Research Article

Study of a Fractional System of Predator-Prey with Uncertain Initial Conditions

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In this manuscript, we study a nonlinear fractional-order predator-prey system while considering uncertainty in initial values. We derive the feasibility region and the boundness of the solution. The suggested model’s equilibrium points and the basic reproduction number are calculated. The stability of equilibrium points is presented. We use the metric fixed point theory to study the existence and uniqueness results concerning the solution of the model. We use the notion of UH-stability to show that the model is Ulam–Hyres type stable. To attain the approximate solution of the proposed model, we construct a method that uses the fuzzy Laplace transform in collaboration with the ADM (Adomian decomposition method). Finally, we simulate our theoretical results using MATLAB to show the dynamics of the considered model.

1. Introduction

A mathematical model is a mathematical representation of a process. The different phenomenon of the real world has been modelled by DEs or by a system of DEs [1]. Various techniques such Taylor series method, variational iteration method, modified variational iteration algorithms, and homotopy perturbation method have been discussed to solve different models [2–5]. Here, we consider the simplest model named as “Lotka–Volterra” or Predator-Prey, presented independently by “Lotka” in (1925) and “Volterra” in (1926) [6]. It refers to the relationship between predator and prey controls, as well as the population development of both species. The model describes the evolution of a biological Predator-Prey relationship through a system of two nonlinear DEs, generalized to a more complex and realistic phenomenon [6–9]. The general system for the dynamic phenomenon is as follows:

\[
\begin{align*}
\frac{dP}{dt} &= P\theta(x, y), \\
\frac{dQ}{dt} &= P\theta(x, y),
\end{align*}
\]

where \(P(t)\) is the prey (rabbits), \(Q(t)\) is the predator (foxes) population at a time \(t\), \(\theta(x, y)\) and \(\theta(x, y)\) are its functions, respectively. The growth rate and the density at time \(t\) have a direct relationship. This is presumed that the populations live in a community, where the age factor is not considered. The prey is abundant in natural resources, and the only threat is the specialist predator, whose growth depends on prey catches. For the prey model, the rate of growth of prey is presumed to be constant, and the particular rate of growth is reduced by an amount equal to the population of the predator [6]. The proposed model of the classical order is governed by
For order, constant; therefore, we extend the model given in (2) to \([15, 26, 27]\). As in the classical model, the coefficients are \([24, 25]\). Recently, fuzzy DEs have been utilized to analyze problems considering fuzziness or uncertainty in initial conditions. To solve such problems, they utilized different techniques. The DEs can model a physical phenomenon more accurately by handling the uncertain situation. Therefore, classical calculus is generalized to fractional and fuzzy Calculus, while DEs are generalized to fractional and fuzzy DEs \([21–23]\). Fuzzy operators are the best tools to model any phenomenon accurately. So, fuzzy operators are the best tools to handle the uncertain situation. Therefore, classical calculus is expanded to fill the study gap. Since we live in an uncertain environment, therefore, we cannot model any phenomenon. The goal of this research is to look at the model under consideration by including fuzziness in the initial values. The model’s basic dynamical features are studied. Fixed point theory and nonlinear analysis are used to derive the existence and stability conclusions of model (3). LADM is used to find the solution to the presented model. To demonstrate the model’s dynamics, the obtained results are simulated for various fractional-order values.

The structure of the paper is as follows: Section 1 deals with the introduction and motivation part of the paper. Section 2 provides the basic concepts of fuzzy fractional calculus. Section 3 deals with stability and feasibility of the equilibrium points. Section 4 provides the existence and uniqueness theory of the proposed model. Section 5 discusses the Ulam–Hyers stability via nonlinear functional analysis. Section 6 gives the general procedure for the solution of the considered model via fuzzy Laplace transform. Section 7 demonstrates the model through 2D and 3D simulations. Section 8 includes conclusion of the manuscript.

\[ \begin{align*}
\frac{dP}{dt} &= aP(t) - bP(t)Q(t); \quad a, b > 0, \\
\frac{dQ}{dt} &= -cQ(t) + dP(t)Q(t); \quad c, d > 0.
\end{align*} \] (2)

From the 20th century, the subject of fractional-order integrals and derivatives has a substantial effect on modeling and simulation due to its nonlocal nature. It is seen that models involving integrals and derivatives of fractional order are more accurate, rather than classical models because fractional operators have long memory and heredity properties. Das and Gupta [13] studied system (2) under fractional differential equations with constant coefficients [13]. Ahmad et al. [14] analyzed the human liver model by using hybrid fractional operators [15]. Alderremy et al. [16] analyzed multi space-fractional Gardner equation in [16]. For more applications of different fractional operators, see [17–20]. It should be noted that, in many branches of mathematics, DEs have been expanded to fill the study gap. Since we live in an uncertain environment, therefore, we cannot model any phenomenon accurately. So, fuzzy operators are the best tools to handle the uncertain situation. Therefore, classical calculus is generalized to fractional and fuzzy Calculus, while DEs are generalized to fractional and fuzzy DEs [21–23]. Fuzzy DEs can model a physical phenomenon more accurately by considering fuzziness or uncertainty in initial conditions. To solve such problems, they utilized different techniques [24, 25]. Recently, fuzzy DEs have been utilized to analyze various models that occur in biology and physics [15, 26, 27]. As in the classical model, the coefficients are constant; therefore, we extend the model given in (2) to fuzzy fractional-order operator with uncertain initial data. For order, \(0 < \alpha \leq 1\) and \(0 \leq q \leq 1\).

\[ \begin{align*}
D^\alpha_t [\tilde{p}(t, q)] &= a\tilde{p}(t, q) - b\tilde{p}(t, q)\tilde{Q}(t, q), \\
D^\alpha_t [\tilde{Q}(t, q)] &= -c\tilde{Q}(t, q) + d\tilde{p}(t, q)\tilde{Q}(t, q),
\end{align*} \] (3)

where \(\tilde{p}(t, q) = \tilde{x}_0,\) \(\tilde{Q}(t, q) = \tilde{y}_0.\) (4)

Definition 1 (see [28]). Consider a fuzzy set of real line, i.e., \(u: \mathbb{R} \rightarrow [0, 1]\). Then, \(u\) is called a fuzzy number if it fulfills the following features:

1. \(u\) is a complete metric space.
2. \(u\) is upper semicontinuous on \(u\).
3. \(u\) is convex \(u(qa + (1 - q)b) \geq (u(a) \wedge u(b))\forall q \in [0, 1], a, b \in \mathbb{R}\).
4. \(u\) is normal for some \(a_0 \in \mathbb{R}; u(a_0) = 1\)

Definition 2 (see [28]). If \(u\) is a fuzzy number, then the parametric form is represented by \([u(q), \bar{u}(q)]\) such that \(0 \leq q \leq 1\) and fulfills the properties which are given as follows:

1. \(u(q)\) is bounded, left continuous, and increasing function in \((0, 1)\) and right continuous at 0.
2. \(\bar{u}(q)\) is bounded, right continuous, and decreasing in \((0, 1)\) and continuous at 0.
3. \(u(q) \leq \bar{u}(q)\)
4. Also, for crisp case, we have \(u(q) \leq \bar{u}(q) = q\).

Definition 3 (see [25]). Let \(E\) contain all fuzzy numbers. We take into consideration a mapping \(\Sigma: E \times E \rightarrow \mathbb{R}\) and take \(g(q), \bar{g}(q)\) and \(b = (\bar{b}(q), \bar{b}(q))\) as two fuzzy numbers. The Hausdorff distance between \(g\) and \(b\) is presented as follows:

\[\Sigma(g, b) = \sup_{q \in [0, 1]} \left[ \max \left| g(q) - b(q), \bar{g}(q) - \bar{b}(q) \right| \right].\] (5)

In \(E\), the metric \(\Sigma\) satisfies the properties which is given as follows:

1. \(\Sigma(g + v, b + v) = \Sigma(g, b)\forall g, v, b \in E\)
2. \(\Sigma(gq, bq) = |q|\Sigma(g, b)\forall g, b, q \in \mathbb{R}\)
3. \(\Sigma(g + \xi, b + c) \leq \rho(g, b) + \Sigma(\xi, c)\forall g, b, \xi, c \in E\)
4. \((E, \Sigma)\) is a complete metric space

Definition 4 (see [28]). Let \(\Sigma: \mathbb{R} \rightarrow E\) be a fuzzy set valued function. Then, \(\Sigma\) is called continuous, if for each \(\varepsilon > 0\) and fixed value of \(\varphi_0 \in [\xi_1, \xi_2]\), the following relation holds:

\[\rho(\varphi(q), \varphi(\varphi_0)) < \varepsilon; \quad \text{whenever} |\varphi - \varphi_0| < \delta.\] (6)
Definition 5 (see [28]). Let \( \mathcal{O} \) be a continuous fuzzy function on \([0, q] \subseteq \mathbb{R} \), then for \( 0 < \omega \leq 1 \), a fuzzy Riemann–Liouville fractional integral is given by
\[
\int_0^t (t - \mathfrak{F})^{\omega - 1} \mathfrak{F}(\omega) d\mathfrak{F}, \quad \mathfrak{F} \in (0, \infty).
\]
(7)

Additionally, if \( \mathcal{O} \in L^F[0, q] \cap C^F[0, q] \), where \( L^F[0, q] \) and \( C^F[0, q] \) are the spaces that contains fuzzy Lebesgue integrable functions and fuzzy continuous, respectively, then
\[
\left[ \int_0^t \mathfrak{F} \mathcal{O}(t) \right]_\omega = \left[ \int_0^t \mathfrak{F} \mathcal{O}(t), \mathfrak{F} \mathcal{O}(t) \right], \quad 0 \leq \omega \leq 1,
\]
(8)

where
\[
\int_0^t \mathfrak{F} \mathcal{O}(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t - \mathfrak{F})^{\omega - 1} \mathfrak{F} \mathcal{O}(t) d\mathfrak{F}, \quad \mathfrak{F} \in (0, \infty),
\]
(9)

Definition 6 (see [28]). Consider a fuzzy function \( \mathcal{O} \in L^F[0, b] \cap C^F[0, b] \) with parametric form \( \mathcal{O} = (\mathcal{O}_0(t), \mathcal{O}_1(t)) \), \( 0 \leq \omega \leq 1 \), then the Caputo fuzzy fractional-order derivative is presented by
\[
\left[ D^\omega \mathcal{O}(t_0) \right]_\omega = \left[ D^\omega \mathcal{O}_0(t_0), D^\omega \mathcal{O}_1(t_0) \right], \quad 0 < \omega \leq 1,
\]
(10)

where
\[
D^\omega \mathcal{O}_0(t_0) = \frac{1}{\Gamma(1 - \omega)} \int_0^t (t - \mathfrak{F})^{1 - \omega} \frac{d}{d\mathfrak{F}} \mathcal{O}_0(\mathfrak{F}) d\mathfrak{F}, \quad \mathfrak{F} = t_0,
\]
(11)

so that the right side integration converges and \( j \) is bracket function of \( \omega \).

Definition 7 (see [28]). The fuzzy Laplace transform (LT) for continuous and fuzzy Riemann-integrable function \( \mathcal{O} \) on \([0, \infty) \) is defined as follows:
\[
L[\mathcal{O}(t)](c) = \int_0^\infty \mathcal{O}(t) e^{-tc} dt.
\]
(12)

Using parametric form of \( \mathcal{O}(t) \), we have for \( 0 \leq \omega \leq 1 \):
\[
\int_0^\infty \mathcal{O}(t; \omega) e^{-\omega c} dt = \left[ \int_0^\infty \mathcal{O}(t; \omega) e^{-\omega c} dt \right] \int_0^\infty \mathcal{O}(t; \omega) e^{-\omega c} \left[ \int_0^\infty \mathcal{O}(t; \omega) e^{-\omega c} dt \right].
\]
(13)

Hence,
\[
L[\mathcal{O}(t; \omega)] = \left[ L[\mathcal{O}(t)], L[\mathcal{O}(t)] \right].
\]
(14)

Theorem 1 (see [21]). The LT of Caputo fractional derivative of \( \mathcal{O}(t) \) is presented as follows:
\[
L[D^\omega \mathcal{O}(t)] = s^\omega L[\mathcal{O}(t)] - s^{\omega - 1} L[\mathcal{O}(0)].
\]
(15)

3. Feasibility and Stability Analysis

In this part, the feasibility and stability of the points of equilibrium will be discussed. To discuss the model’s boundedness and feasibility, we have the following theorem.

Theorem 2. The solution of the proposed model is bound to the feasible region given by
\[
\mathcal{T} = \left\{ (P, Q) \in \mathbb{R}_+^2 : 0 \leq G(t) \leq \frac{a - c}{b - d} \right\}.
\]
(16)

Proof. By adding both equation of (2) and considering \( G(t) = P(t) + Q(t) \), one gets
\[
\frac{dG(t)}{dt} = aP(t) - cQ(t) - bP(t)Q(t) + dP(t)Q(t)
\]
(17)

On using simple integration, we have
\[
G(t) \leq \frac{a - c}{b - d} + C \exp\left(-\left(b - d\right)t\right).
\]
(18)

This implies that \( t \longrightarrow \infty, G(t) \leq (a - c)/b - d \). Consequentially, the result is obtained.

For stability, the equilibrium points for (3) must be found as
\[
D^\omega \mathcal{O}(P = 0(t)) = 0,
\]
(19)

\[
D^\omega \mathcal{O}(Q = 0(t)) = 0.
\]
(20)

We have two points of equilibrium which are as follows: \( E_0 = ((c/d), 0) \) if \( Q = 0 \) and \( E_0 = (0, (a/b)) \) if \( P = 0 \).

Theorem 3. The reproduction number for (1) is \( R_0 = -1 \).

Proof. To find the reproduction number, consider 2nd equation of (1) as \( X = I \),
\[
D^\omega \mathcal{O}(X) = D^\omega \mathcal{O}(I) = -cQ(t) + dP(t)Q(t); \quad c, d > 0,
\]
(20)

\[
D^\omega \mathcal{O}(X) = F - V.
\]

such that \( F = dP(t)Q(t), V = -cQ(t) \) where \( F \) and \( V \) represent the linear and nonlinear terms, respectively. The next step is to determine the next generation matrix \( \mathcal{O} \mathcal{T}^{-1} \), where
\[ \mathcal{F} = \left[ \frac{\partial}{\partial y} (d\tilde{P}(t)\tilde{Q}(t)) \right] = [d\tilde{P}(t)], \]
\[ \gamma' = \left[ \frac{\partial}{\partial Q} (-c\tilde{Q}(t)) \right] = [-c], \]
\[ \gamma^{-1} = \left[ \frac{-1}{c} \right]. \]  
(21)

Then
\[ \mathcal{F}\gamma^{-1} = \left[ \frac{d\tilde{P}(t)}{-c} \right]. \]  
(22)

The leading eigen value is equal to \( R_0 \) at free equilibrium point \( E_0 = (c/d, 0) \) in \( \mathcal{F}\gamma^{-1} \), which can be computed as
\[ \rho(\mathcal{F}\gamma^{-1})_{E_0} = \left[ \frac{dc}{-dc} \right]. \]  
(23)

Similarly, we can compute the reproduction number for 1st equation of (1). Hence, \( R_0 = -1 \). □

**Theorem 4.** If \( R_0 < 1 \), then free equilibrium points of (1) are locally asymptotically stable.

**Proof.** Proof is obvious. □

### 4. Existence Theory

Here, we will establish the existence theory for the model by using fixed point theory. Let us take into consideration the right hand sides of the given system as follows:
\[ \Psi(t, \tilde{P}(t, q), \tilde{Q}(t, q)) = a\tilde{P}(t, q) - \tilde{P}(t, q)\tilde{Q}(t, q), \]
\[ \Xi(t, \tilde{P}(t, q), \tilde{Q}(t, q)) = -\tilde{Q}(t, q) + d\tilde{P}(t, q)\tilde{Q}(t, q). \]  
(24)

Thus, the given system (3) can be written as follows for \( 0 < \Theta \leq 1 \) and \( 0 \leq \Theta \leq 1 \),
\[ \begin{cases} D^\Theta_0[\tilde{P}(t, q)] = \Psi(t, \tilde{P}(t, q), \tilde{Q}(t, q)), \\ D^\Theta_0[\tilde{Q}(t, q)] = \Xi(t, \tilde{P}(t, q), \tilde{Q}(t, q)), \end{cases} \]  
(25)

where
\[ \tilde{P}(0, q) = \tilde{x}_0, \]
\[ \tilde{Q}(0, q) = \tilde{y}_0. \]  
(26)

On using the corresponding integral \( I' \), we have
\[ \begin{cases} \tilde{P}(t, q) = \tilde{P}(0, q) + \frac{1}{\Gamma(\Theta)} \int_0^t (t - \Theta)^{\Theta-1}\Psi(\Theta, \tilde{P}(\Theta, q), \tilde{Q}(\Theta, q))d\Theta, \\ \tilde{Q}(t, q) = \tilde{Q}(0, q) + \frac{1}{\Gamma(\Theta)} \int_0^t (t - \Theta)^{\Theta-1}\Xi(\Theta, \tilde{P}(\Theta, q), \tilde{Q}(\Theta, q))d\Theta. \end{cases} \]  
(27)

We denote Banach space as \( \mathbb{B} = \mathbb{B}_1 \times \mathbb{B}_2 \) under the fuzzy norm as follows:
\[ \|\tilde{P}(t, q), \tilde{Q}(t, q)\| = \max_{t \in [0, T]} \|\tilde{P}(t, q) + \tilde{Q}(t, q)\|. \]  
(28)

Then, we reach
\[ \tilde{N}(t, q) = \tilde{N}_0(t, q) + \frac{1}{\Gamma(\Theta)} \int_0^t (t - \Theta)^{\Theta-1}\Theta(\Theta, \tilde{P}(\Theta, q), \tilde{Q}(\Theta, q))d\Theta, \]  
(29)

where
\[ \tilde{S}(t, q) = \left\{ \begin{array}{l} \tilde{P}(t, q) = \tilde{P}(0, q), \\ \tilde{Q}(t, q) = \tilde{Q}(0, q) \end{array} \right\} \]  
\[ \Theta(\Theta, \tilde{S}(\Theta, q)) = \left\{ \begin{array}{l} \Psi(s, \tilde{P}(s, q), \tilde{Q}(s, q)) \\ \Xi(s, \tilde{P}(s, q), \tilde{Q}(s, q)) \end{array} \right\}. \]

**Theorem 5.** Let assumption C-2 holds, then model (25) possesses at least one solution.

**Proof.** Let us consider a closed and convex fuzzy set \( \mathcal{A} = \{ \tilde{S}(t, q) \in \mathcal{F} : \|\tilde{S}(t, q)\| \leq r \} \subset \mathcal{F} \). Let us define a mapping \( \psi: \mathcal{A} \rightarrow \mathcal{A} \) such that
\[ \psi(\tilde{S}(t, q)) = \tilde{S}_0(t, q) + \frac{1}{\Gamma(\Theta)} \int_0^t (t - \Theta)^{\Theta-1}\Theta(\Theta, \tilde{S}(\Theta, q))d\Theta. \]  
(32)

For any \( \tilde{S}(t, q) \in \mathcal{A} \), we have
\[ \|\psi(\tilde{S}(t, q))\| = \max_{t \in [0, T]} \|\tilde{S}(t, q) + \tilde{S}(t, q)\| \leq \|\tilde{S}_0(t, q)\| + \frac{1}{\Gamma(\Theta)} \int_0^t (t - \Theta)^{\Theta-1}\Theta(\Theta, \tilde{S}(\Theta, q))d\Theta \]
\[ \leq \|\tilde{S}_0(t, q)\| + \frac{1}{\Gamma(\Theta)} \int_0^t (t - \Theta)^{\Theta-1}\Theta(\Theta, \tilde{S}(\Theta, q))d\Theta \]
\[ \leq \|\tilde{S}_0(t, q)\| + \frac{1}{\Gamma(\Theta)} \int_0^t (t - \Theta)^{\Theta-1}[M_s\|\tilde{S}(t, q)\| + N_s]d\Theta \]
\[ \leq \|\tilde{S}_0(t, q)\| + \frac{1}{\Gamma(\Theta) + 1} [M_s\|\tilde{S}(t, q)\| + N_s]. \]  
(33)

The last relation shows that \( \|\psi(\tilde{S}(t, q))\| \leq r \). From the last inequality, we have \( \psi(\mathcal{A}) \subset \mathcal{A} \). It follows that \( \psi \) is bounded operator. Next, the next step is to show that the operator \( \psi \) is completely continuous. For this let \( \phi_1, \phi_2 \in [0, T] \) such that \( \phi_1 < \phi_2 \), then
\[
\| \psi(\bar{S}(t, \varrho))(\phi_2) - \psi(\bar{S}(t, \varrho))(\phi_1) \| = \frac{1}{\Gamma(\omega)} \int_0^{\phi_2} (\phi_2 - \mathcal{F})^\omega \Theta(3, \bar{S}(3, \varrho))d\mathcal{F} \\
\frac{1}{\Gamma(\omega)} \int_0^{\phi_1} (\phi_1 - \mathcal{F})^\omega \Theta(3, \bar{S}(3, \varrho))d\mathcal{F} \leq \left[ \frac{\varrho^2}{2} - \phi_1^2 \right] \frac{M_3|\bar{S}(t, \varrho)| + N_3}{\Gamma(\omega + 1)}.
\]

The last inequality implies that
\[ \| \psi(\bar{S}(t, \varrho))(\phi_2) - \psi(\bar{S}(t, \varrho))(\phi_1) \| \rightarrow 0 \text{ as } \phi_2 \rightarrow \phi_1. \] (35)

**Theorem 6.** Suppose that C-1 holds, then the considered model (25) possesses the unique solution if \( \varrho \omega K_3 < \Gamma(\omega + 1) \).

**Proof.** Let \( S_i(t, \varrho), S_j(t, \varrho) \in \mathbb{B} \), then
\[
\| \psi(S_1(t, \varrho)) - \psi(S_2(t, \varrho)) \| = \max_{t \in [0, T]} \frac{1}{\Gamma(\omega)} \int_0^t (t - \mathcal{F})^\omega \Theta(3, S_i(3, \varrho))d\mathcal{F} \\
\frac{1}{\Gamma(\omega)} \int_0^t (t - \mathcal{F})^\omega \Theta(3, S_j(3, \varrho))d\mathcal{F} \leq \frac{\varrho^2}{\Gamma(\omega + 1)} K_3|S_i(t, \varrho) - S_j(t, \varrho)|.
\]

This implies that \( \| \psi(S_1(t, \varrho)) - \psi(S_2(t, \varrho)) \| \leq (\varrho \omega / \Gamma(\omega + 1)) K_3|S_1(t, \varrho) - S_2(t, \varrho)| \). Hence, \( \psi \) fulfills the contraction condition. Thus, by “Banach fixed point theorem” system (25) possesses the unique solution. \( \square \)

### 5. Ulam–Hyers Stability

We take a small alteration \( \phi \in C[0, T] \) such that \( \phi(0) = 0 \) depends only on the solution \( \psi(S(t, \varrho)) \) as follows:

(i) \( |\phi(t)| \leq \varphi \), for \( \varphi > 0 \)

(ii) \( D^\omega \psi(S(t, \varrho)) = \psi(t, \psi(S(t, \varrho))) + \phi(t) \)

**Lemma 1.** The solution of the perturbed problem
\[ D^\omega \psi(S(t, \varrho)) = \psi(t, \psi(S(t, \varrho))) + \phi(t), \]
\[ \psi(S(0, \varrho)) = \psi(S(0, \varrho)), \] (37)

\[
|\bar{S}(t, \varrho) - \bar{S}(t, \varrho)| = |\bar{S}(t, \varrho) - \left( \bar{S}(t, \varrho) + \frac{1}{\Gamma(\omega)} \int_0^t (t - \mathcal{F})^\omega \psi(3, \bar{S}(3, \varrho))d\mathcal{F} \right)| \\
\leq |\bar{S}(t, \varrho) - \left( \bar{S}(t, \varrho) + \frac{1}{\Gamma(\omega)} \int_0^t (t - \mathcal{F})^\omega \psi(3, \bar{S}(3, \varrho))d\mathcal{F} \right)| \\
+ \frac{1}{\Gamma(\omega)} \int_0^t (t - \mathcal{F})^\omega \psi(3, \bar{S}(3, \varrho))d\mathcal{F} - \frac{1}{\Gamma(\omega)} \int_0^t (t - \mathcal{F})^\omega \psi(3, \bar{S}(3, \varrho))d\mathcal{F} \\
\leq \Omega_{\mathcal{T}, \omega} \varphi + \Delta |\bar{S}(t, \varrho) - \bar{S}(t, \varrho)|. \] (39)

From (39), we get
\[
\| \bar{S}(t, \varrho) - \bar{S}(t, \varrho) \| \leq \frac{\Omega_{\mathcal{T}, \omega}}{1 - \Delta} \varphi. \] (40)

Hence, we infer from (39) that the solution of (38) is UH-stable. Consequently, the model (3) is UH-stable. \( \square \)
6. Derivation of General Procedure for Solution

Here, we derive an algorithm for the solution of the proposed model by fuzzy LADM. Consider system (3) with initial conditions as follows:

\[
\begin{aligned}
D_p^\infty [\bar{P}(t,q)] &= \Psi (t, \bar{P}(t,q), \bar{Q}(t,q)), \\
D_q^\infty [\bar{Q}(t,q)] &= \Xi (t, \bar{P}(t,q), \bar{Q}(t,q))
\end{aligned}
\]

(41)

where

\[
\begin{aligned}
\bar{P}(0,q) &= \bar{x}_0 = (P(0,q), \bar{P}(0,q)), \\
\bar{Q}(0,q) &= \bar{y}_0 = (Q(0,q), \bar{Q}(0,q))
\end{aligned}
\]

(42)

Then, we reach

\[
\begin{aligned}
L[D_p^\infty [\bar{P}(t,q)]] &= L[\Psi (t, \bar{P}(t,q), \bar{Q}(t,q))], \\
L[D_q^\infty [\bar{Q}(t,q)]] &= L[\Xi (t, \bar{P}(t,q), \bar{Q}(t,q))]
\end{aligned}
\]

\[
\begin{aligned}
\bar{P}(t,q) &= \int P(0,q) + \int \Psi (t, \bar{P}(t,q), \bar{Q}(t,q)), \\
\bar{Q}(t,q) &= \int Q(0,q) + \int \Xi (t, \bar{P}(t,q), \bar{Q}(t,q))
\end{aligned}
\]

(43)

The infinite series solution is presented as follows:

\[
\begin{aligned}
\bar{P}(t,q) &= \sum_{k=0}^{\infty} \bar{P}_k(t,q), \\
\bar{Q}(t,q) &= \sum_{k=0}^{\infty} \bar{Q}_k(t,q)
\end{aligned}
\]

(44)

Thus, we reach

\[
\begin{aligned}
L[\bar{P}(t,q)] &= \frac{1}{s} P(0,q) + \frac{1}{s^2} L\left[ \Psi \left( t, \sum_{k=0}^{\infty} \bar{P}_k(t,q), \sum_{k=0}^{\infty} \bar{Q}_k(t,q) \right) \right], \\
L[\bar{Q}(t,q)] &= \frac{1}{s} Q(0,q) + \frac{1}{s^2} L\left[ \Xi \left( t, \sum_{k=0}^{\infty} \bar{P}_k(t,q), \sum_{k=0}^{\infty} \bar{Q}_k(t,q) \right) \right]
\end{aligned}
\]

(45)

Comparison of terms on both sides gives

\[
\begin{aligned}
L[P_0(t,q)] &= \frac{1}{s} P(0,q), \\
L[\bar{P}_0(t,q)] &= \frac{1}{s} \bar{P}(0,q), \\
L[Q_0(t,q)] &= \frac{1}{s} Q(0,q), \\
L[\bar{Q}_0(t,q)] &= \frac{1}{s} \bar{Q}(0,q)
\end{aligned}
\]

(46)

\[
\begin{aligned}
L[P_1(t,q)] &= \frac{1}{s^2} L[\Psi (t, P_0(t,q), Q_0(t,q))], \\
L[\bar{P}_1(t,q)] &= \frac{1}{s^2} L[\Psi (t, \bar{P}_0(t,q), \bar{Q}_0(t,q))], \\
L[Q_1(t,q)] &= \frac{1}{s} \sum_{k=0}^{\infty} \bar{P}_k(t,q), \sum_{k=0}^{\infty} \bar{Q}_k(t,q), \\
L[\bar{Q}_1(t,q)] &= \frac{1}{s} \sum_{k=0}^{\infty} \bar{P}_k(t,q), \sum_{k=0}^{\infty} \bar{Q}_k(t,q)
\end{aligned}
\]

\[
\begin{aligned}
L[P_{m+1}(t,q)] &= \frac{1}{s^2} L[\Psi (t, P_m(t,q), Q_m(t,q))], \\
L[\bar{P}_{m+1}(t,q)] &= \frac{1}{s^2} L[\Psi (t, \bar{P}_m(t,q), \bar{Q}_m(t,q))], \\
L[Q_{m+1}(t,q)] &= \frac{1}{s} \sum_{k=0}^{\infty} \bar{P}_k(t,q), \sum_{k=0}^{\infty} \bar{Q}_k(t,q), \\
L[\bar{Q}_{m+1}(t,q)] &= \frac{1}{s} \sum_{k=0}^{\infty} \bar{P}_k(t,q), \sum_{k=0}^{\infty} \bar{Q}_k(t,q)
\end{aligned}
\]
Taking inverse Laplace transform, we get
\[
\begin{align*}
P_0(t,q) &= P(0, q), \\
P_0(t,q) &= \mathcal{P}(0, q), \\
Q_0(t,q) &= Q(0, q), \\
\overline{Q}_0(t,q) &= \overline{Q}(0, q),
\end{align*}
\]

\[
P_1(t,q) = \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Psi(t, P_0(t,q), Q_0(t,q))] \right],
\]

\[
\begin{align*}
P_1(t,q) &= \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Psi(t, P_0(t,q), \overline{Q}_0(t,q))] \right], \\
Q_1(t,q) &= \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Xi(t, P_0(t,q), Q_0(t,q))] \right], \\
\overline{Q}_1(t,q) &= \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Xi(t, P_0(t,q), \overline{Q}_0(t,q))] \right],
\end{align*}
\]

\[
\vdots
\]

\[
P_{m+1}(t,q) = \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Psi(t, P_m(t,q), Q_m(t,q))] \right], \\
\overline{P}_{m+1}(t,q) = \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Psi(t, \overline{P}_m(t,q), \overline{Q}_m(t,q))] \right],
\]

\[
\begin{align*}
Q_{m+1}(t,q) &= \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Xi(t, P_m(t,q), Q_m(t,q))] \right], \\
\overline{Q}_{m+1}(t,q) &= \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Xi(t, \overline{P}_m(t,q), \overline{Q}_m(t,q))] \right].
\end{align*}
\]

Therefore, we get
\[
\begin{align*}
P(t,q) &= P_0(t,q) + x_1(t,q) + \cdots, \\
\overline{P}(t,q) &= \overline{P}_0(t,q) + \overline{P}_1(t,q) + \cdots, \\
Q(t,q) &= Q_0(t,q) + Q_1(t,q) + \cdots, \\
\overline{Q}(t,q) &= \overline{Q}_0(t,q) + \overline{Q}_1(t,q) + \cdots.
\end{align*}
\]

7. Results and Simulations

In this part, we illustrate the proposed method through simulations; consider the system above under some specific values of parameters as follows:
\[
\begin{align*}
\mathcal{D}_t^\alpha \tilde{P}(t,q) &= a\tilde{P}(t,q) - b\tilde{P}(t,q)\tilde{Q}(t,q), \\
\mathcal{D}_t^\alpha \tilde{Q}(t,q) &= \Xi(t, \tilde{P}(t,q), \tilde{Q}(t,q)),
\end{align*}
\]

where
\[
\begin{align*}
\tilde{P}(0,q) &= (q - 1, 1 - q), \\
\tilde{Q}(0,q) &= (q - 1, 1 - q).
\end{align*}
\]

After applying the proposed method, we get
\[
\begin{align*}
P_0(t,q) &= P(0,q) = q - 1, \\
\overline{P}_0(t,q) &= \overline{P}(0,q) = 1 - q, \\
Q_0(t,q) &= Q(0,q) = q - 1, \\
\overline{Q}_0(t,q) &= \overline{Q}(0,q) = 1 - q.
\end{align*}
\]

Similarly, the next terms are computed as
\[
\begin{align*}
P_1(t,q) &= \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[\Psi(t, P_0(t,q), Q_0(t,q))] \right] = \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[aP_0(t,q) - bP_0(t,q)Q_0(t,q)] \right] \\
&= \mathcal{L}^{-1}\left[ \frac{1}{s^0} \mathcal{L}[a(q - 1) - b(q - 1)^2] \right] = \frac{a(q - 1) - b(q - 1)^2}{\Gamma(\omega + 1)}.
\end{align*}
\]

We can obtain other terms of the series, i.e.,
Figure 1: Graphical presentation of approximate fuzzy solution for specie $\tilde{P}(t)$ up to three terms at various uncertainty and fractional orders for problem (49). Here, the lower and upper cut of $x$ represents the lower and upper solution of $\tilde{P}(t)$.

Figure 2: Graphical presentation of approximate fuzzy solution for specie $\tilde{Q}(t)$ up to three terms at various uncertainty and fractional orders for problem (49). Here, the lower and upper cut of $y$ represents the lower and upper solution of $\tilde{Q}(t)$. 
Figure 3: 3D presentation of approximate fuzzy solution for specie $P(t)$ up to three terms at various uncertainty and fractional orders for problem (49). Here, the lower and upper cut of $x$ represents the lower and upper solution of $P(t)$.

Figure 4: 3D presentation of approximate fuzzy solution for specie $Q(t)$ up to three terms at various uncertainty and fractional orders for problem (49). Here, the lower and upper cut of $y$ represents the lower and upper solution of $Q(t)$.

$$\bar{P}_1(t, \varrho) = \left[ a(1 - \varrho) - b(1 - \varrho)^2 \right] \frac{t^\varphi}{\Gamma(\varphi + 1)}.$$  

$$\bar{Q}_1(t, \varrho) = \left[ d(\varrho - 1)^2 - c(\varrho - 1) \right] \frac{t^\varphi}{\Gamma(\varphi + 1)}.$$  

$$\bar{Q}_1(t, \varrho) = \left[ d(\varrho - 1)^2 - c(\varrho - 1) \right] \frac{t^\varphi}{\Gamma(\varphi + 1)}.$$  

$$\bar{Q}_2(t, \varrho) = -cv_3 \frac{t^\varphi}{\Gamma(2\varphi + 1)} + dv_3 \frac{t^\varphi}{\Gamma(\varphi + 1)} \frac{t^\varphi}{\Gamma(3\varphi + 1)}.$$  

and so on. The unknown terms in the above equations are given as

$$v_1 = a(\varrho - 1) - b(\varrho - 1)^2,$$

$$v_2 = a(1 - \varrho) - b(1 - \varrho)^2,$$

$$v_3 = d(\varrho - 1)^2 - c(\varrho - 1),$$

$$v_4 = d(1 - \varrho)^2 - c(1 - \varrho).$$

Now, we simulate our obtained results to investigate the dynamics of the proposed model for the uncertain initial conditions. In Figures 1 and 2, we plot the fuzzy approximate results up to three terms for the given examples under two different uncertainty values as follows:

We displayed the approximate fuzzy solutions for the considered model for particular fuzzy initial conditions and at the given uncertainty against different fractional orders in Figures 1 and 2. The fuzzy fractional derivative produced global dynamics of the interaction of the two species under fuzzy concept that under taking some values for uncertainty $\varphi = 0.5, 0.9$. Increasing the uncertainty, the dynamics of interaction is also affected as shown in Figures 1 and 2. Such situation is natural and may be observed in daily life. Next, in
In this research, we have used a fuzzy fractional derivative in Caputo sense to extend the fractional predator-prey model. We have discussed the equilibrium points and their stability. Using metric fixed point theory, we were able to derive the existence and uniqueness results for the nonlinear fuzzy fractional predator-prey model. We have shown that the proposed model is HU stable through nonlinear functional analysis. Besides, we have developed a general algorithm for obtaining an approximate solution to the proposed model using an efficient method (fuzzy LADM). If we replace $[q − 1, 1 − q] = 1$ in the obtained numerical results, then we recover the results obtained in the fractional-order model. Thus, our proposed model is the generalization of the fractional model of the predator-prey system. Finally, we simulated numerical results for different fractional order and 0.5 and 0.9 uncertainty values via MATLAB. From figures, we have found that fractional calculus can be combined with the fuzzy theory and thus glorify the global dynamics of species interaction.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding this research work.

References


