategy, which minimizes the expected squared hedging cost in a global time step size. Another local risk-minimization strategy proposed by Föllmer and Sondermann [12] is minimizing the risk on each infinitesimal interval. Heath et al. [13, 14] compared these two optimization strategies and found that mean-variance hedging is far better than local risk-minimization in reducing risk. However, because of the complexity of the mean-variance hedging strategy, it is not easy to explicitly calculate the hedging ratio expression.

In this paper, our work contribution is two-fold. Firstly, we build a mean-variance hedging strategy under a stochastic volatility model driven by exponential Lévy process. Secondly, we develop the closed-form expression for the hedging ratio $\vartheta_t^{\mathrm{MVH}}$ by the correspondence between the Clark-Ocone formula and the martingale representation theorem under variance-optimal equivalent martingale measure Q. To demonstrate the sensitivity of MVH strategy concerning time and strike price, we adopt the FFT algorithm to calculate the conditional expectation contained in $\vartheta_t^{\mathrm{MVH}}$. The underlying assets pricing and volatility processes are approximated with an involved recursive calculation, in which the discrete schemes for Gaussian process and pure jump are defined as [15].

The structure of this paper is as follows. In Section 1, we introduce the setting of the market and model. In Section 2, we describe the theoretical background of the mean-variance hedging strategy under SVJ model. Moreover, we derive the expression of Galtchouk-Kunita-Watanabe (GKW) decomposition under some conditions by using the martingale representation theorem. In Section 3, we introduce some fundamental conceptions of Malliavin calculus in Lévy space and obtain the explicit solution of mean-variance hedging ratio $\vartheta_t^{\mathrm{MVH}}$ by the Clark-Ocone formula under measure Q. Section 4 develops a numerical scheme with the help of FFT and a discrete scheme in [15].

2. Market and Model Settings

Consider a frictionless market with a probability space $(\Omega_W \times \Omega_L, \mathscr{F}, \mathrm{P})$. P is the probability measure of the actual market, and $\mathbb{F} = \{\mathscr{F}_t\}_{t\in T}$, $T \in [0,\infty)$ is the filtration generated by the process of stock price. Ω_W and Ω_L are the sample spaces composed of the one-dimensional Wiener process and pure jump Lévy process, respectively, $\Omega_L = \cup_{n=0}^{\infty}([0,T]\times\mathbb{R}_0)^n$, where $R_0 = \mathbb{R}\backslash\{0\}$. For simplicity, S_t denotes the single risky assets process, B_t is set as the numeraire, and then, the discount price of underlying assets S_t is $X_t = S_t/B_t$. We consider a jump-diffusion stochastic volatility model for this single risky asset, and its stochastic differential equation is given by

$$\begin{aligned} \frac{dX_t}{X_t} &= \mu(t,V_t)dt + \beta(t,V_t)dW_t^x + \int_{\mathbb{R}_0}\gamma_{t,z}\widehat{N}(dt,dz), \\ dV_t &= g(t,X_t,V_t)dt + b(t,X_t,V_t)dW_t^v, \end{aligned} \tag{1}$$

where W_t^x and W_t^y are two standard Brownian motion under measure $P.V_t$ is the volatility of the underlying asset. N is the Poisson random measure, and $N(t,\mathscr{A}) = \sum_{s\le t}1_{\mathscr{A}}(\Delta X_s)$, $\mathscr{A}\in\mathscr{B}(\mathbb{R}_0)$, $\mathscr{B}(\mathbb{R}_0)$ is a Borel field. Denoting the Levy measure of $N by\ v$, $\widehat{N}(dt,dz) = N(dt,dz) - v(dz)dt$ is a compensated Poisson process of $N(dt,dz)$. Functions μ, β, g, and b are assumed to be the predictable processes, $\gamma_{t,z}$ is a stochastic process measurable with respect to the σ-algebra generated by $\mathscr{A}\times(s,u]\times B$, $\mathscr{A}\in\mathscr{F}_s$, $0\le s<u<T$ and $B\in\mathscr{B}(\mathbb{R}_0)$. For the application, we set

$$\begin{aligned} dW_t^x &= \rho dW_t + \sqrt{1-\rho^2}dW_t^*, \\ dW_t^v &= dW_t, \end{aligned} \tag{2}$$

where $dW_t^x dW_t^v = \rho\mathrm{d}t$, $\rho\in[-1,1]$ is the correlation coefficient of W_t^x and W_t^v. W_t^* and W_t are independent. Then, our model in (1) becomes

$$\begin{aligned} \frac{dX_t}{X_t} &= \mu(t,V_t)dt + \rho\beta(t,V_t)dW_t + \sqrt{1-\rho^2}\beta(t,V_t)dW_t^* + \int_{\mathbb{R}_0}\gamma_{t,z}\widehat{N}(dt,dz), \\ dV_t &= g(t,V_t)dt + b(t,V_t)dW_t. \end{aligned} \tag{3}$$

3. Mean-Variance Hedging Strategy

Considering a portfolio $\Theta_t = (\vartheta_t,\eta_t)$, ϑ_t and η_t are two $\mathbb{R}$-valued predictable processes. ϑ_t denotes the number of risk assets (such as the stock). η_t represents the number of riskless (such as the bound). The value process of the portfolio Θ_t can be written as $\Gamma_t(\Theta) = \vartheta_t X_t + \eta_t$, and the cost process is

$$C_t(\Theta) = \Gamma_t(\Theta) - \int_0^t \vartheta_u dX_u, 0\le t\le T. \tag{4}$$

Let $L(X)$ is the space of $\mathbb{R}$-valued predictable X-integrable processes ϑ. Denote the space of all stochastic integral processes $\int_0^T \vartheta_u dX_u$ by Θ. Then, we call the $(\Gamma_0^{\mathrm{MVH}}, \vartheta^{\mathrm{MVH}})$ is a mean-variance optimal strategy for a claim F if it minimizes

$$E\left[(F-\Gamma_T(\Theta))^2\right] = E\left[\left(F-\Gamma_0^{\mathrm{MVH}}-\int_0^T \vartheta_t^{\mathrm{MVH}} dX_t\right)^2\right] = R^{\mathrm{MVH}}, \quad (5)$$

where Γ_0^{MVH} is initial value of the value process, and $\vartheta_t^{\mathrm{MVH}}$ is called mean-variance hedging ratio. Due to the incompleteness of the market, the equivalent martingale measure is not unique, and we can only hope to minimize the global quadratic risk R^{MVH} within the interval $[0,T]$.

In the absence of arbitrage opportunities, we assume that there is an equivalent martingale measure $P_1 \approx P$, and then, X_T under measure P_1 is a martingale. Let $\mathbb{P}$ be the set of all such equivalent martingale measure P_1, and if there is more than one element in $\mathbb{P}$, the market is said to be incomplete. Therefore, finding an appropriate measure P_1 to hedge a given contingent claim is particularly important.

Gourieroux et al. [11] obtain the variance-optimal equivalent martingale measure Q by minimizing $\|(dP_1/dP)\|_{L^2(P)} = \sqrt{1+\mathrm{Var}(dP_1/dP)}$, which is unique in $\mathbb{P}$. Then, the density process of measure Q with respect to measure P can be denoted by $Z_t^Q = E^Q[\mathrm{d}Q/\mathrm{d}P|\mathscr{F}_t]$.$Z_t^Q$ is a martingale under Q, which can be represented by F¨llmer-Schweizer decomposition.

$$Z_t^Q = E^Q\left[Z_T^Q\middle|\mathscr{F}_t\right] = Z_0^Q + \int_0^t \zeta_u^Q dX_u. \quad (6)$$

For a contingent claim $F \in L^2(P)$, the GKW decomposition of F under measure Q is

$$F = E^Q[F] + \int_0^t \xi_t^Q dX_u + L_T^Q, \quad (7)$$

where ξ_t^Q is a $\mathbb{R}^d$-valued predictable process that coincides with the amount of the risky asset in the local risk-minimization strategy at time t. L_T^Q is a martingale under Q and is zero at the $t=0$, and L_T^Q is strongly Q-orthogonal to X_t. The form of hedging ratio $\vartheta_t^{\mathrm{MVH}}$ at time t is proposed by Pham with the following form:

$$\begin{aligned}\vartheta_t^{\mathrm{MVH}} &= \xi_t^Q - \frac{\zeta_t^Q}{Z_t^Q}\left(\Gamma_t^Q - E^Q[F] - \int_0^t \vartheta_u^{\mathrm{mvo}} dX_u\right)\\ &= \xi_t^Q - \zeta_t^Q \int_0^{t-} 1/Z_u^Q dL_u^Q,\end{aligned} \quad (8)$$

where $F \in L^2(P)$ is a contingent claim. From the above equation, it is not difficult to know that the key to solving $\vartheta_t^{\mathrm{MVH}}$ is to compute L_t^Q and ξ_t^Q explicitly. For this purpose, we first give the forms of L_t^Q and ξ_t^Q separately such that we can calculate them in the later sections.

By the theorem given by Pham et al. [16], a martingale X can be decomposed into $X = X_0 + M_t + A_t$, where M_t is a real measurable martingale with the following form:

$$dM_t = X_t\left(\rho\beta(t,V_t)dW_t + \sqrt{1-\rho^2}\,\beta(t,V_t)X_t dW_t^* + \int_{\mathbb{R}_0} \gamma_{t,z}\widehat{N}(dt,dz)\right), \quad (9)$$

A_t is an adapted process with the form as

$$A_t = \int_0^t d\langle M_u\rangle\lambda_u = \int_0^t X_u\mu(u,V_u)du, \quad (10)$$

where

$$\lambda_t = \frac{dA_t}{d\langle M\rangle_t} = \frac{\mu_1(t,V_t)}{X_t\left(\beta^2(t,V_t) + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}. \quad (11)$$

The density process Z_t^Q can be considered as the Radon–Nikodym derivative of Q. According to the concept of Schweizer [17], we can know $\zeta_t^Q/Z_t^Q - \lambda_t$, and the expression of Z_t^Q under model (3) can be written as

$$\begin{aligned} Z_t^Q &= \varepsilon\left(-\int_0^t \lambda_u dM_u - \int_0^t a_u dW_u\right) \\ &= \exp\left(-\int_0^t (\rho c_u + a_u)\mathrm{d}W_t - \int_0^t \sqrt{1-\rho^2}\, c_u W_t^* - \frac{1}{2}\int_0^t \left(c_u^2 + a_u^2\right)\mathrm{d}t \right. \\ &\quad + \int_0^t \int_{\mathbb{R}_0} (\log(1-\theta(u,z)) + \theta(u,z))\nu(\mathrm{d}z)\mathrm{d}u \\ &\quad \left. + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta(u,z))\widehat{N}(\mathrm{d}u,\mathrm{d}z)\right), \end{aligned} \tag{12}$$

where $\varepsilon(x)$ represents the stochastic exponential of x, and the parameters are

$$\begin{aligned} c_t &= \frac{\mu(t,V_t)\beta(t,V_t)}{\beta^2(t,V_t) + \int_{R_0} \gamma_{t,z}^2 \nu(dz)} \\ a_t &= \frac{\partial J(t,V_t)}{\partial V} b(t,V_t), \\ \theta(t,z) &= \frac{\mu(t,V_t)\gamma_{t,z}}{\beta^2(t,V_t) + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)}, \\ J(t,V_t) &= -\log E\left[\exp\left(-\int_t^T \left(\mu(s,V_s)/\beta^2(s,V_s) + \int_{R_0} \gamma_{s,z}^2 \nu\mathrm{d}(z)\right)^2 \mathrm{d}s\right)\right]. \end{aligned} \tag{13}$$

Then, the stochastic differential equation of X_t under measure Q takes the form

$$\begin{aligned} \mathrm{d}X_t &= X_t\left(\rho\beta(t,V_t)\mathrm{d}\widehat{W}_t + \sqrt{1-\rho^2}\,\beta(t,V_t)\mathrm{d}\widehat{W}_t^* + \int_{\mathbb{R}_0} \gamma(t,z)\widehat{N}^Q(\mathrm{d}t,\mathrm{d}z)\right), \\ \mathrm{d}V_t &= g_t^Q(t,V_t)\mathrm{d}t + b(t,V_t)\mathrm{d}\widehat{W}_t, \end{aligned} \tag{14}$$

where $g_t^Q(t,V_t) = g_t(t,V_t) - b^2(t,V_t)\partial J(t,V_t)/\partial V$. The Girsanov's theorem implies that $\widetilde{W}_t = W_t + \int \rho c_t dt$ and $\widetilde{W}_t^* = W_t^* + \int \sqrt{1-\rho^2} c_t dt$ are two standard Brownian motions under measure Q. $\widehat{N}^Q(\mathrm{d}t,\mathrm{d}z) = \theta(t,z)\nu(dz)\mathrm{d}t + \widehat{N}(\mathrm{d}t,\mathrm{d}z)$ represents the compensated Poisson random measure of $\widehat{N}(\mathrm{d}t,\mathrm{d}z)$.

Theorem 1. *The expression of local risk-minimization strategy ξ_t^Q at time t is given by*

$$\xi_t^Q = \frac{\rho\beta(t,V_t)I_t^1 + \sqrt{1-\rho^2}\beta(t,V_t)I_t^2 + \int_{\mathbb{R}_0}\gamma_{t,\varkappa}I_t^3\nu(d\varkappa)}{X_t\left(V_t + \int_{R_0} r_{t,\varkappa}^2\nu(d\varkappa)\right)}, \tag{15}$$

where I_t^1, I_t^3, and I_t^3 are unique adapted processes.

Proof. Due to L_T^Q is a martingale under measure Q, according to the martingale representation theorem, there exist two unique random processes r_1, r_2, such that

$$L_T^Q = L_0^Q + \int_0^T r_1\rho d\widetilde{W}_u + \int_0^T r_1\sqrt{1-\rho^2}d\widetilde{W}_u^* + \int_0^T\int_{\mathbb{R}_0} r_2\widehat{N}^Q(dt,d\varkappa). \tag{16}$$

By the property of L_T^Q strongly Q-orthogonal to X_t, we can obtain

$$\beta(t,V_t)r_1X_t + \int_{R_0} r_2\gamma_{t,\varkappa}\nu(d\varkappa) = 0. \tag{17}$$

Substitute (16) into (7) to get

$$F = E^Q[F] + \int_0^T (r_1 + \xi_t\beta(t,V_t)X_t)\mathrm{d}\widetilde{W}_t + \int_0^T\int_{\mathbb{R}_0}\left(r_2 + \xi_t\gamma_{t,\varkappa}X_t\right)\widehat{N}^Q(\mathrm{d}t,\mathrm{d}\varkappa). \tag{18}$$

Since F is a martingale under measure Q, it can also be described by the martingale representation theorem with the following form:

$$F = E^Q[F] + \int_0^T I_t^1\mathrm{d}\widetilde{W}_t + \int_0^T I_t^2\mathrm{d}\widetilde{W}_t^* + \int_{\mathbb{R}_0} I_t^3\widehat{N}^Q(\mathrm{d}t,\mathrm{d}\varkappa), \tag{19}$$

where I_t^1, I_t^2, and I_t^3 are unique adapted processes. Combine with (18) and (19), we can obtain the following equations:

$$\begin{cases} \rho r_1 + \rho\xi_t^Q\beta(t,V_t)X_t = I_t^1, \\ \sqrt{1-\rho^2}\,r_1 + \sqrt{1-\rho^2}\,\xi_t^Q\beta(t,V_t)X_t = I_t^2, \\ r_2 + \xi_t^Q\gamma_{t,\varkappa}X_t = I_t^3. \end{cases} \tag{20}$$

According to the condition given in (17), ξ_t^Q can be obtained by solving equations (20). □

Remark 1. The processes I_t^1, I_t^2, and I_t^3 are introduced by the martingale representation theorem, and it is hard to calculated them concretely. Next, we will introduce a method that we can obtain the explicit expression of I_t^1, I_t^2, and I_t^3.

4. A Brief of Malliavin Calculus

In the current section, we overview the main concepts of the Malliavin calculus in the Wiener–Poisson space, and for more information, we can refer to [18].

Assume that $Z(t) = \sigma W_t + \int_0^t\int_{\mathbb{R}_0}\varkappa\widehat{N}(\mathrm{d}t,\mathrm{d}\varkappa)$ is a combination of Gaussian and pure jump lévy processes. Considering a set as follows:

$$S_n = \left\{\left(a_1^{k_1},\cdots,a_n^{k_n}\right) \in \prod_{i=1}^n A_{k_i}\colon 0 \le t_1,\cdots,t_n \le T\right\}, \tag{21}$$

where $A_{k_i} = [0,T]$, $k_1,\cdots,k_n = 0$ or 1. There are some definitions as follows:

$$a_i^{k_i} = \begin{cases} t_i, & k_i = 0, \\ (t_i,z), & k_i = 1, \end{cases}, A_{k_i} = \begin{cases} [0,T], & k_i = 0, \\ [0,T]\times\mathbb{R}, & k_i = 1. \end{cases} \tag{22}$$

Let $dP_0 = dW$, $\mathrm{d}P_1(\bullet) = \widetilde{N}(\bullet,\bullet)$, and $d\langle P_0\rangle = dt$, $d\langle P_0\rangle = \nu(d\varkappa)dt$. Then, we have the following definitions.

Definition 1 (N-fold iterated It ô integrals). *Let* $f_{k_1,\cdots,k_n} \in L^2(S_n)$, the n-fold iterated It ô integrals is defined as follows:

$$J_n^{(k_1,\cdots,k_n)}\left(f_{k_1,\cdots,k_n}\right) = \int_{S_n} f_{k_1,\cdots,k_n}\left(a_1^{k_1},\cdots,a_n^{k_n}\right)P_{k_1}\left(\mathrm{da}_1^{k_1}\right)\cdots P_{kn}\left(\mathrm{da}_n^{k_n}\right). \tag{23}$$

It can be known from definition of [19].

$$E\left[I_n^{(k_1,\cdots,k_n)}\left(f_{k_1,\cdots,k_n}\right)I_m^{(k_1,\cdots,k_m)}\left(g_{k_1,\cdots,k_m}\right)\right] = \begin{cases} 0, n\neq m, \\ \left(f_{k_1,\cdots,k_n}, g_{k_1,\cdots,k_m}\right).n = m. \end{cases} \tag{24}$$

Let $\mathscr{H}_n$ be a closed linear subspace on $L^2(P)$ consisting of random variables of the form $I_n^{(k_1,\cdots,k_n)}(f_{k_1,\cdots,k_n})$. The inner product of $I_n^{(k_1,\cdots,k_n)}(f_{k_1,\cdots,k_n})$ and $I_m^{(k_1,\cdots,k_m)}(g_{k_1,\cdots,k_m})$ is 0 if $n\neq m$, and it means that $\mathscr{H}_n$ and $\mathscr{H}_m$ are two orthonormal subspaces of $L^2(P)$.

Definition 2 (Wiener chaos decomposition [20]). $\{\mathscr{H}_n\colon n>0\}$ are dense subspaces in $L^2(P)$, and then, $L^2(P) = \bigoplus_{n=0}^{\infty}\mathscr{H}_n$. If $F\in L^2(P)$, its only expansion is as follows:

$$F = E[F] + \sum_{n=1}^{\infty} F_n, \tag{25}$$

where $E[F]\in\mathscr{H}_0$ and $F_i\in\mathscr{H}_i$, for $i=1,\cdots,n$. F_i series converges in $L^2(P)$.

Definition 3 (Malliavin derivative [21]). Assume

$$\mathbb{D}^{(r)} = \left\{F\in L^2(P), F = EF + \sum_{n=1}^{\infty}\sum_{\substack{k_i=0,1\\ i=0,\cdots n}} J_n^{(k_1,\cdots,k_n)}(f_n)\colon \sum_{n=1}^{\infty}\sum_{\substack{k_i=0,1\\ i=0,\cdots n}}\sum_{j=1}^{n} 1_{(k_j=r)}\int_{A_{k_j}}\left\|f_{k_1,\cdots,k_n}(\cdots,a^r,\cdots)\right\|^2_{L^2(S_n^j)} d\langle P_r\rangle(a^r) < \infty\right\}. \tag{26}$$

For any $F\in\mathbb{D}^{(r)}\subset L^2(P)$, the Malliavin derivative of F can be defined by

$$D_{a^{k_i}}F = \sum_{n=1}^{\infty} nI_{n-1}(f) = \sum_{n=1}^{\infty}\sum_{\substack{k_i=0,1\\ i=0,\cdots n}}\sum_{j=1}^{n} 1_{(k_j=r)}J^{(k_1,\cdots,k_j,\cdots,k_n)}\left(f_{k_1,\cdots,k_n}(\cdots,a^r,\cdots)1_{S_n^j(t)}\right), \tag{27}$$

where

$$S_n^j(t) = \left\{\left(a_1^{k_1},\cdots,\hat{a},\cdots,a_n^{k_n}\right)\in\prod_{i=1}^{n}A_{k_i}\colon 0\le t_1,\cdots,t_n\le T\right\}. \tag{28}$$

Definition 4 (Clark-Ocone formula for Lévy functionals [19]). Assume that $F\in\mathbb{D}^{(r)}, r=0,1$. Then,

$$F = E[F] + \int_0^T E\left[D_sF|\mathscr{F}_s\right]dW_s + \int_0^T\int_{\mathbb{R}_0} E\left[D_{s,z}F|\mathscr{F}_s\right]\widehat{N}(ds,dz). \tag{29}$$

5. The Calculation of Hedging Ratio $\vartheta_t^{\mathrm{MVH}}$

This section will use the related properties of the Malliavin derivative and the Clark-Ocone formula to get the expression of a contingent claim under the equivalent martingale measure. In addition, the correspondence between the martingale representation theorem and the Clark–Ocone formula lead to the closed-form solution of the MVH strategy. For this purpose, we need the following results:

Theorem 2. *Suppose that $F, Z_T^Q \in \mathbb{D}^{(0)}, Z_T^Q$ is defined in (7). Then, the Malliavin derivative of $Z_T^Q F$ at time t is calculated as*

$$D_t\left(Z_T^Q F\right) = Z_T^Q\left[D_t F + F\left(-\left(\rho + \sqrt{1-\rho^2}\right)c_t - a_t + H_1(t)\right)\right], \tag{30}$$

where

$$H_1(t) = -\int_t^T (D_t\rho c_u + D_t\rho a_u)\mathrm{d}\widetilde{W}_u - \int_t^T D_t\sqrt{1-\rho^2}c_u \mathrm{d}\widetilde{W}_u^* - \int_t^T\int_{\mathbb{R}_0} D_t\theta(u,\varkappa)/1-\theta(u,\varkappa)\widehat{N}(\mathrm{d}u,\mathrm{d}\varkappa). \tag{31}$$

Proof. Firstly, by the continuous case product rule of [19], we can deduce that

$$D_t\left(Z_T^Q F\right) = Z_T^Q D_t F + F D_t Z_T^Q. \tag{32}$$

And according to the continuous case operation properties of the Malliavin derivative in [18], we have

$$D_t Z_T^Q = Z_T^Q\left[-\int_t^T (D_t\rho c_u + D_t\rho a_u)d\widetilde{W}_u - \int_t^T D_t\sqrt{1-\rho^2}\,c_u d\widetilde{W}_u^* - \int_t^T\int_{\mathbb{R}_0} D_t\theta(u,z)/1-\theta(u,z)\widehat{N}(du,d\varkappa) - \left(\rho + \sqrt{1-\rho^2}\right)c_t - a_t\right]. \tag{33}$$

In order to simplify the notation, we denote

$$H_1(t) = -\int_t^T (D_t\rho c_u + D_t\rho a_u)d\widetilde{W}_u - \int_t^T D_t\sqrt{1-\rho^2}c_u d\widetilde{W}_u^* - \int_t^T\int_{\mathbb{R}_0} D_t\theta(u,\varkappa)/1-\theta(u,\varkappa)\widehat{N}(\mathrm{d}u,\mathrm{d}\varkappa), \tag{34}$$

Then (33) can be expressed as

$$D_t Z_T^Q = Z_T^Q\left(H_1(t) - \left(\rho + \sqrt{1-\rho^2}\right)c_t - a_t\right), \tag{35}$$

substitute (35) into (32), this theorem is proved. □ □

Theorem 3. *Suppose that $F, Z_T^Q \in \mathbb{D}^{(1)}$, let $Z_T^Q = \exp(\Delta)$ then the Malliavin derivative $D_{t,\varkappa}(Z_T^Q F)$ at time t is*

$$D_{t,\varkappa}\left(Z_T^Q F\right) = Z_T^Q\left[F\left((1-\theta(t,\varkappa))H_2(t,\varkappa) - 1\right) + (1-\theta(t,z))H_2(t,\varkappa)D_{t,\varkappa}F\right], \tag{36}$$

where

$$H_2(t,\varkappa) = \exp\left(D_{t,z}\Delta - \log(1-\theta(t,\varkappa))\right). \tag{37}$$

Proof. According to the discontinuous case product rule in [19], we get

$$D_{t,\varkappa}\left(Z_T^Q F\right) = F D_{t,\varkappa}Z_T^Q + Z_T^Q D_{t,\varkappa}F + D_{t,\varkappa}F D_{t,\varkappa}Z_T^Q. \tag{38}$$

By the chain rule of [12], we have

$$\begin{aligned} D_{t,\varkappa}Z_T^Q &= \exp\left(\Delta + D_{t,\varkappa}\Delta\right) - \exp(F) \\ &= Z_T^Q\left(\exp\left(D_{t,\varkappa}\Delta\right) - 1\right). \end{aligned} \tag{39}$$

Substitute the above equation into (38) and get

$$D_{t,z}\left(Z_T^Q F\right) = Z_T^Q\left(F\left(\exp\left(D_{t,z}\Delta\right) - 1\right) + D_{t,z}F\exp\left(D_{t,z}\Delta\right)\right). \quad (40)$$

Also, by the operation properties of $D_{t,z}$ of [18], we can obtain

$$\begin{aligned} D_{t,z}\Delta = &-\int_t^T \left(D_{t,z}\rho c_u + D_{t,z}a_u\right)dW_u - \int_t^T D_{t,z}\sqrt{1-\rho^2}c_u dW_u^* - \frac{1}{2}\int_t^T D_{t,z}c_u^2 du - \frac{1}{2}\int_t^T D_{t,z}a_u^2 du \\ &+ \int_t^T \int_{\mathbb{R}_0} D_{t,z}[\log(1-\theta(u,z)) + \theta(u,z)]\nu(dz)du \\ &+ \int_t^T \int_{\mathbb{R}_0} D_{t,z}\log(1-\theta(u,z))\widehat{N}(du,dz) + \log(1-\theta(u,z)) \\ = &-\int_t^T \left(D_{t,z}\rho c_u + D_{t,z}a_u\right)d\widetilde{W}_u - \int_t^T D_{t,z}\sqrt{1-\rho^2}c_u d\widetilde{W}_u^* \\ &+ \int_t^T \int_{\mathbb{R}_0} \left[D_{t,z}\theta(u,z) + \log\left(1 - D_{t,z}\theta(u,z)/1-\theta(u,z)\right)(1-\theta(u,z))\right]\nu(dz)du \\ &+ \int_t^T \int_{\mathbb{R}_0} \log\left(1 - D_{t,z}\theta(u,z)/1-\theta(u,z)\right)\widehat{N}^Q(du,dz) + \log(1-\theta(t,z)). \end{aligned} \quad (41)$$

In order to simplify this lengthy formula, we set

$$H_2(t,z) = \exp\left(D_{t,z}\Delta - \log(1-\theta(t,z))\right). \quad (42)$$

Then, $D_{t,z}Z_T^Q$ can be obtained as follows

$$D_{t,z}Z_T^Q = Z_T^Q[(1-\theta(t,z))H_2(t,z)]. \quad (43)$$

By substituting the above equation into (40), we can obtain the expression of $D_{t,z}(Z_T^Q F)$. □

Theorem 4 (The Clark-Ocone formula under measure Q). *If F is a $\mathcal{F}_t$-measurable random variable, $F \in \mathbb{D}^{(r)}$, and $a_t, c_t \in \mathbb{D}^{(r)}$. Suppose the following conditions are true*

(a) $E^Q[|F|] < \infty$,

(b) $E^Q\ [1/2\ \int_0^T\ |D_t F|^2 dt] < \infty,\ E^Q[1/2\ \int_0^T\ |D_{t,z}F|^2 dt] < \infty$,

(c) $E[Z^2(T)F^2\ \int_0^T (-(\rho + \sqrt{1-\rho^2})c_t - a_t + H_1(t))^2 \mathrm{d}t] < \infty$,

(d) $E\ [Z^2(T)\ \int_0^T (F((1-\theta(t,z))\ H(t,z) - 1) + (1 - \theta(t,z))H(t,z)D_{t,z}\ F)^2 dt] < \infty$.

Then, we get the Clark-Ocone formula under measure Q as follows:

$$\begin{aligned} F = E^Q[F] &+ \int_0^T E^Q[(\rho D_t F + C_1 + FH_1(t)) \mid F_t]d\widetilde{W}_t \\ &+ \int_0^T E^Q\left[\sqrt{1-\rho^2}\ (D_t F + FH_1(t) + C_2) \mid F_t\right]d\widetilde{W}_t^* \\ &+ \int_0^T \int_{\mathbb{R}_0} E^Q\left[\left(F(H_2(t,z) - 1) + H_2(t,z)D_{t,z}F\right) \mid F_t\right]\widehat{N}^Q(dt,dz), \end{aligned} \quad (44)$$

where

$$C_1 = -F\left(\rho^2 + \rho\sqrt{1-\rho^2} - \rho\right)c_t - (\rho - 1)a_t, \tag{45}$$

and

$$C_2 = -F\left(\rho + \sqrt{1-\rho^2}\right)c_t - Fa_t + F. \tag{46}$$

Proof. See Appendix A. □

Corollary 1. *Through the correspondence between (18) and (44), and assume that $c_t, a_t, \theta(t, \varkappa)$ are deterministic functions, we can get the following equations:*

(1) $I_t^1 = E^Q[(\rho D_t F + F \quad (-(\rho^2 + \rho\sqrt{1-\rho^2})c_t - \rho a_t + H_1(t))) \mid \mathcal{F}_t]$,

(2) $I_t^2 = E^Q[\sqrt{1-\rho^2} \quad (D_t F + F(-(\rho + \sqrt{1-\rho^2})c_t - a_t + 1 + H_1(t))) \mid \mathcal{F}_t]$,

(3) $I_t^3 = E^Q[(F(H_2(t,\varkappa) - 1) + H_2(t,\varkappa)D_{t,\varkappa}F) \mid \mathcal{F}_t]$.

Theorem 5. *Assume that c_t, a_t, and $\theta(t, \varkappa)$ are deterministic functions, then the closed-form expression of the hedge ratio ϑ_t^{MVH} can be expressed as*

$$\vartheta_t^{MVH} = \xi_t^Q + Z_t^Q \lambda_t \int_0^{t-} \mathrm{d}F_u - \frac{\xi_u^Q \mathrm{d}S_u}{Z_u^Q}. \tag{47}$$

Proof. Substitute the expressions of I_t^1, I_t^2, and I_t^3 into (20) to get the following solution:

$$\xi_t^Q = \frac{\rho\beta I_t^1 X_t + \beta\sqrt{1-\rho^2} I_t^2 X_t + \gamma I_t^3}{X_t^2\beta^2 + X_t\gamma}. \tag{48}$$

Then, combining with (19), r_1 and r_2 can be solved as

$$\begin{cases} r_1 = \dfrac{1}{\rho} I_t^1 - \xi_t \beta X_t, \\ \\ r_2 = I_t^3 - \xi_t \gamma X_t. \end{cases} \tag{49}$$

It means that L_t^Q has the following form:

$$\mathrm{d}L_t^Q = \left(I_t^1 - \rho\xi_t^Q\beta(t,V_t)X_t\right)\mathrm{d}\widetilde{W}_t + \left(\frac{\sqrt{1-\rho^2}}{\rho} I_t^1 - \xi_t^Q\beta(t,V_t)X_t\right)d\widetilde{W}_t^* + \int_{\mathbb{R}_0}\left(I_t^3 - \xi_t^Q\gamma\right)\widehat{N}^Q(dt,d\varkappa) = dF_t - \xi_t^Q dX_t. \tag{50}$$

By substituting (48) and (50) into (8), the closed-form expression of the hedging ratio ϑ_t^{MVH} is obtained. □

6. Numerical Analysis

In this section, we will propose a numerical example to compare across the one-dimensional Lévy model in [22]. We mainly discuss the sensitivity analysis for the hedging ratio ϑ_t^{MVH} with respect to different strike prices and maturity times under these two models. We set F as a European Option with the payoff function $F = (X_T - K)^+$, K is the strike price, and T is maturity time.

6.1. Mathematical Preliminaries. For numerical simulation, we specify the SVJ model as the Heston model coupled with an exponential Lévy process

$$\frac{\mathrm{d}X_t}{X_t} = \mu_1 dt + \rho\sqrt{V_t}dW_t + \sqrt{\left(1-\rho^2\right)V_t}dW_t^* + \int_{\mathbb{R}_0}\gamma_\varkappa \widehat{N}(dt,d\varkappa)\, dV_t = \xi_1(\eta_1 - V_t)dt + \sigma_1\sqrt{V_t}dW_t, \tag{51}$$

where V_t and μ_1 represent volatility and interest rates, ξ_1 and η_1 are the respective corresponding average mean reversion rates and long-run average volatility, σ_1 is the volatility coefficient of the volatility process, $\gamma_\varkappa = e^\varkappa - 1$. The standard Brownian motion W, W_t^* and the Poisson measure $\widehat{N}$ are the same as Section 1. We have calculated the form of (51) under variance-optimal measure Q, as presented in Appendix B.

Theorem 6. *Model (52) under variance-optimal measure Q satisfies the following form:*

$$dX_t = X_t\left(\rho\sqrt{V_t}\,d\widehat{W}_t + \sqrt{\left(1-\rho^2\right)V_t}\,d\widehat{W}_t^* + \int_{\mathbb{R}_0}\gamma_{t,\varkappa}\widehat{N}^Q(dt,d\varkappa)\right),\ dV_t = \left(\xi_1\eta_1 - \left(\xi_1 + \sigma_1^2 B(t)\right)V_t\right)dt + \sigma_1\sqrt{V_t}d\widehat{W}_t, \tag{52}$$

with

$$B(t) = \frac{1}{\sigma_1}\sqrt{2}\sqrt{\varpi}\tan\left(\frac{\sqrt{2}\sqrt{\varpi}\sigma_1(T-t)}{2}\right), \quad (53)$$

where $\widetilde{W}_t = W_t + \int \rho c_t dt$, $\widetilde{W}_t^* = W_t^* + \int \sqrt{1-\rho^2}c_t dt$ *are two Brownian motions, and* $\widehat{N}^{Q}(dt, dz) = \theta(t, \varkappa)\nu(d\varkappa)dt + \widehat{N}(dt, d\varkappa)$ *is a compensated Poisson random measure.*

Proof. See Appendix B.

According to Theorem 5, the discretization on $[0, t]$ for (47) is

$$\vartheta_t^{\mathrm{MVH}} = \xi_t^Q + Z_t^Q\lambda_t \sum_{i=1}^{n} \Delta F_{t_i} - \xi_{t_i}^Q \Delta X_{t_i}/Z_{t_i}^Q, \#(27), \quad (54)$$

where

$$\xi_t^Q = \frac{\rho V_t I_t^1 + \sqrt{1-\rho^2}V_t I_t^2 + \int_{\mathbb{R}_0}\gamma_{t,\varkappa} I_t^3 \nu(d\varkappa)}{X_t\left(V_t + \int_{\mathbb{R}_0} r_{t,\varkappa}^2 \nu(d\varkappa)\right)}. \quad (55)$$

To sum up, we need the following processes:

(a) Observing formulas (54) and (55), ΔF_{t_i}, I_t^1, I_t^2, and I_t^3 can be computed by the FFT algorithm

(b) $Z_{t_i}^Q$ and X_{t_i} can be obtained by iteration

We will deal with them separately.

For processing (a): by the example given by Arai and Imai [22] and properties of conditional expectation, the expressions for I_t^1, I_t^2, and I_t^3 are

$$\begin{cases} I_t^1 = V_t E^Q\left[X_T 1_{\{X_T>k\}} \middle| \mathscr{F}_t\right] + \left(-\left(\rho^2 + \rho\sqrt{1-\rho^2}\right)c_t - \rho a_t\right)E^Q[F|\mathscr{F}_t], \\ I_t^2 = \dfrac{\sqrt{1-\rho^2}}{\rho} I_t^1, \\ I_t^3 = E^Q\left[(e^{\varkappa}X_T - K)^+ - (X_T - K)^+ |\mathscr{F}_t\right]. \end{cases} \quad (56)$$

The expectations contained in the results need to use the FFT algorithm to compute, and we will only take $E^Q[X_T 1_{\{X_T>k\}} | \mathscr{F}_t]$, for example.

Denote the $Y_t = \ln X_t$, and the characteristic function of Y_T can be expressed as

$$\Phi^Q(Y, t; u) = E^Q\left[e^{iuY_T} \middle| \mathscr{F}_t\%\right]. \quad (57)$$

The derivation for the expression of characteristic functions $\Phi^Q(Y, t; u)$ has been shown in Appendix A, and this section only emphasizes the main framework of the FFT algorithm. Let $M_t(k) = E^Q[Y_T 1_{\{Y_T>k\}} | \mathscr{F}_t]$, q_T is the density of the process Y_T, and then,

$$\Phi_t^Q(u) = \Phi^Q(Y, t; u) = \int_{-\infty}^{\infty} e^{iuY} q_T(Y) dY. \quad (58)$$

As Carr and Madan's method in [23] introduces a modified function $m_T^1 = e^{\alpha k}M_T(k)$, $\alpha > 0$. By Fourier transform, we can get

$$\Psi_t^Q(u) = \Psi^Q(Y, t; u) = \int_{-\infty}^{\infty} e^{iuk} m_T(k) dk. \quad (59)$$

We can represent $\Phi^Q(Y, t; u)$ by $\Phi_t(u)$ as follows:

$$\begin{aligned}\Psi_t^Q(u) &= \int_{-\infty}^{\infty} e^{iuk} \int_k^{\infty} e^{-\alpha k} e^{Y_T} q_T \mathrm{dsdk} \\ &= \frac{\Phi_t^Q(u - (\alpha+1)\mathrm{i})}{\alpha - 1 + iu}.\end{aligned} \quad (60)$$

In turn, we apply inverse Fourier transform to get

$$M_t(k) = \frac{e^{-\alpha k}}{\pi}\int_0^{\infty} e^{-iuk}\Psi_t^Q(u)du = \frac{e^{-\alpha k}}{\pi}\int_0^{\infty} e^{-iuk}\Phi_t^Q(u - (\alpha+1)\mathrm{i})/\alpha - 1 + iu\, du. \quad (61)$$

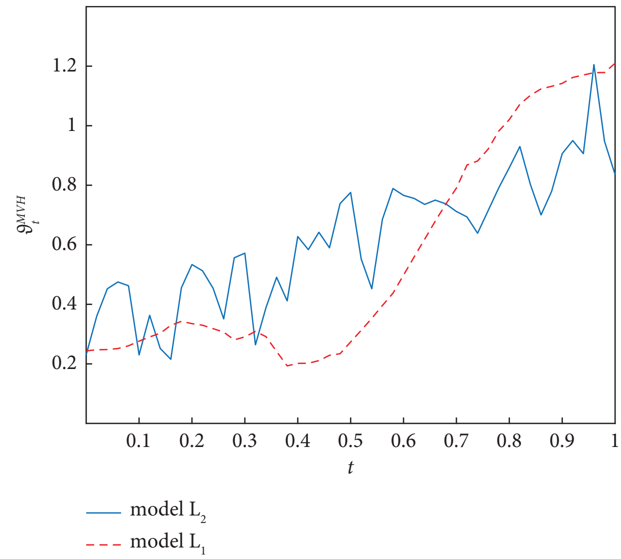

Figure 1: Hedging ratio $\vartheta_t^{\mathrm{MVH}}$ for L_1 and L_2 with respect to time t.

From equations (58)–(61) and Simpson's rule weightings, we consider the discrete version of the $M_t(k)$ as

$$M_t(k_l) = \frac{e^{-\alpha k_l}}{\pi} \sum_{j=1}^{N} e^{-i2\pi/N(j-1)(l-1)} e^{iu_j N\omega/2} \Psi_T(v) \frac{\eta}{3}\left(3 + (-1)^j - \delta_{j-1}\right), \tag{62}$$

where $u_j = (j - N/2)\eta$, η denotes the integration steps, and δ_j is the Kronecker delta function that is unity for $j = 0$ and zero otherwise. The FFT returns N values of $k_l = (l - N/2)\omega$ and for a regular spacing size, $l = 1, \cdots N$.

For processing (b): according to the Euler scheme and method in [15], we get a discretization of the system (52) as

$$\begin{aligned} X_{t_{i+1}} &= X_{t_i} + X_{t_i}\left(\rho\sqrt{V_t}\,\epsilon_1\sqrt{\Delta t} + \sqrt{\left(1-\rho^2\right)V_{t_i}}\,\epsilon_2\sqrt{\Delta t} + \sum_{0\le\tau_1<\cdots<\tau_n<t} K\left(\tau_i, \Lambda\left(\tau_i\right)\right)\right), \\ V_{t_{i+1}} &= V_{t_i} + \left(\xi_1\eta_1 - \left(\xi_1 + \sigma_1^2 F\left(V, t_i\right) B\left(t_i\right)\right) V_{t_i}\right)\Delta t + \sigma_1\sqrt{V_{t_i}}\,\epsilon_1\sqrt{\Delta t}, \end{aligned} \tag{63}$$

where

$$\int_0^t \int_{\mathbb{R}_0} K(s, z)\widehat{N}^{Q}(dt, dz) = \sum_{0\le\tau_1<\cdots<\tau_n<t} K\left(\tau_i, \Lambda\left(\tau_i\right)\right). \tag{64}$$

Denote the countable set of definitions for stationary Poisson point processes $\Lambda(\bullet, \omega)$ by $D_\Lambda(\omega) = \{\tau_1(\omega), \cdots, \tau_n(\omega)\}$. The distance between τ_{i+1} and τ_i follows exponential distribution with parameter λ_1, and $\Lambda(\bullet, \omega)$ follows normal distribution $N(0, \delta_1^2)$. With the same schemes, other variables can also be discretized. □

6.2. Numerical Results. We consider a European call option with the expiry time $T = 1$. The initial price of the underlying asset $X_0 = 1.2$, and $V_0 = 0.12$. For a mean reverting process with stochastic volatility and no jumps, $\mu_1 = 0.05$, $\xi_1 = 0.2$, $\eta_1 = 0.5$, $\rho = 0.6$. For Lévy process, $\gamma = 1$, $m = 0$, and $\delta = 1$. For parameters in a discrete scheme, $\delta_1 = 0.1$, $\lambda_1 = 0.5$, $\Delta_t = 0.01$. We implement our FFT scheme with $\eta = 0.25$ and $N = 256$, which leads to a logarithm strike spacing of $\omega = 8\pi/256$. The modified coefficient is set to $\alpha = 1.5$.

As shown in Figure 1, we learn that the MVH strategy of a European call option is sensitive to time t in the maturity time T. Both models L_1 and L_2 show a gradual increase in their overall trend. The only difference is that the hedge ratio of L_1 changes sharply while that of L_2 increases smoothly. Since there is no essential difference betweenL_1 and L_2 when time t is fixed, we only conducted a sensitivity analysis of MVH strategy on strike price K for L_2. Ordinarily, we obtain $t = 0.5$ and observe that the hedging ratio ϑ_t^{MVH} decreases

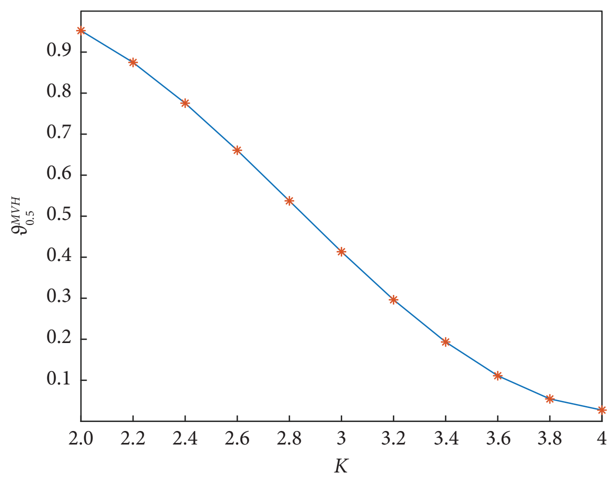

Figure 2: Hedging ratio $\vartheta_t^{\mathrm{MVH}}$ for L_2 with respect to strike price K.

with the increase of strike price K as shown in Figure 2, which is consistent with the experimental situation in literature [13, 23].

7. Conclusion

In this article, we discussed a method of Malliavin calculus to obtain the explicit expression of the hedging ratio ϑ^{MVH} of the mean-variance strategy. We calculated the Clark-Ocone formula under variance-optimal equivalent martingale measure Q. According to the correspondence of the Clark-Ocone formula and the martingale representation theorem, we express the diffusion term coefficients by Malliavin derivatives. By using the FFT algorithm and the discrete scheme for Lévy process, we conducted some numerical analyses for sensitivity analysis of MVH strategy with respect to time t and strike price K.

However, two further extensions of the Malliavin calculus in the mean-variance hedging strategy would be interesting. One possible extension is incorporating stochastic interest into the SVJ model so that the hedging results are more consistent with the real market. Another extension would be to extend the study on the exotic options to make our conclusions more general.

Appendix

A. Proof of Theorem 4

Proof. Let $B(t) = (Z_t^Q)^{-1}$, $G(t) = E^Q[F\mathscr{F}_t]$. From thecorollary ofNunno [19], we can deduce that $G(t) = B(t)E[Z_T^Q F\mathscr{F}_t]$. With the help of the Clark-Ocone formula for Lévy functionals in Definition 2, we can write $G(t)$ as follows

$$\begin{aligned}
G(t) &= B(t)E\left[Z_T^Q F\mathscr{F}_t\right] \\
&\quad + \int_0^T \sqrt{1-\rho^2}E\left[D_u E\left[Z_T^Q F\mathscr{F}_t\right]\mathscr{F}_u\right]dW_u^* \\
&\quad + \int_0^T \int_{\mathbb{R}_0} E\left[D_{u,z} E\left[Z(T)F|\mathscr{F}_t\right]\mathscr{F}_u\right]\widehat{N}(du,dz) \\
&= B(t)E\left[Z_T^Q F\right] + \int_0^t \rho E\left[D_u\left(Z_T^Q F\right)\mathscr{F}_u\right]dW_u \\
&\quad + \int_0^t \int_{\mathbb{R}_0} E\left[D_{u,z}\left(Z_T^Q F\right)|\mathscr{F}_u\right]\widehat{N}(du,dz) \\
&= B(t)U(t)(\mathrm{A.1}).
\end{aligned} \tag{A.1}$$

Apply It ô formula to get

$$dB(t) = B(t)\left((\rho c_t + a_t)d\widetilde{W}_t + \sqrt{1-\rho^2}\, c_t d\widetilde{W}_t^* + \int_{\mathbb{R}_0} \theta(t,\varkappa)/1-\theta(t,\varkappa)\widehat{N}^Q(dt,d\varkappa) \right). \tag{A.2}$$

Therefore, by combining (A.1) and (A.2), we get

$$\begin{aligned} &dG(t)\\ &= B(t)\left(\rho E\left[D_t\left(Z_T^Q F\right)\mathcal{F}_t\right]\left(\rho dW_t + \sqrt{1-\rho^2}\, dW_t^*\right) + \int_{\mathbb{R}_0} E\left[D_{t,\varkappa}\left(Z_T^Q F\right) \Big| \mathcal{F}_t\right]\widehat{N}(dt,d\varkappa)\right)\\ &\quad + U(t)B(t)\left((\rho c_t + a_t)d\widetilde{W}_t + \sqrt{1-\rho^2}\, c_t d\widetilde{W}_t^* + \int_{\mathbb{R}_0} \theta(t,\varkappa)/1-\theta(t,\varkappa)\widehat{N}^Q(dt,d\varkappa)\right)\\ &\quad + \rho(\rho c_t + a_t)B(t)E\left[D_t\left(Z_T^Q F\right)\mathcal{F}_t\right]dt + \left(1-\rho^2\right)E\left[D_t\left(Z_T^Q F\right)\mathcal{F}_t\right]dt\\ &\quad + B(t)\int_{\mathbb{R}_0} \theta(t,\varkappa)/1-\theta(t,\varkappa)E\left[D_t\left(Z_T^Q F\right) \Big| \mathcal{F}_t\right]\nu(d\varkappa)dt\\ &\quad + B(t)\int_{\mathbb{R}_0} \theta(t,\varkappa)/1-\theta(t,\varkappa)E\left[D_{t,\varkappa}\left(Z_T^Q F\right) \Big| \mathcal{F}_t\right]\widehat{N}(dt,d\varkappa)\\ &= B(t)\rho E\left[D_t\left(Z_T^Q F\right)\mathcal{F}_t\right]d\widetilde{W}_t + B(t)\sqrt{1-\rho^2}E\left[D_t\left(Z_T^Q F\right)\mathcal{F}_t\right]d\widetilde{W}_t^*\\ &\quad + G(t)\left((\rho c_t + a_t)d\widetilde{W}_t + \sqrt{1-\rho^2}\, c_t d\widetilde{W}_t{}^*\right)\\ &\quad + B(t)\int_{\mathbb{R}_0} E\left[D_{t,\varkappa}\left(Z_T^Q F\right) \Big| \mathcal{F}_t\right]/1-\theta(t,\varkappa)\widehat{N}^Q(dt,d\varkappa)\\ &\quad + B(t)\int_{\mathbb{R}_0} E\left[Z_T^Q F\theta(t,\varkappa)/1-\theta(t,\varkappa) \mid \mathcal{F}_t\right]\widehat{N}^Q(dt,d\varkappa). \end{aligned} \tag{A.3}$$

By (28) and (34), the above equation can be written as

$$\begin{aligned} &dG(t)\\ &\quad + B(t)E\left[Z_T^Q\left(\sqrt{1-\rho^2}\,\left(D_t F + FH_1(t) + C_2\right)\right)\Big| \mathcal{F}_t\right]d\widetilde{W}_t^*\\ &\quad + B(t)\int_{\mathbb{R}_0} E\left[Z_T^Q\left(F\left(H_2(t,\varkappa)-1\right) + H_2(t,\varkappa)D_{t,\varkappa}F\right)\Big| \mathcal{F}_t\right]\widehat{N}^Q(dt,d\varkappa)\\ &= E^Q\left[\rho D_t F + C_1 + FH_1(t) \,\big|\, \mathcal{F}_t\right]d\widetilde{W}_t\\ &\quad + E^Q\left[\sqrt{1-\rho^2}\,\left(D_t F + C_2 + FH_1(t)\right)\Big| \mathcal{F}_t\right]d\widetilde{W}_t^*\\ &\quad + \int_{\mathbb{R}_0} E^Q\left[\left(F\left(H_2(t,\varkappa)-1\right) + H_2(t,\varkappa)D_{t,\varkappa}F\right)\Big| \mathcal{F}_t\right]\widehat{N}^Q(dt,d\varkappa), \end{aligned} \tag{A.4}$$

where

$$C_1 = -F\left(\rho^2 + \rho\sqrt{1-\rho^2} - \rho\right)c_t - (\rho - 1)a_t, \quad (A.5)$$

and

$$C_2 = -F\left(\rho + \sqrt{1-\rho^2}\right)c_t - Fa_t + F. \quad (A.6)$$

Since F is a martingale under measure Q, we can know that $G(T) = E^Q[F\mathscr{F}_T] = F$ and $G(0) = E^Q[F\mathscr{F}_0] = E^Q[F]$, then the proof is completed. □

B. Proof of Theorem 6

Proof. According the form of Radon-Nikodym derivative given in (12), (12), the parameters can be expressed as

$$\begin{aligned} c_t &= \frac{\mu_1\sqrt{V_t}}{V_t + \int_{\mathbb{R}_0}\gamma_{t,x}^2 \nu(dx)}, \\ a_t &= \frac{\partial J_t}{V}\sigma_1\sqrt{V_t}, \\ \theta(t,x) &= \frac{\mu_1\gamma_{t,x}}{V_t + \int_{\mathbb{R}_0}\gamma_{t,x}^2 \nu(dx)}, \\ J_t &= -\ln E\left[\exp\left(-\int_t^T \mu_1^2/V_s^2 + \int_{\mathbb{R}_0}\gamma_{s,x}^2 \nu d(x) ds\right)\Big|\mathscr{F}_t\right]. \end{aligned} \quad (B.1)$$

Assume that γ is the jumps intensity and the sizes of the jumps are distributed normally with mean m and variance δ^2. Then we can calculate $\int_{\mathrm{R}_0}(e^x - 1)^2 \nu d(x)$ as

$$\int_{\mathbb{R}_0}(e^x - 1)^2 \nu d(x) = \gamma\exp\left(2m + 2\delta^2\right) - \gamma\exp\left(m + \frac{\delta^2}{2} - 1\right). \quad (B.2)$$

Denote the result in (B.2) as γ_1. Let $F(V,t) = E[\exp(-\int_t^T \mu_1^2/V_s^2 + \gamma_1 ds \mid \mathscr{F}_t)]$. From the Feynman-Kac formula, $F(V,t)$ is the solution of the following backward Parabolic partial differential equation with the Cauchy problem:

$$\begin{cases} \dfrac{\partial F}{\partial t} + \xi_1(\eta_1 - V)\dfrac{\partial F}{\partial V} + \dfrac{1}{2}\sigma_1^2 V\dfrac{\partial^2 F}{\partial V^2} - \dfrac{\mu_1^2 F}{V^2 + \gamma_1} = 0, \\ F(V,T) = 0. \end{cases} \quad (B.3)$$

We set $V_t = V$. Because of the affine structure of the volatility process in (B.2), we assume that $F(V,t)$ has a solution as follows:

$$F(V,t) = e^{A(t)+B(t)V_t}. \quad (B.4)$$

We substitute (B.4) into (B.3) to obtain

$$\begin{cases} A_t(t) + \xi_1\eta_1 B(t) = 0, \\ -B_t(t) + \dfrac{1}{2}\sigma_1^2 B^2(t) + \xi_1 B(t) + \dfrac{\mu_1^2}{V(V+\gamma_1)} = \#0\#\#, \end{cases} \quad (B.5)$$

with the boundary conditions $A(T)=0, B(T)=0$. Assume that $V_t=V$ and set $\varpi=\mu_1^2/V^2+\gamma$, and then, the ODE (B.5) can be solved as

$$A(t)$$
$$B(t)=\frac{1}{\sigma_1}\sqrt{2}\sqrt{\varpi}\tan\left(\frac{\sqrt{2}\sqrt{\varpi}\sigma_1(T-t)}{2}\right). \tag{B.6}$$

It means that $J_t=-A(t)-B(t)V_t$, following the stochastic differential equation in (14), and the form model (51) under measure Q can be obtained. □

C. Derivation of the Characteristic Function

In Appendix C we will show the Solutions of characteristic functions (55). According to Levy-It ô formula for Y_t in one-dimensional case, we have the following result:

$$\mathrm{d}Y_t + \int_{\mathbb{R}_0} \varkappa \widehat{N}^Q(\mathrm{d}t, \mathrm{d}\varkappa), \tag{C.1}$$

where $\mu_t^*=-V_t/2+\gamma_2$, $\gamma_2=\int_{\mathbb{R}_0}(z-e^z+1)\nu^Q(dz)$. The dynamic process $\mathrm{d}Y_t$ can be written as $\mathrm{d}Y_t=\mathrm{d}Z_t+\mathrm{d}U_t$, in which $\mathrm{d}Z_t=\mu_t^*\mathrm{d}t+\rho\sqrt{V_t}\mathrm{d}\widehat{W}_t+\sqrt{(1-\rho^2)V_t}\mathrm{d}\widehat{W}_t^*$ and $\mathrm{d}U_t=\int_{\mathbb{R}_0}\varkappa\widehat{N}^Q(\mathrm{d}t,\mathrm{d}\varkappa)$. Because of the independence between $\mathrm{d}Z_t$ and $\mathrm{d}U_t$, we can obtain an expression owing to basic properties of the characteristic function as follows:

$$\Phi^Q(Y,t;u)=\Phi_Z^Q(Y,t;u)\Phi_U^Q(Y,t;u). \tag{C.2}$$

Theorem C.1. *The characteristic function $\Phi^Q(Y,t;u)$ under variance-optimal measure Q is*

$$\Phi^Q(Y,t;u)=eC(t)V_t+D(t)+iuZ_t+(T-t)\varphi_{\text{levy}}^Q, \tag{C.3}$$

where parameters of the formula (37) are constrained as follows:

$$C(t)=\frac{\sqrt{\phi_2}}{\sigma_1{}^2}\tanh\left(\frac{(T-t)\sqrt{\phi_2}}{2}-\frac{\sqrt{\phi_2}\,\operatorname{atanh}\left(\xi_1/\sqrt{\phi_2}+\sigma_1{}^2B(t)/\sqrt{\phi_2}\right)}{\sqrt{\phi_2}}\right)+\frac{B(t)\sigma_1{}^2+\xi_1}{\sigma_1{}^2}, \tag{C.4}$$

$$D(t)=\left(2\rho ui+\frac{2\eta_1\xi_1}{\sigma_1{}^2}\right)\ln\left(\frac{e^{-2\operatorname{atanh}\left(\xi_1/\sqrt{\phi_2}+\sigma_1{}^2B(t)/\sqrt{\phi_2}\right)}+1}{e^{-2\operatorname{atanh}\left(\xi_1/\sqrt{\phi_2}+\sigma_1{}^2B(t)/\sqrt{\phi_2}\right)}e^{x\sqrt{\phi}}+1}\right)+\phi_3, \tag{C.5}$$

in which

$$\phi_2=\sigma_1{}^4B(t)^2+2\sigma_1{}^2\xi_1B(t)+u\sigma_1{}^2i+\xi_1{}^2,$$
$$\phi_3=(T-t)\left(-\rho u\xi_1 i-\eta_1\xi_1B(t)+\rho u\sqrt{\phi_2}\,i-\frac{\eta_1\xi_1{}^2}{\sigma_1{}^2}-\rho\sigma_1{}^2uiB(t)+\frac{\eta_1\xi_1\sqrt{\phi_2}}{\sigma_1{}^2}-\gamma_2ui\right). \tag{C.6}$$

Proof. For $\Phi_Z^Q(Y,t;u)\Phi_U^Q$, we set $M(u,Z,V,t) = E^Q[e^{iuZ_T} \mid Z_t = Z, V_t = V]$ as the moment generating function of Z_T at time t. The generalized Feynman–Kac formula implies the following partial differential difference equations:

$$\frac{\partial M}{\partial t} + \mu_t^* \frac{\partial M}{\partial Z} + \left(\xi_1\eta_1 - \left(\xi_1 + \sigma_1^2 B(t)\right)V_t\right)\frac{\partial M}{\partial V} + \frac{1}{2}V_t\frac{\partial^2 M}{\partial Z^2} + \frac{1}{2}\sigma_1^2 V_t \frac{\partial^2 M}{\partial V^2} + \rho\sigma^2 \frac{\partial^2 M}{\partial Z \partial V} = 0. \tag{C.7}$$

The exponential affine structure for characteristic function can be assumed as follows:

$$M(u,Z,V,t) = e^{C(t)V + D(t) + iuZ}. \tag{C.8}$$

Substitute (C.4) to (C.3) to get

$$\begin{cases} C_t(t) - \frac{1}{2}u^2 + \frac{1}{2}\sigma^2 C^2(t) - \left(\xi_1 + \sigma_1^2 B(t)\right)C(t) - \frac{iu}{2} = 0, \\ D_t(t) + \gamma_2 iu + \xi_1\eta_1 C(t) + \rho\sigma^2 iuC(t) = 0. \end{cases} \tag{C.9}$$

With the boundary condition $C(T) = 0, D(T) = 0$, we solve this ODE as (C.1) and (C.2).

By Lévy–Khintchine theorem, we can compute $\Phi_U^Q(x,t;u) = \exp((T-t)\varphi_{levy}^Q)$, and φ_{levy}^Q as the form of

$$\varphi_{\text{levy}}^Q = \int_{\mathbb{R}_0} \left(e^{\text{ixu}} - 1 - \text{ixu}\right)\nu^Q(d\varkappa) = (1+h)\gamma\left(e^{m\alpha + 1/2\alpha^2\delta^2} - 1 - \alpha m\right) - h\gamma e^{1/2\left(2m+\delta^2\right)}\left(e^{\left(m+\delta^2\right)\alpha + \alpha^2\delta^2/2} - 1 - \alpha\left(m + \delta^2\right)\right). \tag{C.10}$$

Combining (C.4) and (C.5), we obtain the characteristic function $\Phi^Q(Y,t;u)$. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported through the Education Department of Jilin Province (No. JJKH20200030KJ) and the Department of Science and Technology of Jilin Province (No. YDZJ202102ZYTS156).

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