Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 3157217, 12 pages
https://doi.org/10.1155/2022/3157217

*Research Article*

# Dynamical Behaviour of Conformable Time-Fractional Coupled Konno-Oono Equation in Magnetic Field

**M. E. Elbrolosy [1,2] and A. A. Elmandouh [1,3]**

[1] *Department of Mathematics and Statistics, College of Science, King Faisal University, P. O. Box 400, Al-Ahsa 31982, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt*
[3] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

Correspondence should be addressed to M. E. Elbrolosy; melborolosy@kfu.edu.sa

Received 5 May 2022; Accepted 24 May 2022; Published 28 June 2022

Academic Editor: Abdellatif Ben Makhlouf



Our aim is to construct distinct types of novel traveling wave solutions for the nonlinear conformable time-fractional coupled Konno–Oono equation in a magnetic field utilizing the qualitative analysis of the dynamical system that corresponds to the Hamiltonian system. The influences of fractional-order derivative on the wave solutions in addition to equation parameters on the waveform are investigated. Furthermore, He's variational technique has been applied to seek some new solutions. 2D and 3D graphics of the obtained solutions are presented to clarify the physical and mathematical meaning of the waves.

## 1. Introduction

Models based on fractional derivatives have been successful in describing nonlinear physical phenomena. A continuing interest in fractional calculus resides in its applicability. In fields such as physics, mechanics, chemistry, and biology, fractional calculus is extremely useful [1–12]. The properties of derivatives in classical Riemann–Liouville fractional calculus, however, are more complicated than those in usual derivatives. Leibnitz's rule, for instance, cannot be applied. Tarasov had proved several important theorems, in the field of theory, to support his claim that, in general, Leibnitz's rule does not apply for Riemann–Liouville's fractional derivatives [13]. On the other hand, there are some serious mistakes in existed theory such as Jumarie's fractional derivative. Lately, Liu supplied essential development in the field since he provided counterexamples to show that Jumarie's basic formulas are not correct [14, 15]. Respectively, all those results based on Jumarie's formulas are also incorrect. Moreover, Liu [14] verified strictly that the local fractional derivative did not exist at all on any interval. Taking into account these significant results, we should neatly verify the fundamental formulas when the fractional derivative is considered. Conformal fractional derivatives are introduced to overcome the limitations of usual fractional derivatives, which satisfy routine derivative rules and therefore have attracted the attention of numerous scientists. This derivative is particularly useful in describing some local power-law behaviors. Hence, we can learn a lot by studying solutions of conformable fractional differential equations. In other words, the usage of the conformable fractional operator overcomes some restrictions of the different fractional operator's properties such as the chain rule, the derivative of the quotient of two functions, the product of two functions, and the mean value theorem. Thus, it becomes more interesting in describing many physical problems. Now, let us give a short note about the conformable time fractional.

*Definition 1* (see [16]). Let $g$: $[0,\infty] \longrightarrow \mathbb{R}$ be a function, then the conformable fractional derivative of order $\alpha$ is defined as

$$T_{\alpha}(g)(t) = \lim_{\sigma \longrightarrow 0} \frac{g\left(t + \sigma t^{1-\alpha}\right) - g(t)}{\sigma}, \tag{1}$$

for all $t > 0$ and $0 < \alpha \leq 1$.

In what follows, we present some significant properties of the conformable derivatives. Let functions $g_1, g_2$ be two $\alpha$–conformable differential functions. For $t > 0$ and two constants $a$ and $b$, we have the following properties:

(1) $T_\alpha(ag_1 + bg_2) = aT_\alpha(g_1) + bT_\alpha(g_2)$

(2) $T_\alpha(t^\rho) = \rho t^{\rho-\alpha}$ for all $\rho \in \mathbb{R}$

(3) $T_\alpha(g_1 g_2) = g_1 T_\alpha(g_2) + g_2 T_\alpha(g_1)$

(4) $T_\alpha(g_1/g_2) = 1/g_2^2(g_2 T_\alpha(g_1) - g_1 T_\alpha(g_1))$

(5) $T_\alpha(g)(t) = t^{1-\alpha} dg/dt(t)$

(6) If $g\colon (0,\infty) \longrightarrow \mathbb{R}$ is a map which is differentiable and $\alpha$– differentiable and $f$ is another function which is defined in the range of $g$, then $T_\alpha(g^\circ f) = t^{1-\alpha} f'(t) g'(f(t))$

Indeed, fractional calculus was recently utilized to model the majority of physical and engineering processes that, in some cases, provide an adequate description of such models in a comparison of integer-order derivatives. Consequently, this motivates us to study the conformable time-fractional coupled Konno–Oono equation. It takes the form

$$D_t^\alpha p_x(x,t) - 2p(x,t)v(x,t) = 0, D_t^\alpha v(x,t) + 2p(x,t)p_x(x,t) = 0, \tag{2}$$

where $D_t^\alpha$ indicates the conformable time derivative of order $\alpha$. When $\alpha \longrightarrow 1$, system (2) is reduced to the integer-order derivative which has been introduced by Konno and Oono [17].

$$p_{xt}(x,t) - 2p(x,t)v(x,t) = 0, v_t(x,t) + 2p(x,t)p_x(x,t) = 0. \tag{3}$$

Physically, system (3) describes a current-fed string interacting with an external magnetic fed. Geometrically, it describes the parallel transport of points of the arc along the time direction such that the connection is magnetic-valued [17–21]. It has attracted great interest from mathematicians and physicists, and many mathematical approaches have been developed to investigate novel solutions to it. In [22], the modified simple equation method has been utilized for getting some properties of system (3) where some solutions have been obtained like kink solutions and bell-shaped solutions. In [23], the tanh-function method and extended tanh-function method have been applied to construct soliton solutions for system (3). Additional methods have been forced such as a new generalized $(G'/G)$–expansion method [24], the modified $\exp(-\Omega(\xi))$– expansion function method [25], the sine-Gordon expansion method [26], the external trial equation method [27], the generalized exp-function method [28], a modified extended exp-function method [29], the extended simplest equation method [30], the extended Jacobian elliptic function expansion method [31], the functional variable and the two variables $(G'/G, 1/G)$– expansion methods [32], and a new extended direct algebraic method [33]. Recently, some techniques have been imposed such as a generalized $(G'/G)$– expansion method for stochastic Konno–Oono equation that is forced by multiplicative noise term [34] and a unified solver with the aid of probability function distributions [35], for more details about wave solutions for stochastic PDE (see, e.g., [36–38]).

In this work, we study the existence and the type of some novel traveling wave solutions for (2) by the qualitative theory of planar dynamical system which has been applied successfully in various works such as [39–48] before constructing them through the kind of the orbits. For instance, the existence of periodic, homoclinic, and heteroclinic orbits imply to the existence of periodic, solitary, and kink (or antikink) wave solutions. We find the wave solutions by integrating the conserved quantity along with certain parts of the phase orbits corresponding to real wave propagation to avoid the presence of complex solutions which are not desirable in the majority of the physical world. We also study the degeneracy of the solutions throughout the transmission of the phase orbits. This method is summarized in Appendix A. He who is a Chinese mathematician, in his review paper [49] and monograph [50], developed a primitive but intriguing variational approach to the search for solitary solutions to nonlinear wave equations. He's variational method was determined to be highly simple and successful by Tao [51, 52]. References [49, 53–59] provide other uses of He's variational technique.

The paper in the following is organized as follows: Section 2 contains the study of the bifurcation analysis for the Hamiltonian system corresponding to the conformable time-fractional coupled Konno–Oono equation. In Section 3, we construct some new bounded wave solutions for the conformable time-fractional coupled Konno–Oono equation and study the degeneracy of these solutions through the transmission between the phase plane orbits. In Section 4, we apply the variational principle to construct abundant solutions including bright soliton, bright-like soliton, kinky-bright soliton, and periodic solutions. Section 5 is the conclusion.

## 2. Bifurcation Analysis

We consider the wave transformation

$$p(x,t) = \psi(\eta), \eta = a\left(x - \frac{\omega}{\alpha} t^\alpha\right), \tag{4}$$

$$v(x,t) = \phi(\eta), \tag{5}$$

where $a$ and $\omega$ refer to a wave number and a wave velocity respectively. One can easily get

$$D_t^\alpha p_x(x,t) = aD_t^a \psi'(\eta) = -a^2 \omega \psi''(\eta), \tag{6}$$

$$D_t^\alpha v_x(x,t) = -a\omega \phi'(\eta), \tag{7}$$

where $'$ refers to the differentiation with respect to $\eta$. Substituting into (2), we obtain

$$a^2\omega\psi''(\eta)+2\psi(\eta)\phi(\eta)=0, \quad (8)$$

$$-a\omega\phi'(\eta)+2a\psi(\eta)\psi'(\eta)=0. \quad (9)$$

By integrating (9), we have

$$\phi(\eta)=\frac{\psi^2(\eta)-C}{\omega}, \quad (10)$$

where $C$ is an integral constant. Substituting in (8), it follows

$$a^2\omega^2\psi''(\eta)+2\psi^3(\eta)-2C\psi(\eta)=0. \quad (11)$$

Eq. (11) can be expressed as a Hamilton system in the form

$$\psi'(\eta)=z(\eta), z'(\eta)=-\frac{2}{a^2\omega^2}\psi^3(\eta)+\frac{2C}{a^2\omega^2}\psi(\eta), \quad (12)$$

given that $a\omega\neq 0$. System (12) is a one-dimensional Hamiltonian system with a Hamiltonian function

$$H(\psi,z)=\frac{z^2(\eta)}{2}+\frac{1}{2a^2\omega^2}\psi^4(\eta)-\frac{C}{a^2\omega^2}\psi^2(\eta). \quad (13)$$

Due to the Hamiltonian function (13) does not rely explicitly on $\eta$, it is a conserved quantity. Thus, we have

$$\frac{z^2(\eta)}{2}+\frac{1}{2a^2\omega^2}\psi^4(\eta)-\frac{C}{a^2\omega^2}\psi^2(\eta)=h, \quad (14)$$

where $h$ is a constant. Equivalently, the problem of finding a wave solution for conformable time-fractional coupled Konno–Oono equation is reduced to the problem of finding a solution of a Hamiltonian system (12) which describes the one-dimensional motion of a unit mass particle under the influence of the potential forces derived from the potential function.

$$V(\psi)=\frac{1}{2a^2\omega^2}\psi^4(\eta)-\frac{C}{a^2\omega^2}\psi^2(\eta). \quad (15)$$

Notice, the expression (14) consists of two parts. The first is $1/2z^2(\eta)$ which is explained physically as the kinetic energy per unit mass of a particle, and the second term is $V(\eta)$ which is the potential per unit mass of the particle. Hence, the expression (14) characterizes the total energy per unit mass, and consequently, $h$ is the value of the total energy per unit mass. To obtain the traveling wave solutions of (11), we combine Eq. (14) and the first equation in (12), and we obtain the first-order separable differential equation.

$$\frac{d\psi}{\sqrt{-\psi^4+2C\psi^2+2a^2\omega^2h}}=\pm\frac{d\eta}{a\omega}. \quad (16)$$

The integration of Eq. (16) requires appropriate domains of the constants $C$, $a$, $\omega$, and $h$. The domains of these constants can be found by applying distinct methods such as a complete discriminant method [60] for the quartic polynomial $-\psi^4+2C\psi^2+2a^2\omega^2h$ and bifurcation analysis [39–48]. The bifurcation analysis is more significant because it gives the range of those parameters and specifies the kind of the solution before constructing them. For illustration, the existence of periodic, homoclinic, and heteroclinic trajectories refers to the presence of the periodic, solitary, and kink (antikink) solutions. Furthermore, it enables us to examine the dependence of the obtained wave solutions on the given initial conditions for the Hamiltonian system (12). This motivates us to study the bifurcation analysis and examine the phase portrait for the Hamiltonian system (12).

We first find the equilibrium points for the Hamiltonian system (12) by setting $\psi'(\eta)=0$ and $z'(\eta)=0$; that is, they are $(\psi_0,0)$, where $\psi_0$ is the possible real solution for $\psi_0^2(\psi_0^2-C)=0$. Hence, the number of equilibrium points for system (12) is determined according to the value of $C$.

(i) If $C\leq 0$, there is a unique equilibrium point for system (12) which is $O=(0,0)$

(ii) If $C>0$, there are three equilibrium points for system (12) which are $O$ and $P_\pm=(\pm\sqrt{C},0)$

Secondly, we will determine the type of the equilibrium points and explain the phase plane for the Hamiltonian system (see, e.g., [61]). If $C\leq 0$, the equilibrium point $O=(0,0)$ is center as in Figure 1(a). If $C>0$, the equilibrium point $O=(0,0)$ is a saddle point while $P_\pm=(\pm\sqrt{C},0)$ are center points as in Figure 1(b).

Notice, as a result of system (12) being a Hamiltonian, the equilibrium points appears as a critical point for the potential function (15), and their nature is specified; accordingly, these equilibrium points are either local maximum (saddle) or local minimum (center), as in Figure 2.

It is noted that all trajectories of the phase plot are bounded. The following theorem describes the phase portrait of system (12).

**Theorem 1.** *Assume that $a\omega\neq 0$ and $C\in\mathbb{R}$. If $C\leq 0$, system (12) has a family of periodic closed orbits around the center point $O$ as in Figure 1(a), while, for $C>0$, the system has the following:*

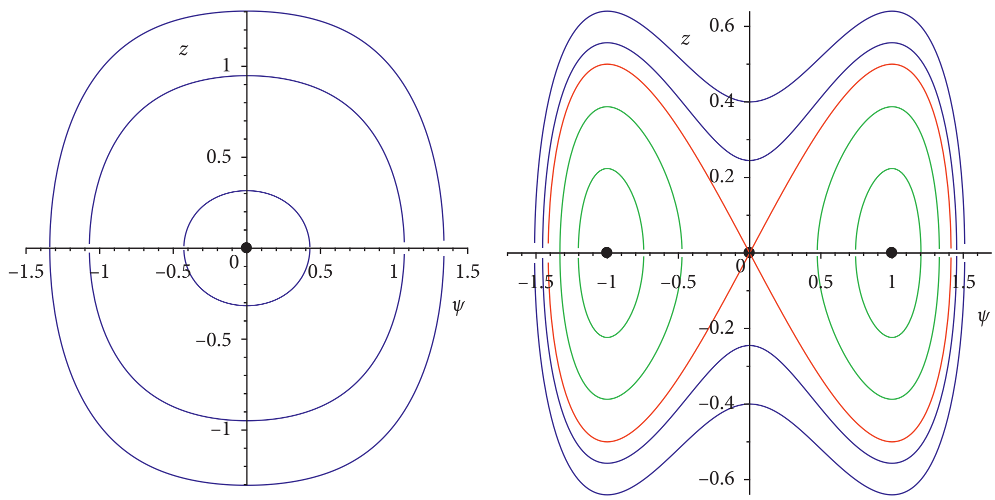
*(i) Two homoclinic orbits to the saddle point $O$, in red, around the two center point points $P_-$ and $P_+$ for $h=0$*

*(ii) Two separated families of periodic closed orbits, in green, around the center points $P_-$ and $P_+$, inside the homoclinic orbits, for $h\in]h_1,0[$, where $h_1=-C^2/2a^2\omega^2$, and*

*(iii) A family of super-periodic closed orbits, in blue, outside the homoclinic orbits, for $h>0$, as in Figure 1(b)*

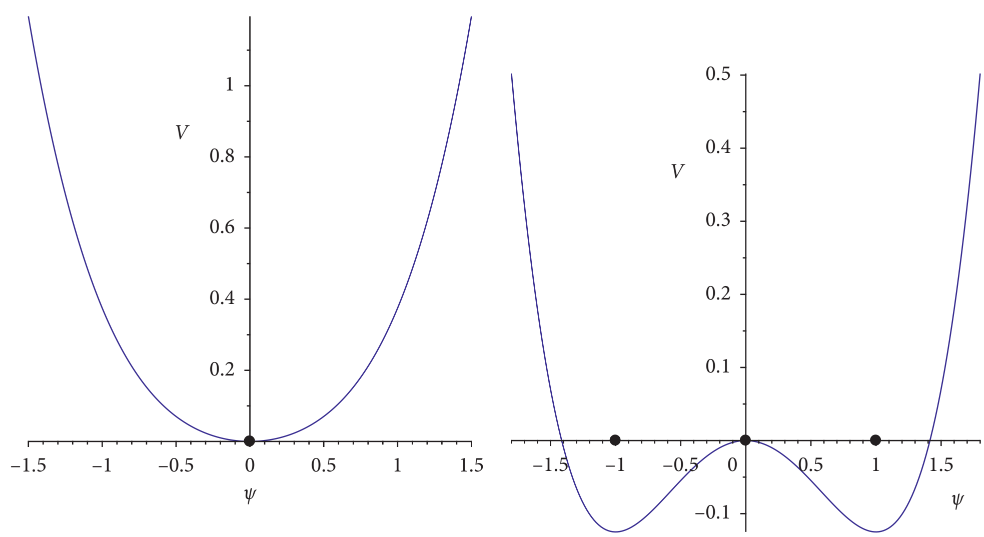
## 3. Classification of All Traveling Wave Solutions

Based on the bifurcation analysis in the previous section, the types of the wave solutions can be classified according to the following theorem:




(a) (b)

FIGURE 1: Phase portrait for the dynamical system (12) in the phase plane $(\psi, z)$. The solid circles indicate the equilibrium points. (a) $C < 0$. (b) $C > 0$.




(a) (b)

FIGURE 2: The potential of the dynamical system (12). (a) $C < 0$. (b) $C > 0$.

**Theorem 2.** *The solutions $\psi(\eta)$ and $\phi(\eta)$ are periodic wave solutions if $(C, h) \in (]-\infty, 0] \times \mathbb{R}^+) \cup (\mathbb{R}^+ \times ]h_1, 0[) \cup (\mathbb{R}^+ \times ]) 0, \infty]$, where $h_1 = -C^2/2a^2\omega^2$. They are solitary wave solutions if $(C, h) \in \mathbb{R}^+ \times \{0\}$.*

Taking into account the bifurcation's constraints on the two parameters $C$, $h$ in the last theorem and the differential form (16), we obtain the following.

### *3.1. The Periodic Wave Solutions*

(a) When $(C, h) \in ]-\infty, 0] \times \mathbb{R}^+$, system (12) admits periodic orbits, given in Figure 1(a) in blue. The periodic wave solutions after integrating (16) are obtained as

$$\psi(\eta) = \pm \frac{a\omega\sqrt{h}}{\sqrt{\sqrt{C^2 + 2a^2\omega^2 h}}} sd\left(\frac{\sqrt{2\sqrt{C^2 + 2a^2\omega^2 h}}}{|a\omega|}\eta, \frac{r_4}{\sqrt{2\sqrt{C^2 + 2a^2\omega^2 h}}}\right), \tag{17}$$

where $r_4$ is the positive root of the polynomial $-\psi^4 + 2C\psi^2 + 2a^2\omega^2 h = 0$ with $C \leq 0$ and $h > 0$ (see, e.g., [62]). Therefore, the solutions of system (2) corresponding to the periodic orbits are given by

$$p(x,t) = \pm \frac{|a\omega|\sqrt{h}}{\sqrt{\sqrt{C^2 + 2a^2\omega^2 h}}} sd\left( \sqrt{2\sqrt{C^2 + 2a^2\omega^2 h}} \left( \frac{x}{\omega} - \frac{t^\alpha}{\alpha} \right), \frac{r_4}{\sqrt{2\sqrt{C^2 + 2a^2\omega^2 h}}} \right), \tag{18}$$

and

$$v(x,t) = \frac{a^2 \omega h}{\sqrt{C^2 + 2a^2\omega^2 h}} sd^2 \left( \sqrt{2\sqrt{C^2 + 2a^2\omega^2 h}} \left( \frac{x}{w} - \frac{t^\alpha}{\alpha} \right), \frac{r_4}{\sqrt{2\sqrt{C^2 + 2a^2\omega^2 h}}} \right) - C. \tag{19}$$

The expressions in (18) and (19) are new elliptic function solutions of conformable time-fractional coupled Konno–Oono equation (2) and are given geometrically in Figures 3 and 4, respectively. Notice, when the fractional-order $\alpha$ tends to one, the expressions (18) and (19) reduce to new elliptic solutions for the integer time order derivative version of the Konno–Oono equation (2). It is noticeable that when $\alpha$ takes values different from one, the periodic solutions lose their periodicity as shown in Figures 3(d) and 4(d).

(b) When $(C,h) \in \mathbb{R}^+ \times ]h_1, 0[$, system (12) admits periodic orbits around the equilibrium points $P_-$ and $P_+$, given in Figure 1(b) in green. The periodic wave solutions after integrating (16) are obtained as

$$\psi(\eta) = \pm r_2 dn\left( \frac{r_2}{a\omega}\eta, \sqrt{1 - \frac{r_1^2}{r_2^2}} \right), \tag{20}$$

where $r_1$ and $r_2$ are the positive roots of the polynomial $-\psi^4 + 2C\psi^2 + 2a^2\omega^2 h = 0$, with $r_1 < r_2, C > 0$, and $h_1 < h < 0$. Therefore, the solutions of system (2) corresponding to these periodic orbits are given by

$$p(x,t) = \pm r_2 dn\left( r_2 \left( \frac{x}{w} - \frac{t^\alpha}{\alpha} \right), \sqrt{1 - \frac{r_1^2}{r_2^2}} \right), \tag{21}$$

and

$$v(x,t) = \frac{1}{\omega} \left[ r_2^2 dn^2 \left( r_2 \left( \frac{x}{w} - \frac{t^\alpha}{\alpha} \right), \sqrt{1 - \frac{r_1^2}{r_2^2}} \right) - C \right]. \tag{22}$$

The expressions in (21) and (22) are new elliptic function solutions of conformable time-fractional coupled Konno–Oono equation (2) and are given geometrically in Figures 5 and 6, respectively. It is worth mentioning that, if $\alpha \longrightarrow 1$, solutions (21) and (22) are also new solutions for the time-integer derivative of coupled Konno–Oono equation (2). It is noticeable that when $\alpha$ takes values different from one, the periodic solutions lose their periodicity as shown in Figures 5(d) and 6(d).

(c) When $(C,h) \in \mathbb{R}^+ \times ]0, \infty[$, system (12) admits super periodic orbits, given in Figure 1(b) in blue. The periodic wave solutions after integrating (16) are obtained as

$$\psi(\eta) = \pm r_3 cn\left( \frac{\sqrt{2C}}{a\omega}\eta, \frac{r_3}{\sqrt{2C}} \right), \tag{23}$$

where $r_3$ is the positive root of the polynomial $-\psi^4 + 2C\psi^2 + 2a^2\omega^2 h = 0$ with $C > 0$ and $h > 0$. Therefore, the solutions of system (2) corresponding to these periodic orbits are given by

$$p(x,t) = \pm r_3 cn\left( \sqrt{2C} \left( \frac{x}{w} - \frac{t^\alpha}{\alpha} \right), \frac{r_3}{\sqrt{2C}} \right), \tag{24}$$

and

$$v(x,t) = \frac{1}{\omega} \left[ r_3^2 cn^2 \left( \sqrt{2C} \left( \frac{x}{w} - \frac{t^\alpha}{\alpha} \right), \frac{r_3}{\sqrt{2C}} \right) - C \right]. \tag{25}$$

The expressions in (24) and (25) are new elliptic function solutions of conformable time-fractional coupled Konno–Oono equation (2). It is remarkable that, if $\alpha$ tends to one, the solutions (24) and (25) are also new solutions for the time-integer derivative of coupled Konno–Oono equation (2). Also, as in the two previous cases, when $\alpha \neq 1$, these solutions also lose its periodicity.

*3.2. The Solitary Wave Solutions.* If $(C,h) \in \mathbb{R}^+ \times \{0\}$, system (12) admits a homoclinic orbit, given in Figure 1(b) in red. The periodic wave solutions after integrating (16) are obtained as

$$\psi(\eta) = \pm \sqrt{2C} \operatorname{sech}\left( \frac{\sqrt{2C}}{a\omega}\eta \right). \tag{26}$$

Therefore, the solitary wave solutions of system (2) corresponding to homoclinic orbit are given by

$$p(x,t) = \pm \sqrt{2C} \operatorname{sech}\left( \sqrt{2C} \left( \frac{x}{w} - \frac{t^\alpha}{\alpha} \right) \right), \tag{27}$$

and

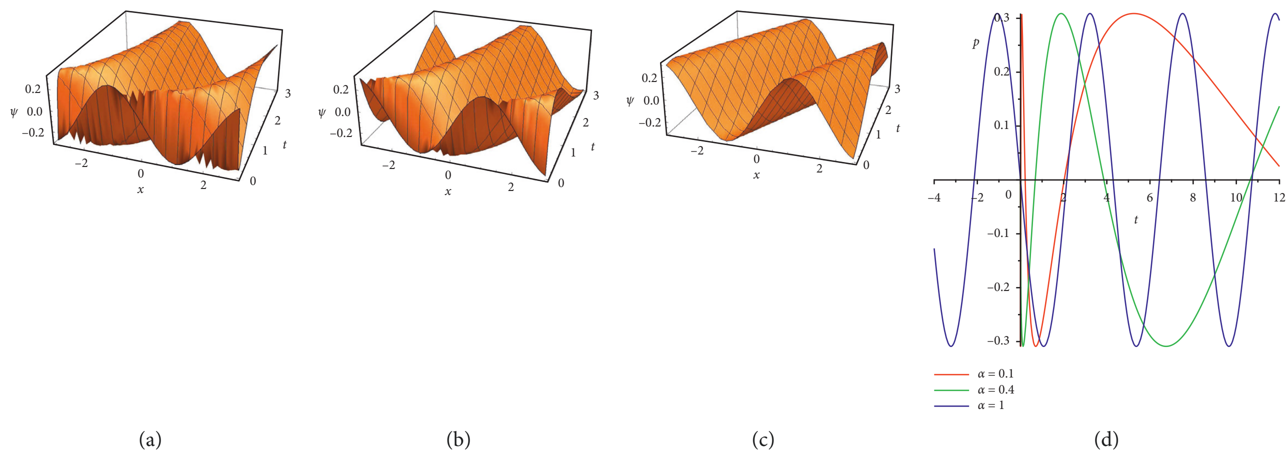



Figure 3: Solution (18) for various values of $\alpha$. (a) $\alpha = 0.1$. (b) $\alpha = 0.4$. (c) $\alpha = 1$. (d) $2-D$ graph.

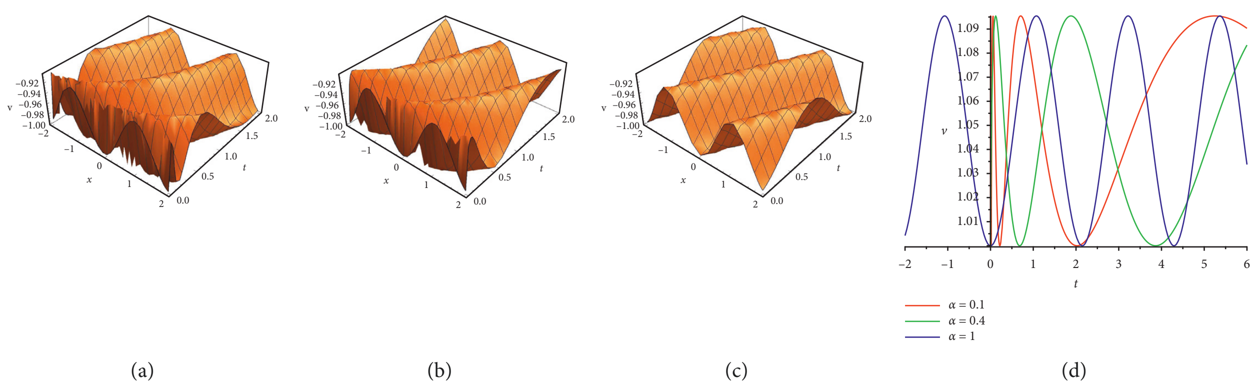



Figure 4: Solution (19) for various values of $\alpha$. (a) $\alpha = 0.1$. (b) $\alpha = 0.4$. (c) $\alpha = 1$. (d) $2-D$ graph.

$$v(x,t) = \frac{C}{\omega}\left[2\operatorname{sech}^2\left(\sqrt{2C}\left(\frac{x}{w} - \frac{t^\alpha}{\alpha}\right)\right) - 1\right]. \tag{28}$$

The expressions in (27) and (28) are new hyperbolic function solutions of conformable time-fractional coupled Konno–Oono equation (2) and are given geometrically in Figures 7 and 8, respectively. When $\alpha$ approaches to one, the solutions (27) and (28) will reduce to a well-known solution for time-integer derivatives coupled Konno- Oono equation (2) [27, 31, 32]. Notice, as outlined in Figures 7(d) and 8(d), the fractional-order derivative influences the Shap of the solitary wave solutions.

*Remark 1. One of the most advantages of the bifurcation analysis is the study of the degeneracy of the obtained solutions through the transmission between the phase orbits. Let us clarify this point.*

(i) It is remarkable that the two periodic families of orbits in green as outlined in Figure 1(b) will degenerate to the homoclinic orbit in red when $h$ tends to zero. Consequently, the corresponding solutions to these orbits also must have this property. When $h \longrightarrow 0$, we have $r_1 = 0, r_2 = \sqrt{2C}$, and hence, the periodic solution (21) and (22) degenerate into the solitary solution (27) and (28). It is worth mentioning that the degeneracy property also proves the consistency of the obtained solutions and its validity.

(ii) The superperiodic orbits in blue will reduce to the homoclinic orbit in red as clarified by Figure 1(b) when $h \longrightarrow 0$. If $h \longrightarrow 0$, we have $r_3 = \sqrt{2C}$. Consequently, the periodic solution (24) and (25) degenerates into the solitary solution (27) and (28).

*Remark 2. The type of wave solutions for Equation(2) depends on the initial conditions. The degeneracy is always studied from the parameter $h$ which is always determined from the initial conditions from the conserved quantity* (14)*. This shows the dependence of the solutions on the initial conditions. For more clarifications, for certain initial conditions making $h \in (h_1, 0)$, we have a periodic solution, and for other initial conditions making $h = 0$, we have a solitary wave solution. Notice that both obtained solutions are completely different from the mathematical and physics point of view.*

## 4. Variational Principal

The variational principle of Eq.(11) can be found by using the semiinverse method [49, 53–55], which has been used widely to construct the needed variational formulations by

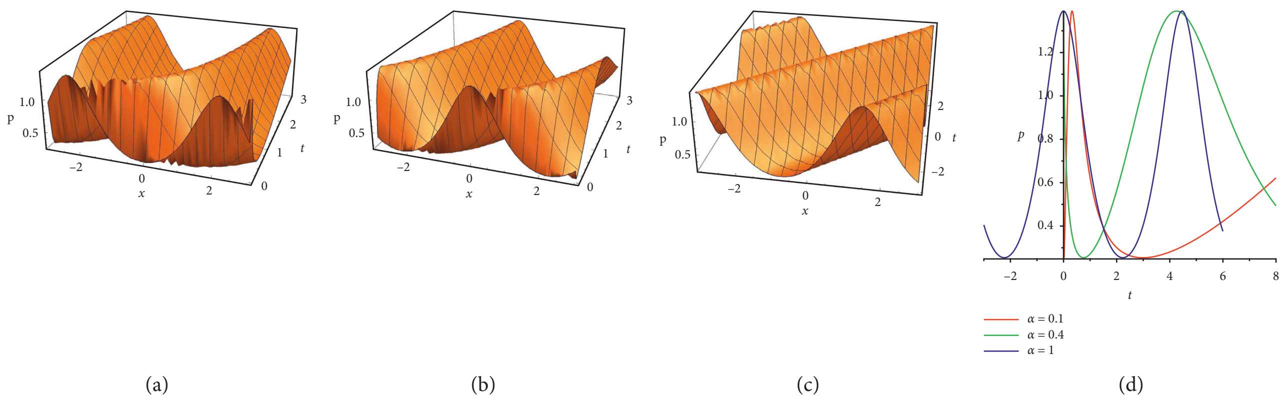



Figure 5: Solution (21) for various values of $\alpha$. (a) $\alpha = 0.1$. (b) $\alpha = 0.4$. (c) $\alpha = 1$. (d) $2-D$ graph.

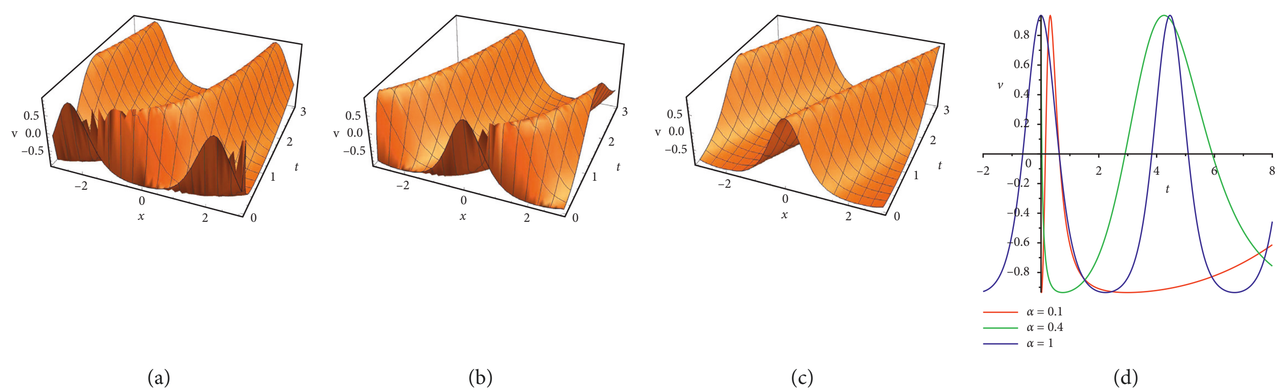



Figure 6: Solution (22) for various values of $\alpha$. (a) $\alpha = 0.1$. (b) $\alpha = 0.4$. (c) $\alpha = 1$. (d) $2-D$ graph.

presenting an undetermined function as a trial functional [56–59].

$$\mathscr{L} = \int \left[ -\frac{1}{2}\psi'^{2}(\eta) + \frac{1}{2a^2\omega^2}\psi^4(\eta) - \frac{c}{a^2\omega^2}\psi(\eta)^2 \right] d\eta. \tag{29}$$

*4.1. Abundant Solutions*

*4.1.1. Bright Soliton.* He's variational method, which is presented in [58], is a powerful method for finding approximated wave solutions. We assume Eq.(11) as a bright soliton solution in the form

$$\psi(\eta) = q\operatorname{sech}(\eta), \tag{30}$$

where $p$ is a constant. Entering the expression (30) into Eq.(29), we get

$$\begin{aligned}\mathscr{L}(q) &= \int_0^\infty \left[ -q^2\left(\frac{1}{2} + \frac{C}{a^2\omega^2}\right)\operatorname{sech}^2(\eta) + \frac{q^2}{2}\left(1 + \frac{q^2}{a^2\omega^2}\right)\operatorname{sech}^4(\eta) \right] d\eta \\ &= -\frac{q^2}{6a^2\omega^2}\left[a^2\omega^2 + 6C - 2q^2\right].\end{aligned} \tag{31}$$

According to the basis of the Ritz-like method, the stationary condition for Eq.(31) gives

$$\frac{\partial \mathscr{L}(q)}{\partial q} = \frac{q}{3a^2\omega^2}\left[4q^2 - \left(a^2\omega^2 + 6C\right)\right] = 0. \tag{32}$$

Solving the last equation for $q$, we get

$$q = \pm\frac{1}{2}\sqrt{a^2\omega^2 + 6C}. \tag{33}$$

Hence, the bright soliton solution in Eq.(30) takes the form

$$\psi(\eta) = \pm\frac{1}{2}\sqrt{a^2\omega^2 + 6C}\operatorname{sech}(\eta). \tag{34}$$

Hence, system (2) admits the solutions

$$\begin{aligned} p(x,t) &= \pm\frac{1}{2}\sqrt{a^2\omega^2 + 6C}\operatorname{sech}\left(a\left(x - \frac{\omega}{\alpha}t^\alpha\right)\right), \\ v(x,t) &= \frac{1}{\omega}\left[\frac{1}{4}\left(a^2\omega^2 + 6C\right)\operatorname{sech}^2\left(a\left(x - \frac{\omega}{\alpha}t^\alpha\right)\right) - C\right]. \end{aligned} \tag{35}$$

*4.1.2. Bright-Like Soliton.* We assume that the solution of Eq.(11) is in the form

$$\psi(\eta) = \frac{q}{1 + \cosh(\eta)}. \tag{36}$$

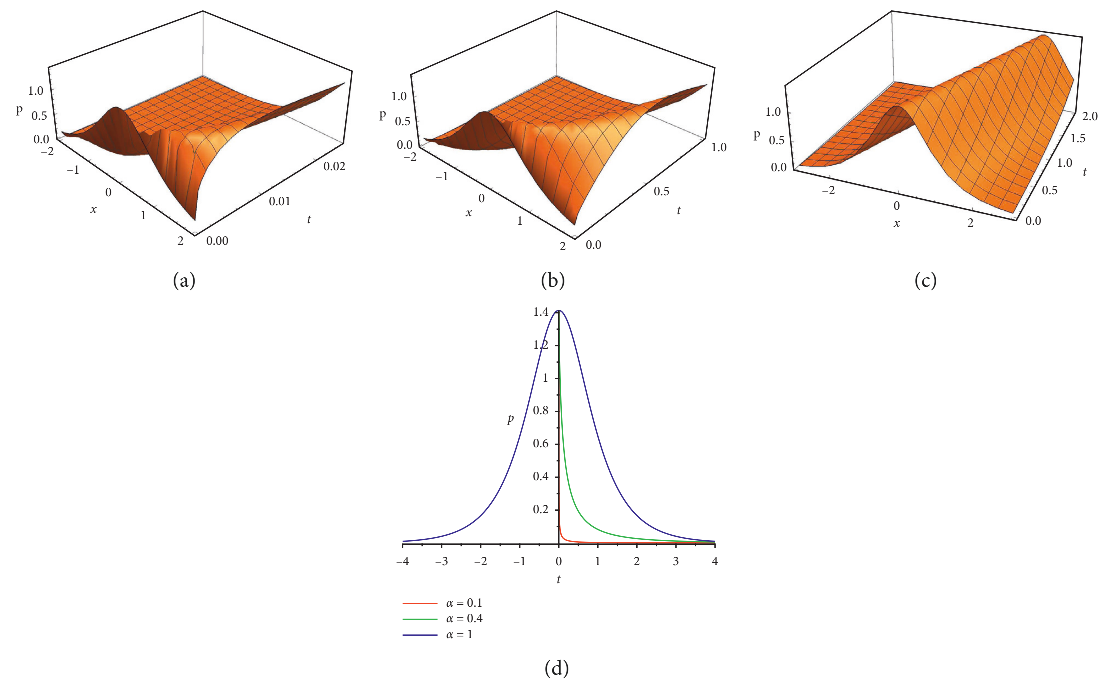



Figure 7: Solution (27) for various values of $\alpha$. (a) $\alpha = 0.1$. (b) $\alpha = 0.4$. (c) $\alpha = 1$. (d) $2-D$ graph.

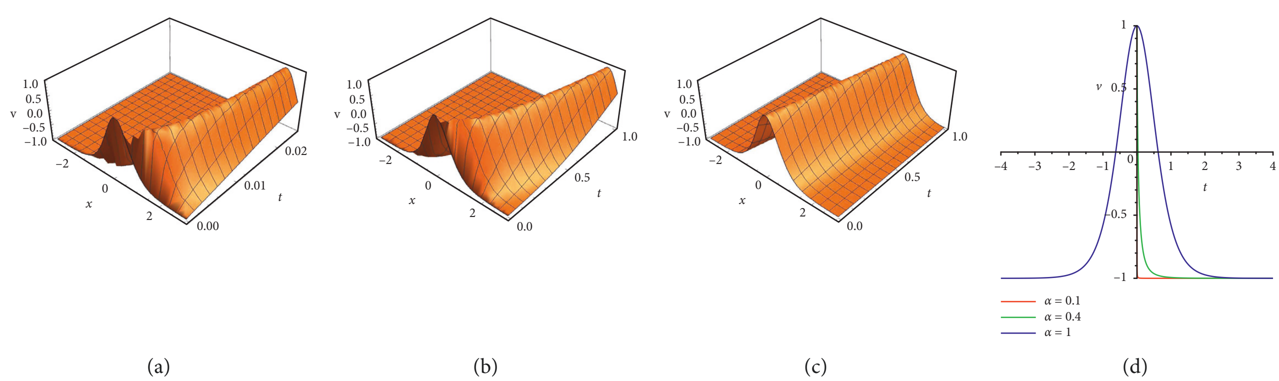



Figure 8: Solution (28) for various values of $\alpha$. (a) $\alpha = 0.1$. (b) $\alpha = 0.4$. (c) $\alpha = 1$. (d) $2-D$ graph.

Inserting Eq.(40) into Eq.(29), we obtain

$$\begin{aligned}\mathcal{J}(q) &= \int_0^{\infty}\left[\frac{-q^2(\cosh(\eta)-1)}{2(1+\cosh(\eta))^3}+\frac{q^4}{2a^2\omega^2(1+\cosh(\eta))^4}-\frac{Cq^2}{a^2\omega^2(1+\cosh(\eta))^2}\right]d\eta \\ &= \frac{\left(6q^2-7a^2\omega^2-70C\right)q^2}{210a^2\omega^2}.\end{aligned} \tag{37}$$

The stationary condition for (37) is

$$\frac{\partial \mathcal{J}(q)}{\partial q} = \frac{\left(12q^2-7a^2\omega^2-70C\right)q}{105a^2\omega^2} = 0, \tag{38}$$

which implies to

$$q = \pm\frac{\sqrt{21a^2\omega^2+210C}}{6}. \tag{39}$$

Thus, the bright dark solution in Eq.(40) takes the form

$$\psi(\eta) = \pm\frac{\sqrt{21a^2\omega^2+210C}}{6(1+\cosh(\eta))}. \tag{40}$$

Hence, system (2) admits the solution

$$p(x,t) = \pm \frac{\sqrt{21a^2\omega^2 + 210C}}{6\left(1 + \cosh\left(a\left(x - \omega/\alpha t^{\alpha}\right)\right)\right)},$$
$$v(x,t) = \frac{1}{\omega}\left[\frac{7a^2\omega^2 + 70C}{12\left(1 + \cosh^2\left(a\left(x - \omega/\alpha t^{\alpha}\right)\right)\right) - C}\right]. \tag{41}$$

*4.1.3. Kinky-Bright Soliton.* We assume a kinky-bright solitons solution of Eq.(11) in the form

$$\psi(\eta) = q\operatorname{sech}^2(\eta). \tag{42}$$

Inserting the expression Eq.(51) into Eq.(29), we obtain

$$\begin{aligned}\mathcal{J}(q) &= \int_0^{\infty}\left[\frac{q^4}{2a^2\omega^2}\operatorname{sech}^8(\eta) + 2q^2\operatorname{sech}^6(\eta) - \left(2 + \frac{C}{a^2\omega^2}\right)q^2\operatorname{sech}^4(\eta)\right]d\eta \\ &= \frac{2\left(12q^2 - 14a^2\omega^2 - 35C\right)q^2}{105\,a^2\omega^2}.\end{aligned} \tag{43}$$

Computing the stationary condition $\partial\mathcal{J}(q)/\partial q = 0$, we have

$$q = \pm \frac{\sqrt{84a^2\omega^2 + 210C}}{12}. \tag{44}$$

Therefore, the kinky-bright soliton in Eq.(51) is

$$\psi(\eta) = \pm \frac{\sqrt{84a^2\omega^2 + 210C}}{12}\operatorname{sech}^2(\eta). \tag{45}$$

Hence, system (2) admits the solution

$$p(x,t) = \pm \frac{\sqrt{84a^2\omega^2 + 210C}}{12}\operatorname{sech}^2\left(a\left(x - \frac{\omega}{\alpha}t^{\alpha}\right)\right),$$
$$v(x,t) = \frac{1}{\omega}\left[\frac{14a^2\omega^2 + 35C}{24}\operatorname{sech}^4\left(a\left(x - \frac{\omega}{\alpha}t^{\alpha}\right)\right) - C\right]. \tag{46}$$

*4.1.4. Periodic Solution.* We assume the solution is in the form

$$\psi(\eta) = q\cos(\eta). \tag{47}$$

Inserting the expression Eq.(47) into Eq.(29), we obtain

$$\mathcal{J}(q) = \int_0^{T/4}\left[\frac{q^4}{2a^2\omega^2}\cos^4(\eta) + q^2\left(\frac{1}{2} - \frac{C}{a^2\omega^2}\right)\cos^2(\eta) - \frac{q^2}{2}\right]d\eta, \tag{48}$$

where $T$ indicates the period of the nonlinear term of Eq.(11); that is, $T = 2\pi$. Thus,

$$\mathcal{J}(q) = \frac{\pi\left(3q^2 - 4a^2\omega^2 - 8C\right)q^2}{32a^2\omega^2}. \tag{49}$$

The stationary condition $\partial\mathcal{J}(q)/\partial q = 0$ implies

$$q = \pm \frac{\sqrt{6a^2\omega^2 + 12C}}{3}. \tag{50}$$

Therefore, the solution in Eq.(47) is

$$\psi(\eta) = \pm \frac{\sqrt{6a^2\omega^2 + 12C}}{3}\cos(\eta). \tag{51}$$

Hence, system (2) admits the solution

$$p(x,t) = \pm \frac{\sqrt{6a^2\omega^2 + 12C}}{3}\cos\left(a\left(x - \frac{\omega}{\alpha}t^{\alpha}\right)\right),$$
$$v(x,t) = \frac{1}{\omega}\left[\frac{2a^2\omega^2 + 4C}{3}\cos^2\left(a\left(x - \frac{\omega}{\alpha}t^{\alpha}\right)\right) - C\right]. \tag{52}$$

## 5. Conclusion

This work is interested in studying the conformable time-fractional coupled Konno–Oono equation in a magnetic field. A certain transformation has been applied to convert this equation into an ordinary differential equation as well as it has been rewritten as a two-dimensional dynamical system. This system is a conservative Hamiltonian system. A qualitative analysis or the qualitative theory of the planar dynamical system has been applied to study the bifurcation and the phase space. This study has enabled us to prove Theorem 2 including the conditions on the parameters that guarantee the existence of periodic and solitary wave solutions before constructing them via the type of the phase plane orbits. Taking into account the bifurcation constraints of the system's parameters, the conserved quantity has been utilized to construct some new solutions for the equation under consideration. Three families of periodic wave solutions and a solitary wave solution have been established. A 2D and 3D for these solutions have been clarified graphically for diverse values of the fractional-order $\alpha$. Or, equivalently, we have outlined the effect of the fractional-order derivative $\alpha$ on the obtained solutions. Moreover, the obtained new solutions have been reduced to new wave solutions for the time-integer derivative coupled Konno–Oono equation. The degeneracy of Jacobi-elliptic wave solutions has been examined via the transmission between the orbits of the phase plane orbits. From another side, He's variational method has been applied to find some new solutions which have been assorted into bright soliton, bright-like soliton, kinky-bright soliton, and periodic solutions.

## Appendix

## A. Wave Solutions via Conserved Quantity

Consider a conformable time-fractional system of partial differential equation taking the form

$$\begin{aligned}&\mathscr{L}\left(D_t^{\alpha}u, u_x, v_x, u_x x, v_x x \ldots\right) = 0,\\ &\mathscr{H}\left(D_t^{\alpha}v, u_x, v_x, u_x x, v_x x \ldots\right) = 0,\end{aligned} \tag{A.1}$$

where $D_t^{\alpha}$ is the time conformal fractional derivative with order $\alpha$, $t$ refers to the time, and $u, v$ are two real-valued functions depending on the two variables $t, x$. To find wave solutions for system (A.1), we follow the next steps:

*Step 1.* Applying the transformation

$$u(x,t) = \psi(\eta), v(x,t) = \phi(\eta), \eta = a\left(x - \frac{\omega}{\alpha}t^{\alpha}\right), \tag{A.2}$$

where $a, \omega$ are arbitrary constants to the system (A.2), we obtain, after some calculations, a system of second-order ordinary differential equations of $\psi$ and $\phi$ which, by substitution, will be rewritten as one degree of freedom Hamiltonian system; that is, it takes the form

$$\psi' = \frac{\partial H}{\partial z} = z, z' = \frac{\partial H}{\partial \psi} = -\frac{\partial V}{\partial \psi}, \tag{A.3}$$

where $z$ is a new variable introduced to convert the second-order differential equation into a dynamical system, $V(\psi)$ is a real-valued function, and $'$ indicates derivative with respect to $\eta$.

*Step 2.* Calculate the Hamiltonian function $H$ as

$$H = \frac{1}{2}z^2 + V(\psi) = h, \tag{A.4}$$

from which we obtain the separable differential equation

$$\frac{d\psi}{\sqrt{2(h - V(\psi))}} = \pm\, d\eta. \tag{A.5}$$

*Step 3.* Find the equilibrium points for the Hamiltonian system (A.3) and apply the qualitative theory for the planar dynamical system to specify their nature

*Step 4.* Find the intervals of real wave propagation which is equivalent to the intervals of real motion of a particle described by the Hamiltonian system (A.3). Integrating both sides of (A.5) along certain intervals of real propagation gives different types of wave solutions $\psi(\eta)$ and hence $\phi(\eta)$ follows.

## Data Availability

No data were used to support the study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

The authors acknowledge the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research at King Faisal University, Saudi Arabia, for financial support under the annual funding track (GRANT847).

## References

[1] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, fields and media*, Springer, Berlin, Heidelberg, 2010.

[2] S. Das, *Functional Fractional Calculus*, Springer, Berlin, Heidelberg, 2011.

[3] R. A. El-Nabulsi, "Path integral formulation of fractionally perturbed Lagrangian oscillators on fractal," *Journal of Statistical Physics*, vol. 172, no. 6, pp. 1617–1640, 2018.

[4] J. Dong and M. Xu, "Space-time fractional Schrödinger equation with time-independent potentials," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 2, pp. 1005–1017, 2008.

[5] S. Ş. Bayın, "On the consistency of the solutions of the space fractional Schrödinger equation," *Journal of Mathematical Physics*, vol. 53, no. 4, 2012.

[6] O. Abu Arqub, "Application of residual power series method for the solution of time-fractional schrödinger equations in one-dimensional space," *Fundamenta Informaticae*, vol. 166, no. 2, pp. 87–110, 2019.

[7] O. Abu Arqub, "Numerical algorithm for the solutions of fractional order systems of dirichlet function types with comparative analysis," *Fundamenta Informaticae*, vol. 166, no. 2, pp. 111–137, 2019.

[8] O. A. Arqub and M. Al-Smadi, "Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions," *Soft Computing*, vol. 24, no. 16, pp. 12501–12522, 2020.

[9] S. Momani, O. Abu Arqub, and B. Maayah, "Piecewise optimal fractional reproducing kernel solution and convergence analysis for the atangana-baleanu-caputo model of the lienard's equation," *Fractals*, vol. 28, no. 08, 2020.

[10] S. Momani, B. Maayah, and O. A. Arqub, "The reproducing kernel algorithm for numerical solution of Van der Pol damping model in view of the Atangana-Baleanu fractional approach," *Fractals*, vol. 28, no. 08, 2020.

[11] M. Al-Smadi, O. Abu Arqub, and S. Momani, "Numerical computations of coupled fractional resonant Schrödinger equations arising in quantum mechanics under conformable fractional derivative sense," *Physica Scripta*, vol. 95, no. 7, 2020.

[12] R. A. El-Nabulsi, "Emergence of quasiperiodic quantum wave functions in Hausdorff dimensional crystals and improved intrinsic Carrier concentrations," *Journal of Physics and Chemistry of Solids*, vol. 127, pp. 224–230, 2019.

[13] V. E. Tarasov, "On chain rule for fractional derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 30, no. 1-3, pp. 1–4, 2016.

[14] C.-s. Liu, "Counterexamples on Jumarie's two basic fractional calculus formulae," *Communications in Nonlinear Science and Numerical Simulation*, vol. 22, no. 1-3, pp. 92–94, 2015.

[15] C.-S. Liu, "Counterexamples on Jumarie's three basic fractional calculus formulae for non-differentiable continuous functions," *Chaos, Solitons & Fractals*, vol. 109, pp. 219–222, 2018.

[16] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[17] K. Konno and H. Oono, "New coupled integrable dispersionless equations," *Journal of the Physical Society of Japan*, vol. 63, no. 2, pp. 377-378, 1994.

[18] K. Konno and H. Kakuhata, "Interaction among growing, decaying and stationary solitons for coupled integrable dispersionless equations," *Journal of the Physical Society of Japan*, vol. 64, no. 8, pp. 2707–2709, 1995.

[19] K. Konno and H. Kakuhata, "Novel solitonic evolutions in a coupled integrable, dispersionless system," *Journal of the Physical Society of Japan*, vol. 65, no. 3, pp. 713–721, 1996.

[20] A. Souleymanou, V. K. Kuetche, T. B. Bouetou, and T. C. Kofane, "Traveling wave-guide channels of a new coupled integrable dispersionless system," *Communications in Theoretical Physics*, vol. 57, no. 1, pp. 10–14, 2012.

[21] M. Khalique, "Exact solutions and conservation laws of a coupled integrable dispersionless system," *Filomat*, vol. 26, no. 5, pp. 957–964, 2012.

[22] K. Khan and M. A. Akbar, "Traveling wave solutions of some coupled nonlinear evolution equations," *ISRN Mathematical Physics*, vol. 2013, Article ID 685736, 8 pages, 2013.

[23] M. Abul Bashar, G. Mondal, K. Khan, and A. Bekir, "Traveling wave solutions of new coupled Konno-Oono equation," *New Trends in Mathematical Science*, vol. 4, no. 2, p. 296, 2016.

[24] M. N. Alam and F. B. M. Belgacem, "New generalized $(G'/G)$-Expansion method applications to coupled konno-oono equation $(G'/G)$*-expansion method applications to coupled Konno-Oono equation*," *Advances in Pure Mathematics*, vol. 06, no. 03, pp. 168–179, 2016.

[25] Z. F. Koçak, H. Bulut, D. A. Koc, and H. M. Baskonus, "Prototype traveling wave solutions of new coupled Konno-Oono equation," *Optik*, vol. 127, no. 22, pp. 10786–10794, 2016.

[26] G. Yel, H. M. Baskonus, and H. Bulut, "Novel archetypes of new coupled Konno-Oono equation by using sine-Gordon expansion method," *Optical and Quantum Electronics*, vol. 49, no. 9, p. 285, 2017.

[27] J. Manafian, I. Zamanpour, and A. Ranjbaran, "On some new analytical solutions for new coupled Konno-Oono equation by the external trial equation method," *Journal of Physics Communications*, vol. 2, no. 1, 2018.

[28] M. M. A. Khater, A. R. Seadawy, and D. Lu, "Dispersive solitary wave solutions of new coupled Konno-Oono, Higgs field and Maccari equations and their applications," *Journal of King Saud University Science*, vol. 30, no. 3, pp. 417–423, 2018.

[29] M. Shakeel, S. T. Mohyud-Din, and M. A. Iqbal, "Modified extended exp-function method for a system of nonlinear partial differential equations defined by seismic sea waves," *Pramana*, vol. 91, no. 2, p. 28, 2018.

[30] M. Torvattanabun, P. Juntakud, A. Saiyun, and N. Khansai, "The new exact solutions of the new coupled Konno-Oono equation by using extended simplest equation method," *Applied Mathematical Sciences*, vol. 12, no. 6, pp. 293–301, 2018.

[31] M. A. E. Abdelrahman and H. A. Alkhidhr, "Fundamental solutions for the new coupled Konno-Oono equation in magnetic field," *Results in Physics*, vol. 19, 2020.

[32] S. Duran, "Solitary wave solutions of the coupled konno-oono equation by using the functional variable method and the two variables $(G'/G,\ 1/G)$ -expansion method $(G'/G,\ 1/G)$ *expansion method*," *Adıyaman University Journal of Science*, vol. 10, no. 2, pp. 585–594, 2020.

[33] S. M. Mirhosseini-Alizamini, H. Rezazadeh, K. Srinivasa, and A. Bekir, "New closed form solutions of the new coupled Konno-Oono equation using the new extended direct algebraic method," *Pramana*, vol. 94, no. 1, p. 52, 2020.

[34] W. W. Mohammed, N. Iqbal, A. Ali, and M. El-Morshedy, "Exact solutions of the stochastic new coupled Konno-Oono equation," *Results in Physics*, vol. 21, 2021.

[35] M. A. E. Abdelrahman and G. Alshreef, "Closed-form solutions to the new coupled Konno-Oono equation and the Kaup-Newell model equation in magnetic field with novel statistic application," *The European Physical Journal Plus*, vol. 136, no. 4, p. 455, 2021.

[36] W. W. Mohammed, M. Alesemi, S. Albosaily, N. Iqbal, and M. El-Morshedy, "The exact solutions of stochastic fractional-space kuramoto-sivashinsky equation by using (G′G)-Expansion method," *Mathematics*, vol. 9, no. 21, p. 2712, 2021.

[37] S. Albosaily, W. W. Mohammed, A. E. Hamza, M. El-Morshedy, and H. Ahmad, "The exact solutions of the stochastic fractional-space Allen-Cahn equation," *Open Physics*, vol. 20, no. 1, pp. 23–29, 2022.

[38] W. W. Mohammed, O. Bazighifan, M. M. Al-Sawalha, A. O. Almatroud, and E. S. Aly, "The influence of noise on the exact solutions of the stochastic fractional-space chiral nonlinear schrödinger equation," *Fractal and Fractional*, vol. 5, no. 4, p. 262, 2021.

[39] A. A. Elmandouh, "Bifurcation and new traveling wave solutions for the 2D Ginzburg-Landau equation," *The European Physical Journal Plus*, vol. 135, no. 8, p. 648, 2020.

[40] M. E. Elbrolosy and A. A. Elmandouh, "Bifurcation and new traveling wave solutions for (2 + 1)-dimensional nonlinear Nizhnik-Novikov-Veselov dynamical equation," *The European Physical Journal Plus*, vol. 135, no. 6, p. 533, 2020.

[41] M. E. Elbrolosy and A. A. Elmandouh, "Dynamical behaviour of nondissipative double dispersive microstrain wave in the microstructured solids," *The European Physical Journal Plus*, vol. 136, no. 9, p. 955, 2021.

[42] A. A. Elmandouh, "Integrability, qualitative analysis and the dynamics of wave solutions for Biswas-Milovic equation," *The European Physical Journal Plus*, vol. 136, no. 6, p. 638, 2021.

[43] M. Al Nuwairan and A. A. Elmandouh, "Qualitative analysis and wave propagation of the nonlinear model for low-pass electrical transmission lines," *Physica Scripta*, vol. 96, no. 9, 2021.

[44] M. E. Elbrolosy, "Qualitative analysis and new soliton solutions for the coupled nonlinear Schrödinger type equations," *Physica Scripta*, vol. 96, no. 12, 2021.

[45] M. E. Elbrolosy, A. A. Elmandouh, and A. A. Elmandouh, "Construction of new traveling wave solutions for the (2+1) dimensional extended Kadomtsev-Petviashvili equation," *Journal of Applied Analysis & Computation*, vol. 12, no. 2, pp. 533–550, 2022.

[46] T. S. Hassan, A. A. Elmandouh, A. A. Attiya, and A. Y. Khedr, "Bifurcation analysis and exact wave solutions for the double-chain model of DNA," *Journal of Mathematics*, vol. 202217 pages, 2022.

[47] M. A. L. Nuwairan, "The exact solutions of the conformable time fractional version of the generalized Pochhammer–Chree equation," *Mathematical Sciences*, pp. 1–12, 2022.

[48] S. Sarkar, P. Kumari Prasad, A. Saha, and K. Kumar Mondal, "Coexisting wave features and various nonlinear waves for Schrödinger equation in superthermal dusty plasma," *Physica Scripta*, vol. 97, no. 4, 2022.
[49] J.-H. He, "Some asymptotic methods for strongly nonlinear equations," *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.
[50] J. H. He, "Non-perturbative methods for strongly nonlinear problems," *dissertation. de-Verlag im Internet GmbH*, 2006.
[51] Z.-L. Tao, "Variational approach to the inviscid compressible fluid," *Acta Applicandae Mathematica*, vol. 100, no. 3, pp. 291–294, 2008.
[52] T. Öziş and A. Yıldırım, "Application of He's semi-inverse method to the nonlinear Schrödinger equation," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 1039–1042, 2007.
[53] J.-H. He, "Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics," *International Journal of Turbo and Jet Engines*, vol. 14, no. 1, pp. 23–28, 1997.
[54] J.-H. He, "Variational principles for some nonlinear partial differential equations with variable coefficients," *Chaos, Solitons & Fractals*, vol. 19, no. 4, pp. 847–851, 2004.
[55] Y. H. Ye and L. F. Mo, "He's variational method for the Benjamin-Bona-Mahony equation and the Kawahara equation," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2420–2422, 2009.
[56] J.-H. He, "Variational approach for nonlinear oscillators," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1430–1439, 2007.
[57] J.-H. He, N. Qie, C.-h. He, and T. Saeed, "On a strong minimum condition of a fractal variational principle," *Applied Mathematics Letters*, vol. 119, 2021.
[58] J.-H. He, W.-F. Hou, N. Qie, K. A. Gepreel, A. H. Shirazi, and H. M. Sedighi, "Hamiltonian-based frequency-amplitude formulation for nonlinear oscillators," *Facta Universitatis – Series: Mechanical Engineering*, vol. 19, no. 2, p. 199, 2021.
[59] J.-H. He, A. García, and A. García, "The simplest amplitude-period formula for non-conservative oscillators," *Reports in Mechanical Engineering*, vol. 2, no. 1, pp. 143–148, 2021.
[60] L. Yang, X. R. Hou, and Z. B. Zeng, "A complete discrimination system for polynomials," *Science in China, Series A E.*vol. 39, no. 6, pp. 628–646, 1996.
[61] M. Lakshmanan and S. Rajasekar, *Nonlinear Dynamics*Springer, Berlin, Heidelberg, 2003.
[62] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*Springer, Berlin, Heidelberg, 1971.
[63] C.-S. Liu, "On the local fractional derivative of everywhere non-differentiable continuous functions on intervals," *Communications in Nonlinear Science and Numerical Simulation*, vol. 42, pp. 229–235, 2017.